PHYSICS OF SYNCHROTRON RADIATION

CAS 2011; Erice, Italy; April 06 - 15, 2011; A. Hofmann

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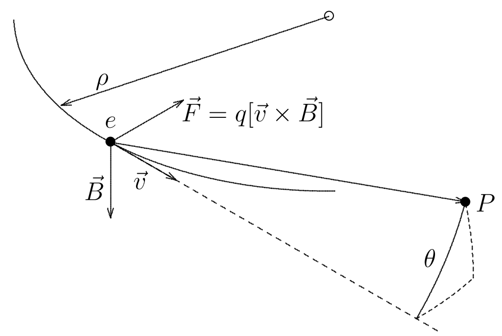
1) INTRODUCTION — QUALITATIVE TREATMENT
What is Synchrotron Radiation

Synchrotron radiation, abbreviated SR, is emitted mainly in the forward direction by a charge e moving with relativistic velocity $\vec{v} = \vec{\beta}c$ on a curved trajectory. This charge undergoes a transverse acceleration which is usually caused by the Lorentz force due to a magnetic field. This force is perpendicular to the velocity and magnetic field vectors $\vec{F} = e[\vec{v} \times \vec{B}]$ and determines the curvature $1/\rho$ of the trajectory. The emitted radiation reaches an observer P at an angle θ with respect to the velocity $\vec{v}$ direction.



Synchrotron radiation creatiton:

- Emitted by a charged relativistic particle
- Transverse acceleration, perpendicular to velocity, usually provided by magnetic field

Resulting in properties:

- Radiation concentrated in forward direction, small opening angle
- High frequency spectrum
- Polarization

Relativity

The charged particle emitting SR moves with relativistic velocity close to a chosen axis, usually z-direction. for the understanding one goes in an **inertial** frame moving with **constant** velocity $\vec{v}^* = \vec{\beta}^* c$ being close to a typical velocity of the particle motion. Lorentz transformation relates coordinates of laboratory $[x,\, y,\, z,\, t]$ and of moving system $[x^*,\, y^*,\, z^*,\, t^*]$

$$x^* = x,\ y^* = y,$$
$$z^* = \gamma^*(z - \beta^* ct),\ ct^* = \gamma^*(ct - \beta^* z)$$

with Lorentz factor γ^* of moving system

$$\gamma^* = \frac{1}{\sqrt{1-\beta^{*2}}}$$

Of particular interest are photon trajectories in the two systems. We take a photon moving in x, z-plane at angle θ

$$x = ct\sin\theta,\ z = ct\cos\theta \text{ in laboratory, and}$$
$$x^* = ct^*\sin\theta^* = \gamma^* ct(1-\beta^*\cos\theta)\sin\theta^* = ct\sin\theta$$
$$z^* = ct^*\cos\theta^* = \gamma^* ct(1-\beta^*\cos\theta)\cos\theta^*$$
$$= \gamma^* ct(\cos\theta - \beta^*) \text{ in moving system}$$

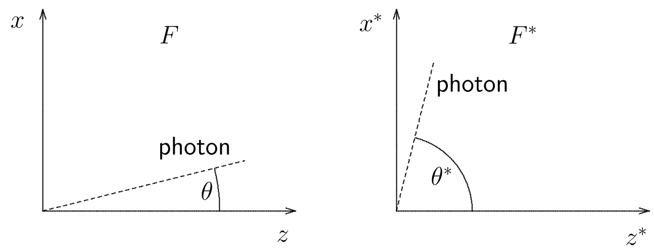
$$\sin\theta^* = \frac{\sin\theta}{\gamma^*(1-\beta^*\cos\theta)},\ \cos\theta^* = \frac{\cos\theta - \beta^*}{1-\beta^*\cos\theta}$$

Photon emitted in moving system at $\theta^* = \pi/2$ appears in laboratory at small angle $\theta_\perp \approx 1/\gamma$

$$\cos\theta_\perp = \beta^*,\ \sin\theta_\perp = \sqrt{1-\beta^{*2}} = 1/\gamma^* \approx \theta_\perp$$

Ultra-relativistic approximation $\beta \approx 1,\ \gamma \gg 1$:

$$\sin\theta \approx \theta,\ 1-\beta = (1-\beta^2)/(1+\beta)) \approx 1/(2\gamma^2)$$



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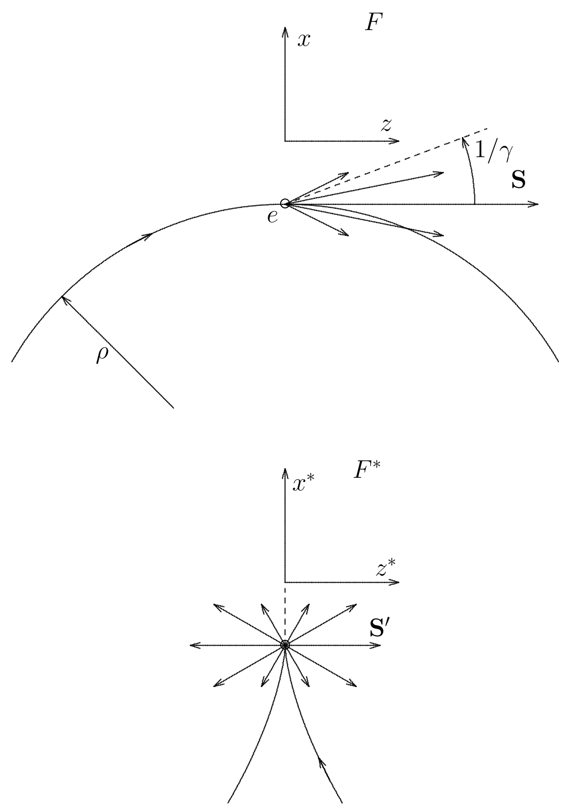
Qualitative Treatment

Opening angle

An electron moves in the laboratory frame F on a circular orbit of radius ρ and radiates. To estimate the opening angle of the radiation we go into a frame F^* which moves for an instant with the electron. The trajectory in F^* is a cycloid with a cusp where the electron is accelerated in the $-x^*$ direction. It emits radiation which is about uniformly distributed in F^*. Going back to the laboratory frame F by a Lorentz transformation, this radiation becomes peaked forward - a photon emitted in F^* along the x^*-axis appears in the laboratory F at an angle $1/\gamma$. The typical opening angle of synchrotron radiation is therefore $\approx 1/\gamma$ being very small for particles with $\gamma \gg 1$.



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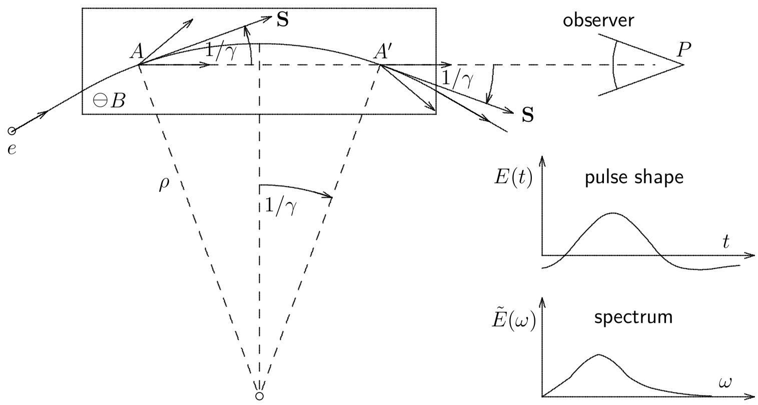
Spectrum

How long is the pulse of electromagnetic field seen by an observer? With the natural opening angle $\approx 1/\gamma$ he sees the pulse first when the e is at A and last when it is at A', i.e. for an arc of angle $\approx 2/\gamma$. The pulse length is the difference in the travel time from A to A' for the electron and photon



$$\Delta t = t_e - t_\gamma = \frac{2\rho}{\gamma\beta c} - \frac{2\rho\sin(1/\gamma)}{c}.$$

For $\gamma \gg 1$, $\theta \ll 1$ and $\sin\theta \approx \theta - \theta^3/6$

$$\Delta t \approx \frac{2\rho}{\gamma\beta c}\left(1-\beta + \frac{\beta}{6\gamma^2}\right) \approx \frac{\rho}{\gamma c}\left(\frac{1}{\gamma^2} + \frac{1}{3\gamma^2}\right) = \frac{4}{3}\frac{\rho}{c\gamma^3},$$

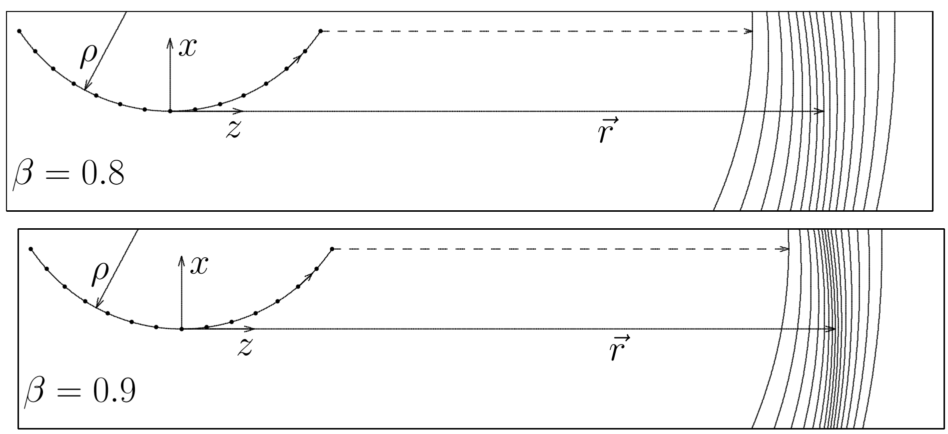
where we used $1-\beta \approx 1/(2\gamma^2)$.

Typical frequency of spactrum $\omega_{typ} \sim \dfrac{1}{\Delta t} = \dfrac{3c\gamma^3}{4\rho}$.

A factor γ^2 is caused by the particle-photon velocity difference, a factor γ due to curvature.
Calculate later critical frequency dividing spectrum in equal parts

$$\omega_c = \frac{3c\gamma^3}{2\rho}.$$

Wavefront



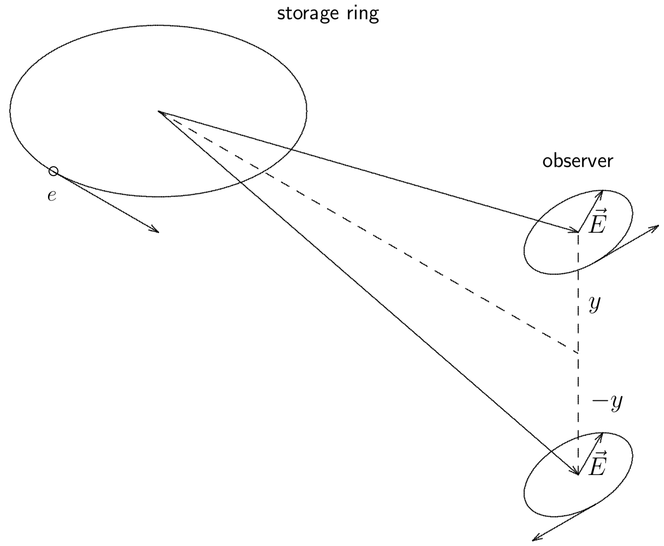
Forward propagation of synchrotron radiation

The above figures show a particle moving on a circular orbit and emitting synchrotron radiation at discrete points marked by a dot. The resulting wave fronts of each such contribution at a fixed time are circles having source points as centers.

In the tangential z-direction the wave fronts emitted close to the origin, $z \approx 0$ have a tight spacing, high frequency. This is determined by the small velocity difference between particle and radiation, which is more pronounced at higher values of γ.

Polarization

In the storage ring lying in a horizontal plane the circulating electrons undergo a horizontal acceleration. We expect therefore that emitted radiation is polarized with the electric field $\vec{E}$ being predominately horizontal. This is exact if the radiation is observed on the plane of the orbit. However, observed from above or below this plane the radiation has also an elliptic polarization with different helicities. This polarization can also be described by a horizontal and a vertical component. In the median plane we have only horizontal polarization. Outside this plane there is horizontal and vertical polarization.



Short, weak magnet

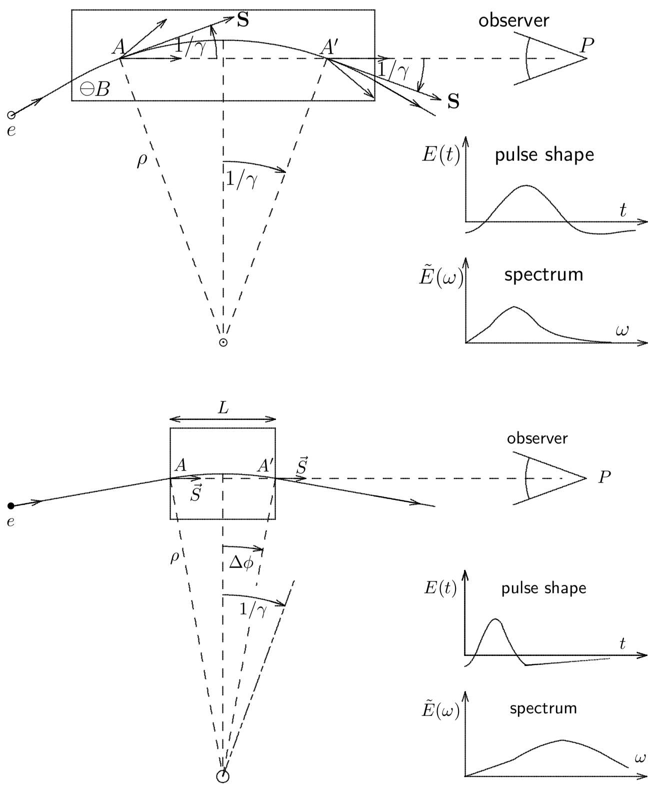
The Spectrum was estimated from the pulse length of observed radiation emitted from a trajectory arc length $\ell > 2\rho/\gamma$. This can be modified by a short and weak magnet of length L deflecting particle by a small angle $\Delta\phi \ll 1/\gamma$, resulting in a pulse length Δt_{sm} seen by the observer

$$\Delta t_{sm} = \frac{L}{\beta c} - \frac{L}{c} = \frac{L}{\beta c}(1-\beta) \approx \frac{L}{2c\gamma^2}$$

neglecting the curvature effect on the particle trajectory length. The corresponding typical frequency is

$$\omega_{sm} \sim \frac{2c\gamma^2}{L}$$

Pulse from a short magnet is truncated which gives a higher frequency spectrum.



Undulator

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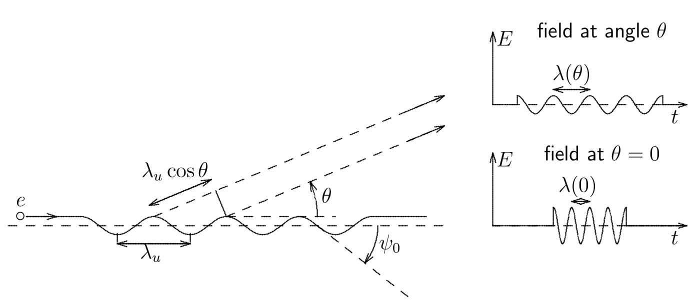
An undulator is a spatially periodic magnet with field $B_y(z)$, period length λ_u, wave number $k_u = 2\pi/\lambda_u$ resulting for weak field, deflecting angle $\psi_0 \ll 1/\gamma \ll 1$, in a harmonic trajectory $x(z)$

$$B(y) = B_0 \cos(k_u z), \quad x(z) = a \cos(k_u z).$$



Each period radiates like short magnet, and is observed at angle θ. Arrival time difference per period

$$\Delta T = \frac{\lambda_u}{\beta c} - \frac{\lambda_u \cos\theta}{c} = \frac{\lambda_u(1-\beta\cos\theta)}{\beta c}$$

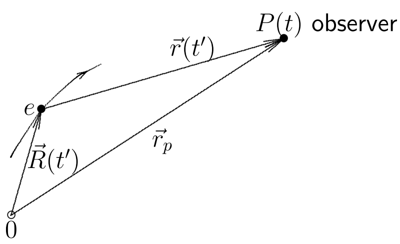
$$\gamma \gg 1 \to \theta \ll 1,\ 1-\beta \approx \frac{1}{2\gamma^2},\ \cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\Delta T = \frac{\lambda_u}{2c\gamma^2}(1+\gamma^2\theta^2),\ \omega_1 = \frac{1}{\Delta T} = \frac{2\pi c}{\lambda_u}\frac{2\gamma^2}{1+\gamma^2\theta^2}$$

The radiation contributions from the N_u periods give interference which is additive at the frequency ω_1 and leads to monochromatic radiation at each angle θ of frequency and width

$$\omega_1 = k_u c \frac{2\gamma^2}{1+\gamma^2\theta^2},\ \frac{\Delta\omega_1}{\omega_1} \approx \frac{1}{N_u}$$

2) POTENTIALS AND FIELDS OF A MOVING CHARGE
Motion



Charge e moving on trajectory $\vec{R}(t')$ emits radiation at time t' being received at time t by observer P at distance r. Fields travel with c giving relation between emission and observation times

$$t = t' + \frac{r(t')}{c}, \quad \text{or } t_p = t' + \frac{r(t') - r_p}{c}.$$

Geometry $\to$ vector relation and derivative

$$\vec{R}(t') + \vec{r}(t') = \vec{r}_p,\ \frac{d\vec{R}}{dt'} = \vec{v} = c\vec{\beta} = -\frac{d\vec{r}}{dt'}$$

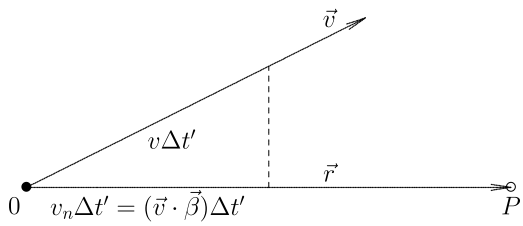
with $\vec{\beta} = \vec{v}/c$. Get derivative of scalar r

$$\vec{r}\frac{d\vec{r}}{dt'} = \frac{1}{2}\frac{d\vec{r}^2}{dt'} = r\frac{dr}{dt'} = -\vec{r}\cdot\vec{v}$$

$$\frac{dr}{dt'} = -\vec{n}\cdot\vec{v} = -c(\vec{n}\cdot\vec{\beta}),\ \frac{d\vec{r}}{dt'} = -c\vec{\beta} = \vec{n}\frac{dr}{dt'}$$

with $\vec{n} = \vec{r}/r$. Differentiate t with respect to t'

$$\frac{dt}{dt'} = 1 + \frac{1}{c}\frac{dr}{dt'} = 1 - \vec{n}\cdot\vec{\beta},\ \frac{dt}{dt'} = (1 - \vec{n}\cdot\vec{\beta}).$$



Distance r source - observer changes with velocity component $v_n = -(\vec{n}\cdot\vec{v})$. Emitted, $\Delta t'$, and observed, Δt, time intervals are related by

$$\Delta t = \Delta t' + \frac{\Delta r}{c} = \Delta t'(1 - \vec{n}\cdot\vec{\beta}).$$

Electromagnetic potentials

Fields $\vec{E}(t)$, $\vec{B}(t)$ created at time t' by a moving charge are best obtained from the electromagnetic scalar and vector potentials V and $\vec{A}$

$$\begin{aligned}\vec{B} &= \mu_0 \,\text{curl}\vec{A} = \mu_0 \left[\nabla \times \vec{A}\right] \\ \vec{E} &= -\,\text{grad}V - \mu_0 \frac{\partial \vec{A}}{\partial t} = -\nabla V - \mu_0 \frac{\partial \vec{A}}{\partial t}\end{aligned}$$

with Lorentz gauge $\nabla \cdot \vec{A} + \epsilon_0 \dot{V} = 0$. Stationary case, charge position or density $\eta(\vec{R})$ are constant and the potentials are for a single charge or a distribution.

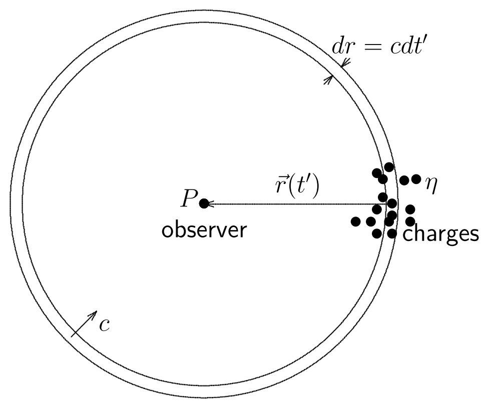
$$V = \frac{e}{4\pi\epsilon_0 r} \;,\quad \vec{A} = \frac{e\vec{v}}{4\pi r}.$$

$$V = \int \frac{\eta(\vec{R})}{4\pi\epsilon_0 r} dx\,dy\,dz \;,\quad \vec{A} = \int \frac{\eta(\vec{R})\vec{v}}{4\pi r} dx\,dy\,dz.$$

For moving charges, the density $\eta(t')$ is evaluated at emission time $t' = t - r(t')/c$ to observe potential at t. The integration

$$V(t) = \frac{1}{4\pi\epsilon_0} \int \frac{\eta\left(\vec{R}(t')\right)}{r(t')} dx\,dy\,dz$$

represents a thin sphere of radius r collapsing with c towards observer P counting all the charges on its way



Potential of a moving charge

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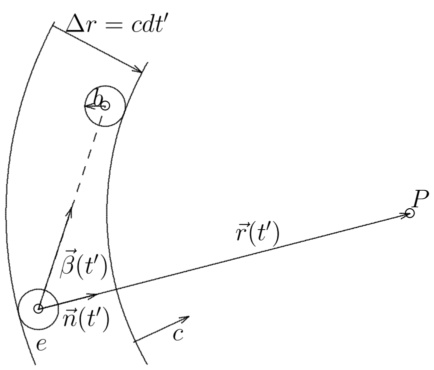


Single charge e of radius b contributes longer to potential if it moves towards observer. Starting with charge at rest:

$$\Delta t'_0 = \frac{2b}{c} \;,\; V = \frac{e}{4\pi\epsilon_0 r}$$

.

If charge moves with $\vec{v}$ (component $v_r = \vec{v}\cdot\vec{n}$ towards observer) the contribution time is

$$\Delta t'_v = \frac{2b}{c - v_r} = \frac{2b}{c - \vec{n}\cdot\vec{v}} = \frac{2b}{c(1 - \vec{n}\cdot\vec{\beta})}$$

The ratio is independent of b

$$\frac{\Delta t'_v}{\Delta t'_0} = \frac{1}{1 - \vec{n}\cdot\vec{\beta}}$$

The electric potential of a moving point charge

$$V(t) = \frac{e}{4\pi\epsilon_0}\left(\frac{1}{r\left(1 - \vec{n}\cdot\vec{\beta}\right)}\right)_{ret.} ,$$

'$ret.$' means evaluation at t' to get V at $t = t' + r(t')/c$. Correspondingly

$$\vec{A}(t) = \frac{e}{4\pi}\left(\frac{\vec{v}}{r\left(1 - \vec{n}\cdot\vec{\beta}\right)}\right)_{ret.} .$$

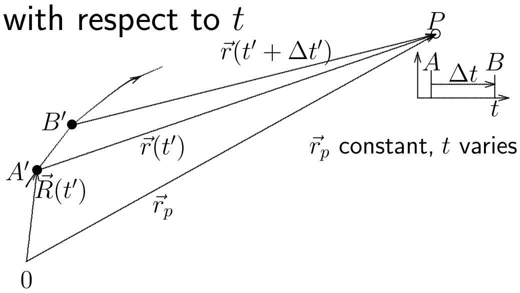
$V(t)$, $\vec{A}(t)$: retarded potentials of moving charge.

Field of a moving charge

$$V(t) = \frac{e}{4\pi\epsilon_0}\left(\frac{1}{r\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)_{ret.} \; , \; \vec{A}(t) = \frac{e}{4\pi}\left(\frac{\vec{v}}{r\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)_{ret.} \; .$$

$$\vec{E} = -\nabla V - \mu_0\frac{\partial \vec{A}}{\partial t} \; , \; \vec{B} = \mu_0 \text{ curl}\vec{A} = \mu_0\left[\nabla\times\vec{A}\right].$$

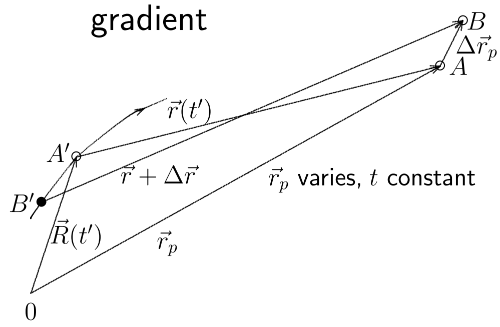
derivative with respect to t



$$\Delta t = \Delta t'(1-\vec{n}\cdot\vec{\beta}) \; , \; \frac{\partial\vec{A}}{\partial t} = \frac{1}{1-\vec{n}\cdot\vec{\beta}}\frac{\partial\vec{A}}{\partial t'}$$

A change Δt of observation time changes emission time t' and location $\vec{R}(t')$

gradient



$$\Delta V = \Delta\vec{r}_p\cdot\nabla V.$$

Change $\Delta\vec{r}_p$ of observation point changes emission location and time $\vec{R}(t'),\ t'$.

These changes make the calculation complicated and we give only the results.

Liénard-Wiechert equation

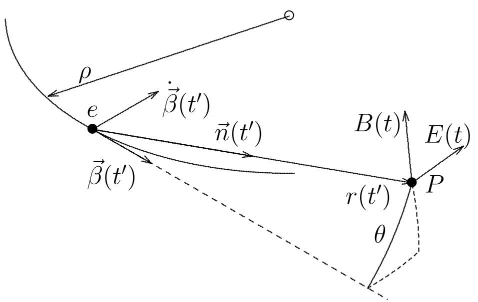
The potentials: $V(t)=\frac{e}{4\pi\epsilon_0}\left(\frac{1}{r\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)_{ret.}$, $\vec{A}(t) = \frac{e}{4\pi}\left(\frac{\vec{v}}{r\left(1-\vec{n}\cdot\vec{\beta}\right)}\right)_{ret.}$

and Maxwell's equations: $\vec{E}=-\nabla V - \mu_0\frac{\partial\vec{A}}{\partial t}$, $\vec{B} = \mu_0 \text{ curl}\vec{A} = \mu_0\left[\nabla\times\vec{A}\right]$.

give 'Liénard-Wiechert' fields of moving charge

$$\vec{E}(t)=\frac{e}{4\pi\epsilon_0}\left(\frac{(1-\beta^2)(\vec{n}-\vec{\beta})}{r^2(1-\vec{n}\cdot\vec{\beta})^3} + \frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.}$$

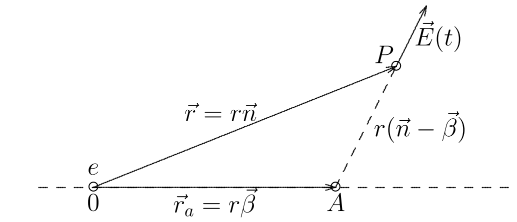
$$\vec{B}(t)=\frac{[\vec{n}\times\vec{E}]}{c}, \quad \vec{B}\perp \text{ on } \vec{E} \text{ and } \vec{n}.$$



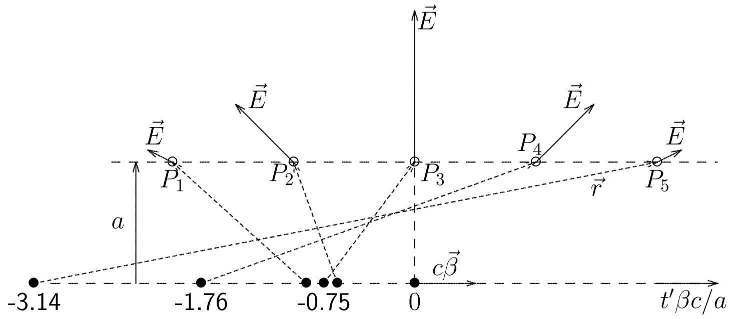
Discussion

- $\vec{E}$ and $\vec{B}$ are perpendicular to each other.
- Two terms,
  'near field' $\propto 1/r^2$,
  'far field' $\propto 1/r$ and $\propto \dot{\beta}$
  dominates at large distances.
- For stationary charge $\vec{\beta}=\dot{\vec{\beta}}=0$ we recover Coulomb's law
  $$\vec{E} = \frac{e\vec{n}}{4\pi\epsilon_0 r^2} = \text{const.}.$$

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0}\left(\frac{(1-\beta^2)(\vec{n}-\vec{\beta})}{r^2(1-\vec{n}\cdot\vec{\beta})^3} + \frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.}$$



- $\dot{\vec{\beta}} = 0$, $\vec{\beta} = (0, 0, \beta)$=constant leaves only first term giving field of uniformly moving charge e. At emission time $t' = 0$, e is at origin 0 and emits fields which reach observer P at time $t = r/c$ and location $r\vec{n}$, while e has moved by $t\beta c = r\beta$ and reached point A at $r\vec{\beta}$. The vector $r(\vec{n}-\vec{\beta})$ from A to P has same direction as $\vec{E}$ which seems to come from A but has been emitted earlier at 0.

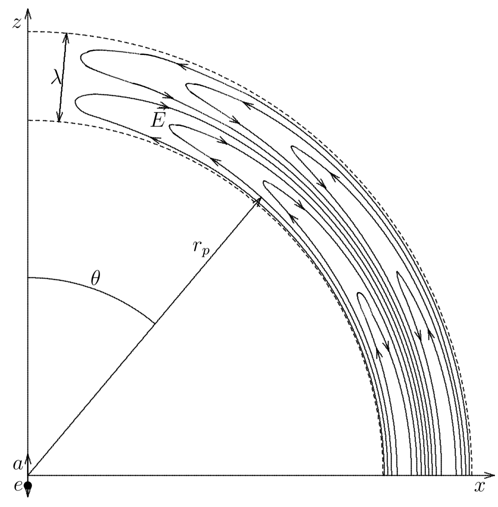


For given emission time and location the field arriving later at P was obtained. More useful and difficult is field calculation for given observation time t and location P.
This requires the search for emission parameters t' and $r(t')$ first, illustrated for $t = 0$ at 5 observers P_i.

The field of uniformly moving charge is best obtained by Lorentz transforming Coulomb field of stationary charge. This also illustrates that the 'near-field' does not radiate energy.

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0}\left(\frac{(1-\beta^2)(\vec{n}-\vec{\beta})}{r^2(1-\vec{n}\cdot\vec{\beta})^3} + \frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.}, \quad \vec{B}(t) = \frac{[\vec{n}\times\vec{E}]}{c}$$

- The 'far-field' has to be included for all radiation processes. One application is the radiation emitted by an oscillation dipole.



It is shown by field lines which follow direction of $\vec{E}$, end only at charges and have a density proportional to field strength.
The 'far-field' decays like $1/r$, slower than 'near-field' and dominates at large distances. Its fields $\vec{E}$ and $\vec{B}$ are perpendicular to $\vec{r}$ giving energy flux, Poynting vector $\vec{S} = [\vec{E}\times\vec{B}]/\mu_0$ in $\vec{r}$-direction, while 'near-field' does not contribute. In synchrotron radiation the emitted power and its distribution in frequency, angle and polarization is of interest. We will exclude the 'near-field'. However, this neglects some physics, like radial electric field components necessary to close field lines, condition div$\vec{E} = 0$ is not satisfied.

Fourier transform of the field

We take the 'far-field' only

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0 c}\left(\frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.}$$

Applications more interested in spectrum than time domain, Fourier transform

$$\tilde{\vec{E}}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(t)e^{-i\omega t}dt.$$

Integration involves t to get spectrum seen by observer but expression for $\vec{E}$ is evaluated at t'. Formal variable transform $t = t' + r(t')/c,\ dt = (1-\vec{n}\cdot\vec{\beta})dt'$

$$\tilde{\vec{E}}(\omega) \propto \int_{-\infty}^{\infty}\frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{r(1-\vec{n}\cdot\vec{\beta})^2}\cdot$$

$$e^{-i\omega(t'+\frac{r(t')}{c})}dt' = \int\frac{dU}{dt'}V\,dt'.$$

For $\gamma \gg 1$, radiation is observed for small angle $\approx 1/\gamma \ll 1$ or trajectory length $\approx \Delta l \sim 2\rho/\gamma$ while $\vec{n}$ and r change little.

$$\int\frac{dU}{dt'}V\,dt' = UV - \int U\frac{dV}{dt'}dt'$$

$$U = \frac{\left[\vec{n}\times[\vec{n}\times\vec{\beta}]\right]}{1-\vec{n}\cdot\vec{\beta}},$$

$$\frac{dV}{dt'} = -i\omega(1-\vec{n}\cdot\vec{\beta})e^{-i\omega(t'+\frac{r(t')}{c})}$$

$$\tilde{\vec{E}}(\omega) = \frac{e}{4\pi\sqrt{2\pi}\epsilon_0 cr}\left[\frac{\left[\vec{n}\times[\vec{n}\times\vec{\beta}]\right]}{1-\vec{n}\cdot\vec{\beta}}e^{-i\omega(t'+\frac{r(t')}{c})}\Bigg|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty}\left[\vec{n}\times[\vec{n}\times\vec{\beta}]\right]e^{-i\omega(t'+\frac{r(t')}{c})}dt'\right]$$

First term is not 'seen' at limits, neglected.

$$\tilde{\vec{E}} = \frac{i\omega e}{4\pi\epsilon_0\sqrt{2\pi}cr}\int_{-\infty}^{\infty}\left[\vec{n}\times[\vec{n}\times\vec{\beta}]\right]e^{-i\omega(t'+\frac{r(t')}{c})}dt'$$

Spectrum is easier to calculate than $\vec{E}(t)$

3) RADIATED POWER AND ENERGY

The angular power distribution

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0}\left(\frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.} \quad \vec{B}(t) = \frac{[\vec{n}\times\vec{E}]}{c}.$$

Poynting vector $\vec{S}$ gives **received** energy dU per area $r^2 d\Omega$ (solid angle Ω), and **observer time** dt

$$\vec{S}(t) = \frac{[\vec{E}(t)\times\vec{B}(t)]}{\mu_0} = \frac{\left[\vec{E}\times[\vec{n}\times\vec{E}]\right]}{\mu_0 c}$$

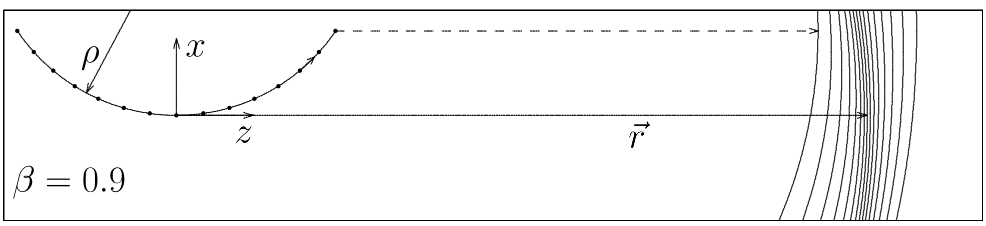
$$= \frac{\vec{n}E^2 - (\vec{n}\cdot\vec{E})\vec{E}}{\mu_0 c} = \frac{\vec{n}E^2}{\mu_0 c} = \frac{\vec{n}dU}{r^2 d\Omega dt}$$

'far-field' $\vec{E}\perp\vec{n}$, $\vec{S} = E^2\vec{n}/(\mu_0 c)$.
and $[\vec{a}\times[\vec{b}\times\vec{c}]] = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.
The **radiated** energy per area and **emission time** dt' is

$$\frac{dU}{r^2 d\Omega dt'} = \frac{dU}{r^2 d\Omega dt}\frac{dt}{dt'} = \vec{n}\cdot\vec{S}(1-\vec{n}\cdot\vec{\beta})$$

$$\frac{dP}{d\Omega} = \frac{d^2U}{d\Omega dt'} = S(1-\vec{n}\cdot\vec{\beta})r^2$$

$$\frac{dP}{d\Omega} = \frac{(1-\vec{n}\cdot\vec{\beta})r^2E^2}{\mu_0 c}$$

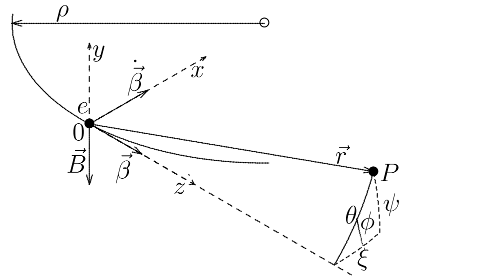


Transverse acceleration

For angular emitted power distribution $dP/d\Omega$ we need electric field $\vec{E}(t)$

$$\frac{dP}{d\Omega} = \frac{(1-\vec{n}\cdot\vec{\beta})r^2E^2}{\mu_0 c},$$

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0}\left(\frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.}$$



$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{\beta} = \beta\,(0,0,1)\,,\ \dot{\vec{\beta}} = (\beta^2 c/\rho)(1.0.0)$$

$$\dot{\vec{\beta}}\ \perp\ \vec{\beta},\ 1-\vec{n}\cdot\vec{\beta} = 1-\beta\cos\theta$$

The triple vector product

$$\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right] = (\vec{n}\cdot\dot{\vec{\beta}})(\vec{n}-\vec{\beta}) - (1-\vec{n}\cdot\vec{\beta})\dot{\vec{\beta}}$$

$$= [1-\beta\cos\theta-\sin^2\theta\cos^2\phi, -\sin^2\theta\cos\phi\sin\phi, -\sin\theta, \cos\phi(\cos\theta-\beta)]$$

Ultra-relativistic approximation $\gamma \gg 1, \theta \ll 1, \beta \approx 1$

$$1-\beta \approx \frac{1}{2\gamma^2},\ 1-\beta\cos\theta \approx \frac{1+\gamma^2\theta^2}{2\gamma^2}$$

$$\vec{E} = -\frac{e\gamma^4}{\pi\epsilon_0 r_p\rho}\frac{[1-\gamma^2\theta^2 cos(2\phi),\ -\gamma^2\theta^2\sin(2\phi),\ 0]}{(1+\gamma^2\theta^2)^3}$$

$$\frac{dP}{d\Omega} = P_0\frac{3\gamma^2}{\pi}\frac{(1-\gamma^2\theta^2 cos(2\phi))^2 + (\gamma^2\theta^2\sin(2\phi))^2}{(1+\gamma^2\theta^2)^5}$$

$$= \frac{dP_\sigma}{d\Omega} + \frac{dP_\pi}{d\Omega}$$ two polarization modes

$$P_0 = \frac{2r_0 c m_0 c^2\gamma^4}{3\rho^2},\ r_0 = \frac{e^2}{4\pi\epsilon_0 m_0 c^2}$$ classical e radius

cas11x-19

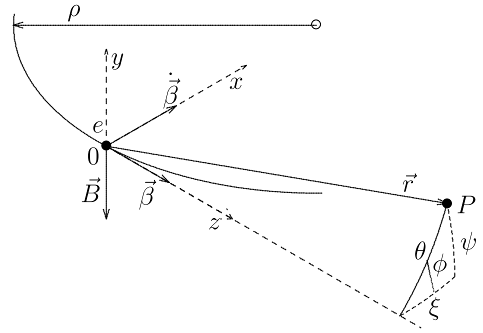
Instantaneous angular distribution

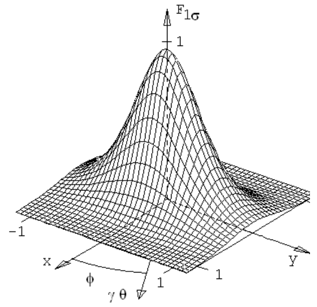
Instantaneous angular distribution is observed in weak undulator radiation, but not in ordinary SR since light beacon sweeps horizontally integrating over angle ξ.

$$\frac{dP}{d\Omega} = P_0\frac{3\gamma^2}{\pi}\frac{(1-\gamma^2\theta^2 cos(2\phi))^2 + (\gamma^2\theta^2\sin(2\phi))^2}{(1+\gamma^2\theta^2)^5}$$

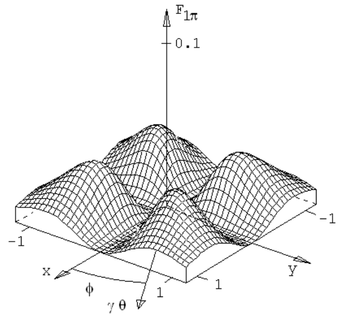
$$P_0 = \frac{2r_0 c m_0 c^2\gamma^4}{3\rho^2},\ P_I = \frac{4\pi r_0 m_0 c^2\gamma^4 I}{3e\rho^2}$$

P_0 power emitted by **one** electron, P_I by current I





horiz. pol.



vert. pol., $\times$ 10

Polarization

$$\frac{P_\sigma}{P_0} = \frac{7}{8},\ \frac{P_\pi}{P_0} = \frac{1}{8}$$

Variance

$$\langle\theta^2\rangle = \frac{1}{\gamma^2}$$

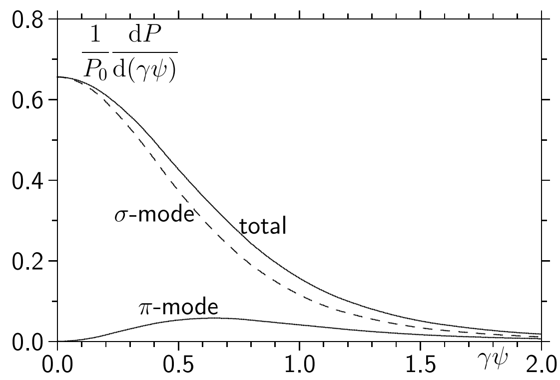
cas11x-20

SR angular distribution

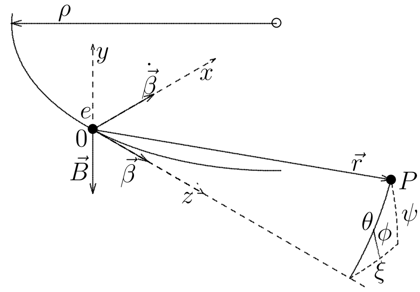
Instantaneous angular distribution not seen in SR, light beacon sweeps and integrates horizontally. Introduce angles $\xi = \theta\cos\phi$ (x-direction) and $\psi = \theta\sin\phi$ (y-direction), substitute in $dP/d\Omega$, and integrate over ξ get vertical distribution of ordinary SR.

$$\frac{dP}{d\Omega} = P_0 \frac{3\gamma^2}{\pi} \frac{(1-\gamma^2\theta^2 cos(2\phi))^2 + (\gamma^2\theta^2\sin(2\phi))^2}{(1+\gamma^2\theta^2)^5}$$

$$\frac{dP}{d\psi} = P_0 \frac{21\gamma}{32(1+\gamma^2\psi^2)^{5/2}} \left(1 + \frac{5\gamma^2\psi^2}{7(1+\gamma^2\psi^2)}\right)$$



Two terms inside bracket correspond to horizontal (σ) and vertical polarization (π) polarization modes. $\langle\gamma^2\psi^2\rangle = 5/8$



$$\frac{dP}{d\Omega}(\theta,\,\phi) \rightarrow \frac{dP}{d\Omega}(\xi,\,\psi)$$

integrate over ξ to get $dP/d\psi$.

4) SYNCHROTRON RADIATION — BASIC PHYSICS

The radiation field

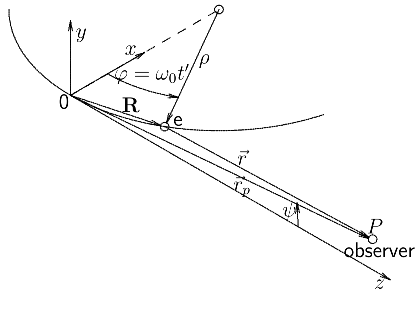
Circular orbit, arc, radius ρ, angular velocity $\omega_0 \approx c/\rho$, position, velocity of particle

$$\vec{R}(t') = \rho\left((1-\cos(\omega_0 t'))\ ,0,\ \sin(\omega_0 t')\right)$$
$$\vec{\beta}(t') = \beta\left(\sin(\omega_0 t')\ ,0,\ -\cos(\omega_0 t')\right).$$

Observation point P in yz-plane at large r_p

$$\vec{n} = (0, \sin\psi, \cos\psi) = \mathrm{const}$$

For $\gamma \gg 1 \rightarrow \psi \ll 1$. Situation is independent on 'sweeping' angles $\varphi = \omega_0 t' = \beta c t'/\rho$.



$$\left[\vec{n} \times [\vec{n} \times \vec{\beta}]\right] \approx \beta\,(-\omega t', \psi, 0)$$

and the term in the exponent by

$$t' + \frac{r(t')}{c} = t' + \frac{\vec{n}\cdot(\vec{r}_p - \vec{R})}{c}$$
$$= t' + \frac{\vec{n}\cdot\vec{r}_p}{c} - \frac{\rho}{c}\sin(\omega_0 t')\cos\psi$$
$$\approx \frac{\vec{n}\cdot\vec{r}_p}{c} + \frac{t'}{2\gamma^2}(1+\gamma^2\psi^2) + \frac{c^2 t'^3}{6\rho^2}.$$

A term of order t'^3 included since first order term is divided by γ^2 and can be very small. Constant phase factor $\vec{n}\cdot\vec{r}_p$ of no interest and omitted. The expression $\tilde{\vec{E}}(\omega)$ gives

$$\tilde{\vec{E}}(\omega) = \frac{i\omega e}{4\pi\sqrt{2\epsilon_0\pi}cr}\int_{-\infty}^{\infty}\left[\vec{n}\times[\vec{n}\times\vec{\beta}]\right]\cdot e^{-i\omega(t'+\frac{r(t')}{c})}dt'$$

$$\left[\vec{n}\times[\vec{n}\times\vec{\beta}]\right] \approx \beta\,(-\omega t', \psi, 0)$$

$$t' + \frac{r(t')}{c} = \frac{t'}{2\gamma^2}(1+\gamma^2\psi^2) + \frac{c^2 t'^3}{6\rho^2}.$$

Fourier transformed field $\tilde{\vec{E}}(\omega)$

$$\tilde{\vec{E}}(\omega) = \frac{i\omega e}{4\pi\sqrt{2\pi}\epsilon_0 c r}\int_{-\infty}^{\infty}\left[\vec{n}\times[\vec{n}\times\vec{\beta}]\right]\cdot e^{-i\omega(t'+\frac{r(t')}{c})}dt'$$

$$\tilde{E}_x = \frac{-\omega e}{4\pi\sqrt{2\pi}\epsilon_0 c r}\cdot \int \omega_0 t' \sin\left(\omega t'\left(\frac{1+\gamma^2\psi^2}{2\gamma^2}+\frac{c^2 t'^2}{6\rho^2}\right)\right)dt',$$

$$\tilde{E}_y = \frac{i\omega e}{4\pi\sqrt{2\pi}\epsilon_0 c r}\cdot \int \psi \cos\left(\omega t'\left(\frac{1+\gamma^2\psi^2}{2\gamma^2}+\frac{c^2 t'^2}{6\rho^2}\right)\right)dt'.$$

$$u = t'\left(\omega c^2/2\rho^2\right)^{1/3}, \quad \omega_c = 3c\gamma^3/(2\rho)$$

Substitute u, use critical frequency ω_c

$$\tilde{E}_x = \frac{-e\gamma}{\sqrt{2\pi}\epsilon_0 c r}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{1}{3}}\cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} u \sin\left(\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)u + \frac{u^3}{3}\right)du$$

$$\tilde{E}_y = \frac{ie\gamma^2\psi}{\sqrt{2\pi}\epsilon_0 c r}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}\cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)u + \frac{u^3}{3}\right)du.$$

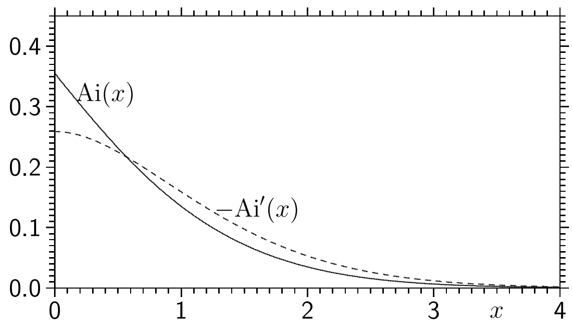
These are expressions for the Airy function $\mathrm{Ai}(v)$ and its derivative $\mathrm{Ai}'(v)$

$$\mathrm{Ai}(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\left(vt+\frac{t^3}{3}\right)dt$$

$$\mathrm{Ai}'(v) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} t\sin\left(vt+\frac{t^3}{3}\right)dt.$$

$$\tilde{E}_x(\omega) = \frac{e\gamma}{\sqrt{2\pi}\epsilon_0 c r}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{1}{3}}\mathrm{Ai}'\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\right)$$

$$\tilde{E}_y(\omega) = \frac{ie\gamma}{\sqrt{2\pi}\epsilon_0 c r}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}\gamma\psi\,\mathrm{Ai}\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\right)$$



Since the functions Ai and Ai' are finite over the range of interest we expect that the field $\tilde{\vec{E}}(\omega)$ vanishes for small ω.

Expressing $\tilde{\vec{E}}(\omega)$ by Airy functions profits frome their well documented properties, like being finite in region of interest, figure. They are related to modified Bessel functions which are also used to describe SR.

$$\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{1/3}\left(\frac{2x^{3/2}}{3}\right)$$

$$\mathrm{Ai}'(x) = -\frac{1}{\pi}\frac{x}{\sqrt{3}}K_{2/3}\left(\frac{2x^{3/2}}{3}\right).$$

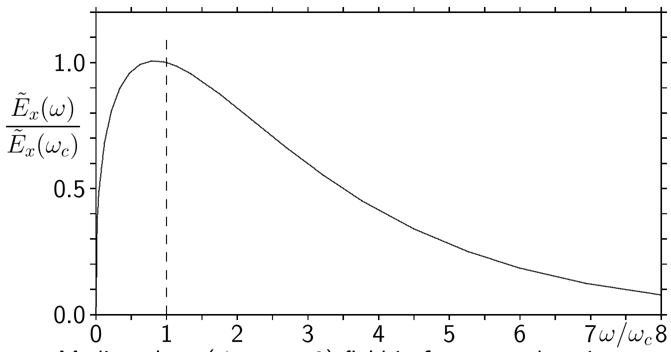
The x-component of the field is real and y-component imaginary. This indicates 90^0 phase shift between the two leading to elliptic polarization at $\psi \neq 0$.

$$\tilde{E}_x(\omega) = \frac{e\gamma}{\sqrt{2\pi}\epsilon_0 c r}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{1}{3}} \mathrm{Ai}'\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\right)$$

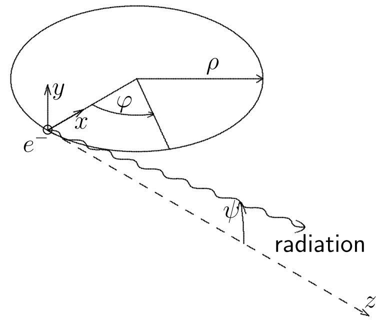
$$\tilde{E}_y(\omega) = \frac{ie\gamma}{\sqrt{2\pi}\epsilon_0 c r}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}} \gamma\psi\mathrm{Ai}\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\right)$$

$$\omega_c = \frac{3c\gamma^3}{2\rho} = \frac{3\omega_0}{2}\gamma^3$$

critical frequency ω_c and angular velocity ω_0
$\omega_0 = \omega_{rev.}$ only for a closed circle orbit.



Median plane ($\psi = y = 0$) field in frequency domain



Field has smooth frequency dependence, vanishes at $\omega \to 0,\ \omega \to \infty$ and is characterized by critical frequency ω_c being close to maximum field value. The horizontal component $\tilde{E}_x(\omega)$ is symmetric in ψ while the vertical component $\tilde{E}_y(\omega)$ is antisymmetric and vanishes in the median plane.

Spectral angular energy distribution

The power per unit solid angle is integrated over time

$$\frac{dP}{d\Omega} = \frac{d^2U}{d\Omega dt'} = S(1-\vec{n}\cdot\vec{\beta})r^2 = \frac{(1-\vec{n}\cdot\vec{\beta})r^2|\vec{E}|^2}{\mu_0 c}$$

to give angular energy distribution

$$\frac{dU}{d\Omega} = \int_{-\infty}^{\infty}\frac{dP}{d\Omega}dt' = \int_{-\infty}^{\infty}\frac{dP}{d\Omega}\frac{dt}{1-\vec{n}\cdot\vec{\beta}}$$
$$= \frac{r^2}{\mu_0 c}\int_{-\infty}^{\infty}|\vec{E}^2(t)|dt$$

Express observed $E(t)$ by inverse Fourier transform

$$\vec{E}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}(\omega)\mathrm{e}^{i\omega t}dt$$

$$\frac{dU}{d\Omega} = \frac{r^2}{2\pi\mu_0 c}\int dt\int\int\tilde{\vec{E}}(\omega)\tilde{\vec{E}}(\omega')\mathrm{e}^{i(\omega+\omega')t}d\omega d\omega'$$

with δ-function representation: $\int_{-\infty}^{\infty}\mathrm{e}^{iat}dt = 2\pi\delta(a)$
we obtain for the integration over t

$$\frac{dU}{d\Omega} = \frac{r^2}{\mu_0 c}\int\int\tilde{\vec{E}}(\omega)\tilde{\vec{E}}(\omega')\cdot \delta(\omega+\omega')d\omega d\omega'$$
$$= \frac{r^2}{\mu_0 c}\int_{-\infty}^{\infty}\tilde{\vec{E}}(\omega)\tilde{\vec{E}}(-\omega)d\omega.$$

Since $\vec{E}(t)$ is real: $\tilde{\vec{E}}(-\omega) = \tilde{\vec{E}}^*(\omega)$
giving angular energy distribution

$$\frac{dU}{d\Omega} = \frac{r^2}{\mu_0 c}\int_{-\infty}^{\infty}\left|\tilde{\vec{E}}(\omega)\right|^2 d\omega = 2\int_0^{\infty}$$

$$\frac{d^2U}{d\Omega d\omega} = \frac{2r^2\left|\tilde{\vec{E}}(\omega)\right|^2}{\mu_0 c}.$$

This spectral angular energy distribution contains most of the relevant properties of radiation.

The spectral angular power distribution

Fourier transformed fields and angular spectral energy and power distribution

$$\tilde{E}_x(\omega) = \frac{e\gamma}{\sqrt{2\pi}\epsilon_0 c r_p}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{1}{3}} \mathrm{Ai}'\Big(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\Big)$$

$$\tilde{E}_y(\omega) = \frac{ie\gamma}{\sqrt{2\pi}\epsilon_0 c r}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}} \gamma\psi\mathrm{Ai}\Big(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\Big)$$

$$\frac{d^2U}{d\Omega d\omega} = \frac{2r^2\left|\tilde{\vec{E}}(\omega)\right|^2}{\mu_0 c} = \frac{2r^2\left(\tilde{E}_x^2 + \tilde{E}_y^{\,2}\right)}{\mu_0 c}$$

$$= \frac{U_0\gamma}{\omega_c}\left[F_\sigma(\omega,\psi) + F_\pi(\omega,\psi)\right], \quad \text{with}$$

$$F_{s\sigma} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{2/3} \mathrm{Ai}'^2\Big(\left(\tfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\Big)$$

$$F_{s\pi} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{4/3} \gamma^2\psi^2\mathrm{Ai}^2\Big(\left(\tfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\Big)$$

$$F_s(\omega,\psi) = F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)$$

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d(\gamma\psi)\int_0^\infty F_s(\omega,\psi)\frac{d\omega}{\omega_c} = 1$$

with energy loss per turn U_0 and critical frequency ω_c

$$U_0 = \frac{4\pi r_0 m_0 c^2\gamma^4}{3\rho},\ \omega_c = \frac{3c\gamma^3}{2\rho}$$

Charge on a closed circular orbit has a revolution time T_0 and radiates a constant power $P_0 = U_0/T_0$

$$T_0 = \frac{2\pi\rho}{c},\ P_0 = \frac{2r_0 c m_0 c^2\gamma^2}{3\rho^2}$$

also valid for local curvature $1/\rho$ and instantaneous radiated power. However, in ring with straight sections $T_{rev.} > T_0$ and P_0 refers to power while in the magnet,

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_0\gamma}{\omega_c}F(\omega,\,\psi)$$

Synchrotron radiation field in time domain

Liénard–Wiechert expression, relation between time scales

$$\vec{E}(t_p) = \frac{e}{4\pi c\epsilon_0}\left(\frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{r((1-\vec{n}\cdot\vec{\beta}))^3}\right)_{\text{ret}}$$

$$t_p = t' + \frac{r(t') - r_p}{c}.$$

$r \approx r_p$ in denominator, same approximation as before

$$E_x(t_p) = -\frac{e\omega_0\gamma^4}{\pi\epsilon_0 c r_p}\frac{\left(1 - 4\sinh^2\left(\frac{1}{3}\mathrm{Arcsinh}\tau\right)\right)}{(1+\gamma^2\psi^2)^2\left(1+4\sinh^2\left(\frac{1}{3}\mathrm{Arcsinh}\tau\right)\right)^3}$$

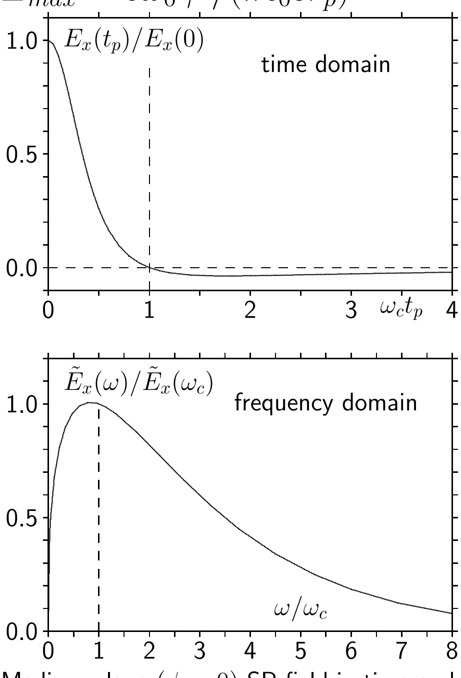
$$E_y(t_p) = -\frac{e\omega_0\gamma^4}{\pi\epsilon_0 c r_p}\frac{4\gamma\psi\sinh\left(\frac{1}{3}\mathrm{Arcsinh}\tau\right)}{(1+\gamma^2\psi^2)^{5/2}\left(1+4\sinh^2\left(\frac{1}{3}\mathrm{Arcsinh}\tau\right)\right)^3}$$

$$\tau = \frac{2\omega_c t_p}{(1+\gamma^2\psi^2)^{3/2}}$$

Symmetry properties

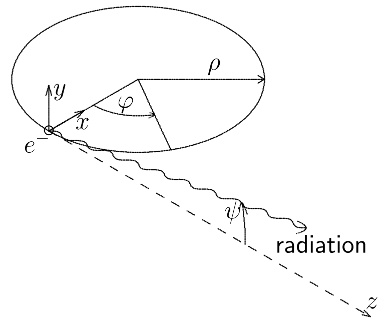
$$E_x(-t_p,\psi) = E_x(t_p,\psi)\ ,\quad E_y(-t_p,\psi) = -E_y(t_p,\psi)$$
$$E_x(t_p,-\psi) = E_x(t_p,\psi)\ ,\quad E_y(t_p,-\psi) = -E_y(t_p,\psi).$$

E_x has maximum at $t_p = 0$, $\psi = 0$ where E_y vanishes $E_{max} = e\omega_0\gamma^4/(\pi\epsilon_0 c r_p)$



Medium plane ($\psi = 0$) SR field in time and frequency domain

Periodic motion and line spectrum



SR field was calculated for charge moving on circular arc of angle $\Delta\varphi = \omega_0\beta\Delta t' > 2\gamma$. Take now full circular orbit, radius ρ, angular frequency $\omega_0 = \beta c/\rho = 2\pi/T_0$. Periodic motion gives line spectrum with $\omega_n = n\omega_0$.

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0 rc}\left(\frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.}$$
$$= \sum_{-\infty}^{\infty}\vec{E}_n e^{i\omega_n t},\ \vec{E}_n = \frac{1}{T_0}\int_0^{T_0}\vec{E}(t)e^{-i\omega t}dt.$$

Emission and observation have same period $\Delta t' = \Delta t = T_0$. Approximation distance > source size, non-relativistic $r \gg \rho$, ultra-relativistic $r \gg \rho/\gamma$. Only horizontal field

$$E_{n\phi} = \frac{e\omega_0\beta}{4\pi\epsilon_0 cr_p}nJ_n'(n\beta\cos\psi)$$
$$\frac{dP_{n\sigma}}{d\Omega} = P_s\frac{3}{4\pi\gamma^4}n^2J_n'^2(n\beta\cos\psi)$$

Compare with previous $dP/d\omega \approx P_n/\omega_0$, $\gamma \gg 1$, Bessel function for large order n

$$J_n'(n\beta\cos\psi) \approx -2\left(\frac{n}{2}\right)^{\frac{1}{3}}\mathrm{Ai}'\left(\left(\frac{n}{2\gamma^3}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\right)$$

The emission of SR quanta makes a spread of energy and revolution frequency which destroys line spectrum.

5) SYNCHROTRON RADIATION — PROPERTIES

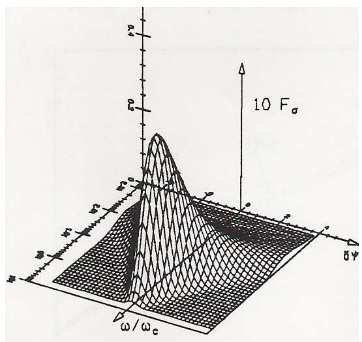
The spectral angular power distribution . .

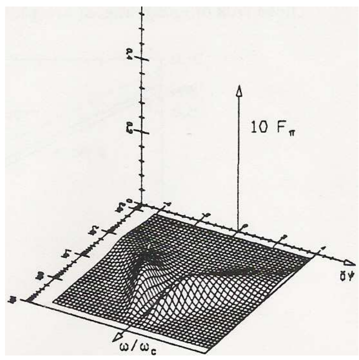
Gives most general description of synchrotron radiation.

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_0\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right],\quad \text{with}$$
$$P_0 = \frac{2r_0cm_0c^2\gamma^2}{3\rho^2} = \text{ power radiated per electron}$$
$$F_{s\sigma} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{2/3}\mathrm{Ai}'^2\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\right)$$
$$F_{s\pi} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{4/3}\gamma^2\psi^2\mathrm{Ai}^2\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\right)$$

these functions are normalized distributions in frequency and angles as shown in the figures.

- Vertical opening angle ψ decreases with frequency ω
- Horizontal polarization (top) is largest in median plane, $\psi = 0$, and has a maximum around $\omega = \omega_c$
- Vertical polarization (bottom) vanishes at $\psi = 0$.





Angular power distribution

Plotting spectral angular power distribution for few fixed frequencies ω gives detailed information about angular distribution.

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_0\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right], \quad \text{with}$$

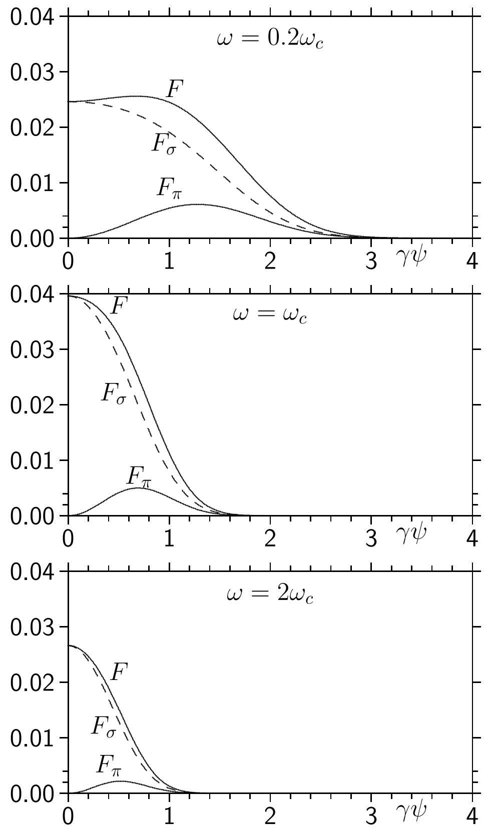
$$F_{s\sigma} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{2/3} \mathrm{Ai}'^2\Big(\left(\tfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\Big)$$

$$F_{s\pi} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{4/3} \gamma^2\psi^2\mathrm{Ai}^2\Big(\left(\tfrac{3\omega}{4\omega_c}\right)^{2/3}(1+\gamma^2\psi^2)\Big)$$

Integrating spectral angular power density over frequency gives angular power distribution of total radiation found before

$$\frac{dP}{d\Omega} = \frac{P_0\gamma}{\omega_c}\int_0^\infty (F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi))d\omega$$

$$= \frac{P_0}{2\pi}\frac{21}{32}\frac{\gamma}{(1+\gamma^2\theta^2)^{5/2}}\left[1 + \left(\frac{5}{7}\right)\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\right]$$



Spectral distribution

Integrating spectral angular power density over angle ψ gives spectral distribution.

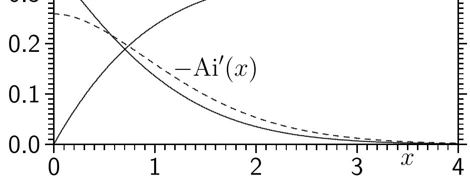
$$\frac{dP}{d\omega} = \int \frac{d^2P}{d\Omega d\omega}d\Omega = \frac{P_0}{\omega_c}S\left(\frac{\omega}{\omega_c}\right)$$

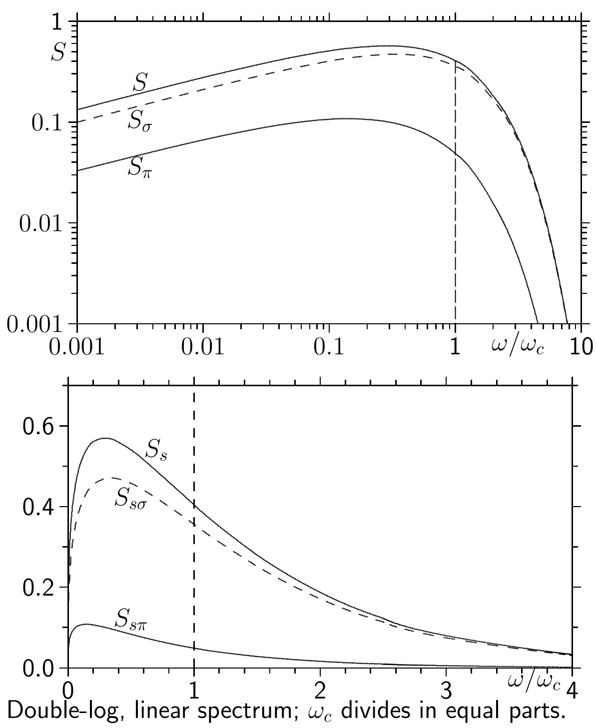
$$= \frac{P_0}{\omega_c}\left[S_\sigma\left(\frac{\omega}{\omega_c}\right) + S_\pi\left(\frac{\omega}{\omega_c}\right)\right],$$

$$S_\sigma\left(\frac{\omega}{\omega_c}\right) = \frac{27\omega}{16\omega_c}\left[-3\frac{\mathrm{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \mathrm{Ai}(z')dz'\right]$$

$$S_\pi\left(\frac{\omega}{\omega_c}\right) = \frac{27\omega}{16\omega_c}\left[-\frac{\mathrm{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \mathrm{Ai}(z')dz'\right]$$

$$z = \left(\frac{3\omega}{\omega_c}\right)^{3/2}$$





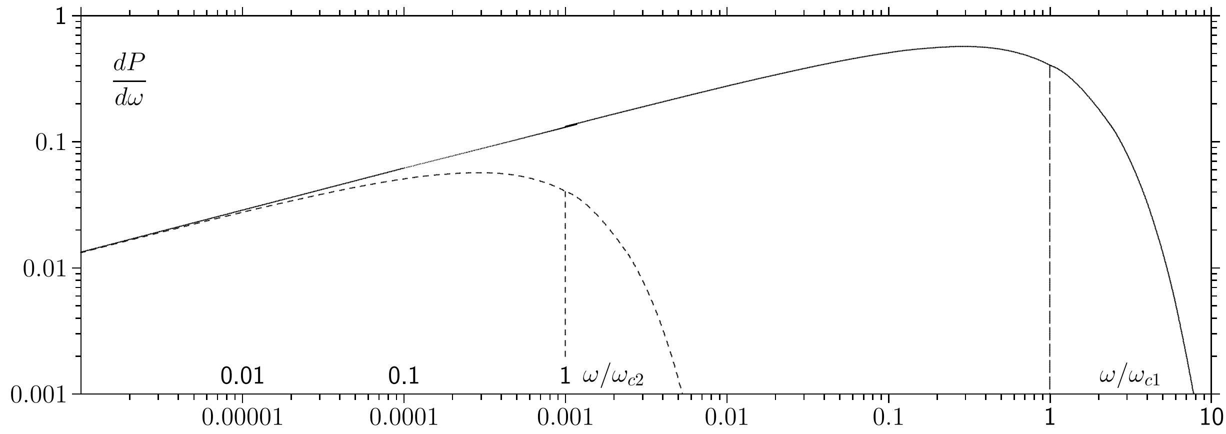
Double-log, linear spectrum; ω_c divides in equal parts.

Comparing spectral power density for a given orbit radius but different energies.

$$P_0 = \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2},\ \omega_c = \frac{3c\gamma^3}{2\rho}$$

$$\frac{dP}{d\omega} = \frac{P_0}{\omega_c} S\left(\frac{\omega}{\omega_c}\right) = \frac{4r_0 m_0 c^2 \gamma}{9\rho} S\left(\frac{\omega}{\omega_c}\right)$$

Increasing energy scales spectral density $\propto \gamma$ and frequency range $\propto \gamma^3$ total radiated power $\propto \gamma^4$. The lower end of the spectrum is independent of γ. Important for beam diagnostic with visible SR.



Spectral power densities for energy change by a factor 10

SR properties at small frequencies

Spectral angular power distribution of SR

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_0\gamma}{\omega_c}\left[F_{s\sigma}(\omega,\psi) + F_{s\pi}(\omega,\psi)\right]$$

$$F_{s\sigma} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}} \mathrm{Ai}'^2\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\right)$$

$$F_{s\pi} = \frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{4}{3}} \gamma^2\psi^2 \mathrm{Ai}^2\left(\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2)\right)$$

For $\omega/\omega_c \ll 1$ the argument of the Airy function are small and will not influence these function unless term $\gamma^2\psi^2$ becomes much larger than 1

$$\left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}(1+\gamma^2\psi^2) \approx \left(\tfrac{3\omega}{4\omega_c}\right)^{\frac{2}{3}}\gamma^2\psi^2 = \left(\frac{\omega}{2\omega_0}\right)^{\frac{2}{3}}\psi^2$$

use $\omega_c = 3\omega_0\gamma^3/2$, angular velocity $\omega_0 = \beta c/\rho$. This argument does no longer depend on γ.

Factors in front of Airy functions

$$\frac{P_0\gamma}{\omega_c}\frac{9}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{2}{3}} = \frac{2r_0 m_0 c^2}{\pi\rho}\left(\frac{\omega}{2\omega_0}\right)^{\frac{2}{3}}$$

$$\frac{P_0\gamma}{\omega_c}\frac{9\gamma^2\psi^2}{2\pi}\left(\frac{3\omega}{4\omega_c}\right)^{\frac{4}{3}} = \frac{2r_0 m_0 c^2\psi^2}{\pi\rho}\left(\frac{\omega}{2\omega_0}\right)^{\frac{4}{3}}$$

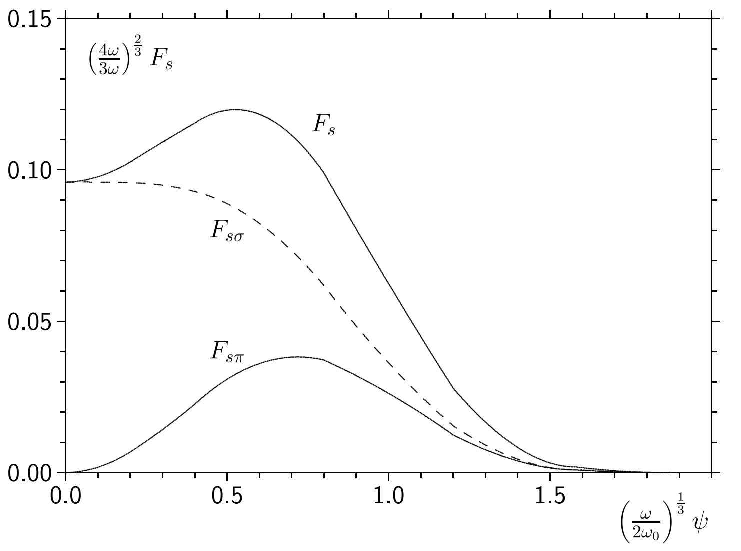
giving spectral power density expression for $\omega \ll \omega_c$ without γ, i.e. in ring of fixed ρ the SR properties are independent of γ at low frequencies, where diagnostics is done.

$$\frac{d^2P}{d\Omega d\omega} = \frac{2r_0 m_0 c^2}{\pi\rho}\left(\frac{\omega}{2\omega_0}\right)^{\frac{2}{3}} \cdot \left[\mathrm{Ai}'^2\left(\left(\tfrac{\omega}{2\omega_0}\right)^{\frac{2}{3}}\psi^2\right) + \left(\frac{\omega}{2\omega_0}\right)^{\frac{2}{3}}\psi^2\mathrm{Ai}^2\left(\left(\tfrac{\omega}{2\omega_0}\right)^{\frac{2}{3}}\psi^2\right)\right]$$

Spectral angular power density at low frequencies

$$\frac{d^2P}{d\Omega d\omega} = \frac{2r_0 m_0 c^2}{\pi\rho}\left[\left(\frac{\omega}{2\omega_0}\right)^{2/3} \mathrm{Ai}'^2\left(\left(\tfrac{\omega}{2\omega_0}\right)^{2/3}\psi^2\right) + \left(\frac{\omega}{2\omega_0}\right)^{4/3}\psi^2 \mathrm{Ai}^2\left(\left(\tfrac{\omega}{2\omega_0}\right)^{2/3}\psi^2\right)\right]$$

Angular distribution at low frequencies

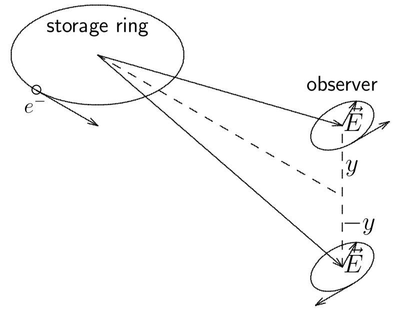


In diagnostics an image of the beam cross section is formed with visible synchrotron radiation. Its vertical angular distribution is used to calculated resolution limitation by diffraction.
RMS angles at small frequencies are

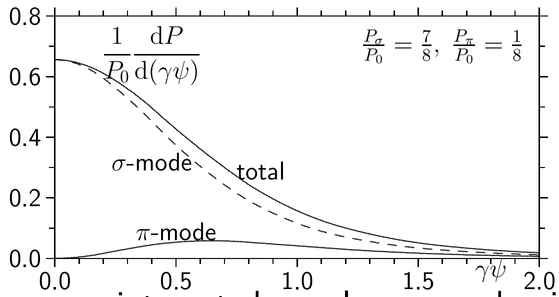
$$\sqrt{\langle\psi^2\rangle_\sigma} = 0.756\left(\frac{\omega_0}{\omega}\right)^{1/3}$$
$$\sqrt{\langle\psi^2\rangle_\pi} = 1.014\left(\frac{\omega_0}{\omega}\right)^{1/3}$$
$$\sqrt{\langle\psi^2\rangle} = 0.828\left(\frac{\omega_0}{\omega}\right)^{1/3}$$

Polarization

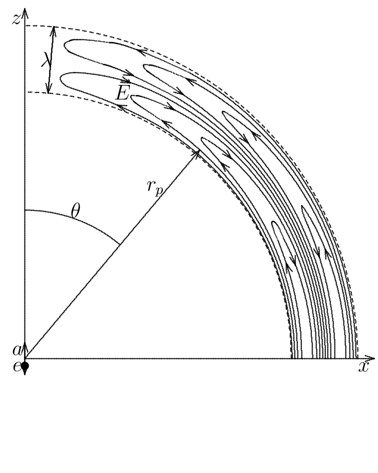
Power has no direction but its parts, P_σ, P_π, originate from two orthogonal electric field components.



$$\frac{d^2P}{d\Omega d\omega} = \frac{2r^2\left|\vec{\tilde{E}}(\omega)\right|^2}{2\pi\mu_0\rho} = \frac{2r^2\left(\tilde{E}_x^2 + \tilde{E}_y^{\,2}\right)}{2\pi\mu_0\rho}$$
$$= \frac{P_0\gamma}{\omega_c}\left[F_{s\sigma} + F_{s\pi}\right].$$



Frequency integrated angular power density

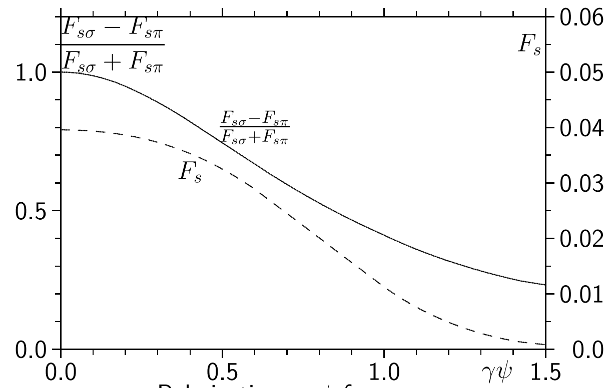


polarization degree: $\dfrac{\frac{d^2P_\sigma}{d\Omega d\omega} - \frac{d^2P_\pi}{d\Omega d\omega}}{\frac{d^2P_\sigma}{d\Omega d\omega} + \frac{d^2P_\pi}{d\Omega d\omega}} = \dfrac{F_{s\sigma} - F_{s\pi}}{F_{s\sigma} + F_{s\pi}}$

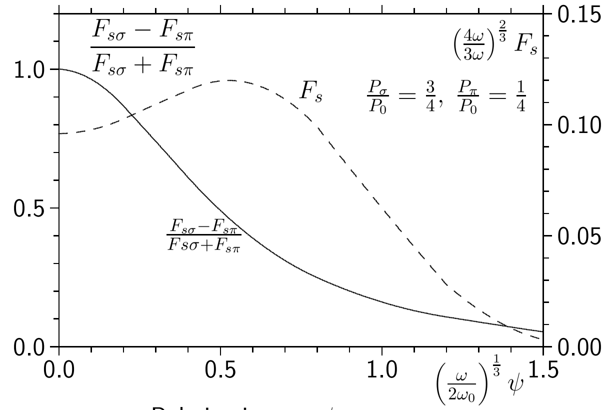
Polarization degree vs. ψ and ω

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_0\gamma}{\omega_c}[F_\sigma(\omega,\psi)) + F_\pi(\omega,\psi)]$$

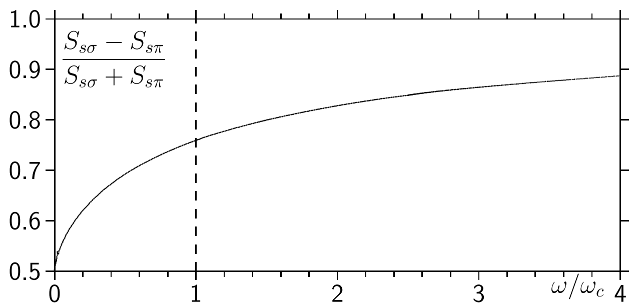
$$\text{pol. deg.} = \frac{\frac{d^2P_\sigma}{d\Omega d\omega} - \frac{d^2P_\pi}{d\Omega d\omega}}{\frac{d^2P_\sigma}{d\Omega d\omega} + \frac{d^2P_\pi}{d\Omega d\omega}} = \frac{F_{s\sigma} - F_{s\pi}}{F_{s\sigma} + F_{s\pi}}$$



Polarization vs ψ for $\omega = \omega_c$



Polarization vs. ψ at $\omega \ll \omega_c$



Polarization of angle integrated distribution vs. ω

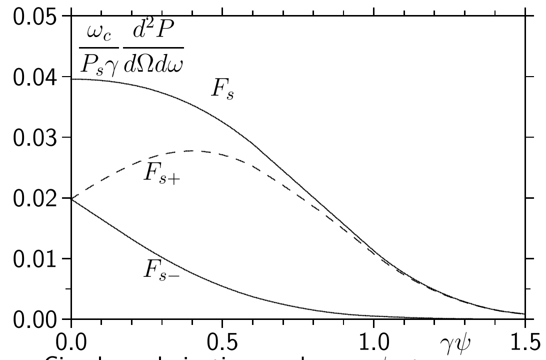
Elliptic polarization

In elliptic polarization there is a phase shift between horizontal and vertical oscillations, resulting in rotating field vector $\vec{E}$. Fourier transformed field becomes complex. For $\pi/2$ phase shift between the two, $\tilde{E}_x$ is real and $\tilde{E}_y = -i|\tilde{E}_y|$. Define new orthogonal components for vector representing $\pm$-rotations

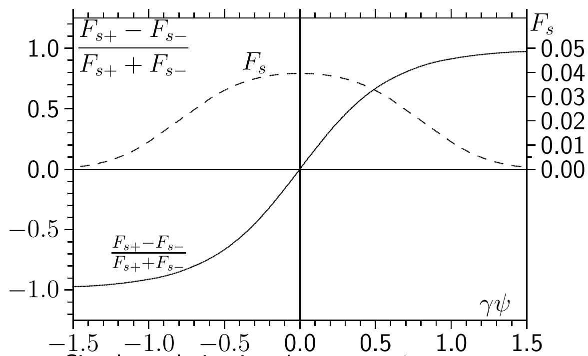
$$\tilde{\vec{E}} = [\tilde{E}_x,\ \tilde{E}_y] = [\tilde{E}_+,\ \tilde{E}_-]$$
$$\tilde{E}_+ = (\tilde{E}_x + i\tilde{E}_y)/\sqrt{2} = (\tilde{E}_x + |\tilde{E}_y|)/\sqrt{2}$$
$$\tilde{E}_- = (\tilde{E}_x - i\tilde{E}_y)/\sqrt{2} = (\tilde{E}_x - |\tilde{E}_y|)/\sqrt{2}$$
$$\tilde{E}_+^2 + \tilde{E}_-^2 = \tilde{E}_x^2 + \tilde{E}_y^2$$

Power has two parts due to $\tilde{E}_+$, $\tilde{E}_-$

$$\frac{d^2P}{d\Omega d\omega} = \frac{P_0\gamma}{\omega_c}[F_{s+}(\omega,\psi) + F_{s-}(\omega,\psi)]$$
$$F_{s\pm} = \frac{1}{2}\left(F_{s\sigma} + F_{s\pi} \pm 2\frac{\psi}{|\psi|}\sqrt{F_{s\sigma}F_{s\pi}}\right)$$



Circular polarization modes vs. ψ at $\omega = \omega_c$



Circular polarization degree vs. ψ at $\omega = \omega_c$

Photon distribution

The synchrotron radiation is emitted in quanta (photons) with energy $E_\gamma = \hbar\omega$, where $\hbar = h/2\pi$ and $h = 6.6262 \cdot 10^{-34}$ Js, is Planck's constant. The photon number emitted per second $\dot{n}$ is related to emitted power $P = \dot{n}(E_\gamma)\, E_\gamma$. With critical photon energy $E_{\gamma c} = \hbar\omega_c$ we convert the angular spectral power density to the angular spectral photon flux

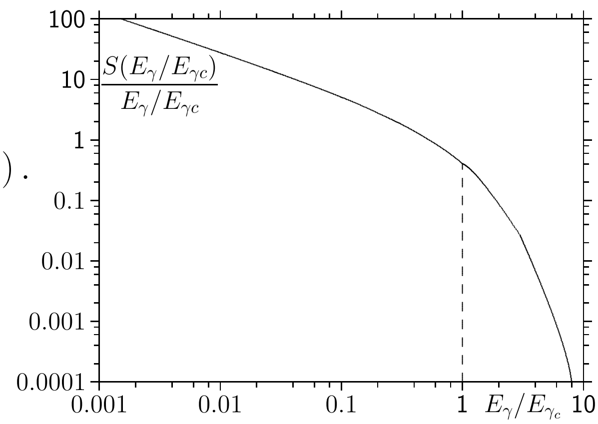
$$\frac{d^2\dot{n}}{d\Omega dE_\gamma/E_\gamma} = \frac{d^2P}{d\Omega\hbar d\omega} = \frac{P_0\gamma}{E_{\gamma c}}\left(F_\sigma(\omega,\psi) + F_\pi(\omega,\psi)\right).$$

giving the number of photon flux per solid angle into a *relative* photon energy band width $\Delta E_\gamma/E_\gamma$. Integrating this over solid angle gives the photon energy distribution

$$\frac{d\dot{n}}{dE_\gamma/E_\gamma} = \frac{1}{\hbar}\frac{dP}{d\omega} = \frac{P_0}{E_{\gamma 0}}\left[S_{s\sigma}\left(\frac{E_\gamma}{E_{\gamma c}}\right) + S_{s\pi}\left(\frac{E_\gamma}{E_{\gamma c}}\right)\right].$$

This has the form of the power spectrum. Sometimes photon distribution in absolute energy band is more relevant

$$\frac{d\dot{n}}{dE_\gamma} = \frac{P_0\gamma}{E_{\gamma c}^2}\frac{S_\sigma\left(\frac{E_\gamma}{E_{\gamma c}}\right) + S_\pi\left(\frac{E_\gamma}{E_{\gamma c}}\right)}{\left(\frac{E_\gamma}{E_{\gamma c}}\right)}.$$



Integrating the photon distribution gives the number $\dot{n}$ of photons radiated by an electron per second

$$\dot{n} = \frac{P_0}{E_{\gamma c}}\int_0^\infty \frac{S_\sigma\left(\frac{E_\gamma}{E_{\gamma c}}\right) + S_\pi\left(\frac{E_\gamma}{E_{\gamma c}}\right)}{\left(\frac{E_\gamma}{E_{\gamma c}}\right)} d(E_\gamma/E_{\gamma c}).$$

$$\dot{n} = \frac{15\sqrt{3}}{8}\frac{P_0}{E_{\gamma_c}},\ \dot{n}_\sigma = \frac{12\sqrt{3}}{8}\frac{P_0}{E_{\gamma_c}},\ \dot{n}_\pi = \frac{3\sqrt{3}}{8}\frac{P_0}{E_{\gamma c}}.$$

We are interested in the variance $< E_\gamma^2 >$ of the photon energy.

$$< E_{\gamma\sigma}^2 > = \frac{25}{54}E_{\gamma c}^2,\ < E_{\gamma\pi}^2 > = \frac{5}{27}E_{\gamma_c}^2$$
$$< E_\gamma^2 > = \frac{11}{27}E_{\gamma c}^2.$$

Back to the number of photons expressing P_0 and ϵ_c

$$\frac{P_0}{E_{\gamma c}} = \frac{\frac{2r_0cm_0c^2\gamma^4}{3\rho^2}}{\frac{3\hbar c\gamma^3}{2\rho}} = \frac{2e^2\gamma}{9\rho\epsilon_0 h} = \frac{4\omega_0\alpha_f}{9},$$

with angular velocity $\omega_0 = c/\rho$ and fine structure constant α_f is

$$\alpha_f = \frac{e^2}{2ch\epsilon_0} = \frac{1}{137.036}$$

and obtain for the number of photons radiated per second

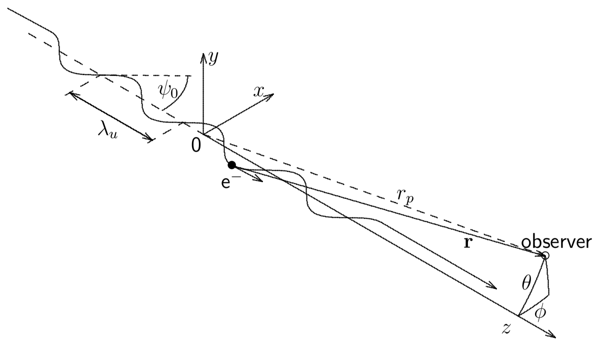
$$\dot{n} = \frac{5}{2\sqrt{3}}\gamma\omega_0\alpha_f.$$

or in one revolution of photons

$$n/\mathrm{rev.} = \frac{5\pi}{\sqrt{3}}\gamma\alpha_f = 0.0662\gamma.$$

6) UNDULATOR RADIATION

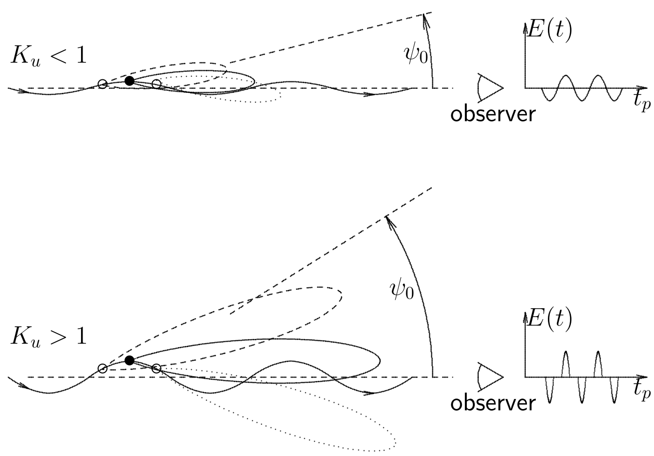
Introduction



An undulator is spatially periodic magnet designed to provide quasi-monochromatic radiation. We start with a plane undulator with a mid plane field $B(z) = B_y(z) = B_0 \cos(k_u z)$, N_u periods of length λ_u, $k_u = 2\pi/\lambda_u$ resulting approximately in a harmonic particle trajectory $x(z) = a\cos(k_u z)$, amplitude a, maximum angle ψ_0

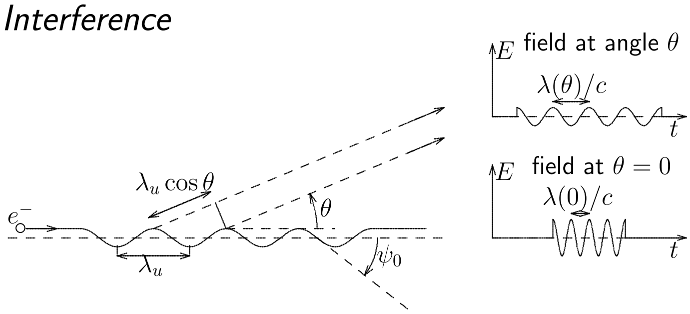
$$a = \frac{eB_0}{m_0 c\gamma k_u^2} \;\;,\;\; \psi_0 = \frac{eB_0}{m_0 c \gamma k_u} = \frac{K_u}{\gamma}$$

The undulator parameter $K_u = \gamma\psi_0$, ratio between deflection angle ψ_0 and radiation opening angle $\approx 1/\gamma$. For $K_u < 1$ emitted light is modulated weakly. For $K_u > 1$ strong modulation gives harmonics.



Qualitative treatment

Interference



Arrival time difference from each period is

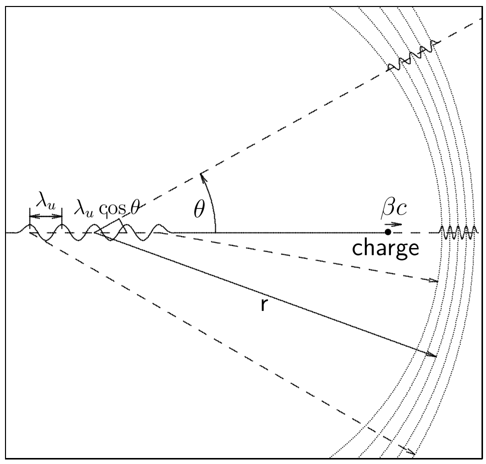
$$\Delta T = \frac{\lambda_u}{\beta c} - \frac{\lambda_u \cos\theta}{c} = \frac{\lambda_u(1-\beta\cos\theta)}{\beta c}$$

$$\gamma \gg 1,\ \theta \ll 1,\ 1-\beta \approx \frac{1}{2\gamma^2}$$

$$\Delta T = \frac{\lambda_u}{2c\gamma^2}(1+\gamma^2\theta^2)\ ,\ \omega_1 = \frac{2\pi c}{\lambda_u}\frac{2\gamma^2}{1+\gamma^2\theta^2}$$

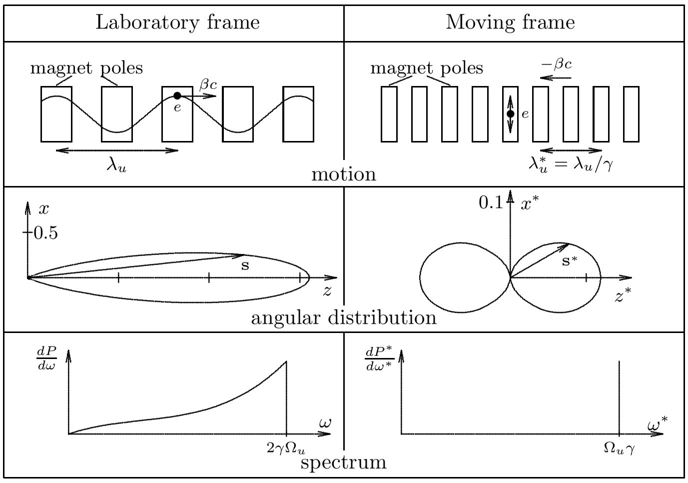
Sinusoidal particle trajectory increases its path-length with amplitude, neglected here. This makes time difference between photons and electrons larger and lowers radiation frequency.

Wave front



The radiation emission from each undulator produces a wave front. The smaller the difference between photon and electron velocity $c(1-\beta)$, the denser the wave fronts are. At large angles this difference becomes larger $c(1-\beta\cos\theta)$.

Weak undulator in laboratory and moving frame



LS:$B_y = B_0 \cos(k_u z)$

LT:$k_u z = k_u\gamma(z^* + \beta c t^*) = (k_u^* z^* - \Omega_u^* t^*)$

$E_x^* = \gamma(E_y - \beta c B_y); B_y^* = \gamma(B_y + \beta c E_y)$

$k_u^* = \gamma k_u,\ \Omega_u^* = \gamma\Omega_u,\ E_x^* = -\gamma\beta c B_y$

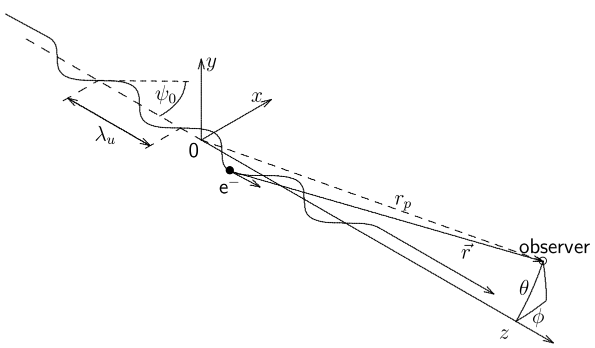
MS:$E_x^* = -\beta c B_y^* = -\gamma\beta c B_0 \cos(k_u^* z^* + \Omega_u^* t^*)$

$\sim$ EM wave with $v^* = \dot{z}^* = \Omega_u^*/k_u^* = -\beta c$

The particle going through a weak undulator has very small transverse momentum resulting in a nearly constant longitudinal velocity v_z. Going into a frame moving with this velocity the particle motion consists of a harmonic transverse oscillation. In this frame the undulator moves with $-v_z$ and its period is shortened to be $\lambda_u^* = \lambda_u/\gamma$. The particle oscillates therefore with a frequency $\Omega_u^* = 2\pi c\gamma/\lambda_u$ and emits radiation with typical dipole pattern. Going back in laboratory system this gets peaked forward and Doppler shifted resulting in angle dependent frequency

$$\omega_1 = \frac{\Omega_u^*}{\gamma(1-\beta\cos\theta)} \approx \frac{2\pi c}{\lambda_u}\frac{2\gamma^2}{1+\gamma^2\theta^2}$$

Particle motion



Plane harmonic undulator with $N_u \gg 1$, period length λ_u, total length $L_u = N_u\Lambda_u$ and field in $y = 0$ plane

$$B(z) = B_y(z) = B_0\cos(k_u z) \ \ ,\ k_u = 2\pi/\lambda_u$$

For moderate fields the particle trajectory is

$$\frac{1}{|\rho|} = \frac{eB_y(z)}{m_0 c\gamma} \approx -\frac{d^2x}{dz^2}$$

$$x''(z) = -\frac{eB_0}{m_0 c\gamma}\cos(k_u z)$$
$$x'(z) = \frac{eB_0}{m_0 c\gamma k_u}\sin(k_u z) = \frac{K_u}{\gamma}\sin(k_u z)$$
$$x(z) = \frac{eB_0}{m_0 c\gamma k_u^2}\cos(k_u z) = a\cos(k_u z).$$

The maximum trajectory angle and excursions are

$$\psi_0 = \frac{eB_0}{m_0 c\gamma k_u} = \frac{K_u}{\gamma}\ ,\quad a = \frac{eB_0}{m_0 c\gamma k_u^2}$$

$K_u = \gamma\psi_0$ is the ratio between the deflection angle ψ_0 and the natural opening angle $\approx 1/\gamma$ of the radiation. For $K_u < 1$ emitted light is only modulated weakly. For $K_u > 1$ the strong modulation produced higher harmonics.

In strong undulator transverse motion affects longitudinal one giving drift velocity $\langle v_z \rangle = \beta^* c < \beta c$. Trajectory length s_u over a period

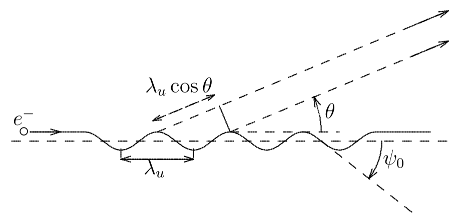
$$
\begin{aligned}
x(z) &= a\cos(k_u z) \\
x' &= -a k_u \sin(k_u z) \\
s_u &= \int_0^{\lambda_u} \sqrt{1 + x'^2(z)}\,dz \\
&= \int_0^{\lambda_u} \sqrt{1 + a^2 k_u^2 \sin^2(k_u z)}\,dz \\
&\approx \lambda_u \left(1 + \frac{K_u^2}{4\gamma^2}\right), \quad a = \frac{eB_0}{m_0 c \gamma k_u^2} = \frac{K_u}{\gamma k_u}
\end{aligned}
$$

Since βc is constant we get for the drift velocity

$$\langle v_z \rangle = \beta^* c = \beta c \frac{\lambda_u}{s_u} = \frac{\beta c}{1 + \frac{K_u^2}{4\gamma^2}} \approx c\left(1 - \frac{K_u^2}{4\gamma^2}\right)$$

normalized drift parameter β^*, γ^* depend on K_u

$$\beta^* = 1 - \frac{K_u^2}{4\gamma^2}\ , \ \gamma^* = \frac{1}{\sqrt{1-\beta^{*2}}} \approx \frac{\gamma}{\sqrt{1 + K_u^2/2}}$$



Calculate arrival difference by period we take drift velocity $\beta^* c$ for electron

$$\Delta T = \frac{\lambda_u}{\beta^* c} - \frac{\lambda_u \cos\theta}{c} = \frac{\lambda_u(1 - \beta^* \cos\theta)}{\beta^* c}$$

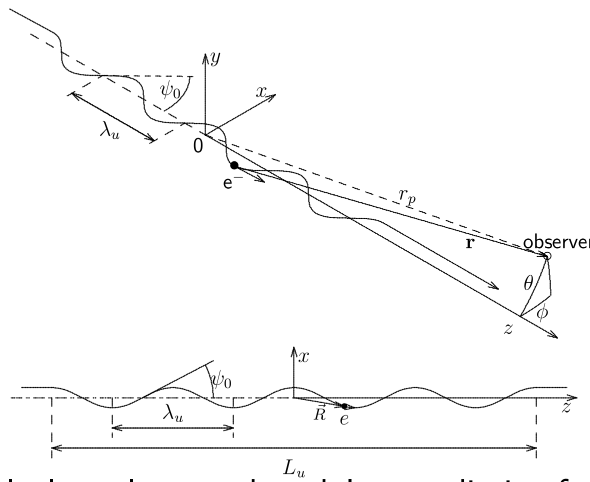
$$\gamma^* \gg 1,\ \theta \ll 1,\ 1 - \beta^* \approx \frac{1}{2\gamma^{*2}}$$

$$
\begin{aligned}
\omega_1 &= \frac{2\pi c}{\lambda_u} \frac{2\gamma^{*2}}{(1 + \gamma^{*2}\theta^2)} \\
&= \frac{2\pi c}{\lambda_u} \frac{2\gamma^2}{(1 + K_u^2/2 + \gamma^2\theta^2)}
\end{aligned}
$$

K_u-dependence can adjust frequency

Weak undulator radiation

Radiation field



Calculate plane weak undulator radiation from

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0} \left(\frac{\left[\vec{n} \times [(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}]\right]}{c r (1 - \vec{n}\cdot\vec{\beta})^3} \right)_{ret.} .$$

Chose origin in undulator center and use $\Omega_u = k_u \beta c$ to get trajectory

$$
\begin{aligned}
\vec{R}(t') &= \left[\frac{K_u}{\beta\gamma k_u} x \cos(\Omega_u t'),\ 0,\ \beta c t'\right] \\
\vec{\beta}(t') &= \left[-\frac{K_u}{\gamma}\sin(\Omega_u t'),\ 0,\ \beta\right] \\
\dot{\vec{\beta}}(t') &= \left[-\frac{K_u k_u \beta}{\gamma}\cos(\Omega_u t'),\ 0,\ 0\right] \\
\vec{r}_p &= [\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta] \\
\vec{r}(t') &= \vec{r}_p - \vec{R}(t') \\
&\approx r_p \left[\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta - \frac{\beta c t'}{r_p}\right] \\
r(t') &\approx r_p - \beta c t' \quad \text{for } K_u < 1.
\end{aligned}
$$

For the time scale relation

$$t = t' + \frac{r(t')}{c},\ t_p = t - \frac{r_p}{c} = t'(1 - \beta\cos\theta)$$

observer time t_p without fixed delay r_p/c.

$$\vec{E}(t) = \frac{e}{4\pi\epsilon_0}\left(\frac{\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]}{cr(1-\vec{n}\cdot\vec{\beta})^3}\right)_{ret.}$$

Observe radiation from large distance $r_p \gg N_u\lambda_u = L_u$, neglect change of unit vector $\vec{n} \approx \vec{r}_p/r_p$ and distance $r \approx r_p$ in denominator. Assume $K_u < 1$, ultra-relativistic case $\gamma \gg 1,\ \theta \ll 1$.

$$1-\vec{n}\cdot\vec{\beta} = 1-\beta\cos\theta \approx \frac{1+\gamma^2\theta^2}{2\gamma^2}$$

$$\begin{aligned}
\vec{\beta}(t') &\approx [0,\ 0,\ \beta] \\
\dot{\vec{\beta}}(t') &= \left[-\frac{K_u k_u \beta}{\gamma}\cos(\Omega_u t'),\ 0,\ 0\right] \\
\vec{n} &\approx \left[\theta\cos\phi,\ \theta\sin\phi,\ 1-\theta^2/2\right] \\
r &\approx r_p - \beta c t'\cos\theta \approx r_p \\
t_p &\approx t'\frac{1+\gamma^2\theta^2}{2\gamma^2}
\end{aligned}$$

We get for the triple vector product

$$\left[\left[\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]\right] = \frac{cK_u k_u\cos(\Omega_u t')}{2\gamma^3}\cdot$$
$$\left[1+\gamma^2\theta^2-2\gamma^2\theta^2\cos^2\phi,\ -2\gamma^2\theta^2\cos\phi\sin\phi,\ 0\right].$$

This equation is evaluated at the emission time t' but we like to relate it to the observer time t_p

$$\Omega_u t' = \Omega_u\frac{2\gamma^2}{1+\gamma^2\theta^2}t_p = \omega_1 t_p \text{ with } \omega_1 = \frac{2\gamma^2\Omega_u}{1+\gamma^2\theta^2}.$$

We also use

$$\sin(2\phi) = 2\cos\phi\sin\phi\ ,\quad \cos(2\phi) = 2\cos^2\phi - 1$$

get electric time domain field emitted by weak plane undulator

$$\vec{E}(t) = \frac{2eK_u\Omega_u\gamma^3}{\pi c\epsilon_0 r_p}\cos(\omega_1 t_p)\cdot$$
$$\frac{\left[(1-\gamma^2\theta^2\cos(2\phi)),\ -\gamma^2\theta^2\sin(2\phi),\ 0\right]}{(1+\gamma^2\theta^2)^3}.$$

$$\vec{E}(t) = \frac{eK_u\Omega_u\gamma^3}{\pi c\epsilon_0 r_p}\cos(\omega_1 t_p)\cdot$$
$$\frac{\left[(1-\gamma^2\theta^2\cos(2\phi)),\ -\gamma^2\theta^2\sin(2\phi)\right]}{(1+\gamma^2\theta^2)^3}.$$

$$\omega_1 = \frac{\Omega_u 2\gamma^2}{1+\gamma^2\theta^2} = \frac{\omega_{10}}{1+\gamma^2\theta^2}$$

Fourier transform gives frequency domain

$$\begin{aligned}
\tilde{\vec{E}}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-N_u/2}^{N_u/2}\vec{E}(t)dt \\
&= \frac{eK_u\Omega_u\gamma^3}{\pi\sqrt{2\pi}c\epsilon_0 r_p}\frac{\pi N_u}{\omega_1}\frac{\sin(\Delta\omega\pi N_u/\omega_1}{\Delta\omega\pi N_u/\omega_1}\cdot \\
&= \frac{\left[(1-\gamma^2\theta^2\cos(2\phi)),\ -\gamma^2\theta^2\sin(2\phi)\right]}{(1+\gamma^2\theta^2)^3}
\end{aligned}$$

$\Delta\omega = \omega-\omega_1,\ |\Delta\omega/\omega \ll 1$ Apart from time or frequency dependence this was found before for transverse acceleration

$$P_0(t') = \frac{2r_0 c m_0 c^2\gamma^4}{3\rho^2(t')},\ P_u = \langle P_0(t')\rangle = \frac{P_0}{2}$$

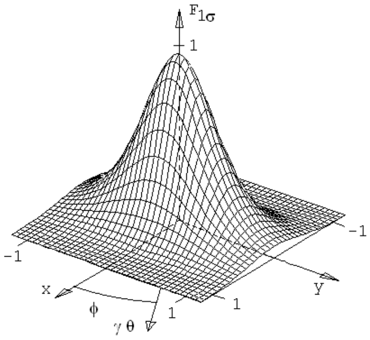
Dividing energy distribution by passage time in undulator L_u/c gives average power

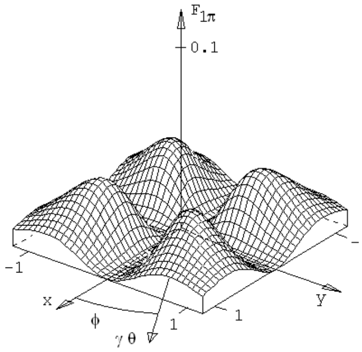
$$\frac{d^2U}{d\Omega d\omega} = \frac{2r^2\left|\tilde{\vec{E}}(\omega)\right|^2}{\mu_0 c}.\ \rightarrow\ \frac{d^2P}{d\Omega d\omega} = \frac{2r^2\left|\tilde{E}(\omega)\right|^2}{\mu_0 L_u}.$$

$$\frac{d^2P_u}{d\Omega d\omega} = P_u\gamma^2\left(F_{u\sigma}(\theta,\phi)+F_{u\pi}(\theta,\phi)\right)f_N(\Delta\omega)$$

$F_{u\sigma},\ F_{u\pi}$ are normalized distributions of horizontal and vertical polarizations and $f_N(\Delta\omega)$ a spectral function

Spectral angular power distribution





$$\frac{d^2 P_u}{d\Omega d\omega} = P_u \gamma^2 \left(F_{u\sigma}(\theta,\phi) + F_{u\pi}(\theta,\phi) \right) f_N(\Delta\omega)$$

$$\omega_1 = \Omega_u 2\gamma^2/(1+\gamma^2\theta^2)$$

$$f_N(\Delta\omega) = \frac{N_u}{\omega_1}\left(\frac{\sin(\Delta\omega\pi N_u/\omega_1)}{\Delta\omega\pi N_u/\omega_1}\right)^2$$

$$\Delta\omega = \omega - \omega_1,\ \int_{-\infty}^{\infty} f_N(\Delta\omega)d\Delta\omega = 1$$

$$F_{u\sigma} = \frac{3}{\pi}\frac{(1-\gamma^2\theta^2\cos(2\phi))^2}{(1+\gamma^2\theta^2)^5}$$

$$F_{u\pi} = \frac{3}{\pi}\frac{(\gamma^2\theta^2\sin(2\phi))^2}{(1+\gamma^2\theta^2)^5}$$

Due to θ , ω_1 relation distribution in angle contains one in frequency obtained by integrating over ϕ.

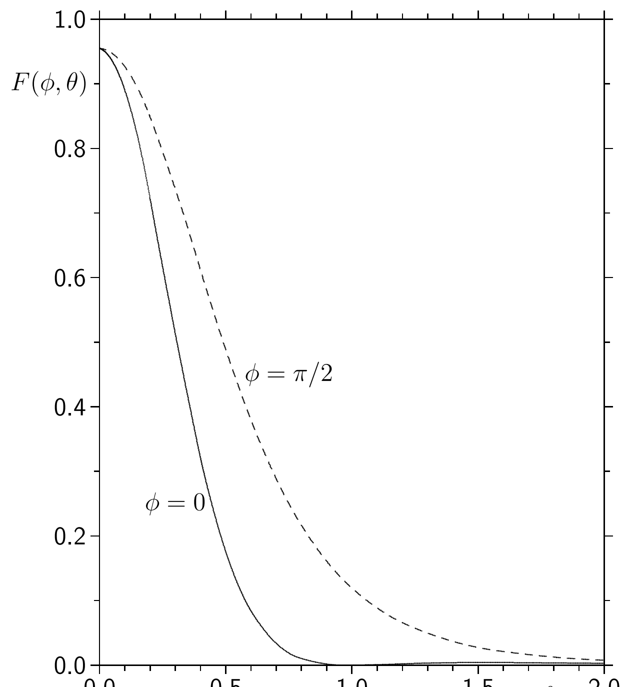
Angular distribution

We make horizontal ($\phi = 0$) and vertical ($\phi = \pi/2$) cuts through the angular distribution of weak undulator radiation

$$F_{u\sigma}(\theta,0) = \frac{2}{\pi}\frac{1-\gamma^2\theta^2}{(1+\gamma^2\theta^2)^5}$$

$$F_{u\sigma}(\theta,\pi/2) = \frac{2}{\pi}\frac{1}{(1+\gamma^2\theta^2)^4}.$$

In the horizontal plane the intensity vanishes at $\gamma\theta = \pm 1$. This point corresponds to the axis of the oscillating dipole in the moving system along which no radiation is emitted. The vertical polarization $F_{u\pi}$ vanishes in these two planes.



Spectrum

In undulators with many periods $N_u \gg 1$ to every radial angle θ belongs a frequency ω_1

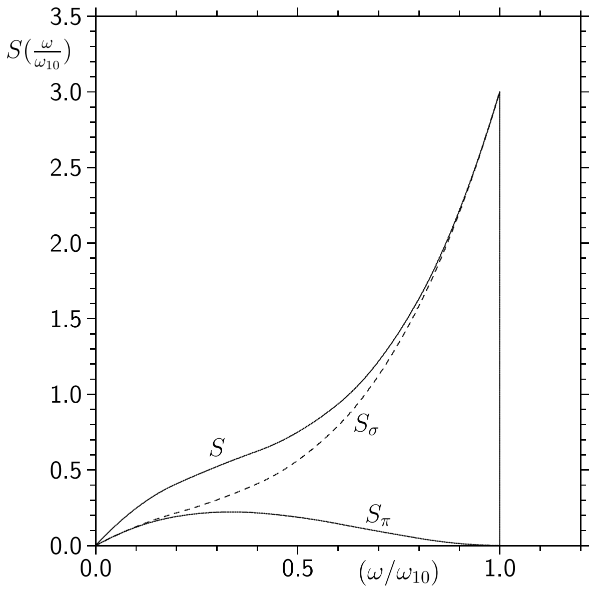
$$\omega_1 = \frac{\Omega_u 2\gamma^2}{1+\gamma^2\theta^2} = \frac{\omega_{10}}{1+\gamma^2\theta^2}.$$

Integrating the angular distribution ϕ we get a distribution $dP_u/d\theta$ which can be converted in the spectrum

$$d\omega_1 = -\frac{2\omega_{10}\gamma^2\theta d\theta}{1+\gamma^2\theta^2} = -2\omega_{10}\gamma^2\left(\frac{\omega_1}{\omega_{10}}\right)^2\theta d\theta$$

$$\frac{dP_u}{d\omega_1} = \frac{1}{2\omega_{10}}\left(\frac{\omega_{10}}{\omega_1}\right)^2\frac{dP_u}{\theta d\theta} = \frac{1}{2\omega_{10}}\left(\frac{\omega_{10}}{\omega_1}\right)^2\int_0^{4\pi}\frac{dP_u}{d\Omega}d\phi.$$

This can mono chromatize radiation by collimation in θ if angular spread of beam is small. Integration gets spectral distribution.



$$\frac{dP_{u\sigma}}{d\omega} = \frac{3P_u}{\omega_{10}}\frac{\omega_1}{\omega_{10}}\left[\frac{1}{2} - \frac{\omega_1}{\omega_{10}} + \frac{3}{2}\left(\frac{\omega_1}{\omega_{10}}\right)^2\right]$$

$$\frac{dP_{u\pi}}{d\omega} = \frac{3P_u}{\omega_{10}}\frac{\omega_1}{\omega_{10}}\left[\frac{1}{2} - \frac{\omega_1}{\omega_{10}} + \frac{1}{2}\left(\frac{\omega_1}{\omega_{10}}\right)^2\right]$$

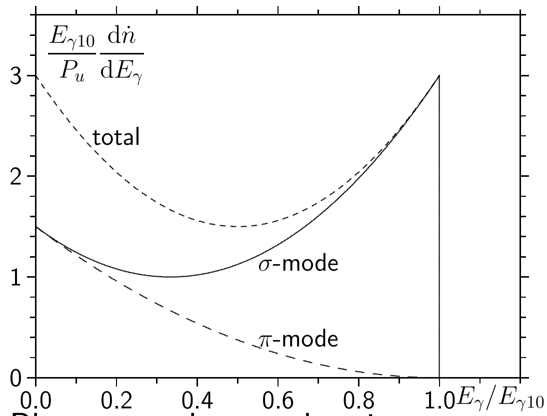
Photon energy distribution

Relation between photon energy and radiation frequency $E_\gamma = \hbar\omega,\ E_{\gamma 10} = \hbar\omega_{10}$ convert spectral angular distribution and spectrum into corresponding photon fluxes per **relative** energy band

$$\frac{d\dot{n}_u}{dE_\gamma/E_\gamma} = \frac{1}{\hbar}\frac{dP}{d\omega} = \frac{3P_0}{E_{\gamma 10}}\frac{E_{\gamma 1}}{E_{\gamma 10}}\left[1 - 2\frac{E_{\gamma 1}}{E_{\gamma 10}} + 2\left(\frac{E_{\gamma 1}}{E_{\gamma 10}}\right)^2\right]$$

or per **absolute** energy band

$$\frac{d\dot{n}_u}{dE_\gamma} = \frac{3P_u}{E_{\gamma 10}^2}\left[1 - 2\frac{E_{\gamma 1}}{E_{\gamma 10}} + 2\left(\frac{E_{\gamma 1}}{E_{\gamma 10}}\right)^2\right] = \frac{3}{2}\frac{\dot{n}_u}{E_{\gamma 10}}\left[1 - 2\frac{E_{\gamma 1}}{E_{\gamma 10}} + 2\left(\frac{E_{\gamma 1}}{E_{\gamma 10}}\right)^2\right]$$



Photon numbers and variances

$$\dot{n}_u = \frac{2P_u}{E_{\gamma 10}},\ \langle E_{\gamma\sigma}^2\rangle = \frac{21}{60}E_{\gamma 10}^2$$

$$\frac{n_u}{\text{period}} = \frac{\dot{n}_{u\sigma}\lambda_u}{c} = \frac{2\pi\alpha_f K_u^2}{3} < 1$$

$\alpha_f =$ fine struc. const. $\approx 1/137$

Collimating central angle

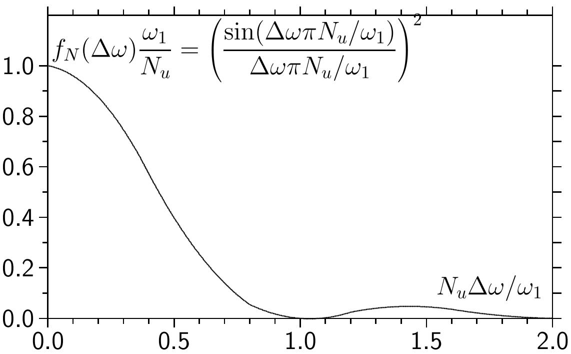
At each θ UR has narrow peak at frequency

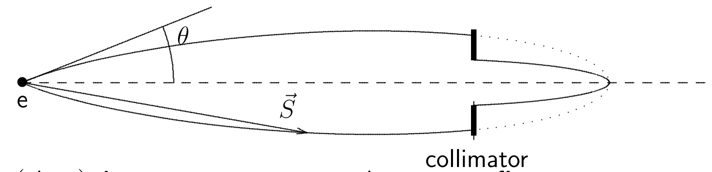
$$\omega_1 = \Omega_u \frac{2\gamma^2}{1+\gamma^2\theta^2} = \frac{\omega_{10}}{1+\gamma^2\theta^2}$$

with $\Omega_u = ck_u,\ k_u = 2\pi/\lambda_u,\ \omega_{10} = \Omega_u 2\gamma^2$. Mono chromatize by collimating at $\theta \approx 0$

$$F_{u\sigma} = \frac{3}{\pi}\frac{(1-\gamma^2\theta^2\cos(2\phi))^2}{(1+\gamma^2\theta^2)^5} \approx \frac{3}{\pi}$$

$$F_{u\sigma} = \frac{3}{\pi}\frac{(\gamma^2\theta^2\sin(2\phi))^2}{(1+\gamma^2\theta^2)^5} \approx 0$$





$f_N(\Delta\omega)$ has maximum at $\Delta\omega = 0$, first minimum at $\Delta\omega/\omega_{10} = 1/N_u$ giving bandwidth due to finite N_u

$$(\Delta\omega/\omega_{10})_N \approx 1/N_u.$$

Electrons with angular deviation θ_e give

$$\omega = \frac{\omega_{10}}{1+\gamma^2\theta_e^2} \text{ or } \left(\frac{\Delta\omega}{\omega_{10}}\right)_\theta = 1 - \frac{\omega}{\omega_{10}} \approx \gamma^2\theta_e^2.$$

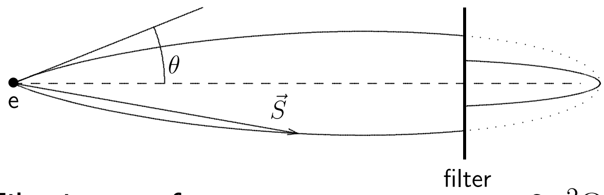
Balance both effects with undulator condition

$$\theta_e \leq \frac{1}{\gamma\sqrt{N_u}}$$

small e-emittance, parallel beam. Electron γ

$$(\Delta\omega/\omega_{10})_\gamma \approx 2\Delta\gamma/\gamma.$$

Filtering central frequency



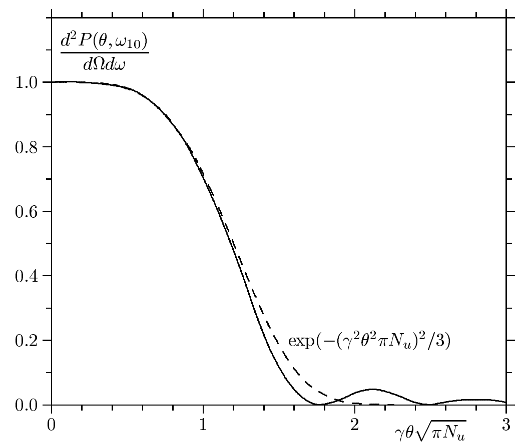
Filtering at frequency $\omega = \omega_{10} = 2\gamma^2\Omega_0$ selects central part of angular distribution where $\gamma\theta \ll 1,\ F_{u\sigma} \approx 3/\pi,\ F_{u\pi} \approx 0$

$$\Delta\omega = \omega - \omega_1 = \omega_{10} - \frac{\omega_{10}}{1+\gamma^2\theta^2} \approx \omega_{10}\gamma^2\theta^2$$

$$\frac{d^2P(\omega_{10})}{d\Omega d\omega} \approx P_u\gamma^2\frac{3}{\pi}\frac{N_u}{\omega_{10}}\left(\frac{\sin(\gamma^2\theta^2\pi N_u)}{\gamma^2\theta^2\pi N_u}\right)^2.$$

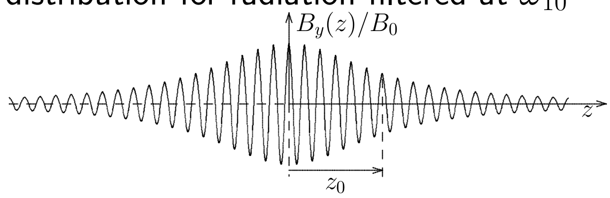
It has a divergent RMS value due to abrupt termination of undulator field. An exponential fit gives equivalent value

$$\theta_{\text{RMS-eq}} \approx \frac{1}{\gamma}\sqrt{\frac{\sqrt{3\pi}}{\pi^2 N_u}} = \frac{0.558}{\gamma}\frac{1}{\sqrt{N_u}}$$

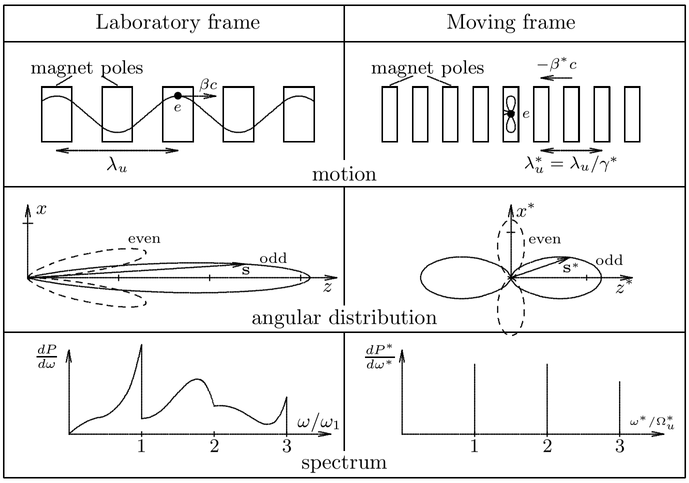


Angular distribution of UR for $\omega = \omega_{10}$

Undulator field modulated by a Lorentzian $B = B_0\cos(k_u z)/(1+z^2/z_0^2)$ gives Gaussian θ-distribution for radiation filtered at ω_{10}



Strong undulator in laboratory and moving frame



$$\omega_n = \frac{n\Omega_u^*}{\gamma^*(1-\beta^*\cos\theta)} \approx \frac{2\gamma^2(2\pi c/\lambda_u)}{1+K_u^2/2+\gamma^2\theta^2}$$

Most undulators in operation are strong, provide more power, give high frequency harmonics, can be tuned. But complicated, only qualitatively treated and properties summarized.

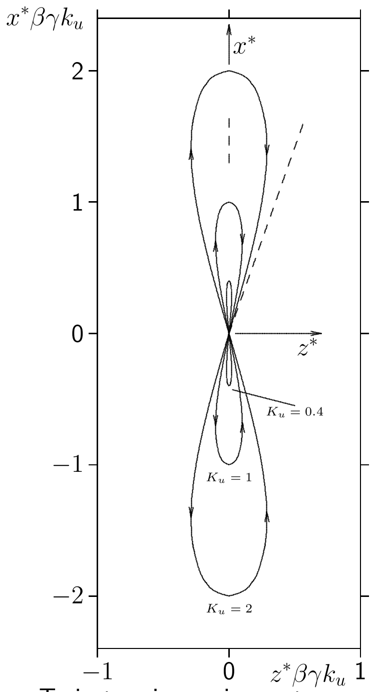
Deflects particle by > natural angle $\psi_0\gamma = K_u > 1$. Transverse motion modulates longitudinal one, reduces drift velocity $\langle v_z\rangle = \beta^* c < \beta c$

$$v^2 = \beta^2 c^2 = \dot{x}^2 + \dot{z}^2$$

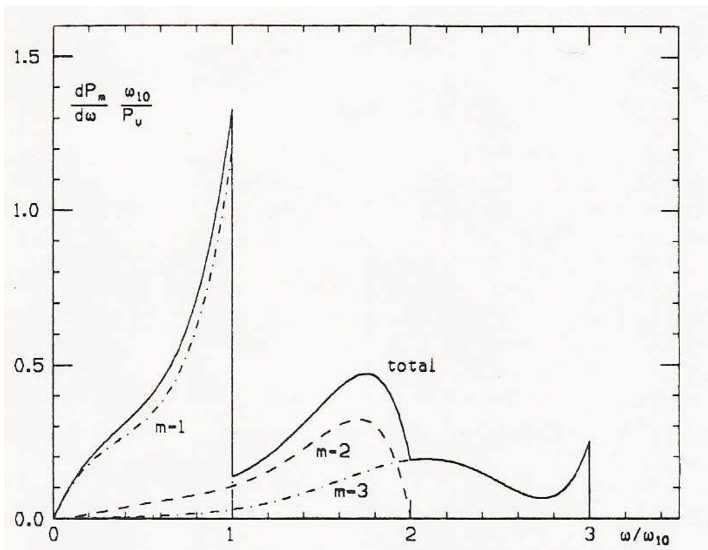
$$\beta^* \approx \left(1 - \frac{K_u^2}{4\gamma^2}\right) \ , \ \gamma^* \approx \frac{\gamma}{\sqrt{1+K_u^2/2}}.$$

In frame moving with $\beta^* c$ transverse component of particle motion contains odd and longitudinal one even harmonics of $\Omega_u^* = \gamma^* 2\pi c/\lambda_u t$. In this frame radiation is emitted in odd harmonics mainly in z^*-direction and even ones mainly perpendicular to it. Back in laboratory the odd harmonics are around z-axis and even ones around cones of opening angle $1/\gamma^*$. Frequencies are Doppler shifted

Spectrum



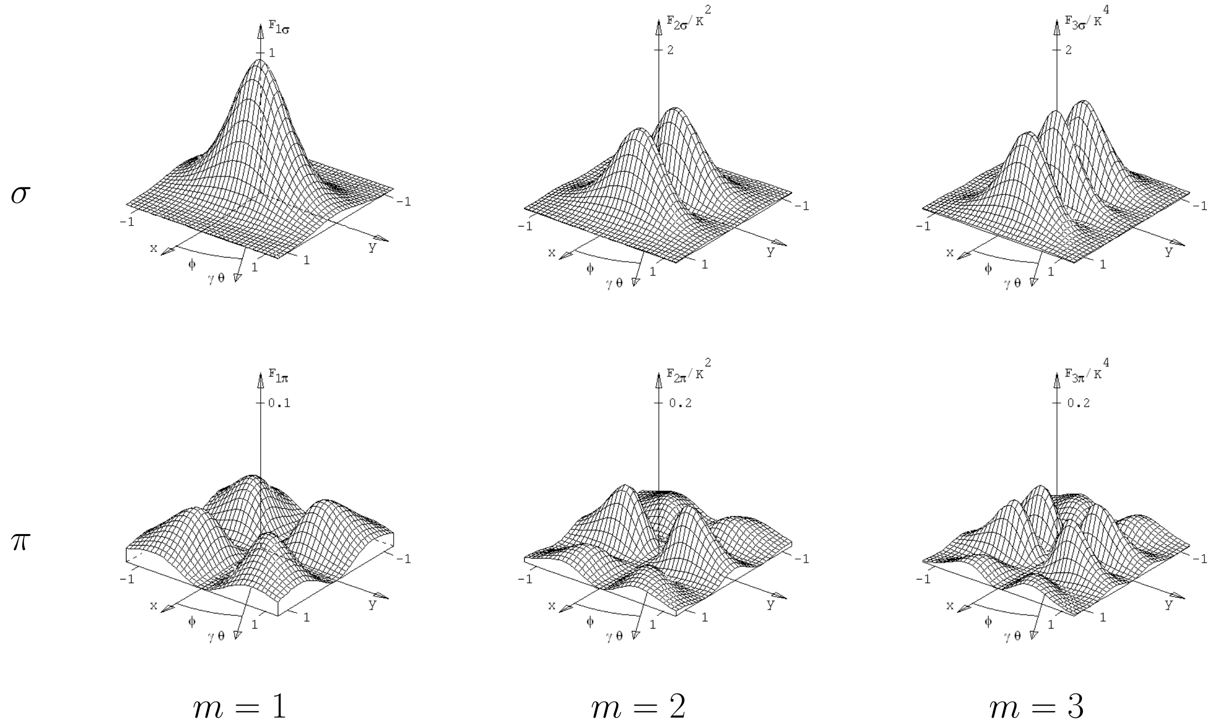
Trajectory in moving system



Strong undulator spectrum

Trajectory in moving system is composed of x-oscillation with odd and z-oscillation having even harmonics. For each the highest frequency is radiated along z-axis.

Angular power density of first 3 harmonics, each to its lowest power in K_u



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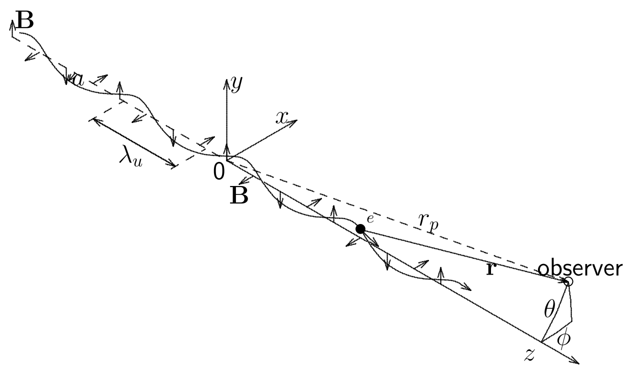
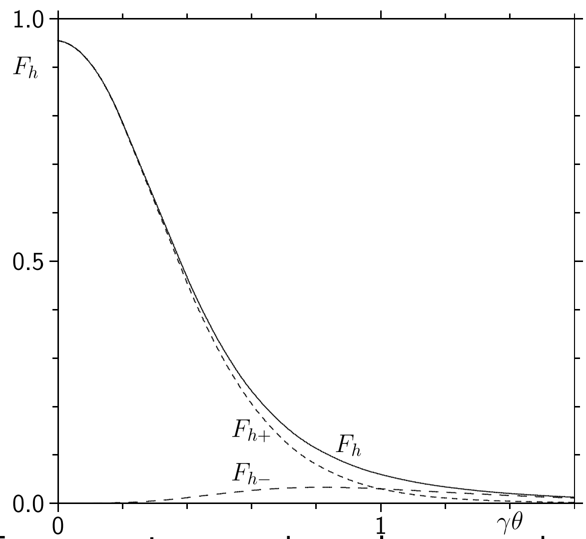
Helical undulators

Helical undulator has close to axis a transverse magnetic field with spatial period λ_u of constant absolute value but a direction rotating around z-coordinate, resulting in a helix trajectory and circularly polarized radiation. In the weak case, $K_u < 1$, it can be treated as a horizontal and a vertical undulator being longitudinally shifted by $\lambda_u/4$

$$\vec{B} = [B_x,\ B_y,\ B_z] = B_0[-\sin(k_u z), \cos(k_u z), 0]$$





Frequency integrated angular power density of total and two helicity modes radiated by a helical undulator.

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7) SR OPTICS AND BEAM DIAGNOSTICS

Beam diagnostics

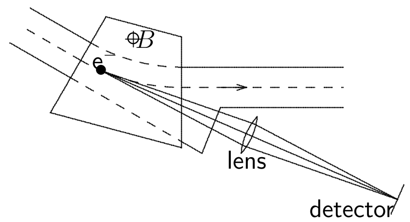
Introduction

Synchrotron radiation is used to measure the properties of the electron beam. Forming an image and measuring the opening angle of the radiation gives the cross section and transverse momentum distribution of the electrons. The longitudinal bunch shape is directly obtained from the time structure of the radiation.

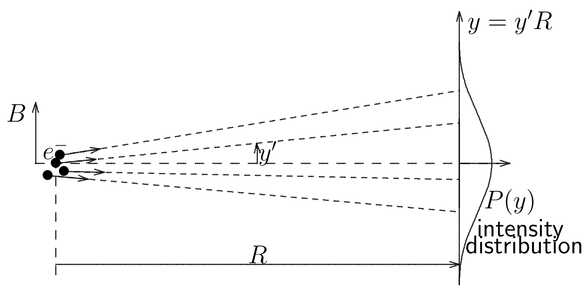
Bending magnets and undulators are used as radiation sources. The resolution is limited by diffraction due to the small opening angle. This can be improved by using short wavelength. However, for imaging visible light is used for practical reason.

Here we concentrate on the basic physics of these measurements and not on their techniques.

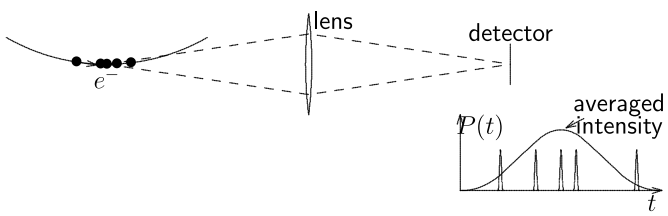
Types of measurements



Imaging with SR gives beam cross section



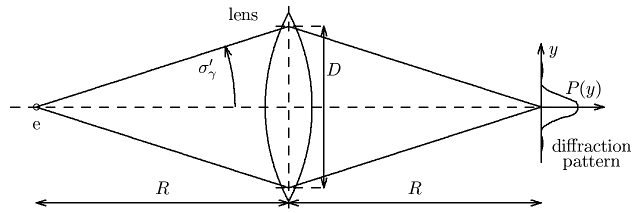
Direct observation gives angular spread (vertical for SR)



Time structure of SR gives bunch length

Imaging electron beam with SR

Qualitative diffraction



We use synchrotron radiation to form a 1:1 image of beam cross section with a single lens. Due to the small vertical opening angle of $\approx 1/\gamma$ only the central part of the lens is illuminated. Similar to optical imaging with limited lens aperture D this leads to **diffraction** limiting the resolution d (half image size)

$$d \approx \frac{\lambda}{2D/R},$$

For SR from long magnets the vertical opening (RMS) angle at low frequency $\omega \ll \omega_c$ (visible) $\sigma'_\gamma = \psi_{\sigma \mathrm{RMS}} \approx 0.41(\lambda/\rho)^{1/3}$ and take $D \approx 4\sigma_\gamma R$ we find

$$d \approx 0.3(\lambda^2\rho)^{1/3}.$$

The resolution improves with small λ and ρ. It can be quite limited in large machines

machine	ρ	λ	σ'_γ	d
	m	nm	m rad	mm
EPA (CERN)	1.43	400	2.7	0.018
LEP (CERN)	3096	400	0.21	0.24

Quantitative diffraction

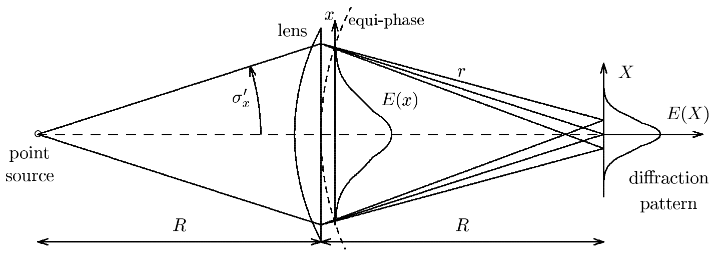
Forming a 1:1 image with SR with a lens at distance R transforms the angular radiation distribution $\tilde{E}(x', y')$ in a spacial field distribution at the lens

$$\tilde{E}(x, y) = \tilde{E}(Rx'_\gamma, Ry'_\gamma).$$

All rays between source and image have the same optical length and a sphere of radius R around the image point is an equi-phase surface on which each point is a radiation source (Huygens) of strength $\propto \tilde{E}(x, y)$ which propagates to the image giving

$$\delta\tilde{E}(X, Y) \propto \tilde{E}(x, y)\mathrm{e}^{i(kr - \omega t)}$$

$$k = 2\pi/\lambda$$



$$r = \sqrt{R^2 - 2xX - 2yY + X^2 + Y^2}$$
$$\approx R\left(1 - \frac{xX}{R^2} - \frac{yY}{R^2} + \ldots\right).$$

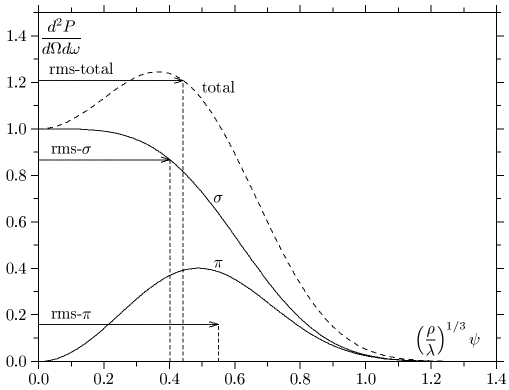
Fraunhofer diffraction, with $x' = x/R$ get image field

$$kr = kR - \frac{xXk}{R} - \frac{yYk}{R} = kR - x'Xk - y'Yk$$

$$\tilde{E}(X, Y) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}(Rx', Ry')\mathrm{e}^{-i\left(x'kX + y'kY\right)}dx'dy'.$$

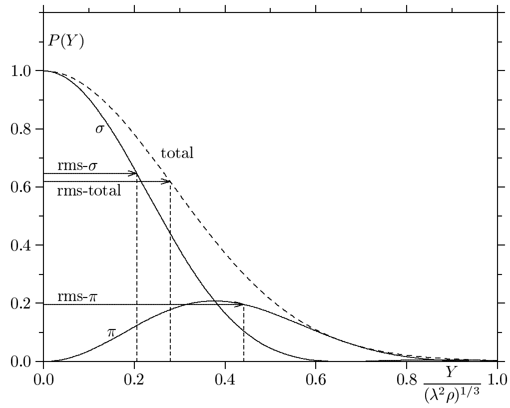
The field distribution at the image is the Fourier transform of the angular distribution.

Fraunhofer diffraction of SR



SR vertical angular distr. at $\omega \ll \omega_c$

$$\sigma'_{y,\sigma} = 0.4097\,(\lambda/\rho)^{\frac{1}{3}}$$
$$\sigma'_{y,\pi} = 0.5497\,(\lambda/\rho)^{\frac{1}{3}}$$
$$\sigma'_{y,tot} = 0.4488\,(\lambda/\rho)^{\frac{1}{3}}$$

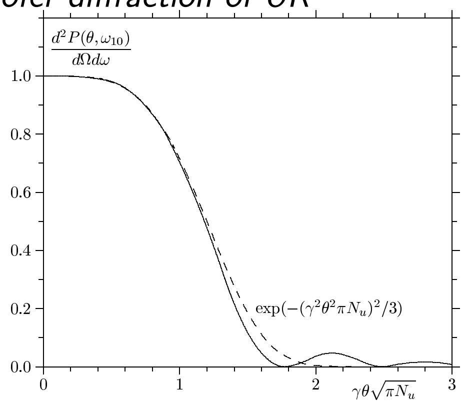


SR vertical diffraction at $\omega \ll \omega_c$

$$\sigma_{Y,\sigma} = 0.206(\lambda^2\rho)^{1/3}$$
$$\sigma_{Y,\pi} = 0.429(\lambda^2\rho)^{1/3}$$
$$\sigma_{Y,tot.} = 0.279(\lambda^2\rho)^{1/3}$$

This confirms the simple calculation done before. Selecting the σ-mode with a filter improves the resolution.

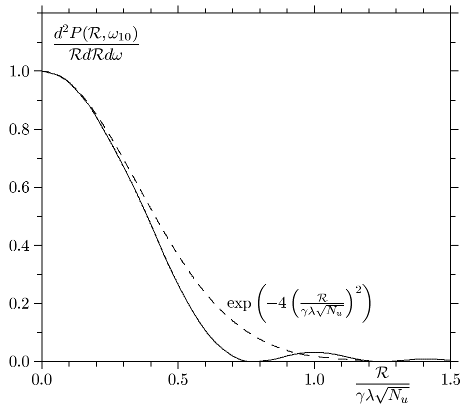
Fraunhofer diffraction of UR



Angular distribution of UR at ω_{10}

θ_{rms} diverges but exponential fit has

$$\theta_{\text{RMS-eq}} \approx \frac{1}{\pi\gamma}\sqrt{\frac{\sqrt{3\pi}}{N_u}} = \frac{0.558}{\gamma}\frac{1}{\sqrt{N_u}}$$
$$\sigma'_x = \sigma'_y = \frac{\theta_{\text{RMS-eq}}}{\sqrt{2}}$$



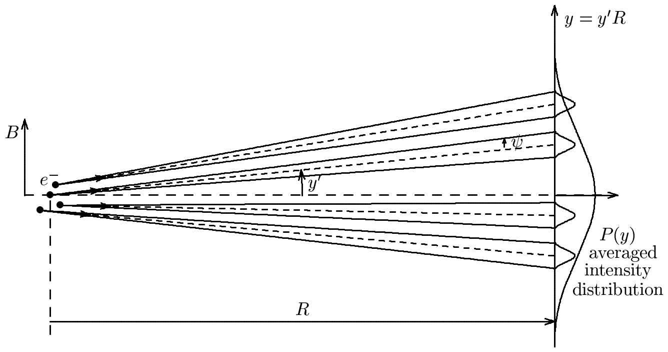
Diffraction for UR at ω_{10}

$\mathcal{R}_{\text{RMS}}$ diverges but the exponential fit has

$$\mathcal{R}_{\text{RMS-eq}} \approx \frac{\gamma\lambda_{10}\sqrt{N_u}}{2}$$
$$\sigma_X = \sigma_Y = \frac{\mathcal{R}_{\text{RMS-eq}}}{\sqrt{2}} = 0.354\gamma\lambda_{10}\sqrt{N_u}$$

which agrees approximately with the simple calculation.

Direct observation



Observing SR directly gives angular spread of the particles in the beam. Neglecting the natural opening angle of SR, the emitted photons have same emittance and propagation as the electrons in a straight section which can be described by the Twiss parameter $\beta_w, \alpha_w, \gamma_w$,

$$\alpha_y = -\beta_y'/2,\ \gamma_y = (1+\alpha_y^2)/\beta_y.$$

From their values at source we obtain $\beta_y(R)$ on screen

$$\beta_y(R) = \beta_y(0) - 2\alpha_y(0)R + \gamma_y(0)R^2.$$

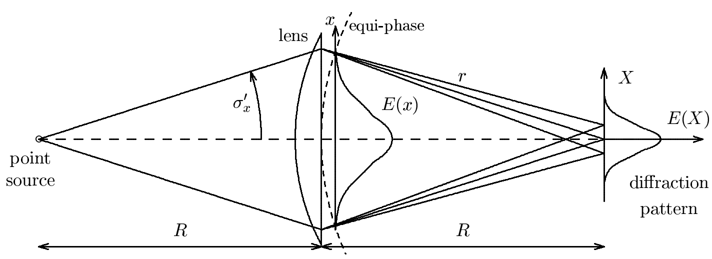
Measured spot size $\sigma_{my}(R)$ on screen is a convolution between this idealized photon beam having $\sigma_{\gamma y}(R) = \sqrt{\epsilon_y \beta_y(R)}$ and its natural angular distribution with opening angle $\sigma'_{\gamma y}$

$$\sigma_{my}^2(R) = R^2\sigma'^2_{\gamma y} + \sigma^2_{\gamma y}(R).$$

Deconvolution gives electron emittance

$$\epsilon_y = \frac{\sigma_{my}^2 - R^2\sigma'_{\gamma y}}{\beta_y(R)}$$

Emittance of a photon beam



Imaging photons of Gaussian angular distr.

$$\tilde{E}(x') = E_0 \mathrm{e}^{-\frac{x'^2}{2\sigma_E'^2}} \rightarrow \tilde{E}_{lens}(x) = E_1 \mathrm{e}^{-\frac{x^2}{2(R\sigma'_E)^2}}$$

$$P(x') \propto |\tilde{E}(x')|^2 = E_0^2 \mathrm{e}^{-\frac{x'^2}{\sigma_E^2}} = E_0^2 \mathrm{e}^{-\frac{x'^2}{2\sigma_\gamma'^2}}$$

With $P \propto |E|^2$ RMS angles of field and power (photons) are related $\sigma'_E = \sqrt{2}\sigma'_\gamma$. Image field:

$$\tilde{E}(X) \propto \int \mathrm{e}^{-\frac{x'^2}{2(\sigma'_E)^2}} \cos(kXx')dx' = \sqrt{2\pi}\sigma'_E \mathrm{e}^{-\frac{(kX\sigma'_E)^2}{2}}$$

Gaussian image with RMS values of field, power $\sigma_E = \sqrt{2}\sigma_\gamma$, $\sigma_P = \sigma_\gamma = 1/(2k\sigma'_\gamma)$. The product $\sigma_\gamma\sigma'_\gamma$ is the photon emittance

$$\epsilon_\gamma = \sigma_\gamma\sigma'_\gamma = \frac{1}{2k} = \frac{\lambda}{4\pi}$$

Emittances of other distributions larger

source	SR	UR
σ_γ	$0.21(\lambda^2\rho)^{1/3}$	$\sim \frac{\lambda\gamma\sqrt{2N_u}}{4}$
σ'_γ	$0.41\left(\frac{\lambda}{\rho}\right)^{1/3}$	$\sim \frac{1}{\pi\gamma}\sqrt{\frac{\sqrt{3}\pi}{2N_u}}$
ϵ_γ	$1.06\frac{\lambda}{4\pi}$	$\sim 1.75\frac{\lambda}{4\pi}$
$\sigma_\gamma/\sigma'_\gamma$	$0.5(\lambda\rho^2)^{1/3}$	$\sim L_u$

Measuring $\sigma_{x/y}$ and $\sigma'_{x/y}$

Emittance $\epsilon_y = \sigma_y \sigma'_y / \sqrt{1+\alpha_y^2} \approx \sigma_y \sigma'_y$ can be obtained by measuring σ_y and σ'_y has the limitation $\epsilon_\gamma \geq \lambda/4\pi$ but some corrections can be made by a deconvolution. The direct observation can easily be made with x-rays which improves the resolution.

Measuring $\sigma_{x/y}$ or $\sigma'_{x/y}$

If only one of the two is measured the emittance is obtained with the lattice functions

$$\epsilon_y = \frac{\sigma_w^2}{\beta_y} = \frac{\sigma'^2_y}{\gamma_y} \approx \sigma'^2_y \beta_y \ , \ \frac{\sigma_y}{\sigma'_y} = \frac{\beta_y}{\sqrt{1+\alpha_y^2}}.$$

$$\text{where } \alpha_y = -\frac{1}{2}\beta'_y \ , \ \gamma_y = \frac{1+\alpha_y}{\beta_y}.$$

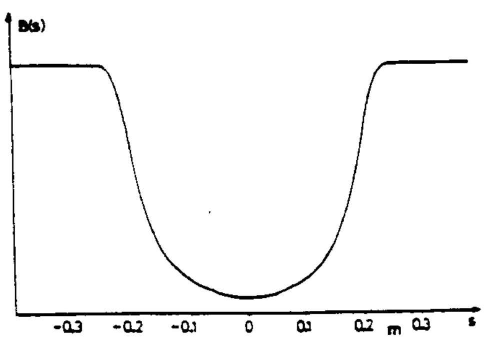
Diffraction limits the accuracy of the σ_y measurement and the finite opening angle the one of the σ'_y measurements. Since for a fixed ϵ_y

$$\sigma_y = \sqrt{\epsilon_w \beta_y} \ , \ \sigma'_y = \sqrt{\epsilon_{\gamma y} \gamma_y} \approx \sqrt{\epsilon_y / \beta_y}$$

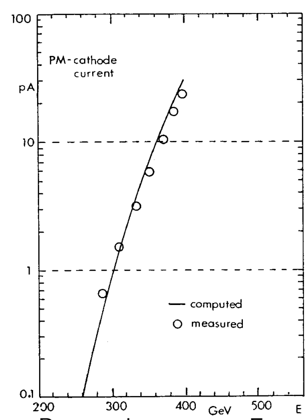
a high β_y is desired in the first and a low β_y in the second measurement. In practice the limited knowledge of the lattice function is often the largest error. It is helpful to measure beta at the source, e.g. by varying a quadrupole and observing the tune change.

Proton beam profile with 'anti short magnet'

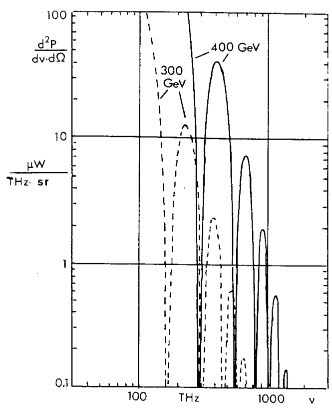
SR from SPS protons at 315 GeV is weak and low frequency ($\lambda_c = 82\,\mu$m) bad resolution. Short magnet of $L_{sm} \approx 0.135$ m gives visible light. The 'short magnet' is replaced by 'short gap' between dipoles resulting in about spectrum being proportional to Fourier transformed magnetic field $B(s)$. Measured power accepted by detector vs. proton energy agrees with calculation



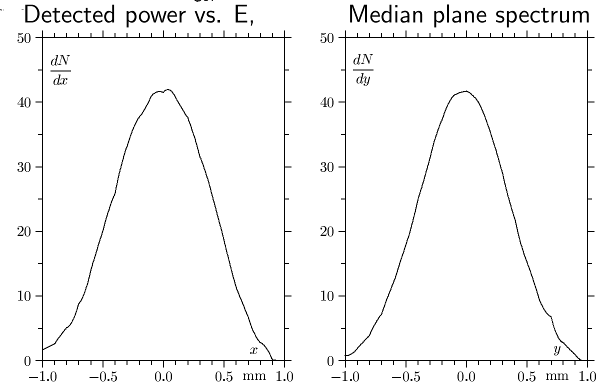
Gap field $B(s)$



Detected power vs. E,



Median plane spectrum



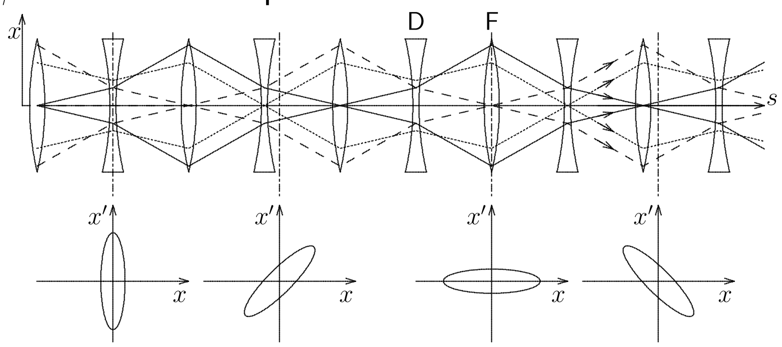
Beam profiles

8) APPLICATIONS

Many particles

Spatial coherence

A circulating particle is kept close to nominal orbit by lenses and its excursions x_i are observed in different locations each turn i. They are on ellipses with different orientations but constant area $A = \pi\epsilon_x$ and emittance ϵ_x. Quantum excitation and damping determine the emittance a single or many electrons. In F-quads maximum of excursion $\hat{x}$ is large and of angles small and their ratio is β-function. In D-quads is reversed.



Ideally electron emittance is comparable to the one of radiation, $\epsilon_x = \epsilon_y \approx \lambda/(4\pi)$ difficult for short wave length. Furthermore, the two should be matched, beta function equal size/angle ratio of radiation

$$(\sigma_y/\sigma_y')_\gamma \approx (\sigma_y/\sigma_y')_e \approx \beta_y$$
$$SR: \beta_y \approx 0.5(\lambda\rho^2)^{1/3} \approx \rho/\gamma$$
$$UR: \beta_x = \beta_y \approx L_u$$

Both cases demand $\beta_{y/x} \approx$ source length giving a small number for SR but for UR about the same as for optimal aperture.

If conditions are satisfied we have spatially coherent radiation, diffraction limit, can not be improved.

Temporal coherence

Radiation from diffraction limited beam, observed at any point, comes from all electrons; spatial coherence. If also bunch length σ_s is shorter than λ, field contributions from all N particles ad up giving a power N times larger than for longer bunch: temporal coherence.

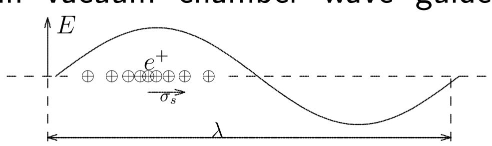
inc. : $\sigma_s > \lambda,\ P = \sum P_i = NP_i$

coh. : $\sigma_s < \lambda,\ E = \sum E_i = NE_i$

$$P \propto N^2 E_i^2 = N^2 P_i$$

Radiation at $\lambda > \sigma_s$ is enhanced.

But, long waves don't propagate in vacuum chamber wave guide.

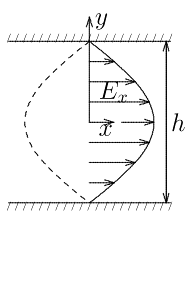


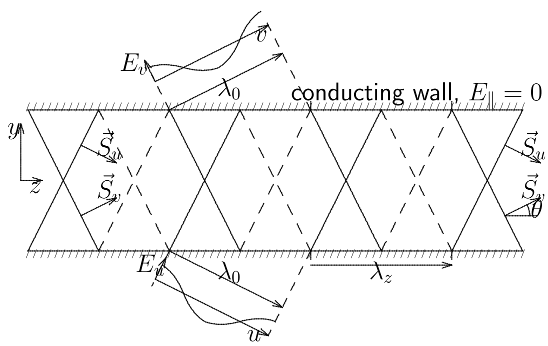
The small SR opening angle makes this occurs at shorter wavelengths than expected from 'cut-off'. Conducting plates at $y = \pm h/2$ impose there $E_x = 0$ for wave propagating in z-direction.

$$\vec{k} = [k_y,\ k_z] = k[\sin\theta,\ \cos\theta],\ k = 2\pi/\lambda$$
$$\lambda_y = \frac{2h}{n},\ k_y = \frac{2\pi}{\lambda} = \frac{\pi n}{h} = k\sin\theta, \lambda = \frac{2h\sin\theta}{n}$$

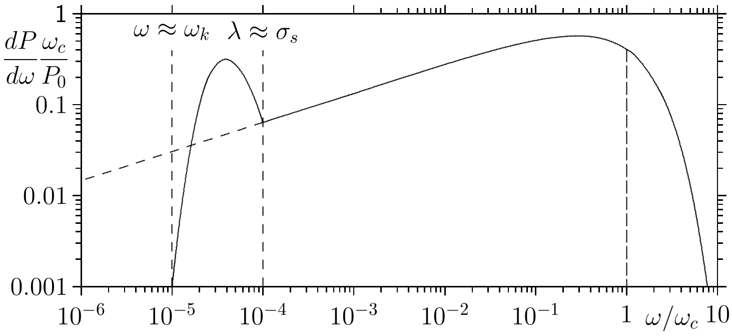
$n = 1$ lowest mode, opening angle θ at $\omega \ll \omega_c$

$$\theta \approx 0.41(\lambda/\rho)^{1/3} \ll 1,\ \lambda = 2h\theta = 2\cdot 0.41(\lambda/\rho)^{1/3}$$
$$\lambda_{co} = 0.82^{3/2}\sqrt{h^3/\rho} = 0.74\sqrt{h^3/\rho}.$$

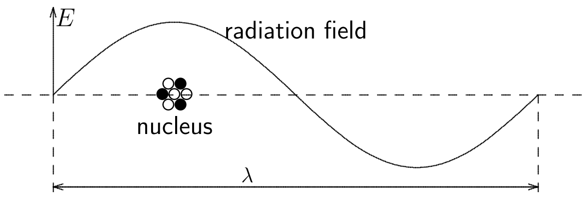




If $\lambda_{co} > \sigma_s$ lower part of spectrum is enhanced



Reducing η_c=revolution time vs. momentum change $\Delta T_{rev}/T_{rev} = \eta_c \Delta p/p_0$, gives short bunches, difficult. However, Z protons in ions radiate coherently, but smaller γ. Lead ions have twice damping rate of protons in same field.



Quantum excitation of SR establishes a spread in particle energy and through the dependence of revolution frequency on energy, also a finite bunch length

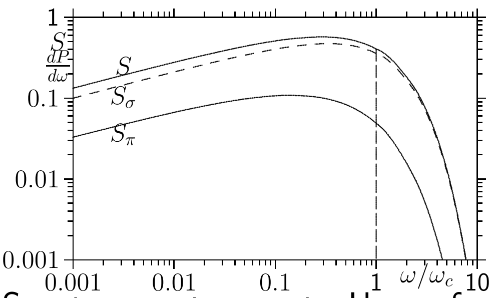
$$\sigma_\epsilon^2 = \frac{\langle \Delta E^2 \rangle}{E^2} = \frac{55\sqrt{3}}{96} \frac{h\gamma^2}{2\pi m_0 c\rho}$$

$$\sigma_s = \frac{c|\eta_c|}{\omega_s}\sigma_\epsilon = \sqrt{\frac{-2\pi E_0 \eta_c}{\omega_{rev}\omega_{RF} e\hat{V}\cos\phi_s}}\sigma_\epsilon$$

RF-voltage $\hat{V}$, frequency ω_{RF}, momentum compaction α_c, $\eta_c = \alpha_c - 1/\gamma^2$. Increasing ρ, $\hat{V}$, decreasing α_s hard. Coherent radiation from short bunches seen in linacs, some in storage rings. Coherent infrared radiation is interesting. Bunch current stable periodic signal, Radiation at $\lambda > \sigma_s$ has line spectrum at ω_{RF}/ω_{rev}. High stability, long coherence length.

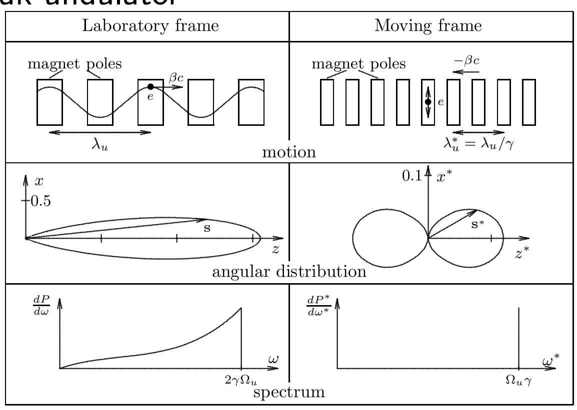
Quantum effects

Electron circulating in storage ring resemble atom, but in very high quantum state where classical treatment is justified. But, photon emission gives finite energy spread and emittance, clearly visible. Quantum effect influence macroscopic engineering.



Spectrum not correct. Upper frequencies unlimited and reach photon energies $E_\gamma = \hbar\omega > E_e$ exceeding the electron energy. We should consider photon momentum, as in Compton back scattering.

Used laboratory and moving frame to explain weak undulator

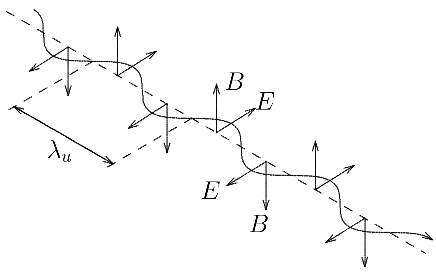


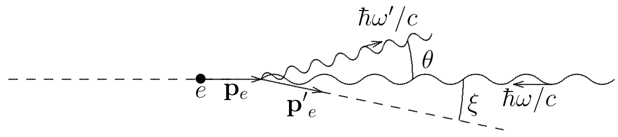
$$E_x^* = -\beta c B_y^* = -\gamma\beta c B_0 \cos(k_u^* z^* + \Omega_u^* t^*)$$

In moving system the undulator is like an EM wave, but with velocity βc, doing Compton scattering on the electron. Compare this with real case.

Compton scattering

The photon momentum, is taken into account in Compton back scattering. In some experiments a laser beam collides head-on with electrons of a storage ring. This is an electromagnetic wave with fields perpendicular to each other and direction of propagation and period length λ. It resembles undulator field but 2 differences: laser beam has E and B-field both deflecting the electron and move against electrons emitting radiation at twice the frequency compared to static undulator of same period length.





Energy and momentum conservation with $\gamma,\ \gamma' \gg 1$

$$\begin{aligned} m_0c^2\gamma + \hbar\omega &= m_0c^2\gamma' + \hbar\omega' \\ m_0c\beta\gamma - \frac{\hbar\omega}{c} &= m_0c\beta'\gamma' \cos\xi + \frac{\hbar\omega'}{c}\cos\theta \\ 0 &= -m_0c\beta'\gamma'\sin\xi + \frac{\hbar\omega'}{c}\sin\theta \end{aligned}$$

$$\omega' = \omega\frac{4\gamma^2}{1+\gamma^2\theta^2+4\hbar\omega\gamma/m_0c^2}$$

Quantum correction for laser, high field and harmonics correction for undulator

$$\omega_1 = m\frac{2\pi c}{\lambda_u}\frac{2\gamma^2}{(1+K_u^2/2+\gamma^2\theta^2)}$$