Double-real radiation in top quark pair hadroproduction

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MC, arXiv: 1101.0642 [hep-ph]

Heavy Particles at the LHC, January 6th 2011

Cross Sections for Top Production

$$d\sigma^{\mathrm{NNLO}}_{t\bar{t}+X} = d\sigma^{VV} + d\sigma^{RV} + d\sigma^{RR}$$

MC '11

1-loop amplitudes
from tt+j:

Dittmaier, Uwer, Weinzierl '07
Bevilacqua, MC, Papadopoulos, Worek '10
Melnikov, Schulze '10

Subtraction terms: (long list of papers)

Bern et al. '94 – '99, Catani & Grazzini '00

MC, Mitov, Moch '07 '08
MC '08
Bonciani, Ferroglia, Gehrmann, Maitre,
von Manteuffel, Studerus '08 – '10

Ferroglia, Neubert,
Pecjak, Yang '09

Kniehl, Korner, Merebashvili, Rogal '08

Virtual Point

Two-loop amplitude for gg -> tt MC, Bärnreuther, in preparation

Bonciani, Ferroglia, Gehrmann, von Mantuffel, Studerus '10

Ferroglia, Neubert, Pecjak, Yang '09

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
A_{LC}	10.74942557	18.69389337	156.8237244	262.1482588	12.72180680
A	10.74942557	18.69389337	-156.8237244	262.1482588	12.72180680
B	-21.28599123	-55.99039551	-235.0412564	1459.833288	-509.6019155
C		-6.199051597	-68.70297402	-268.1060373	804.0981895
D			94.08660818	-130.9619794	-283.3496755
E_l		-12.54099650	18.20646589	27.95708293	-112.6060988
E_h			0.012907497	11.79259573	-47.68412574
F_l		24.83365643	-26.60868620	-50.75380859	125.0537955
F_h			0.0	-23.32918072	132.5618962
G_l			3.099525798	67.04300456	-214.1081462
G_h				0.0	-179.3374874
H_l			2.388761238	-5.452031425	3.632861953
H_{lh}				-0.004302499	-3.945712447
H_h					0.00439856
I_l			-4.730220272	10.81032548	-7.182940516
I_{lh}				0.0	7.780900470
I_h					0.0
A^{Poles}	10.749	18.694	−156.82	262.15	
B^{Poles}	−21.286	−55.990	−235.04	1459.8	
C^{Poles}		−6.1991	−68.703	−268.11	
D^{Poles}			94.087	−130.96	
E_l^{Poles}		−12.541	18.207	27.957	
E_h^{Poles}			0.012908	11.793	
F_l^{Poles}		24.834	−26.609	−50.754	
F_h^{Poles}			0.0	−23.329	
G_l^{Poles}			3.0995	67.043	
G_h^{Poles}				0.0	
H_l^{Poles}			2.3888	−5.4520	
H_{lh}^{Poles}				−0.0043025	
H_h^{Poles}					
I_l^{Poles}			−4.7302	10.810	
I_{lh}^{Poles}				0.0	
I_h^{Poles}					

Cross Sections for Top Production

$$\sigma^{RR}_{ab\to t\bar{t}cd}(s, m^2, \mu^2 = m^2, \alpha_s, \epsilon) = \frac{\alpha_s^4}{m^2} f_{ab\to t\bar{t}cd}(\beta, \epsilon)$$

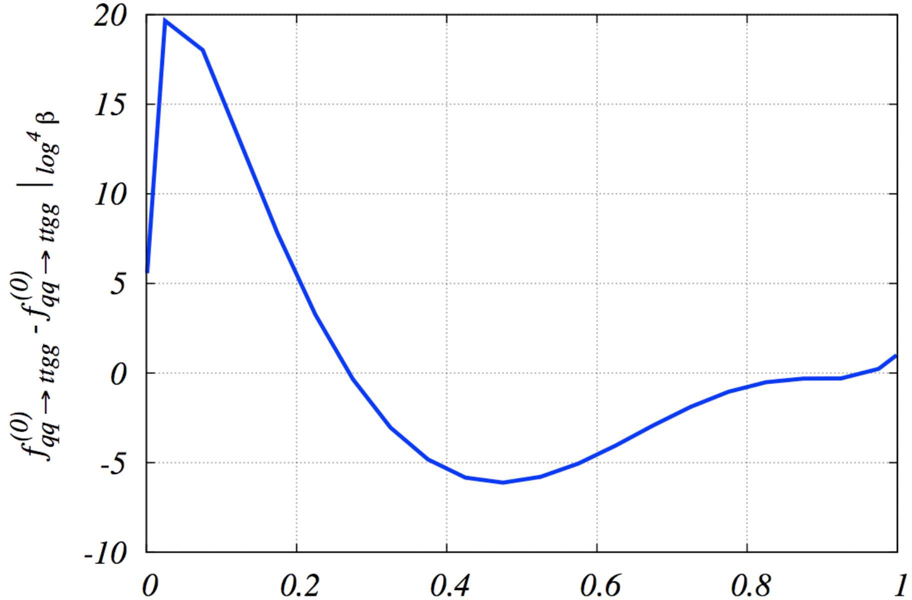
$$\beta = \sqrt{1 - \frac{4m^2}{s}}$$

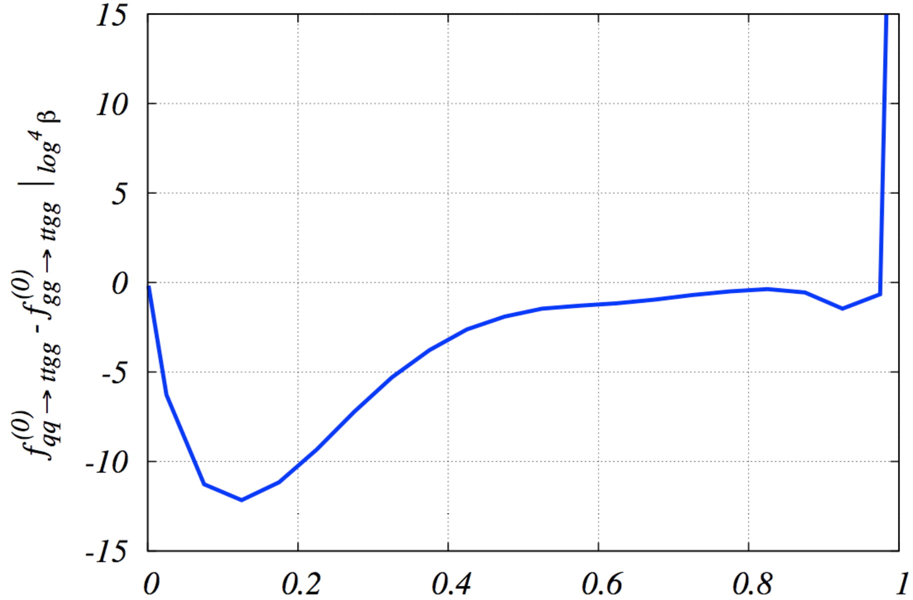
- $gg \to t\bar{t}gg$
- $gg \to t\bar{t}q\bar{q}$
- $q\bar{q} \to t\bar{t}gg$
- $q\bar{q} \to t\bar{t}q'\bar{q}', \; q' \neq q$

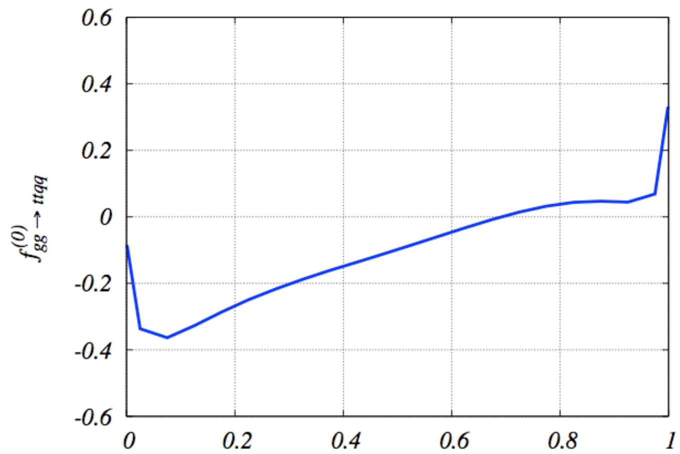
β	ϵ^{-1}			ϵ^{0}		
0.001	$+2.947 \times 10^{0}$	$\pm$	1.4×10^{-3}	$+5.218 \times 10^{1}$	$\pm$	2.9×10^{-2}
0.025	$+1.196 \times 10^{1}$	$\pm$	4.4×10^{-3}	$+1.143 \times 10^{2}$	$\pm$	4.6×10^{-2}
0.075	$+1.293 \times 10^{1}$	$\pm$	4.8×10^{-3}	$+8.660 \times 10^{1}$	$\pm$	3.6×10^{-2}
0.125	$+1.127 \times 10^{1}$	$\pm$	4.5×10^{-3}	$+5.975 \times 10^{1}$	$\pm$	3.0×10^{-2}
0.175	$+9.161 \times 10^{0}$	$\pm$	4.1×10^{-3}	$+3.949 \times 10^{1}$	$\pm$	2.9×10^{-2}
0.225	$+7.062 \times 10^{0}$	$\pm$	4.3×10^{-3}	$+2.454 \times 10^{1}$	$\pm$	2.6×10^{-2}
0.275	$+5.149 \times 10^{0}$	$\pm$	3.1×10^{-3}	$+1.375 \times 10^{1}$	$\pm$	1.9×10^{-2}
0.325	$+3.469 \times 10^{0}$	$\pm$	2.7×10^{-3}	$+6.118 \times 10^{0}$	$\pm$	1.6×10^{-2}
0.375	$+2.063 \times 10^{0}$	$\pm$	2.4×10^{-3}	$+9.998 \times 10^{-1}$	$\pm$	1.5×10^{-2}
0.425	$+9.289 \times 10^{-1}$	$\pm$	2.1×10^{-3}	-2.245×10^{0}	$\pm$	1.3×10^{-2}
0.475	$+7.292 \times 10^{-2}$	$\pm$	2.2×10^{-3}	-3.973×10^{0}	$\pm$	1.3×10^{-2}
0.525	-5.197×10^{-1}	$\pm$	2.7×10^{-3}	-4.575×10^{0}	$\pm$	1.1×10^{-2}
0.525	-5.197×10^{-1}	$\pm$	2.7×10^{-3}	-4.575×10^{0}	$\pm$	1.1×10^{-2}
0.575	-8.661×10^{-1}	$\pm$	1.7×10^{-3}	-4.393×10^{0}	$\pm$	1.1×10^{-2}
0.625	-9.995×10^{-1}	$\pm$	1.6×10^{-3}	-3.711×10^{0}	$\pm$	8.9×10^{-3}
0.675	-9.525×10^{-1}	$\pm$	1.5×10^{-3}	-2.772×10^{0}	$\pm$	8.9×10^{-3}
0.725	-7.695×10^{-1}	$\pm$	1.9×10^{-3}	-1.823×10^{0}	$\pm$	8.5×10^{-3}
0.775	-5.075×10^{-1}	$\pm$	1.1×10^{-3}	-1.023×10^{0}	$\pm$	5.6×10^{-3}
0.825	-2.310×10^{-1}	$\pm$	8.5×10^{-4}	-5.028×10^{-1}	$\pm$	4.5×10^{-3}
0.875	-2.577×10^{-2}	$\pm$	6.8×10^{-4}	-3.085×10^{-1}	$\pm$	3.5×10^{-3}
0.925	$+1.082 \times 10^{-3}$	$\pm$	5.2×10^{-4}	-2.996×10^{-1}	$\pm$	2.6×10^{-3}
0.975	-2.722×10^{-1}	$\pm$	3.0×10^{-4}	$+2.370 \times 10^{-1}$	$\pm$	1.5×10^{-3}
0.999	-2.158×10^{-1}	$\pm$	1.9×10^{-4}	$+9.970 \times 10^{-1}$	$\pm$	1.5×10^{-3}

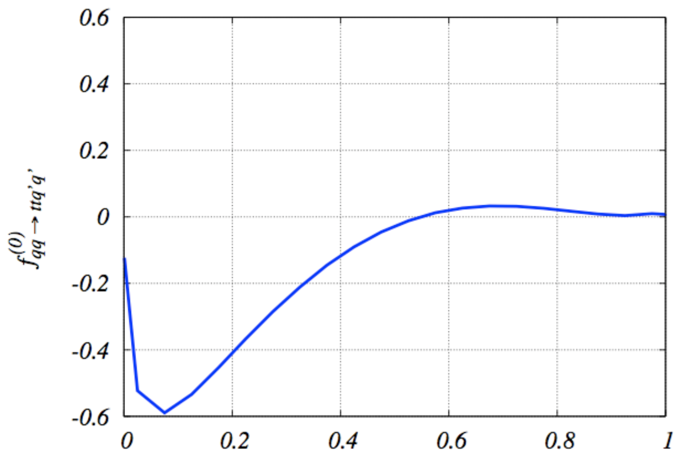
β	$f^{(0)}\vert_{10^{-7}} - f^{(0)}\vert_{10^{-6}}$	$\Delta f^{(0)}\vert_{10^{-7}} + \Delta f^{(0)}\vert_{10^{-6}}$	$f^{(\epsilon,0)}$
0.001	$+1.9 \times 10^{-2}$	4.8×10^{-2}	-3.7×10^{-8}
0.025	$+6.5 \times 10^{-2}$	8.8×10^{-2}	-3.0×10^{-5}
0.075	$+8.5 \times 10^{-2}$	7.1×10^{-2}	-3.9×10^{-4}
0.125	$+8.8 \times 10^{-2}$	6.6×10^{-2}	-1.3×10^{-3}
0.175	$+7.9 \times 10^{-2}$	5.5×10^{-2}	-2.7×10^{-3}
0.225	$+6.3 \times 10^{-2}$	4.8×10^{-2}	-4.8×10^{-3}
0.275	$+5.9 \times 10^{-2}$	3.8×10^{-2}	-6.6×10^{-3}

Cross Sections for Top Production









Cross Sections for Top Production

$$\sigma^{RR}_{gg\to t\bar{t}gg} = 10C_A^2\frac{1}{\epsilon^4}\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{gg\to t\bar{t}} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right),$$

$$\sigma^{RR}_{q\bar{q}\to t\bar{t}gg} = 2C_F(C_A+4C_F)\frac{1}{\epsilon^4}\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{q\bar{q}\to t\bar{t}} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right),$$

$$\sigma^{RR}_{gg\to t\bar{t}q\bar{q}} = \mathcal{O}\left(\frac{1}{\epsilon^3}\right),$$

$$\sigma^{RR}_{q\bar{q}\to t\bar{t}q'\bar{q}'} = \mathcal{O}\left(\frac{1}{\epsilon^3}\right),$$

$$\sigma^{VV}_{gg\to t\bar{t}} = (4C_A^2+4C_A^2)\frac{1}{\epsilon^4}\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{gg\to t\bar{t}} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

$$\sigma^{VV}_{q\bar{q}\to t\bar{t}} = (4C_F^2+4C_F^2)\frac{1}{\epsilon^4}\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{q\bar{q}\to t\bar{t}} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

$$\sigma^{RV}_{gg\to t\bar{t}g} = -18C_A^2\frac{1}{\epsilon^4}\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{gg\to t\bar{t}} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right),$$

$$\sigma^{RV}_{q\bar{q}\to t\bar{t}g} = -2C_F(C_A+8C_F)\frac{1}{\epsilon^4}\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{q\bar{q}\to t\bar{t}} + \mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

$$\sigma^{RR}_{gg\to t\bar{t}gg} = \frac{12500}{3}C_A^2\log^4\beta\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{gg\to t\bar{t}} + \mathcal{O}\left(\log^3\beta\right),$$

$$\sigma^{RR}_{q\bar{q}\to t\bar{t}gg} = \frac{2500}{3}C_F(C_A+4C_F)\log^4\beta\left(\frac{\alpha_s}{4\pi}\right)^2\sigma^B_{q\bar{q}\to t\bar{t}} + \mathcal{O}\left(\log^3\beta\right)$$

leading threshold behavior

cancellation of leading divergences

$$\sigma^{\mathrm{NNLO}}_{gg\to t\bar{t}+X} = \mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

$$\sigma^{\mathrm{NNLO}}_{q\bar{q}\to t\bar{t}+X} = \mathcal{O}\left(\frac{1}{\epsilon^3}\right)$$

Subtraction Schemes

- NLO
  1) Catani-Seymour (smooth interpolation between limits, remapping of phase space allows for arbitrary phase space generators)
  2) FKS (Frixione-Kunszt-Signer) (decomposition of phase space according to collinear singularities, energy-angle parameterization, residue subtraction)
- NNLO general and successful
  1) Sector Decomposition (Binoth,Heinrich '04, Anastasiou,Melnikov,Petriello '05)
  2) Antenna Subtraction (Gehrmann De-Ridder, Gehrmann, Glover '05)
- NNLO special for colourless particles – Catani, Grazzini '07
- NNLO in the making (main problem - integration of subtraction terms)
  1) "generalized" Catani-Seymour – Weinzierl '03 (many others unfinished)
  2) "generalized" FKS – Somogyi, Trocsanyi, Del Duca '06
  3) "new stuff with variable change" - Anastasiou, Herzog, Lazopoulos '10

STRIPPER (SecToR ImProved Phase space for real Radiation) MC '10

STRIPPER Ideas

About the phase space:

1. parameterization of the massless system with energies and angles modified to allow for a description of all collinear singular configurations with only two variables

2. level 1 decomposition into sectors allowing for only one type of collinear singularities

3. level 2 decomposition into sectors defining the order of singular limits

About the subtraction terms:

1. Subtraction at the endpoint derived from known soft and collinear limits of QCD amplitudes

2. No analytic integration of the subtraction terms

Phase Space

$$d\Phi_4 = \frac{d^{d-1}k_1}{(2\pi)^{d-1}2k_1^0}\frac{d^{d-1}k_2}{(2\pi)^{d-1}2k_2^0}\frac{d^{d-1}q_1}{(2\pi)^{d-1}2q_1^0}\frac{d^{d-1}q_2}{(2\pi)^{d-1}2q_2^0}(2\pi)^d\delta^{(d)}(k_1+k_2+q_1+q_2-p_1-p_2)$$

$$\begin{aligned}
p_1^\mu &= \frac{\sqrt{s}}{2}(1,0,0,1)\ ,\\
p_2^\mu &= \frac{\sqrt{s}}{2}(1,0,0,-1)\ ,\\
n_1^\mu &= \frac{\sqrt{s}}{2}\beta^2(1,0,\sin\theta_1,\cos\theta_1)\ ,\\
n_2^\mu &= \frac{\sqrt{s}}{2}\beta^2(1,\sin\phi\sin\theta_2,\cos\phi\sin\theta_2,\cos\theta_2)\ ,\\
k_1^\mu &= \hat{\xi}_1\, n_1^\mu\ ,\\
k_2^\mu &= \hat{\xi}_2\, n_2^\mu\ ,
\end{aligned}$$

$$\begin{aligned}
\hat{\eta}_{1,2} &= \frac{1}{2}(1-\cos\theta_{1,2})\ ,\\
\eta_3 &= \frac{1}{2}(1-\cos\theta_3)\\
&= \frac{1}{2}(1-\cos\phi\sin\theta_1\sin\theta_2-\cos\theta_1\cos\theta_2)\\
&= \frac{1}{2}(1-\cos(\theta_1-\theta_2)+(1-\cos\phi)\sin\theta_1\sin\theta_2)\ ,
\end{aligned}$$

$$\zeta = \frac{1}{2}\frac{(1-\cos(\theta_1-\theta_2))(1+\cos\phi)}{1-\cos(\theta_1-\theta_2)+(1-\cos\phi)\sin\theta_1\sin\theta_2}$$

$$\eta_3 = \frac{(\hat{\eta}_1-\hat{\eta}_2)^2}{\hat{\eta}_1+\hat{\eta}_2-2\hat{\eta}_1\hat{\eta}_2-2(1-2\zeta)\sqrt{\hat{\eta}_1(1-\hat{\eta}_1)\hat{\eta}_2(1-\hat{\eta}_2)}}$$

all collinear limits with only two variables

$$d\Phi_4 = d\Phi_3(p_1+p_2;k_1,k_2)\ d\Phi_2(Q;q_1,q_2)$$

$$\begin{aligned}
d\Phi_3(p_1+p_2;k_1,k_2) &= \frac{\pi^{2\epsilon}}{8(2\pi)^5\Gamma(1-2\epsilon)}s^{2-2\epsilon}\beta^{8-8\epsilon}\ (\zeta(1-\zeta))^{-\frac{1}{2}-\epsilon}\\
&\times\ (\hat{\eta}_1(1-\hat{\eta}_1))^{-\epsilon}(\hat{\eta}_2(1-\hat{\eta}_2))^{-\epsilon}\frac{\eta_3^{1-2\epsilon}}{|\hat{\eta}_1-\hat{\eta}_2|^{1-2\epsilon}}\ \hat{\xi}_1^{1-2\epsilon}\hat{\xi}_2^{1-2\epsilon}\\
&\times\ d\zeta\ d\hat{\eta}_1\ d\hat{\eta}_2\ d\hat{\xi}_1\ d\hat{\xi}_2\ .
\end{aligned}$$

$d\mu_{\eta\xi}$

Level 1 Decomposition

$$
\begin{aligned}
1 = & \\
& +\theta_1(k_1)\theta_1(k_2) \\
& +\theta_2(k_1)\theta_2(k_2) \Big\} \text{ triple-collinear sector} \\
& +\theta_1(k_1)\theta_2(k_2)(1-\theta_3(k_1,k_2)) \\
& +\theta_2(k_1)\theta_1(k_2)(1-\theta_3(k_1,k_2)) \Big\} \text{ double-collinear sector} \\
& +(\theta_1(k_1)\theta_2(k_2)+\theta_2(k_1)\theta_1(k_2))\theta_3(k_1,k_2)\} \text{ single-collinear sector}
\end{aligned}
$$

most difficult → triple-collinear sector

non-trivial only because of soft-collinear divergences → double-collinear sector

trivial, because NLO type
attach to first sector (contains same divergences) → single-collinear sector

top quark pair production

general case

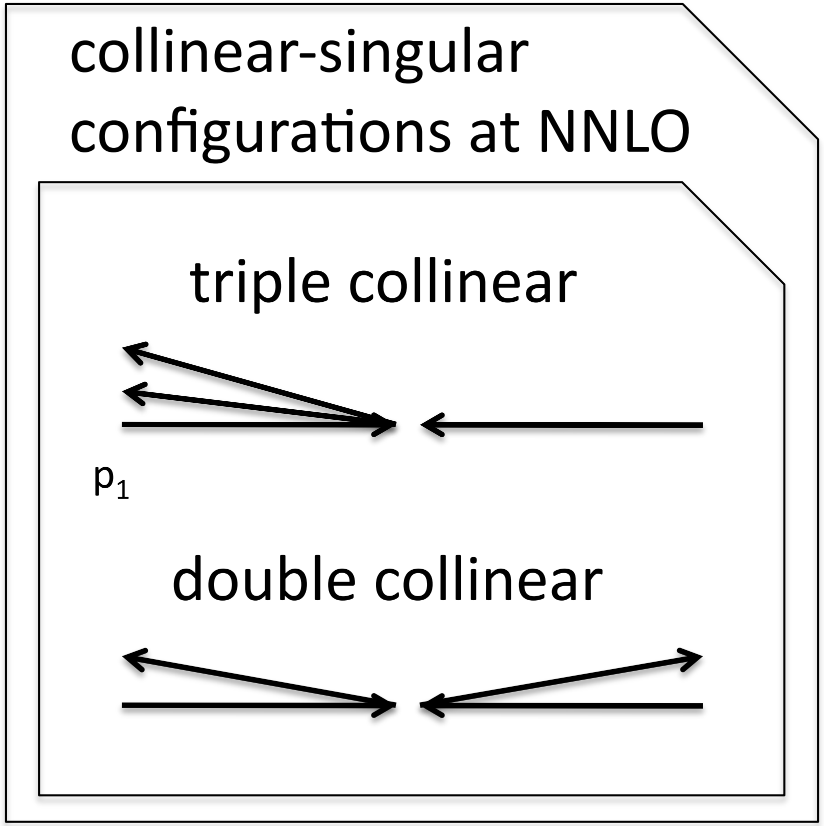
$$1 = \sum_{\substack{\text{pairs} \\ i,j \in \mathcal{F}}} \sum_{\substack{k \in \mathcal{I} \cup \mathcal{F} \\ k \notin \{i,j\}}} \left[\theta_{ij,k} + \sum_{\substack{l \in \mathcal{I} \cup \mathcal{F} \\ l \notin \{i,j,k\}}} \theta_{ij,kl} \right]$$

$$d_{ij} = \left[\left(\frac{2E_i}{\sqrt{s}} \right) \left(\frac{2E_j}{\sqrt{s}} \right) \right]^{\alpha} (1-\cos\theta_{ij})^{\beta},$$

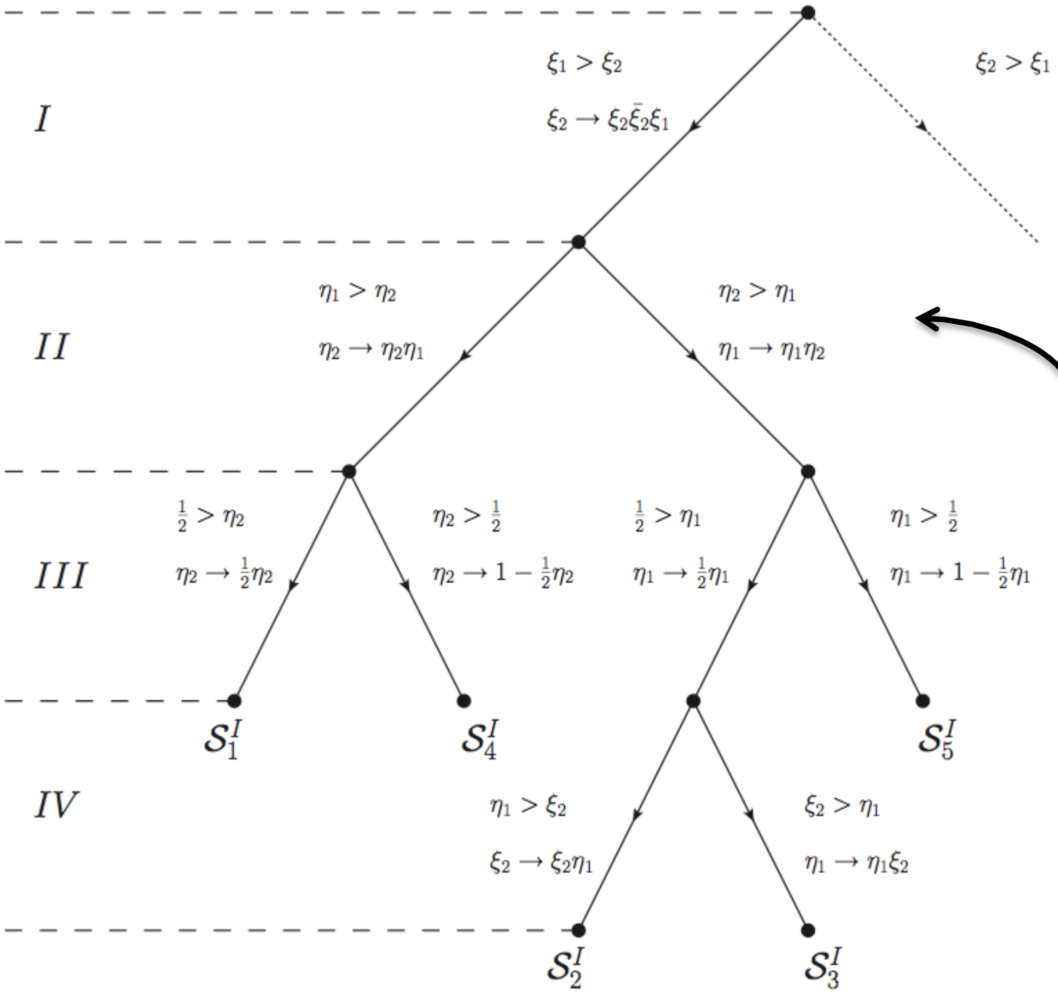
$$d_{ijk} = \left[\left(\frac{2E_i}{\sqrt{s}} \right) \left(\frac{2E_j}{\sqrt{s}} \right) \left(\frac{2E_k}{\sqrt{s}} \right) \right]^{\alpha} \left[(1-\cos\theta_{ij})(1-\cos\theta_{ik})(1-\cos\theta_{jk}) \right]^{\beta}$$

$$\theta_{ij,k} = \frac{1}{\mathcal{D}} \frac{h_{ij,k}}{d_{ijk}}, \qquad \theta_{ij,kl} = \frac{1}{\mathcal{D}} \frac{h_{i,k}}{d_{ik}} \frac{h_{j,l}}{d_{jl}}$$

$$\mathcal{D} = \sum_{\substack{\text{pairs} \\ i,j \in \mathcal{F}}} \sum_{\substack{k \in \mathcal{I} \cup \mathcal{F} \\ k \notin \{i,j\}}} \left[\frac{h_{ij,k}}{d_{ijk}} + \sum_{\substack{l \in \mathcal{I} \cup \mathcal{F} \\ l \notin \{i,j,k\}}} \frac{h_{i,k}}{d_{ik}} \frac{h_{j,l}}{d_{jl}} \right]$$



Level 2 Decomposition



I) factorization of the soft singularities;

II, III) factorization of the collinear singularities;

IV) factorization of the soft-collinear singularities.

triple-collinear sector

Example from S_1^I

$$s_{156} = -\beta^2(\hat{\eta}_1\hat{\xi}_1 + \hat{\eta}_2\hat{\xi}_2 - \beta^2\hat{\xi}_1\hat{\xi}_2\eta_3)$$

$$-\frac{1}{2}\beta^2\eta_1\xi_1\left(2 + \eta_2\xi_2\bar{\xi}_2 - 2\beta^2\xi_1\xi_2\eta_{31}\bar{\xi}_2\right)$$

remove II & III in the double-collinear sector

Subtraction Terms

$$d\mu_{\eta\xi} = \eta_1^{a_1+b_1\epsilon}\eta_2^{a_2+b_2\epsilon}\xi_1^{a_3+b_3\epsilon}\xi_2^{a_4+b_4\epsilon}\ \mu_{\mathcal{S}}^{\text{reg}}\ d\eta_1 d\eta_2 d\xi_1 d\xi_2$$

$$\sigma_{\mathcal{O}} = \sum_{\mathcal{S}} \sigma_{\mathcal{O}}^{(\mathcal{S})} \qquad \sigma_{\mathcal{O}}^{(\mathcal{S})} = \int d\zeta\ d\eta_1\ d\eta_2\ d\xi_1\ d\xi_2\ d\cos\theta_Q\ d\phi_Q\ d\cos\rho_Q\ \Sigma_{\mathcal{O}}^{(\mathcal{S})}$$

$$\Sigma_{\mathcal{O}}^{(\mathcal{S})} = \frac{1}{2s}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{3\epsilon} \mu_\zeta\ \mu_{\mathcal{S}}^{\text{reg}}\ \mu_2\ \theta_{\mathcal{S}}\ F_J \frac{1}{\eta_1^{1-b_1\epsilon}}\frac{1}{\eta_2^{1-b_2\epsilon}}\frac{1}{\xi_1^{1-b_3\epsilon}}\frac{1}{\xi_2^{1-b_4\epsilon}}\ \mathfrak{M}_{\mathcal{S}}$$

$$\mathfrak{M}_{\mathcal{S}} = \eta_1^{1+a_1}\eta_2^{1+a_2}\xi_1^{1+a_3}\xi_2^{1+a_4}\ \overline{|\mathcal{M}_4|^2}$$

$$\int_0^1 \frac{d\lambda}{\lambda^{1-b\epsilon}} f(\lambda) \longrightarrow \int_0^1 d\lambda \left[\frac{f(0)}{b\epsilon} + \frac{f(\lambda)-f(0)}{\lambda^{1-b\epsilon}}\right]$$

apply four times

$$\Sigma_{\mathcal{O}}^{(\mathcal{S})} \longrightarrow \left[\Sigma_{\mathcal{O}}^{(\mathcal{S})}\right]$$

Subtraction Terms

$$\mathbf{X} \subseteq \{\eta_1, \eta_2, \xi_1, \xi_2\}$$

$$\lim_{\mathbf{X}\to\mathbf{0}} \mathfrak{M}_{\mathcal{S}} = g^2 \langle \mathcal{M}_3|\mathbf{V}|\mathcal{M}_3\rangle \quad \text{or} \quad \lim_{\mathbf{X}\to\mathbf{0}} \mathfrak{M}_{\mathcal{S}} = g^4 \langle \mathcal{M}_2|\mathbf{V}|\mathcal{M}_2\rangle$$

Example:

$$\mathfrak{R}_{\mathcal{S}} = \eta_1^{1+a_1}\eta_2^{1+a_2}\xi_1^{1+a_3}\xi_2^{1+a_4}$$

$$\boxed{\hat{\eta}_1 = \hat{\eta}_2 = 0}$$

$$\mathbf{V}^{ss'}_{a_1 a_5 a_6} = \lim_{\mathbf{X}\to\mathbf{0}} \mathfrak{R}_{\mathcal{S}} \frac{4\hat{P}^{ss'}_{a_1 a_5 a_6}}{s^2_{156}}$$

$$x_1 = -1\,, \quad x_5 = \beta^2\hat{\xi}_1\,, \quad x_6 = \beta^2\hat{\xi}_2\,,$$

$$k^{\mu}_{\perp 1} = 0\,, \quad k^{\mu}_{\perp 5} = \beta^2\hat{\xi}_1\sqrt{\hat{\eta}_1}\,\bar{k}^{\mu}_{\perp 5}\,, \quad k^{\mu}_{\perp 6} = \beta^2\hat{\xi}_2\sqrt{\hat{\eta}_2}\,\bar{k}^{\mu}_{\perp 6}(\hat{\eta}_1,\hat{\eta}_2)\,,$$

$$\begin{aligned} \bar{k}^{\mu}_{\perp 5} &= (0,0,1,0)\,, \\ \bar{k}^{\mu}_{\perp 6}(\hat{\eta}_1,\hat{\eta}_2) &= \frac{1}{\hat{\eta}_1+\hat{\eta}_2-2(1-2\zeta)\sqrt{\hat{\eta}_1\hat{\eta}_2}} \\ &\times \Big(0,\ 2|\hat{\eta}_1-\hat{\eta}_2|\sqrt{\zeta(1-\zeta)},\ 2\sqrt{\hat{\eta}_1\hat{\eta}_2}-(\hat{\eta}_1+\hat{\eta}_2)(1-2\zeta),\ 0\Big) \end{aligned}$$

Two-Particle Phase Space

$$q_i^0 = \frac{Q^0 q_i'^0 + \vec{Q}\cdot\vec{q_i'}}{\sqrt{Q^2}},$$

$$\vec{q_i} = \vec{q_i'} + \left(q_i'^0 + \frac{\vec{Q}\cdot\vec{q_i'}}{Q^0+\sqrt{Q^2}}\right)\frac{\vec{Q}}{\sqrt{Q^2}}$$

Boost into CM system

$$q_1'^{\,0} = q_2'^{\,0} = \tfrac{1}{2}\sqrt{Q^2},$$

$$\vec{q_1'} = -\vec{q_2'} = \tfrac{1}{2}\sqrt{Q^2+\beta^2-1}$$

$$\times \left(\sin\rho_Q \sin\phi_Q \sin\theta_Q\ \vec{n}^{(d-4)}, \cos\rho_Q \sin\phi_Q \sin\theta_Q, \cos\phi_Q \sin\theta_Q, \cos\theta_Q\right)$$

five dimensional vectors

$$\vec{n}^{(d-4)} = (\vec{0}^{(d-5)}, 1)$$

$$d\Phi_2(Q;q_1,q_2) = \frac{(4\pi)^\epsilon \Gamma(1-\epsilon)}{8(2\pi)^2\Gamma(1-2\epsilon)}\left(Q^2\right)^{-\epsilon}\left(\sqrt{1-\frac{4m^2}{Q^2}}\right)^{1-2\epsilon}\left(1-\cos^2\theta_Q\right)^{-\epsilon}\left(\sin^2\phi_Q\right)^{-\epsilon}$$
$$\times \frac{4^{1+\epsilon}\Gamma(-2\epsilon)}{\Gamma^2(-\epsilon)(1-\cos^2\rho_Q)^{1+\epsilon}}\, d\cos\theta_Q\, d\phi_Q\, d\cos\rho_Q$$

ordinary four-vectors

$$\frac{4^{1+\epsilon}\Gamma(-2\epsilon)}{\Gamma^2(-\epsilon)(1-\cos^2\rho_Q)^{1+\epsilon}} = \delta(1-\cos\rho_Q) + \frac{4^{1+\epsilon}\Gamma(-2\epsilon)}{\Gamma^2(-\epsilon)}\left[\frac{1}{(1-\cos^2\rho_Q)^{1+\epsilon}}\right]_+$$

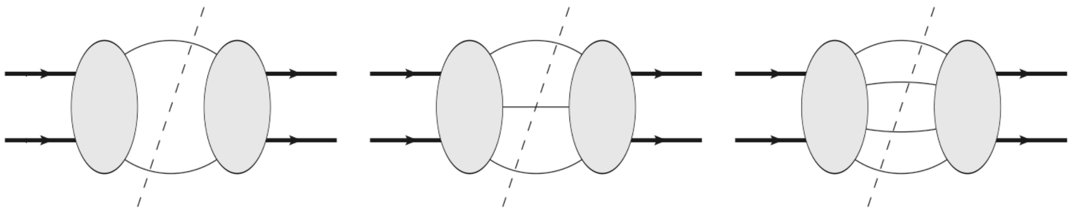
Matrix Elements

amplitude	spin correlated lines	color correlated lines	amplitude	spin correlated lines	color correlated lines
$gg \to t\bar{t}gg$			$q\bar{q} \to t\bar{t}gg$		
$gg \to t\bar{t}q\bar{q}$			$q\bar{q} \to t\bar{t}q'\bar{q}'$		
$gg \to t\bar{t}g$			$q\bar{q} \to t\bar{t}\bar{q}'q'$		
$gg \to t\bar{t}g$	1		$q\bar{q} \to t\bar{t}g$		
$gg \to t\bar{t}g$	2		$q\bar{q} \to t\bar{t}g$	5	
$gg \to t\bar{t}g$	5		$q\bar{q} \to t\bar{t}g$		(1,2)
$gg \to t\bar{t}g$		(1,2)	$q\bar{q} \to t\bar{t}g$		(1,3)
$gg \to t\bar{t}g$		(1,3)	$q\bar{q} \to t\bar{t}g$		(1,4)
$gg \to t\bar{t}g$		(1,4)	$q\bar{q} \to t\bar{t}g$		(2,3)
$gg \to t\bar{t}g$		(2,3)	$q\bar{q} \to t\bar{t}g$		(2,4)
$gg \to t\bar{t}g$		(2,4)	$q\bar{q} \to t\bar{t}$		
$gg \to t\bar{t}$			$q\bar{q} \to t\bar{t}$		(1,2)
$gg \to t\bar{t}$	1		$q\bar{q} \to t\bar{t}$		(1,3)
$gg \to t\bar{t}$	2		$q\bar{q} \to t\bar{t}$		(1,2)(1,2)
$gg \to t\bar{t}$	1,2		$q\bar{q} \to t\bar{t}$		(1,2)(1,3)
$gg \to t\bar{t}$		(1,2)	$q\bar{q} \to t\bar{t}$		(1,3)(1,3)
$gg \to t\bar{t}$		(1,3)			
$gg \to t\bar{t}$		(1,2)(1,2)	$\bar{q}q \to t\bar{t}$		
$gg \to t\bar{t}$		(1,2)(1,3)			
$gg \to t\bar{t}$		(1,3)(1,3)	$gq \to t\bar{t}q$		
$gg \to t\bar{t}$	1	(1,2)	$qg \to t\bar{t}q$		
$gg \to t\bar{t}$	1	(1,3)	$\bar{q}g \to t\bar{t}\bar{q}$		

Timings

process	matrix element [msec]	using HELAC [msec]	including subtraction [msec]	in quadruple precision [msec]
$gg \to t\bar{t}gg$	13	53	18	450
$q\bar{q} \to t\bar{t}gg$	0.71	6.5	0.81	27
$gg \to t\bar{t}q\bar{q}$	0.71	6.3	0.97	32
$q\bar{q} \to t\bar{t}q'\bar{q}'$	0.015	0.52	0.041	1.5

Cut graphs



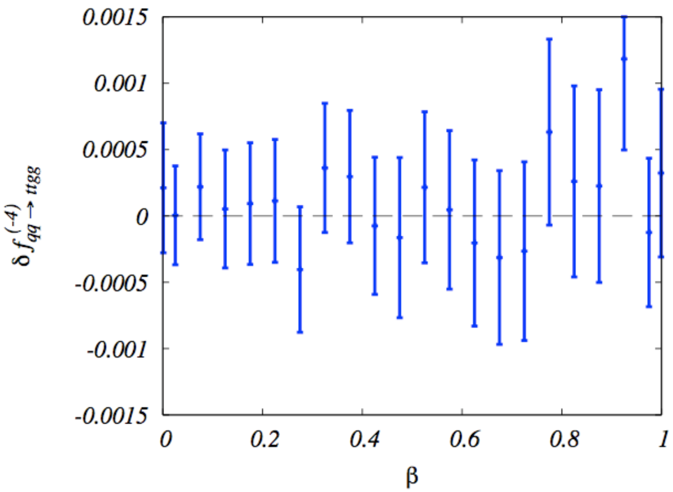
Cutoff

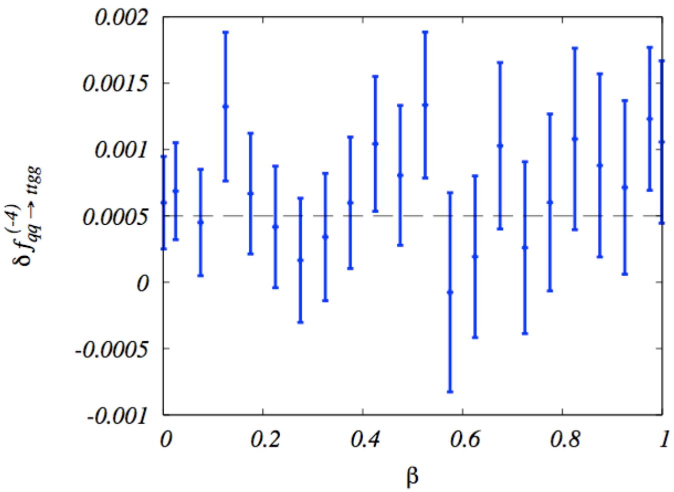
$$\eta_1\ \eta_2\ \xi_1\ \xi_2 > \Delta$$

$$\delta\Phi_n(\Delta) = \int_0^1 \prod_{i=1}^{n} d\alpha_i\ \theta\left(\prod_{j=1}^{n} \alpha_j < \Delta\right) = \Delta \sum_{i=0}^{n-1} \frac{1}{i!} \log^i\left(\frac{1}{\Delta}\right)$$

10^{-7}

10^{-6}





ε-Contribution Example

Analytic result:

$$\frac{\mu^{6\epsilon}}{P_4(s,\epsilon)}\int d\Phi_4\left(\frac{k_1\cdot q_1}{s}\right)^2 = \Big(1.4553533 + 10.106976\,\epsilon + 34.956382\,\epsilon^2 + 80.126733\,\epsilon^3 + 136.49975\,\epsilon^4\Big)\times 10^{-9}\,,$$

Two parts of the numeric result:

$$\frac{\mu^{6\epsilon}}{P_4(s,\epsilon)}\int d\Phi_4^{(d|\epsilon)}\left(\frac{k_1\cdot q_1}{s}\right)^2 = \Big((1.45536\pm0.000014) + (10.1056\pm0.000085)\,\epsilon + (34.9482\pm0.00035)\,\epsilon^2 + (80.1058\pm0.0017)\,\epsilon^3 + (136.473\pm0.0084)\,\epsilon^4\Big)\times 10^{-9}\,,$$

$$\frac{\mu^{6\epsilon}}{P_4(s,\epsilon)}\int d\Phi_4^{(\epsilon)}\left(\frac{k_1\cdot q_1}{s}\right)^2 = \Big((0.001413\pm0.000011)\,\epsilon + (0.008145\pm0.000056)\,\epsilon^2 + (0.02064\pm0.00013)\,\epsilon^3 + (0.02469\pm0.00020)\,\epsilon^4\Big)\times 10^{-9}$$

Sum:

$$\frac{\mu^{6\epsilon}}{P_4(s,\epsilon)}\int d\Phi_4\left(\frac{k_1\cdot q_1}{s}\right)^2 = \Big((1.45536\pm0.000014) + (10.1070\pm0.000086)\,\epsilon + (34.9564\pm0.00035)\,\epsilon^2 + (80.1265\pm0.0017)\,\epsilon^3 + (136.498\pm0.0084)\,\epsilon^4\Big)\times 10^{-9}\,.$$

LO Cross Sections

$$k_{1,2} \cdot p_{1,2} > 10^{-4}\ s\ ,\ k_1 \cdot k_2 > 10^{-6}\ s$$

$$E_{\rm CM} = 400\ {\rm GeV}\ ,\ m_t = 172.6\ {\rm GeV}\ ,\ \alpha_s(m_t) = 0.107639510785815$$

$$\begin{aligned}
\sigma_{gg\to t\bar{t}gg} &= (7.75 \pm 0.018) \times 10^{-2}\ {\rm nb}\ ,\\
\sigma_{gg\to t\bar{t}q\bar{q}} &= (4.81 \pm 0.019) \times 10^{-4}\ {\rm nb}\ ,\\
\sigma_{q\bar{q}\to t\bar{t}gg} &= (2.86 \pm 0.0065) \times 10^{-2}\ {\rm nb}\ ,\\
\sigma_{q\bar{q}\to t\bar{t}q'\bar{q}'} &= (3.55 \pm 0.010) \times 10^{-4}\ {\rm nb}\ ,
\end{aligned}$$

HELAC

$$\begin{aligned}
\sigma_{gg\to t\bar{t}gg} &= (7.76 \pm 0.039) \times 10^{-2}\ {\rm nb}\ ,\\
\sigma_{gg\to t\bar{t}q\bar{q}} &= (4.77 \pm 0.025) \times 10^{-4}\ {\rm nb}\ ,\\
\sigma_{q\bar{q}\to t\bar{t}gg} &= (2.88 \pm 0.0078) \times 10^{-2}\ {\rm nb}\ ,\\
\sigma_{q\bar{q}\to t\bar{t}q'\bar{q}'} &= (3.54 \pm 0.019) \times 10^{-4}\ {\rm nb}\ .
\end{aligned}$$

Pointwise Convergence

$gg \to t\bar{t}gg$

$\mathcal{S}_4^I$

$$s = 1\ ,\ \beta = \frac{1}{2}\ ,\ \zeta = \frac{1}{3}\ ,\ \eta_1 = \frac{1}{4}\ ,\ \eta_2 = \frac{1}{5} \times 10^{-40}\ ,\ \xi_1 = \frac{1}{6}\ ,\ \xi_2 = \frac{1}{7} \times 10^{-40}\ ,$$

$$\cos\theta_Q = \frac{1}{8}\ ,\ \phi_Q = \frac{1}{9}\ ,\ \cos\rho_Q = 1\ .$$

$$\begin{aligned} \Sigma^{(\mathcal{S}_4^I)} &= 2.92 \times 10^{79} + 1.14003 \times 10^{82}\ \epsilon + 2.22528 \times 10^{84}\ \epsilon^2 \\ &\quad + 2.89548 \times 10^{86}\ \epsilon^3 + 2.82539 \times 10^{88}\ \epsilon^4\ . \end{aligned}$$

$$\begin{aligned} \left[\Sigma^{(\mathcal{S}_4^I)}\right] &= 9.98769 \times 10^{-6}\ \frac{1}{\epsilon^4} + 0.0000869195\ \frac{1}{\epsilon^3} + 0.00676374\ \frac{1}{\epsilon^2} + 0.641465\ \frac{1}{\epsilon} \\ &\quad + 40.9821 + 1961.23\ \epsilon + 75083.4\ \epsilon^2 + 2.39952 \times 10^6\ \epsilon^3 + 6.61228 \times 10^7\ \epsilon^4 \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{4} \left(1 - \frac{\Sigma_{\text{approx}}^{(\mathcal{S}_4^I)}\big|_{\epsilon^i}}{\Sigma^{(\mathcal{S}_4^I)}\big|_{\epsilon^i}}\right) \epsilon^i &= -2.68 \times 10^{-41} - 2.68 \times 10^{-41}\ \epsilon - 2.68 \times 10^{-41}\ \epsilon^2 \\ &\quad -2.67 \times 10^{-41}\ \epsilon^3 - 2.67 \times 10^{-41}\ \epsilon^4\ . \end{aligned}$$

Double-Soft Limit

Catani & Grazzini '00

$$\langle a_1, a_2; \mu_1, \mu_2 | \mathcal{M}_{g,g,c_1,\dots,c_n}(q_1, q_2, p_1, \dots, p_n)\rangle \simeq g^2 \mu^{2\epsilon} J^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) | \mathcal{M}_{c_1,\dots,c_n}(p_1, \dots, p_n)\rangle$$

$$\begin{aligned} J^{\mu_1\mu_2}_{a_1 a_2}(q_1, q_2) &= \frac{1}{2}\{J^{\mu_1}_{a_1}(q_1), J^{\mu_2}_{a_2}(q_2)\} + i f_{a_1 a_2 a_3} \sum_{i=1}^{n} T_i^{a_3} \left\{ \frac{p_i^{\mu_1} q_1^{\mu_2} - p_i^{\mu_2} q_2^{\mu_1}}{(q_1 \cdot q_2)\,[p_i \cdot (q_1 + q_2)]} \right. \\ &\quad \left. - \frac{p_i \cdot (q_1 - q_2)}{2\,[p_i \cdot (q_1 + q_2)]} \left[\frac{p_i^{\mu_1} p_i^{\mu_2}}{(p_i \cdot q_1)(p_i \cdot q_2)} + \frac{g^{\mu_1 \mu_2}}{q_1 \cdot q_2} \right] \right\}, \end{aligned}$$

$$\left[J^{a_1 a_2}_{\mu\nu}(q_1, q_2)\right]^\dagger d^{\mu\sigma}(q_1) d^{\nu\rho}(q_2) J^{a_1 a_2}_{\sigma\rho}(q_1, q_2) = \frac{1}{2}\left\{\mathbf{J}^2(q_1), \mathbf{J}^2(q_2)\right\} - C_A \sum_{i,j=1}^{n} \mathbf{T}_i \cdot \mathbf{T}_j \, \mathcal{S}_{ij}(q_1, q_2) + \dots$$

$$\mathcal{S}_{ij}(q_1, q_2) = \mathcal{S}^{m=0}_{ij}(q_1, q_2) + \left(m_i^2 \, \mathcal{S}^{m\neq 0}_{ij}(q_1, q_2) + m_j^2 \, \mathcal{S}^{m\neq 0}_{ji}(q_1, q_2) \right)$$

New:

$$\begin{aligned} \mathcal{S}^{m\neq 0}_{ij}(q_1, q_2) &= -\frac{1}{4\, q_1 \cdot q_2 \; p_i \cdot q_1 \; p_i \cdot q_2} + \frac{p_i \cdot p_j \; p_j \cdot (q_1 + q_2)}{2\, p_i \cdot q_1 \; p_j \cdot q_2 \; p_i \cdot q_2 \; p_j \cdot q_1 \; p_i \cdot (q_1 + q_2)} \\ &\quad - \frac{1}{2\, q_1 \cdot q_2 \; p_i \cdot (q_1 + q_2) \; p_j \cdot (q_1 + q_2)} \left(\frac{(p_j \cdot q_1)^2}{p_i \cdot q_1 \; p_j \cdot q_2} + \frac{(p_j \cdot q_2)^2}{p_i \cdot q_2 \; p_j \cdot q_1} \right) \end{aligned}$$

Conclusions

Maybe, we can finish the NNLO top program at some point

and maybe, one can use the same tools for other applications