4.5. The custodial symmetry.

(Sikivie,Susskind,Voloshin,Zakharov,1980)

The tree-level relation $\rho = M_W^2/(M_Z^2 \cos^2\theta_w) = 1$ is the result of an (approximate) symmetry.

In any theory of electroweak interactions which conserves the electric charge and has an approximate global $SU(2)$ symmetry under which A_m^a transform as a triplet, $\rho = 1$ at tree-level.

Approximate means : in the limit of $g' = 0$ and in the absence of the Yukawa couplings.

Proof: The gauge boson mass matrix is then of the form

$$\begin{pmatrix} M^2 & 0 & 0 & 0 \\ 0 & M^2 & 0 & 0 \\ 0 & 0 & M^2 & m_1^2 \\ 0 & 0 & m_1^2 & m_2^2 \end{pmatrix} \tag{62}$$

No photon mass $\rightarrow M^2 m_2^2 - m_1^4 = 0$. The $W_3 - A$ mass matrix is then of the form : (homework)

$$\begin{pmatrix} M_W^2 & \pm M_W \sqrt{M_Z^2 - M_W^2} \\ \pm M_W \sqrt{M_Z^2 - M_W^2} & M_Z^2 - M_W^2 \end{pmatrix} \tag{63}$$

It is then easy to check that $M_W = \cos\theta_w M_Z$.

The Higgs potential $V(\Phi^\dagger\Phi)$ is invariant under an $SO(4)$ symmetry. Indeed,

$$\Phi = \begin{pmatrix} \Phi_1 + i\Phi_2 \\ \Phi 3 + i\Phi_4 \end{pmatrix} \quad , \quad \Phi^\dagger\Phi = \sum_{i=1}^{4} \Phi^2 \qquad \rightarrow$$

$SO(4) = SU(2)_L \times SU(2)_R$ symmetry. The Higgs vev

$$\Phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \text{ breaks } SO(4) \rightarrow SO(3) \;=\; SU(2)_D$$

Other Higgs representations ? **Homework** :

Consider Higgs triplets. Show that the Higgs vev generate the breaking $SO(3) \rightarrow SO(2)$. In this case there is no custodial symmetry and $\rho \neq 1$.

A useful parametrization :

$$\mathcal{H} = \begin{pmatrix} i\tau_2\Phi^* & \Phi \end{pmatrix} = \begin{pmatrix} \Phi_0^* & \Phi_+ \\ -\Phi_+^* & \Phi_0 \end{pmatrix} \,, \Phi^\dagger\Phi = Tr\mathcal{H}^\dagger\mathcal{H}$$

$V(\Phi^\dagger\Phi)$ is invariant under $\mathcal{H} \to U_L \mathcal{H} U_R^\dagger$, with $U_{L,R}$ unitary matrices implementing $SU(2)_L \times SU(2)_R$ transformations. Symmetry breaking

$$\langle\mathcal{H}\rangle = \frac{v}{\sqrt{2}} I_{2\times 2} \quad \text{breaks} \quad SU(2)_L \times SU(2)_R \to SU(2)_D$$

$U(1)_Y$ and Yukawas break the custodial symmetry. However

$$\mathcal{L}_{\mathsf{Yuk}} = h \begin{pmatrix} \bar{t}_L & \bar{b}_L \end{pmatrix} \mathcal{H} \begin{pmatrix} t_R \\ b_R \end{pmatrix}$$

is invariant under $SU(2)_D$ (if $h_t = h_b$).

A one-loop computation in the SM gives

$$\delta\rho = \frac{3g^2(m_t^2 - m_b^2)}{64\pi^2 M_W^2} - \frac{3g^2}{32\pi^2}\ln\frac{m_H}{M_Z} + \cdots$$

where $\cdots$ are subleading contributions from the SM (or eventual new physics contributions, see lectures Bogdan) that are smaller than 10^{-3}.

5. QUANTUM CORRECTIONS AND RENORMALIZATION.

5.1. UV divergences and regularization.

Perturbation theory in QFT is plagued with UV divergences. We have to keep an UV cutoff Λ in computing physical quantities. There are three cases that arise :

- Super-renormalizable theories : only a finite number of Feynman diagrams diverge.
- Renormalizable theories : a finite number of amplitudes diverge. Divergences at all orders in pert. theory.
- Non-renormalizable theories : All amplitudes are divergent at a certain order in perturbation theory.

• In (super)renormalizable theories, UV divergences can be absorbed into rescaling of fields and redefinitions of the various couplings and masses. Taking the couplings/masses from experience, the UV cutoff disappears from physical quantities $\rightarrow$ the theory is predictive at any energy scale.

• In non-renormalizable theories, we need an infinite number of couplings and masses in order to absorbe UV divergences. We would need an infinite amount of experimental data to determine all these couplings $\rightarrow$ at high-energies $E > \Lambda$ the theory looses its predictive power. At low-energy the theory is perfectly predictive.

- 5.2. Relevant, marginal and irrelevant couplings

Consider a scalar theory of the form

$$S_\Lambda = \int d^4x \left(\frac{1}{2}(\partial\phi)^2 + \frac{m^2\phi^2}{2} + \sum_n \lambda_n \phi^n\right) \ , \quad (64)$$

where S_Λ is the euclidian action defined with a cutoff Λ. The couplings λ_n have (classical) mass dimensions $[\lambda_n] = 4 - n$. Let us consider the theory with two different maximal euclidian momenta/cutoffs:

i) $0 < p < \Lambda$

ii) $0 < p < \Lambda' = \epsilon \Lambda$, where $\epsilon < 1$.

The theory ii) has therefore a lower cutoff.

It is interpreted as a theory where the high-momenta of theory i) were integrated out. The theory i) has the action (64). In the theory ii) the cutoff can be redefined to be the same as in i) with the help of a scale transformation

$$x' = \epsilon\ x \quad , \quad p' = \epsilon^{-1} p \quad , \quad \phi' = \epsilon^{-1} \phi \tag{65}$$

In terms of the rescaled field and coordinates, the action of theory ii) become (**homework**)

$$S_{\Lambda'} = \int d^4x' \left(\frac{1}{2}(\partial'\phi')^2 + \frac{m'^2(\phi')^2}{2} + \sum_n \lambda'_n (\phi')^n \right) , \tag{66}$$

where

$$m'^2 = \frac{1}{\epsilon^2} m^2 \quad , \quad \lambda'_n = \epsilon^{n-4} \lambda_n \qquad (67)$$

Notice that the new mass and couplings scale with their classical dimension. We see therefore that the mass and couplings with positive dimension grow in the IR, whereas couplings with negative dimension decrease in the IR. It is said that

$$[\lambda_n] > 0 \quad \longrightarrow \quad \text{relevant coupling}$$

$$[\lambda_n] = 0 \quad \longrightarrow \quad \text{marginal coupling}$$

$$[\lambda_n] < 0 \quad \longrightarrow \quad \text{irrelevant coupling}$$

5.3. (Non)renormalizability and couplings dims.

There is a straight connection between renormalizability and the three type of couplings above:

- relevant couplings $\longrightarrow$ super-renormalizability.
- marginal couplings $\longrightarrow$ renormalizability.
- irrelevant couplings $\longrightarrow$ non-renormalizability.

It is easy to argue for this by dimensional arguments.

Take some simple examples.

a) - Relevant coupling

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2\phi^2}{2} - \lambda_3\phi^3 \, . \qquad (68)$$

The coupling has dimension $[\lambda_3] = +1$, so it is relevant.

At one-loop, the UV divergent terms lead to (Hw:)

$$\delta\mathcal{L}_1 \sim \lambda_3 \Lambda^2 \phi + \lambda_3^2 \phi^2 \ln \Lambda \ ,$$

which are both of super-renormalizable type. The first lead to mass renormalization, whereas the second leads to a scalar tadpole.

At two loops, the only UV divergences are a cosmological constant and a scalar tadpole. At three loops, there is only a log UV divergence in the cosmological constant. No UV divergences exist at higher loops.

Dim. argument : The highest UV divergent term in the coupling is the three-loop vacuum energy

$$\lambda_3^4 \ln \Lambda \tag{69}$$

Higher loops have higher powers in λ_3 and cannot contribute to the UV divergent terms in the effective lagrangian

Obs: $1/m^2$ terms are IR, not UV contributions.

b) - Irrelevant coupling

$$\mathcal{L} \;=\; \frac{1}{2}(\partial\phi)^2 \;-\; \frac{m^2\phi^2}{2} \;-\; \lambda_6\phi^6 \;. \tag{70}$$

The coupling has dimension $[\lambda_6] = -2$, so it is irrelevant. At one-loop, the UV divergent terms in the eight-point amplitude lead to (Homework:)

$$\Gamma^{(8)}_{1-\text{loop}}(p_i) \; \sim c \; \lambda_6^2 \ln \Lambda + \cdots \; .$$

To cancel this divergence, one has to add a new coupling to the original action

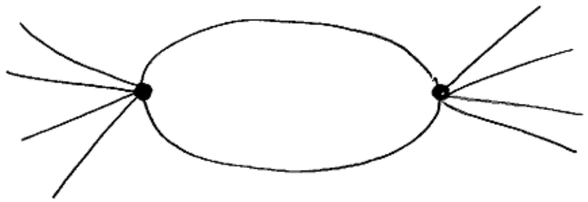
$$\delta \mathcal{L}_1 \sim \lambda_8 \phi^8 \; ,$$

and to adjust the coupling λ_8 such that

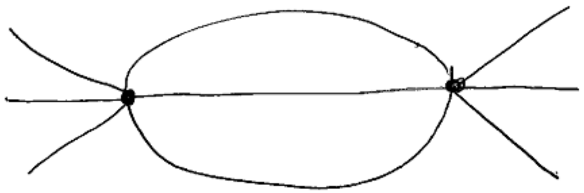
$$\lambda_8 \; + \; c \; \lambda_6^2 \ln \Lambda \; = \; \text{finite}$$

At two-loops, we get new new UV divergences, like the one in the six-point amplitude, prop. to

$$\Gamma^{(6)}_{2-\text{loops}}(p_i) \; \sim \; c' \; (p_i p_j) \lambda_6^2 \ln \Lambda \; ,$$



One-loop divergent vertex in the φ^6 theory asking for adding λ_8 and $\delta\mathcal{L}_1$



Two-loop divergent vertex, asking for adding λ'_8 and $\delta\mathcal{L}_2$

which can be canceled by adding another coupling

$$\delta\mathcal{L}_2 \sim \lambda_8' \, \phi^4(\partial\phi)^2 \ ,$$

such that

$$\lambda_8' + c' \, \lambda_6^2 \ln\Lambda = \text{finite}$$

The UV divergences proliferate at higher loop orders, generating an infinite tower of operators of higher and higher dimension.

Dimensional argument: Terms of the type $\lambda_6^n\phi^{4+2n}\ln\Lambda$, $\lambda_6^n(\partial\phi)^2\phi^{2n}\ln\Lambda$ have the correct dimension to be generate for any n. Predictivity at high-energy is lost.

• However, let us define $\lambda_6 \sim 1/M^2$. Then :

In the IR $E < M$, the effect of non-renormalizable operators on physical quantities is prop. to some power or E/M and/or m/M, so their effects is negligible.

Effective theories with cutoff Λ (ex. General relativity, $\Lambda = M_P$) are predictive at energies $E << \Lambda$.

Another viewpoint: for $\mathcal{L}_{\text{int}} = \sum_n \lambda_n \phi^n$, leading cross-section for $2 \to 2$ particle scattering is

$$\sigma \ = \ \sum_n c_n \lambda_n^2 E^{2n-10} \ \sim \frac{1}{E^2} \sum_n c_n (\frac{E}{M})^{2n}$$

for $\lambda_n \sim 1/M^{n-4} \to$ predictive power lost for $E \geq M$.

Ex. 1 : Coupling renormalization for ϕ^4 theory.

Consider the ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

and compute the four-point function at one-loop

$$\Gamma(k_1k_2k_3k_4) = -i\lambda_0 + \frac{(-i\lambda_0)^2}{2} \times$$

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_0^2} \frac{i}{(p-k_1-k_2)^2 - m_0^2} + \text{two crossing terms}$$

After the Wick rotation to euclidian momenta

$$\Gamma(k_1k_2k_3k_4) = -i\lambda_0 + \frac{i\lambda_0^2}{2}\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m_0^2} \frac{1}{(p-k_1-k_2)^2 + m_0^2}$$

$$+ \text{two crossing terms}$$

The integral is log divergent in the UV. There are various ways to "renormalize" the integral. Here is a simple way : Define

$$V(s) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m_0^2} \frac{1}{(p-k_1-k_2)^2+m_0^2}$$
$$= \int_{p^2 \geq \mu^2}^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{p^4} + \text{finite} \,,$$

where the energy scale μ is arbitrary. We find (Hw)

$$\Gamma(k_1k_2k_3k_4) = -i\lambda_0 + \frac{3i\lambda_0^2}{16\pi^2} \ln\frac{\Lambda}{\mu} + \text{finite} = -i\lambda(\mu) + \text{finite}$$

What is the physical interpretation of this manipulation?

i) λ_0 is not a physical parameter. It can be chosen to depend on Λ such that

$$\lambda(\mu) = \lambda_0(\Lambda) - \frac{3\lambda_0^2}{16\pi^2} \ln \frac{\Lambda}{\mu}$$

is independent of Λ.

ii) Any value of μ leads to the same physical result. λ_0 is independent of μ? Therefore

$$\frac{d\lambda}{d \ln \mu} = \frac{3\lambda^2}{16\pi^2} = \beta(\lambda) \qquad (71)$$

describes the renormalization group equation (RGE) of λ at one-loop. (71) is then a differential eq., whose solution is (homework)

$$\lambda(\mu) \;=\; \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{16\pi^2}\ln\frac{\mu}{\mu_0}}$$

There is an equivalent prescription : add a local "counterterm" to the lagrangian

$$\mathcal{L} \;+\; \delta\mathcal{L} \;=\; \mathcal{L}_0 \;,$$

which cancels the UV divergence.

In renormalizable theories, a finite number of counterterms are needed in order to render the theory UV finite. In non-renormalizable theories, an infinite number of counterterms are needed.

Ex. 2 : QED and running of fine structure constant.

We use here the counterterm method for the renormalization of QED. In this case

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{mn}^2 + \bar{\Psi}(i\gamma^m\partial_m - q\gamma^m A_m - M)\Psi \\
\delta\mathcal{L} &= -\frac{1}{4}(Z_3-1)F_{mn}^2 + (Z_2-1)\bar{\Psi}i\gamma^m\partial_m\Psi \\
&-(Z_1-1)q\bar{\Psi}\gamma^m A_m\Psi - (Z_M-1)M\bar{\Psi}\Psi \\
\mathcal{L}_0 &= \mathcal{L} + \delta\mathcal{L} = -\frac{1}{4}(F_{mn}^0)^2 + \bar{\Psi}_0(i\gamma^m\partial_m - q_0\gamma^m A_m^0 - M_0)\Psi_0
\end{aligned}
$$

The relations between bare and renormalized quantities are then

$$A_m^0 = Z_3^{1/2}\, A_m \quad , \quad \Psi_0 = Z_2^{1/2}\, \Psi$$
$$M_0 = \frac{Z_M}{Z_2}\, M \quad , \quad q_0 = \frac{Z_1}{Z_2 Z_3^{1/2}}\, q$$

In QED $Z_1 = Z_2$ (Ward identity) $\Rightarrow$ $q_0 = Z_3^{-1/2} q$. The RG running can be found from

$$\mu \frac{\partial}{\partial \mu} q_0 = 0 \Rightarrow (Hw) \quad \beta(q) = \mu \frac{\partial q}{\partial \mu} = q \frac{\partial \ln Z_3^{1/2}}{\partial \ln \mu}$$

By an explicit computation we find

$$Z_3 = 1 - \frac{\alpha}{3\pi} \ln \frac{\Lambda}{\mu} + \text{ finite }, \tag{72}$$

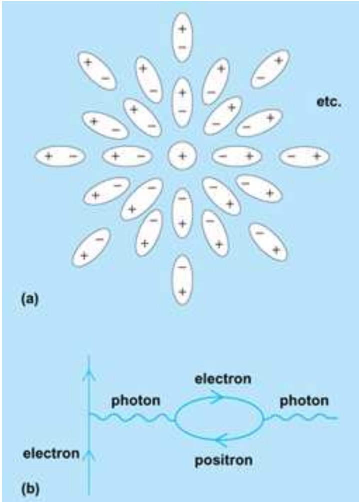
where μ is an arbitrary, renormalization scale.

Then we find ($\alpha = q^2/(4\pi)$)

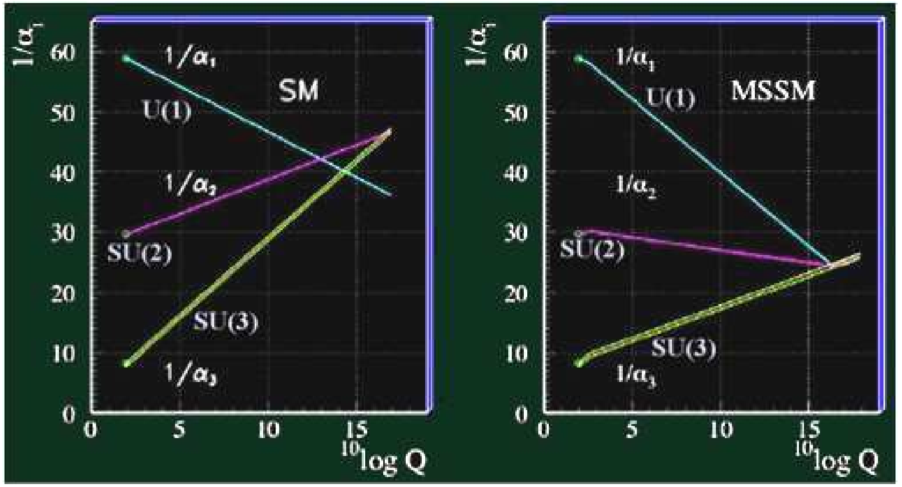
$$\beta(q) \;=\; \frac{q^3}{24\pi^2} \qquad \Rightarrow \qquad \frac{1}{\alpha(Q)} = \frac{1}{\alpha(\mu)} - \frac{1}{3\pi}\ln\frac{Q}{\mu}$$

The fine structure coupling increases with energy !

Screening of electric charge by vacuum polarization

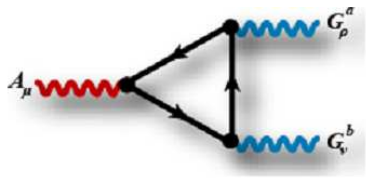


The strong coupling α_3 is anti-screened due to gluon self-interactions



Tendency of unification of couplings at high energy ?

5.4. Global and gauge anomalies



Symmetries of the classical action can have anomalies at the quantum level, generated by one-loop triangle diagrams.

For global symmetries, this does not creates problems.

Consider to start with

$$\mathcal{L} = \bar{\Psi} i \gamma^m D_m \Psi - M \bar{\Psi} \Psi$$

For $M \to 0$, the model has symmetry $U(1)_V \times U(1)_A$.