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Field Theory and the Standard Model

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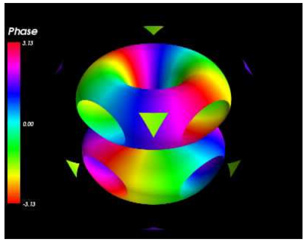
CERN European School, Cheile Gradistei.

Outline

1. Quantum fields and Symmetries.

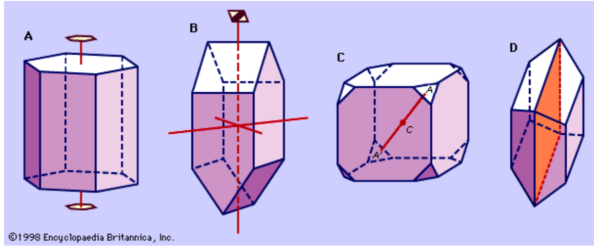
Symmetries are fundamental in our understanding in nature.

- Continuous spacetime symmetries, ex. rotations:



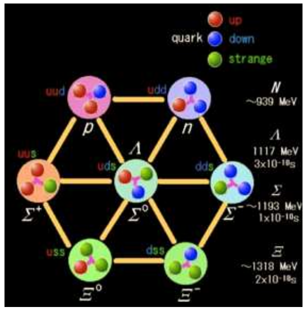
Atomic orbital

- Discrete symmetries in crystals



- Continuous and discrete internal symmetries in particle physics :

Ex. the eightfold way : $SU(3)$ Gell-Mann classification of hadrons



The importance of symmetries in nature is to a large extent due to the Noether theorem : To any continuous symmetry corresponds a conserved charge.

Examples :

Symmetry	Conserved charge
Time translation	Energy
Space translation	Momentum
Rotations	Angular momentum
Phase rotations wave function	Electric charge

- Symmetries are manifest in the spectrum and interactions. Their study greatly simplifies the dynamics.
- In nature, local symmetries determine the fundamental interactions !

1.2. Quantization and perturbation theory.

We start from Schrodinger versus interaction/Heisenberg picture in Quantum Mechanics.

$$H \quad = \quad H_0 \quad + \quad H_{int}$$

free hamiltonian $\nearrow$ $\qquad\qquad$ $\nwarrow$ interaction

Schrodinger eq. is

$$i\frac{d\ |\Psi_S(t)\rangle}{dt} \ = \ (H_0 \ + \ H_{int})\ |\Psi_S(t)\rangle$$

time dep. $\nearrow$ $\qquad\qquad$ $\nwarrow$ time-indep. operators.

In the interaction picture

$$|\Psi_I(t)\rangle \ = \ e^{iH_0 t}\,|\Psi_S(t)\rangle \quad , \quad H_{int}(t) \ = \ e^{iH_0 t}\ H_{int}(t)\ e^{-iH_0 t}$$

the Schrodinger eq. becomes (Ex:)

$$i\frac{d\ |\Psi_I(t)\rangle}{dt} \ = \ H_{int}(t)\ |\Psi_I(t)\rangle$$

We define the evolution operator by

$$|\Psi_I(t)\rangle \ = \ U(t,t_i)|\Psi_I(t_i)\rangle \quad , \quad U(t_i,t_i) \ = \ 1$$

Ex: U satisfies the eq.

$$i\ \frac{\partial U(t,t_i)}{\partial t} \ = \ H_{int}(t)\ U(t,t_i)$$

It can be shown that (Ex:)

$$U(t,t_i) \ = \ T\ e^{-i\int_{t_i}^{t} dt' H_{int}(t')}$$

where the time-ordered product is defined as

$$T A(t_1)B(t_2) \ = \ \theta(t_1-t_2)A(t_1)B(t_2) + \theta(t_2-t_1)B(t_2)A(t_1)$$

The S-matrix is defined as

$$S \ = \lim_{t\to\infty, t_i\to-\infty} U(t,t_i) \ = \ T\ e^{-i\int dt H_{int}(t)} \ = \ T\ e^{i\int d^4x \mathcal{L}_{int}(x)}$$

QFT $\nearrow$

whereas transition amplitudes are

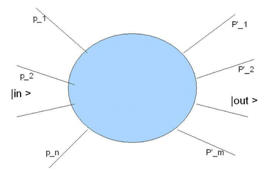
$$S_{if} \ = \ \langle\Psi_f|S|\Psi_i\rangle \ = \ \langle p'_1\cdots p'_m|\ S\ |p_1\cdots p_n\rangle$$

$$= \ \langle p'_1\cdots p'_m,\ \text{out}\ |\ p_1\cdots p_n,\ \text{in}\ \rangle \ = \ \text{no interaction term}$$

$$+\ i\ (2\pi)^4\ \delta^4(\sum_{j=1}^{m} p'_j - \sum_{i=1}^{n} p_i)\ \mathcal{A}_{if}$$

Feynman rules are given for the matrix $\mathcal{A}_{if}$.

Scattering amplitude $\quad \langle p'_1\cdots p'_m,\ \text{out}\ |\ p_1\cdots p_n,\ \text{in}\ \rangle$



Let us consider for illustration a scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \;=\; \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$
$$= \mathcal{L}_0 + \mathcal{L}_{int} \quad , \quad \text{where} \quad \mathcal{L}_{int} \;=\; -\,\frac{\lambda}{4!}\phi^4$$

• Metric convention $\eta_{mn} = diag(1,-1,-1,-1)$. Conjugate momentum : $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$ and hamiltonian

$$H = \int d^3\mathbf{x} \left[\dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L}\right] \;=\; \int d^3\mathbf{x}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right]$$
$$= H_0 + H_{int} \quad , \quad \text{where}$$

$$\begin{cases} H_0 \;=\; \int d^3\mathbf{x}\,\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right] \\ H_{int} \;=\; \int d^3\mathbf{x}\,\frac{\lambda}{4!}\phi^4 \end{cases}$$

Eqs. and solutions for the free-theory :

$$(\Box + m^2)\;\phi(x) \;=\; 0 \quad \Rightarrow$$
$$\phi(x) \;=\; \int \frac{\mathbf{d^3k}}{(2\pi)^{3/2}\sqrt{2\omega_k}}\;\left(\,e^{ikx}\;a^\dagger_{\mathbf{k}} \;+\; e^{-ikx}\;a_{\mathbf{k}}\,\right)$$

where $k_0 = \omega_k = \sqrt{\mathbf{k}^2 + m^2}$. The solution $\phi(x)$ is the operator in the Heisenberg picture. Quantization proceeds as usual:

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] \;=\; \delta^3(\mathbf{k}-\mathbf{k}') \qquad \rightarrow \qquad [\phi(t,\mathbf{x},\pi(t,\mathbf{y})] \;=\; i\delta^3(\mathbf{x}-\mathbf{y})$$

The one-particle states are

$$|\mathbf{k}\rangle \;=\; a^\dagger_{\mathbf{k}}|0\rangle \;\Rightarrow\; \langle\mathbf{k}'|\mathbf{k}\rangle \;=\; \delta^3(\mathbf{k}-\mathbf{k}')$$

and the energy/hamiltonian

$$H_0 \;=\; \int d^3\mathbf{k}\;\omega_k(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}) \qquad (1)$$

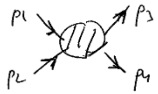
is one of a collection of quantum oscillators. Therefore (no interaction in the asymptotic past and future)

$$\begin{cases} |\psi_i\rangle \;=\; |p_1p_2\cdots p_n\rangle = a^\dagger_{\mathbf{p_1}}\cdots a^\dagger_{\mathbf{p_n}}\;|0\rangle \\ |\psi_f\rangle \;=\; |p'_1p'_2\cdots p'_m\rangle = a^\dagger_{\mathbf{p'_1}}\cdots a^\dagger_{\mathbf{p'_m}}\;|0\rangle \end{cases}$$

Feynman rules in perturbation theory then follow from the expanding in powers of the interaction

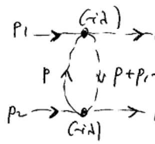
$$\langle p'_1\cdots p'_m|\;S\;|\;p_1\cdots p_n\;\rangle \;=\; \langle 0|a_{\mathbf{p'_m}}\cdots a_{\mathbf{p'_1}}\;T\;e^{i\int d^4x\mathcal{L}_{int}(x)}a^\dagger_{\mathbf{p_1}}\cdots a^\dagger_{\mathbf{p_n}}\;|0\rangle$$

Ex. $2 \to 2$ scattering at 1-loop order (95)

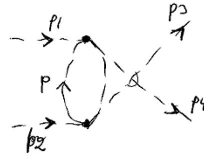


$=$

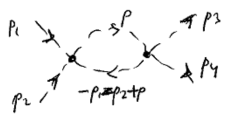
$-i\lambda \;+$



$$\frac{1}{2}(-i\lambda)^2\int\frac{d^4p}{(2\pi)^4}\;\frac{1}{p^2-m^2}\;\frac{1}{(p+p_1-p_3)^2-m^2} \;+$$



$$\frac{1}{2}(-i\lambda)^2\int\frac{d^4p}{(2\pi)^4}\;\frac{1}{p^2-m^2}\;\frac{1}{(p+p_1-p_4)^2-m^2} \;+$$



$$\frac{1}{2}(-i\lambda)^2\int\frac{d^4p}{(2\pi)^4}\;\frac{1}{p^2-m^2}\;\frac{1}{(p-p_1-p_2)^2-m^2}$$

Obs Integrals have a log UV divergence. We will

Perturbation theory is now one of the cornerstones of QFT. The anomalous magnetic moment of the electron was computed for the first time by Schwinger at one-loop in 1948 (the factor below, $\frac{\alpha}{2\pi}$, is engraved on Schwinger's tombstone). Today it is known up to four-loops !

$$a_e = \frac{g-2}{2} = \frac{\alpha}{2\pi} + \cdots$$
$$a_e^{\text{exp}} = (1159652185.9 \pm 3.8) \times 10^{-12} \,,$$
$$a_e^{\text{th}} = (1159652175.9 \pm 8.5) \times 10^{-12}$$

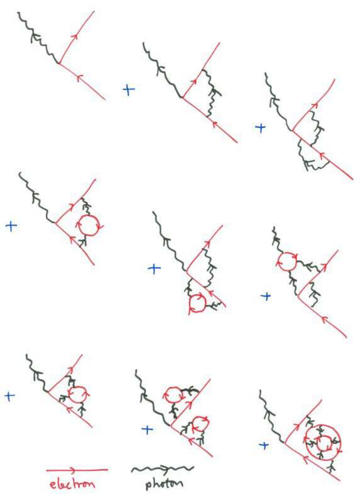
The agreement is very impressive !

There are however still mysteries. For the muon, the measure value at BNL disagrees by 3.4 σ from the theoretical SM calculation

$$a_\mu^{\text{th}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}}$$
$$a_\mu^{\text{exp}} \simeq 0,00116592089$$

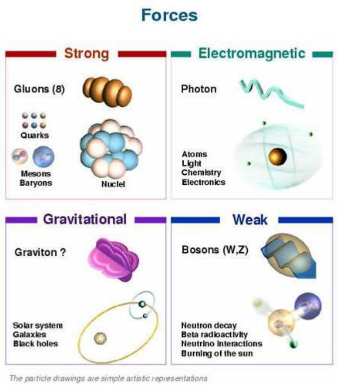
It is likely that the hadronic contribution is not known accurately enough. This is a very hot research topic nowdays.

Feynman diagrams: electron magnetic moment



2. Gauge theories.

The four fundamental interactions in nature



have a common feature: they are gauge interactions.

2.1. Gauge invariance of Schrödinger eq.

Simplest example of gauge symmetry: particle mass m and charge q in quantum mechanics, hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qV \ , \qquad (2)$$

where the vector $\mathbf{A}$ and the scalar V potential are related to the electric/magnetic fields via

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A} \ . \qquad (3)$$

Maxwell eqs. invariant under gauge transformations

$$\mathbf{A}' = \mathbf{A} + \nabla\alpha \ , \ V' = V - \frac{\partial \alpha}{\partial t} \ . \qquad (4)$$

The Schrödinger eq. is covariant, with $H = H(\mathbf{A}, V)$, $H' = H(\mathbf{A}', V')$

$$i\hbar\frac{\partial \Psi}{\partial t} = H\Psi \ \rightarrow \ i\hbar\frac{\partial \Psi'}{\partial t} = H'\Psi' \qquad (5)$$

if the wave function transforms as

$$\Psi'(\mathbf{r}, t) \ = \ e^{\frac{iq\alpha}{\hbar}} \ \Psi(\mathbf{r}, t) \ . \qquad (6)$$

• The mean value of any physically measurable quantity is gauge invariant, ex. $P(\mathbf{r}) = |\Psi|^2 = |\Psi'|^2$.

Exercice: Defining the velocity operator $\mathbf{v} = \frac{1}{m}(\mathbf{p} - q\mathbf{A})$, check that $\langle\Psi|\mathbf{v}|\Psi\rangle = \langle\Psi'|\mathbf{v}'|\Psi'\rangle$.

Gauge principle : Postulate that physical laws are invariant under (4)+ (6) → the hamiltonian is determined to be (2). (6) + (4) define an $U(1)$ transformation. Therefore, $U(1)$ gauge invariance determines the electromagnetic interaction.

2.2. From Dirac and Maxwell eqs. to QED.

Maxwell eqs. in terms of $A_m = (\mathbf{A}, V)$ are invariant under gauge transformations

$$A_m \rightarrow A'_m = A_m - \partial_m \alpha \ . \qquad (7)$$

Relativistic spin 1/2 fermion described by the Dirac eq. $(i\gamma^m \partial_m - M)\Psi = 0$.

Gauge invariance postulate : physics invariant under (7), supplemented with

$$\Psi(x) \rightarrow \Psi'(x) = e^{iq\alpha(x)}\Psi(x) \ . \qquad (8)$$

Dirac eq. not invariant unless we replace the derivative with a covariant derivative

$$D_m\Psi \equiv (\partial_m + iqA_m)\Psi \ \rightarrow$$
$$(D_m\Psi)' = (\partial_m + iqA'_m)\Psi' = e^{iq\alpha(x)}D_m\Psi(x) \ . \qquad (9)$$

Dirac eq. in an electromagnetic field becomes

$$(i\gamma^m D_m - M)\Psi = (i\gamma^m \partial_m - q\gamma^m A_m - M)\Psi = 0 \ . \qquad (10)$$