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QCD and advanced Monte Carlo tools

CERN-FNAL HCP summer school – Lecture 2

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What to keep in mind

- QCD is an asymptotically-free QFT, supported by hadron spectroscopy and high-energy experiments
- Colour gives QCD amplitudes peculiar features
- Perturbative techniques can be used, but are not sufficient: long-distance effects (hadrons) always contribute to physical observables
- To deal with them, one must introduce (at least) hadron-parton duality, infrared safety, factorization theorems

Hadron-parton duality

Inclusive hadronic observables can be expressed in terms of quark and gluon degrees of freedom. More precisely

$$\int ds\, w(s)\, O^{hadron}(s) = \int ds\, w(s)\, O^{parton}(s)$$

with $w(s)$ a weight function of some energy scale s, peaked at $s = s_0$ (s_0 is a characteristic *large* scale of the process). In practice one always uses *local* hadron-parton duality, for which

$$w(s) = \delta(s - s_0)$$

In other words: compute your observables in terms of quarks and gluons, and assume the results would be the same if you were able to perform a hadron-level computation (e.g., jets, with $s_0 = p_T(jet)$)

Infrared safety

An observable $\mathcal{O}$ is infrared safe if the functions $\mathcal{O}_n(k_1, \cdots, k_n)$ that define it in terms of parton momenta have the following properties:

$$\mathcal{O}_n(k_1, \cdots, k_i, \cdots, k_n) \xrightarrow{E_i \to 0} \mathcal{O}_{n-1}(k_1, \cdots, k_n)$$

$$\mathcal{O}_n(k_1, \cdots, k_i, \cdots, k_j, \cdots, k_n) \xrightarrow{k_i \| k_j} \mathcal{O}_{n-1}(k_1, \cdots, k_i + k_j, \cdots, k_n)$$

which can be iterated as many times as necessary

Translation: an observable must be insensitive to emissions of soft partons, or to collinear splittings of partons

- IR-safe observables: thrust, p_T of single-inclusive and hardest jet,...
- IR-unsafe observables: number of gluon jets, y of the hardest jet,...

Factorization theorems

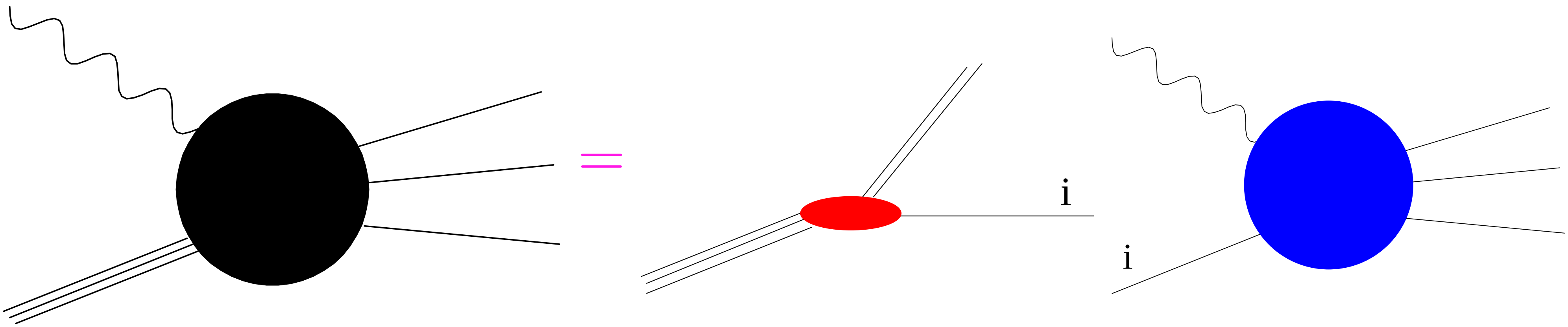
$$
\begin{aligned}
d\sigma_{H_1 H_2}(P_1, P_2) &= \sum_{ij} \int dx_1 dx_2 f_i^{(H_1)}(x_1, \mu^2) f_j^{(H_2)}(x_2, \mu^2) \\
&\times \; d\hat{\sigma}_{ij}(x_1 P_1, x_2 P_2; \alpha_S(\mu^2), \mu^2) \\
d\sigma_{eH}(P) &= \sum_{i} \int dx f_i^{(H)}(x, \mu^2) d\hat{\sigma}_{ei}(xP; \alpha_S(\mu^2), \mu^2)
\end{aligned}
$$

- The partonic cross sections $d\hat{\sigma}_{ij}$, $d\hat{\sigma}_{ei}$ are computable in perturbation theory
- The PDFs f_i must be extracted from data

Intuitive physical picture (Born & Oppenheimer): phenomena at different time scales (hadronization and hard scattering) factorize

Factorization theorems are, apart from the case of DIS, formally unproved. They are however largely accepted, and stand countless tests

$$d\sigma_{eH}(P) = \sum_i \int dx\, f_i^{(H)}(x,\mu^2)\, d\hat{\sigma}_{ei}(xP;\alpha_S(\mu^2),\mu^2)$$



The timescale $1/M$ for binding the hadron is much larger than the timescale $1/Q$ for the hard scattering $\Longrightarrow$ incoherent scatterings

- μ arbitrarily separates hard from soft scales
- In practice: pull out a parton with a random fraction z of the hadron momentum, scatter it with the photon. Ignore the hadron remnants
- There are “leakages”, ie corrections of type $(1/Q)^p$
- Intuitively clear that f doesn't depend on the nature of hard scattering

Final-state version (fragmentation)

The idea: partons produced in the hard collision move fast away from each other. Each of them will eventually pick up (at large p_T) the missing colour and flavour from the vacuum to create an observable hadron

Example: b hadroproduction. The single-inclusive p_T spectrum of the b-flavoured hadron is:

$$\frac{d\hat{\sigma}_{ij\to H_b}}{dp_T(H_b)} = \int \frac{dz}{z} D^{b\to H_b}(z,\epsilon) \frac{d\hat{\sigma}_{ij\to b}}{dp_T(b)}, \qquad p_T(H_b) = z p_T(b)$$

- $d\hat{\sigma}_{ij\to H_b}$ is convoluted with the PDFs to get $H_1 H_2 \longrightarrow H_b$
- The *fragmentation function* $D^{Q\to H_Q}$ is analogous to the PDFs: it cannot be computed in pQCD, but is universal
- One tipically uses e^+e^- to fit the parameter(s) ϵ; the functional form in z must be guessed (Peterson, Kartvelishvili,...)

PERTURBATION THEORY AT WORK

$$e^+e^- \longrightarrow \text{hadrons}$$

Let's see in practice the way in which hadron-parton duality, infrared safety and factorization theorems work

The simplest case is the total hadronic rate in e^+e^- collisions

- Hadron-parton duality $\Longrightarrow$ compute the total *partonic* rate
- Total rate is (trivially) infrared safe

Remember that it's actually more convenient to compute:

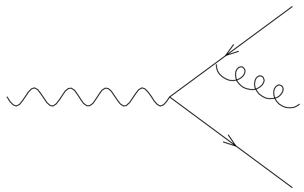
$$R = \frac{\sigma(e^+e^- \to \mathrm{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \ldots\right]$$

- Is this result systematically improvable, in the sense of perturbation theory? This is what we expect from the β_{QCD} computation

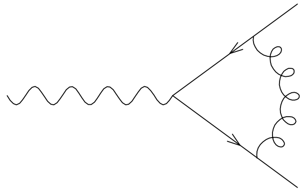
Perturbative corrections to R

At the first order beyond Born (*i.e. next-to-leading order*, NLO), there are two classes of corrections (as in QED)

- Real contribution: all Feynman diagrams with an additional (wrt Born) parton in the final state



- Virtual contribution: all one-loop Feynman diagrams that can be obtained from Born diagrams

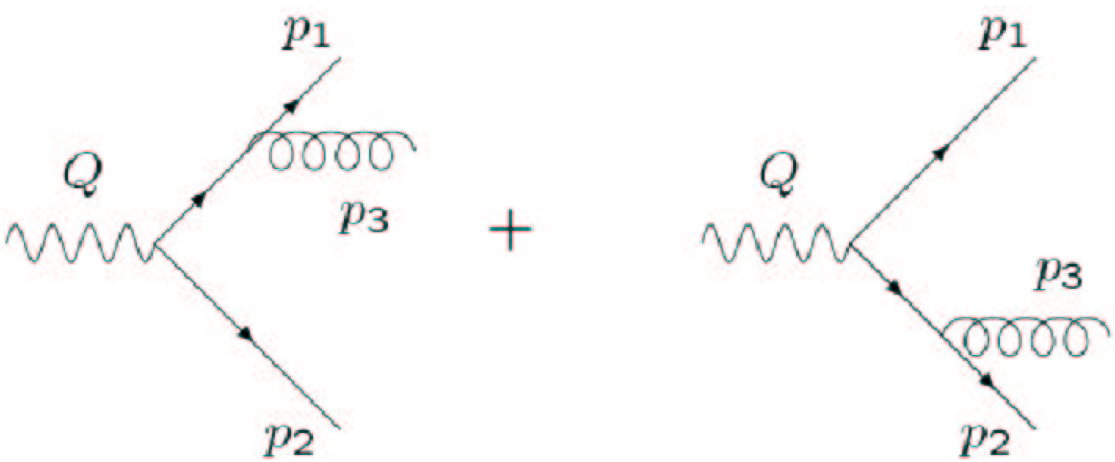


R and V don't interfere: diagrams have different number of legs

$$\text{real} = g_S \mathcal{A}_R \qquad \text{virtual} = g_S^2 \mathcal{A}_V$$

$$|\mathcal{A}_{NLO}|^2 = |\mathcal{A}_{LO}|^2 + \alpha_S \left(|\mathcal{A}_R|^2 + 2\Re(\mathcal{A}_{LO}\mathcal{A}_V^\star) \right) + \mathcal{O}(\alpha_S^2)$$

Real contribution



$$x_i = \frac{2p_i \cdot Q}{Q^2} = \frac{2E_i}{\sqrt{s}}$$

$$p_1 + p_2 + p_3 = Q \implies$$

$$x_1 + x_2 + x_3 = 2$$

Phase space and matrix element:

$$d\Phi_{q\bar{q}g} = \frac{s}{32(2\pi)^5}\delta(2 - x_1 - x_2 - x_3)dx_1 dx_2 dx_3 d\Omega$$

$$|\mathcal{A}_R|^2 = |\mathcal{A}_{LO}|^2 \, C_F \frac{\alpha_S}{2\pi} \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}$$

which lead to

$$\sigma_R = \int d\Phi_{q\bar{q}g} \, |\mathcal{A}_R|^2 = \infty$$

It is instructive to see why this is divergent

$$
\begin{aligned}
1 - x_1 &= x_2 \frac{E_3}{\sqrt{s}} (1 - \cos\theta_{23}) = \frac{(p_2 + p_3)^2}{Q^2} \\
1 - x_2 &= x_1 \frac{E_3}{\sqrt{s}} (1 - \cos\theta_{13}) = \frac{(p_1 + p_3)^2}{Q^2}
\end{aligned}
$$

The divergences of the matrix elements are at

$$
\begin{array}{rcll}
x_1 \longrightarrow 1 \;\&\; x_2 \longrightarrow 1 & \Longleftrightarrow & E_3 \longrightarrow 0 & \text{soft} \\
x_1 \longrightarrow 1 & \Longleftrightarrow & \theta_{23} \longrightarrow 0 & \text{collinear} \\
x_2 \longrightarrow 1 & \Longleftrightarrow & \theta_{13} \longrightarrow 0 & \text{collinear}
\end{array}
$$

This clarifies that the divergences are not physical: we are pushing pQCD beyond its range of applicability, since parton energies or parton-pair invariant masses are comparable to hadron masses $\Longrightarrow$ confinement effects can't be neglected

In other words: we are trying to resolve partons in a regime where the concept of parton is not particularly meaningful – s_0 is not large

- Go home and throw hadron-parton duality (and pQCD) in the bin?

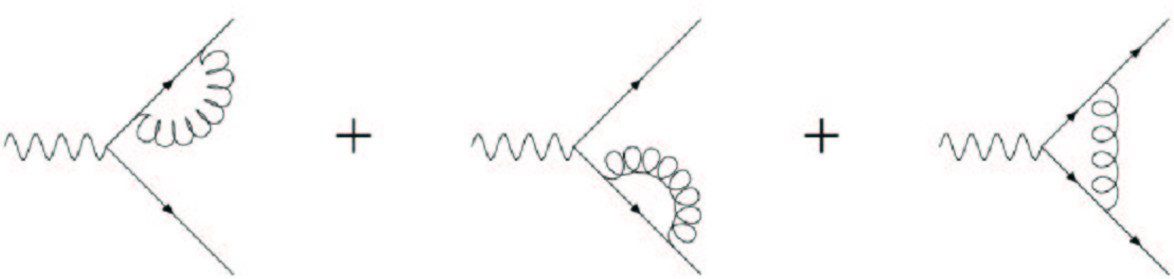
Not yet: what the previous computation tells us is that the cross section for the production of $q\bar{q}g$ is not a meaningful quantity in perturbation theory

But this cross section is just one of the contributions to $e^+e^- \longrightarrow$ hadrons at $\mathcal{O}(\alpha_S)$ – we still have to consider the virtual contribution

So before throwing everything away, we have to prove that soft/collinear emissions are dominant also after adding virtual corrections

Note that what we've got is not peculiar of QCD: you get the same if you compute $\mu^+\mu^-\gamma$ production in QED

Virtual contribution



$$x_i = \frac{2p_i \cdot Q}{Q^2} = \frac{2E_i}{\sqrt{s}}$$

$$p_1 + p_2 = Q \Longrightarrow$$

$$x_1 = 1, \; x_2 = 1$$

One can easily see that

$$\sigma_V = \int d\Phi_{q\bar{q}} \Re(\mathcal{A}_{LO} \mathcal{A}_V^\star) = -\infty$$

- Physical meaning: we are trying to compute the probability of having *exactly* two quarks in the final state
- As in QED, this quantity diverges order-by-order in PT. The result to all orders, however, is not the same as in QED, owing to the different behaviour of the running coupling

$$\sigma_R + \sigma_V = \infty - \infty = ?$$

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! *Regularize* R and V contributions before summing them $\longrightarrow$ in QCD, this usually means computing the integrals in $d = 4 - 2\epsilon$ dimensions

$$\int^1 \frac{dx}{1-x} = -\log(0) \quad \xrightarrow{\text{regularization}} \quad \int^1 \frac{dx(1-x)^{-2\epsilon}}{1-x} = -\frac{1}{2\epsilon}$$

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$$\int^1 \frac{dx}{1-x} = -\log(0) \xrightarrow{\text{regularization}} \int^1 \frac{dx(1-x)^{-2\epsilon}}{1-x} = -\frac{1}{2\epsilon}$$

$$\sigma_R = \sigma_{LO} C_F \frac{\alpha_S}{2\pi}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2\right) + \mathcal{O}(\epsilon)$$

$$\Longrightarrow \quad \sigma_V = \sigma_{LO} C_F \frac{\alpha_S}{2\pi}\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2\right) + \mathcal{O}(\epsilon)$$

$$\lim_{\epsilon \longrightarrow 0} (\sigma_R + \sigma_V) = \frac{\alpha_S}{\pi}\sigma_{LO}$$

The singularities are gone! So we can obtain

$$R = N \sum_f Q_f^2 \left(1 + \frac{\alpha_S}{\pi}\right) + \mathcal{O}(\alpha_S^2)$$

This is a small correction ($< 5\%$), and improves the comparison to data – we have proven that the total rate is insensitive to soft/collinear emissions

Physical meaning: soft/collinear real configurations are kinematically degenerate with virtual configurations. Thus, it looks like finite quantities are obtained by summing over degenerate (ie non-resolvable) partonic configurations

This is in fact true, and true to all orders:

Kinoshita-Lee-Nauenberg (KLN) theorem: in the computation of inclusive (enough) quantities, infrared divergences cancel, and the result is finite

And this can indeed be checked by explicit computations $\longrightarrow$

$$R = R_{LO}\left[1 + \frac{\alpha_S}{\pi} + 1.411\left(\frac{\alpha_S}{\pi}\right)^2 - 12.8\left(\frac{\alpha_S}{\pi}\right)^3\right] + \mathcal{O}(\alpha_S^4)$$

The new terms improve further the agreement with data

This is a huge success! Keep in mind we have used several highly non trivial ingredients

- *Asymptotic freedom*
- *Hadron-parton duality*
- *Infrared safety*

and we have also verified that the KLN theorem works

What to take home

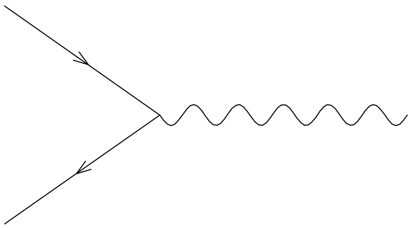
- When considering perturbative corrections, IR divergences appear
- Certain observables are finite, ie insensitive to the IR sector. For this to happen, real and virtual contributions to the perturbative corrections must both be considered at the NLO
- Perturbative corrections are larger than in QED, but still under control; a pQCD program makes sense...
- ... but one always needs hadron-parton duality, infrared safety, factorization theorems (large distance unavoidable)

The distinction between contributions of real and virtual origin is simplistic, since it stems from an NLO picture

Beyond NLO, one can have mixed cases

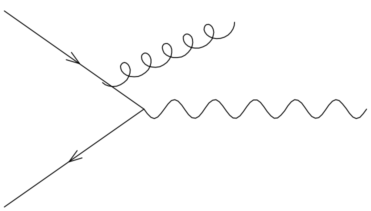
$\longrightarrow$

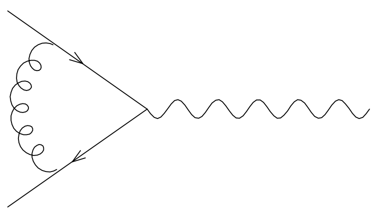
LO



$$\left|\mathcal{A}^{(0)}_{q\bar{q}'}\right|^2$$

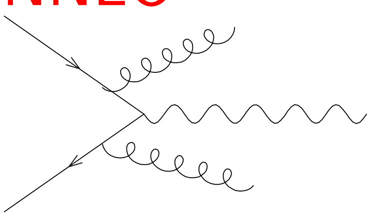
NLO

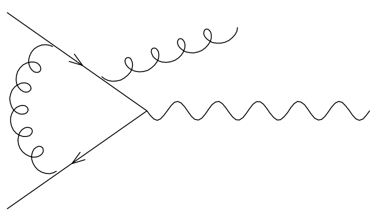


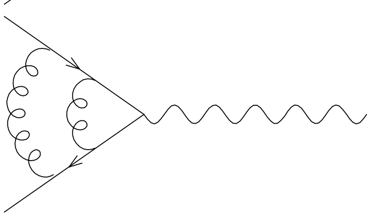


$$\left|\mathcal{A}^{(0)}_{q\bar{q}'}\right|^2, \;\; \Re\left(\mathcal{A}^{(0)}_{q\bar{q}'}\mathcal{A}^{(1)^\star}_{q\bar{q}'}\right)$$

NNLO







$$\left|\mathcal{A}^{(0)}_{q\bar{q}'}\right|^2, \;\; \Re\left(\mathcal{A}^{(0)}_{q\bar{q}'}\mathcal{A}^{(1)^\star}_{q\bar{q}'}\right),$$

$$\Re\left(\mathcal{A}^{(0)}_{q\bar{q}'}\mathcal{A}^{(2)^\star}_{q\bar{q}'}\right), \;\; \left|\mathcal{A}^{(1)}_{q\bar{q}'}\right|^2$$

and so forth...

It is clear that these computations grow rapidly in complexity. For non-trivial observables, analytical results cannot be obtained.
How, then, can one check KLN cancellation without encountering numerical instabilities?

The key point is that singularities must arise from soft and/or collinear (collectively called unresolved) configurations. One must therefore study the behaviour of the contribution due to $n+m$ partons:

$$d\sigma^{(n+m)} \sim \mathcal{O}_{n+m}\mathcal{M}^{(n+m)}d\phi_{n+m}$$

when m of these become unresolved. One finds that

$$\mathcal{O}_{n+m} \longrightarrow \mathcal{O}_n$$
$$\mathcal{M}^{(n+m)} \longrightarrow \mathcal{M}^{(n)}\mathcal{K}^{(m)}$$
$$d\phi_{n+m} \longrightarrow d\phi_n d\phi_m$$

The conditions:

$$\mathcal{O}_{n+m} \longrightarrow \mathcal{O}_n\,, \qquad\qquad d\phi_{n+m} \longrightarrow d\phi_n d\phi_m$$

are IR safety, and kinematic factorization (intuitively easy to grasp, and not difficult to prove formally). One then follows the logic:

$$\int \mathcal{O}_n \mathcal{M}^{(n)} d\phi_n \qquad \text{difficult, finite} \quad \Longrightarrow \quad \text{numerical}$$

$$\int \mathcal{K}^{(m)} d\phi_m \qquad \text{“easy”, divergent} \quad \Longrightarrow \quad \text{analytical}$$

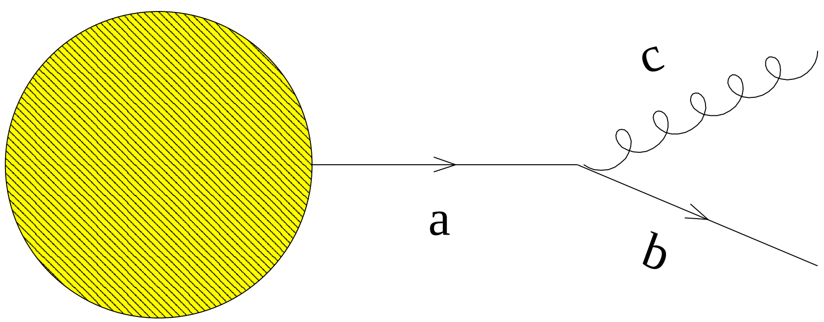
So the problem of determining the divergences (or the leading behaviour of an observable) largely boils down to proving:

$$\mathcal{M}^{(n+m)} \longrightarrow \mathcal{M}^{(n)} \mathcal{K}^{(m)}$$

Let's consider the simplest cases: collinear and soft limits at the NLO

Collinear limit

Use the following (Sudakov) parametrization of momenta



$$k_b = zk_a + k_T + \zeta_b n$$

$$k_c = (1-z)k_a - k_T + \zeta_c n$$

$$k_b^2 = 0 \;\Rightarrow\; \zeta_b = -\frac{k_T^2}{2zn \cdot k_a}$$

$$k_c^2 = 0 \;\Rightarrow\; \zeta_c = -\frac{k_T^2}{2(1-z)n \cdot k_a}$$

The collinear limit is $k_T \longrightarrow 0$. Hence, one keeps only the dominant terms in $1/k_T$ in the matrix elements. By direct computation:

$$\mathcal{M}^{(n+1)} = \frac{g^2}{k_b \cdot k_c}\left(C_F \frac{1+z^2}{1-z}\right)\mathcal{M}^{(n)} + \mathcal{O}\left(\frac{1}{\sqrt{k_T^2}}\right)$$

Note that $k_T^2 \sim k_b \cdot k_c$

As predicted by the general formula, the residue of the divergence is the Born (which depends only on the momenta of the resolved partons), times a simple kernel. At the NLO, one has the following cases

$$q \to q(z)g(1-z) \quad \Longrightarrow \quad P_{qq}(z) = C_F \frac{1+z^2}{1-z}$$

$$g \to q(z)\bar{q}(1-z) \quad \Longrightarrow \quad P_{qg}(z) = T_R \left(z^2 + (1-z)^2\right)$$

$$q \to g(z)q(1-z) \quad \Longrightarrow \quad P_{gq}(z) = C_F \frac{1+(1-z)^2}{z} = P_{qq}(1-z)$$

$$g \to g(z)g(1-z) \quad \Longrightarrow \quad P_{gg}(z) = C_A \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right)$$

$$C_F = \frac{4}{3}\,, \quad C_A = 3\,, \quad T_R = \frac{1}{2}$$

called the (unsubtracted) *Altarelli-Parisi splitting kernels*

History of AP kernels

$$P_{ab} = \sum_{i=0}^{N} \left(\frac{\alpha_S}{4\pi}\right)^i P_{ab}^{(i)}$$

- $P_{ab}^{(0)}$: Altarelli, Parisi (1977) (those just shown)
- $P_{ab}^{(1)}$: Curci, Furmanski, Petronzio (1980)
- $P_{ab}^{(2)}$: Moch, Vermaseren, Vogt (2004)

The calculation of $P_{ab}^{(2)}$ is the toughest ever performed in perturbative QCD, with 10^6 lines of dedicated algebraic code, and 20 man-year of work

- One loop $\Longrightarrow$ 18 Feynman diagrams
- Two loops $\Longrightarrow$ 350 Feynman diagrams
- Three loops $\Longrightarrow$ 9607 Feynman diagrams

Note:

$$P_{qq}(z) \xrightarrow{z\to 1} C_F \frac{2}{1-z} \longrightarrow \infty \qquad\qquad P_{gg}(z) \xrightarrow{z\to 1} C_A \frac{2}{1-z} \longrightarrow \infty$$

- This is the soft singularity
- If two partons are collinear, nothing prevents them to also become soft
- Does this imply that the collinear residue is able to describe soft singularities as well?

No!

(Technically, the Sudakov parametrization of momenta adopted before is not suited to study the soft limit, since in this limit *both* $k_T \to 0$ *and* $z \to 1$)

Soft limit

In the soft limit, one just rescales the soft momentum

$$k \longrightarrow \lambda k$$

and then keeps only the most singular terms in $1/\lambda$. One obtains

$$\mathcal{M}^{(n+1)} = \frac{1}{2} g^2 \sum_{ij} \frac{k_i \cdot k_j}{(k_i \cdot k)(k_j \cdot k)} \mathcal{M}_{ij}^{(n)} + \text{less singular}$$

The kinematic factor in front of the Born cross section is known in QED: it's the *eikonal factor*. Note that, in the c.m. frame of the system $k_i + k_j$, we have

$$\frac{k_i \cdot k_j}{(k_i \cdot k)(k_j \cdot k)} \propto \frac{1}{E^2} \frac{1}{1 - \cos^2 \theta}$$

The eikonal therefore has the leading soft singularity (which is obvious by construction), and also the collinear singularities wrt to the partons from which the gluon is emitted

Soft-gluon insertion rules

By looking at the expression we got before, it is easy to understand that the soft limit is actually easier to interpret at the amplitude (rather than at the amplitude squared) level. One introduces the soft current

$$\vec{J}_i^{\mu} = g\vec{Q}_i \, \frac{k_i^{\mu}}{k_i \cdot k}$$

for the emission of a gluon of momentum k from a parton i (quark, antiquark or gluon) of momentum k_i

When squaring the amplitudes, one will obtain the typical structure

$$\sum_{ij} \vec{J}_i^{\mu} \cdot \vec{J}_j^{\nu} \, (-g^{\mu\nu} + \text{gauge terms}) = -g^2 \sum_{ij} \vec{Q}_i \cdot \vec{Q}_j \, \frac{k_i \cdot k_j}{(k_i \cdot k)(k_j \cdot k)}$$

Hence:

$$\mathcal{M}^{(n+1)} = \left|\mathcal{A}^{(n+1)}\right|^2 \longrightarrow \sum_{ij} \frac{k_i \cdot k_j}{(k_i \cdot k)(k_j \cdot k)} \mathcal{M}_{ij}^{(n)}$$

$$\mathcal{M}_{ij}^{(n)} = -2 < \mathcal{A}^{(n)} | \vec{Q}_i \cdot \vec{Q}_j | \mathcal{A}^{(n)} >$$

- $\mathcal{M}_{ij}^{(n)}$ are called colour-linked Born amplitudes squared. They are s.t.:

$$\mathcal{M}_{ij}^{(n)} = \mathcal{M}_{ji}^{(n)} \,, \qquad\qquad \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathcal{M}_{ij}^{(n)} = 2C(i)\mathcal{M}^{(n)}$$

- The colour thus introduce a highly non trivial structure in the soft limit
- The soft-collinear limit computed with soft-gluon insertions coincides with that computed via Altarelli-Parisi

I stress again that the structure is simpler at the amplitude level.
By considering dual amplitudes, there is actually an amazing further simplification

The soft limit of a given dual amplitude is:

$$\widehat{\mathcal{A}}^{(n+1)}(\ldots k_k, k, k_l, \ldots) \xrightarrow{k\rightarrow 0} g^2 \left(\frac{k_l \cdot \epsilon}{k_l \cdot k} - \frac{k_k \cdot \epsilon}{k_k \cdot k} \right) \widehat{\mathcal{A}}^{(n)}(\ldots k_k, k_l, \ldots)$$

with ϵ the polarization vector of the soft gluon

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Keep in mind that dual amplitudes are orthogonal when $N \to \infty$. Thus, at the leading order in N there is a one-to-one correspondence between the kinematically-dominant behaviour of the cross section (i.e., an amplitude squared), and the colour structure of dual amplitudes

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- There is a similar result: the only collinear singularities of a dual amplitude are due to adjacent partons

Keep in mind that dual amplitudes are orthogonal when $N \to \infty$. Thus, at the leading order in N there is a one-to-one correspondence between the kinematically-dominant behaviour of the cross section (i.e., an amplitude squared), and the colour structure of dual amplitudes

This property is crucially important for the correct treatment of soft singularities in Event Generators. The order in which partons appear in a dual amplitude is called colour flow in EvG's

Summary

- The rich IR structure of QCD is both a curse and a resource
- It complicates exact computations, but allows one to attack all-order problems, and guess asymptotic behaviour
- Matrix elements have factorization properties in the soft and collinear limits, which match those of phase spaces
- The picture is simpler if dual amplitudes are considered

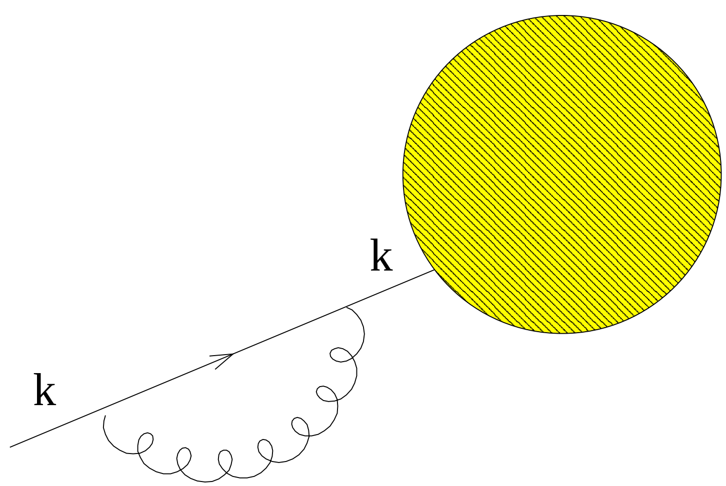
Consider now the case a process with an initial-state hadron: DIS

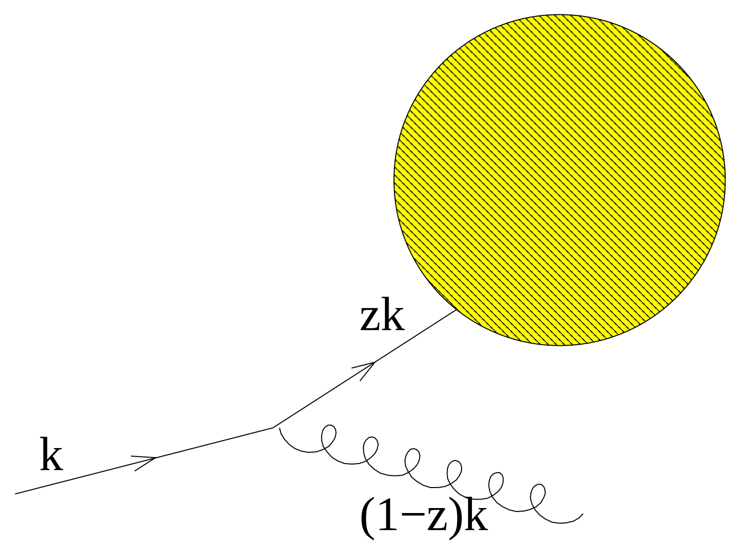
Remember that the leading order is Feynman's parton-model formula

$$d\sigma_{ep}(K) = \sum_q \int dx f_q(x) d\sigma_{eq}(xK)$$

with $d\sigma_{eq}$ the LO cross section for $eq \to eX$

Following what done before, we consider NLO corrections to $d\sigma_{eq}$





$$d\sigma_R + d\sigma_V = \frac{\alpha_S}{2\pi}\int dk_T^2 dz\, C_F \frac{1+z^2}{1-z}\frac{1}{k_T^2}\left(d\sigma^{(0)}(zk_a) - d\sigma^{(0)}(k_a)\right)$$

Finite for $z \rightarrow 1$ (soft), but *divergent* for $k_T \rightarrow 0$ (collinear)!

The real kinematic is not degenerate with the virtual one in the collinear limit. This does not happen in the case of final-state emissions

Tentative conclusion: the parton model *does not survive* radiative corrections

If so, pQCD can only be used for final-state hadrons

But there is a way out, which implies replacing the naive parton model by its QCD equivalent, the factorization theorem

Recovering the parton model

Exclude the collinear divergence with a cutoff $\mu_0 \ll Q$. Inserting the partonic cross section into the parton model we get after the k_T integration

$$d\sigma^{(NLO)}(K) = \frac{\alpha_S}{2\pi} \log \frac{Q^2}{\mu_0^2} \int dy dz f(y) P(z) d\sigma^{(0)}(yzK)$$

and with some algebra

$$d\sigma(K) \equiv d\sigma^{(0)}(K) + d\sigma^{(NLO)}(K) = \int dy \hat{f}(y, \mu^2, \mu_0^2) d\hat{\sigma}(yK, \mu^2, Q^2)$$

with $\mu_0 \ll \mu \sim Q$

$$\begin{aligned} \hat{f}(y, \mu^2, \mu_0^2) &= f(y) + \frac{\alpha_S}{2\pi} \log \frac{\mu^2}{\mu_0^2} \int_y^1 \frac{dz}{z} P(z) f(y/z) \\ d\hat{\sigma}(K, \mu^2, Q^2) &= d\sigma^{(0)}(K) + \frac{\alpha_S}{2\pi} \log \frac{Q^2}{\mu^2} \int_0^1 dz P(z) d\sigma^{(0)}(zK) \end{aligned}$$

By deriving $\hat{f}$ w.r.t. μ you get the Altarelli-Parisi equations!

(Note: it is $\hat{f}$ that is usually denoted by f)

It is now manifest that the *divergence is independent of the process* (as for final-state emissions). Consequences

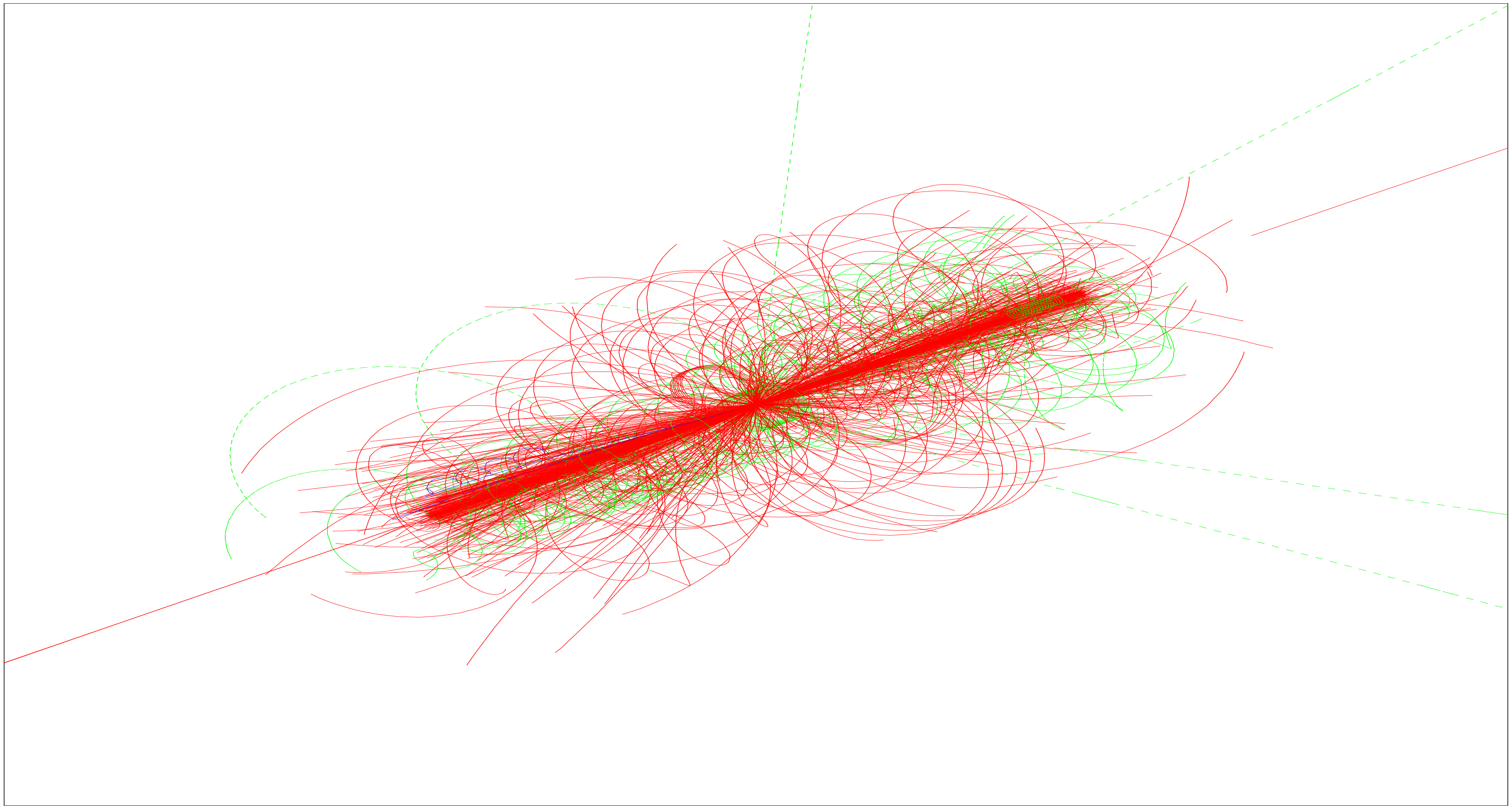
- ♦ PDFs acquire a dependence upon mass scales: scaling violations
- ♦ PDFs cannot be expanded in perturbation theory
- ♦ Parton cross sections do have a perturbative expansion

The key assumption: Nature will kill the $\log \mu_0$ divergence in the PDFs (smearing typical of long-distance phenomena). We cannot compute PDFs, but we can extract them from data

Parton model is formally recovered. An all-order proof of these QCD-improved formulae gives a *factorization theorem*

APPLICATIONS TO LHC PHYSICS

The aim: predict/describe this



$H \to ZZ \to 4\mu$ as simulated by ATLAS

in the best possible way

Before going on, keep in mind the factorization theorem:

$$
\begin{aligned}
d\sigma_{H_1 H_2}(P_1, P_2) &= \sum_{ij} \int dx_1 dx_2 f_i^{(H_1)}(x_1, \mu^2) f_j^{(H_2)}(x_2, \mu^2) \\
&\times \; d\hat{\sigma}_{ij}(x_1 P_1, x_2 P_2; \alpha_S(\mu^2), \mu^2)
\end{aligned}
$$

This implies that hadronic cross sections are the incoherent sums of parton-parton cross sections, which from the pQCD viewpoint are identical to e^+e^- ones (except for the fact that partons are strongly-interacting)

Hence, what was discussed so far applies without modifications to LHC physics

A complete description must account for two ingredients:

1) the hard process: all the high-p_T stuff, plus particles at small relative p_T or with small energies

2) the rest: this is generally low-p_T stuff, and includes

- the underlying event
- the pile-up, ie other pp collisions

Truth be told, there's no unambiguous separation between 1) and 2), since to a certain extent it is always definition dependent

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1) the hard process: all the high-p_T stuff, plus particles at small relative p_T or with small energies

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Two different approaches

♦ Event Generators: aim at giving a description as realistic as possible, including all the details of 1) and 2)

Examples: HERWIG, PYTHIA, ARIADNE, ...

♦ Cross Section Integrators: don't include 2), and are only able to give predictions for infrared-safe observables resulting from 1)

Examples: MCFM, ResBos, ...

For both Event Generators and Cross Section Integrators, the simulation of the hard process proceeds schematically as follows

- Hard subprocess: *only* large-p_T particles, parton-level. Two partons pulled out of the incoming hadrons scatter and produce few (2–6) particles

- Radiation: adds more partons. Equivalent to considering *higher-order corrections* in perturbative QCD

- Hadronization: converts incoming partons into scattering hadrons, and outgoing partons into observed particles

Strategies

- For Hadronization
  1. Use factorization theorems $\longrightarrow$ Cross Section Integrators
  2. Use phenomenological models at mass scales where pQCD is not applicable $\longrightarrow$ Event Generators

- For Higher-order Corrections
  1. Compute exactly the result to a given order in α_S
  2. Estimate the dominant effects to all orders in α_S

  Cross Section Integrators may implement 1, 2, or a combination of the two. Event Generators always implement 2, possibly combined with 1

In the following, I'll talk about Event Generators