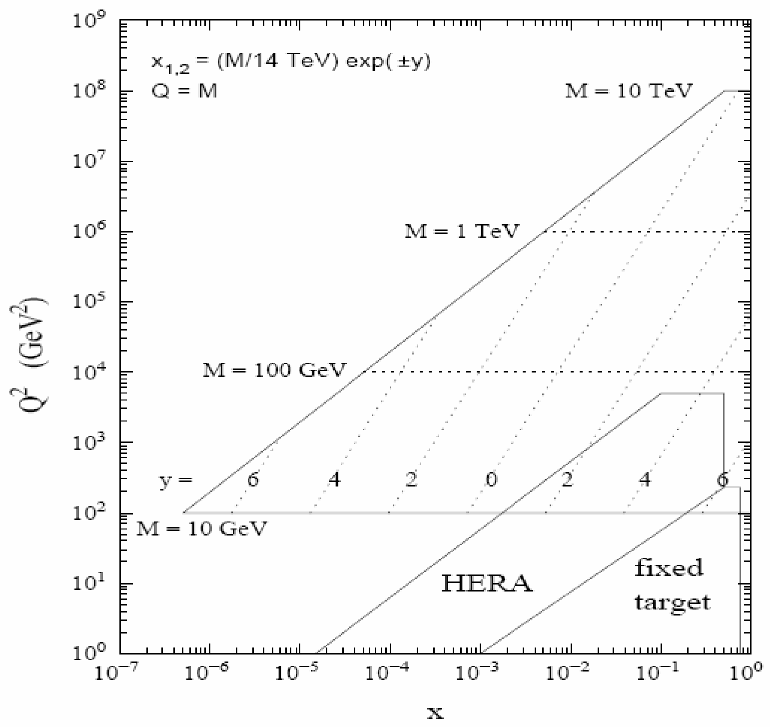
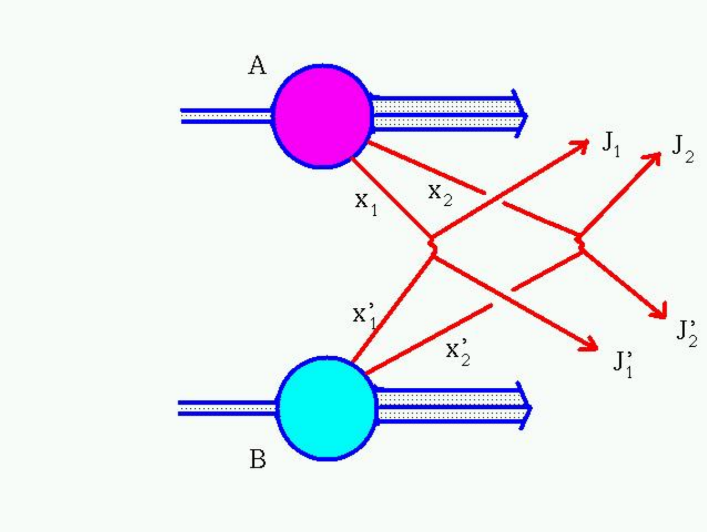
Parton correlations and multi-parton exclusive cross sections

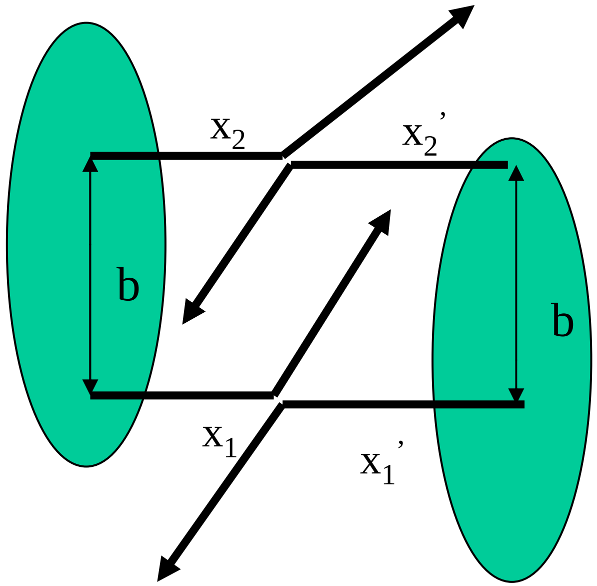
G. Calucci, E. Cattaruzza, A. Del Fabbro and D. Treleani

The rapid increase of the parton flux at small *x* will produce large rates of events with multiple parton collisions in high energy hadronic interactions





As the scale of the hard interactions is very small in comparison with the hadron size, in a double parton collisions the non-perturbative component is factorized into a function which depends on two fractional momenta and on the relative transverse distance b



The inclusive double parton-scattering cross-section, for two parton processes A and B in a pp collision, is given by

$$\sigma^D_{(A,B)} = \frac{m}{2}\sum_{i,j,k,l}\int D_{ij}(x_1,x_2;b)\hat{\sigma}^A_{ik}(x_1,x_1')\hat{\sigma}^B_{jl}(x_2,x_2')D_{kl}(x_1',x_2';b)dx_1dx_1'dx_2dx_2'd^2b$$

where $D_{ij}(x_1,x_2,b)$ are the double parton distribution functions

If partons are not correlated in *x*, the two-body parton distribution is given by

$$D_{ij}(x_1, x_2; b) = D_i(x_1) D_j(x_2) F_i^j(b)$$

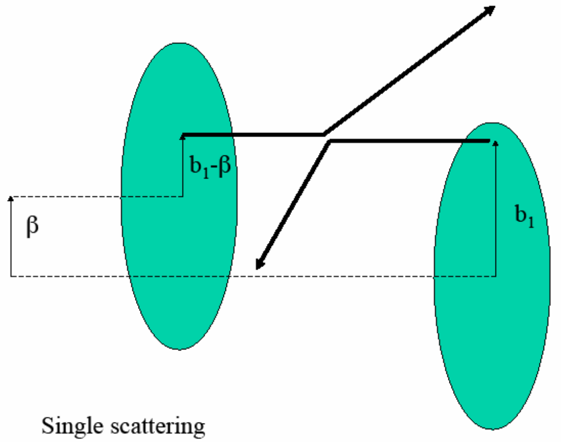
and the expression of the cross section is

$$\sigma^D_{(A,B)} = \frac{m}{2} \sum_{ijkl} \Theta^{ij}_{kl}\, \hat{\sigma}_{ij}(A)\, \hat{\sigma}_{kl}(B)$$

$$\boxed{\Theta^{ij}_{kl} = \int d^2b\, F^i_k(b)\, F^j_l(b)}$$

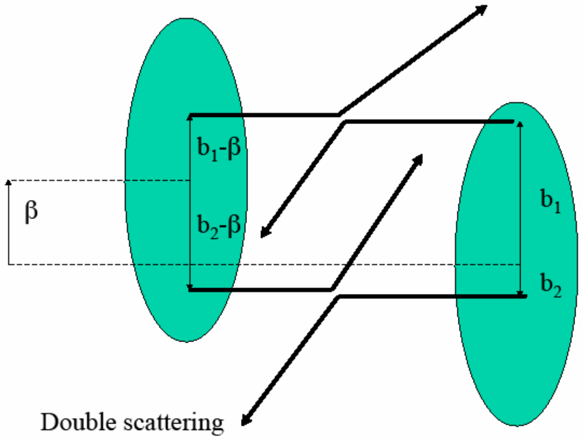
are geometrical coefficients with dimension an inverse cross section and depending directly on the transverse correlation of the different kinds of partons in the hadron structure

Neglecting all longitudinal correlations and assuming independence on the flavor indices one may easily write the N-parton inclusive cross section σ_N



Single scattering

$$\sigma_S = \int_{p_t^c} D(x) f(b) \hat{\sigma}(x,x') D(x') f(b-\beta) d^2b d^2\beta dx dx'$$



Double scattering

$$\begin{aligned}\sigma_D &= \frac{1}{2!}\int_{p_t^c} D(x_1) f(b_1) \hat{\sigma}(x_1,x_1') D(x_1') f(b_1-\beta) d^2b_1 dx_1 dx_1' \times \\ &\quad \times D(x_2) f(b_2) \hat{\sigma}(x_2,x_2') D(x_2') f(b_2-\beta) d^2b_2 dx_2 dx_2' d^2\beta \\ &= \int \frac{1}{2!}\Big(\int_{p_t^c} D(x) f(b) \hat{\sigma}(x,x') D(x') f(b-\beta) d^2b dx dx'\Big)^2 d^2\beta\end{aligned}$$

$$\sigma_N = \int \frac{1}{N!}\Big(\int_{p_t^c} D(x) f(b) \hat{\sigma}(x,x') D(x') f(b-\beta) d^2b dx dx'\Big)^N d^2\beta$$

The integrand $\frac{1}{N!}\Big(\int_{p_t^c} D(x)f(b)\hat{\sigma}(x,x')D(x')f(b-\beta)d^2b\,dx\,dx'\Big)^N$ is dimensionless

and *after normalization* it may be understood as the probability to have *N* parton collisions:

$$\frac{(\sigma_S F(\beta))^N}{N!}e^{-\sigma_S F(\beta)} = \mathrm{P}_N(\beta) \qquad \sigma_S F(\beta) = \int_{p_t^c} D(x)f(b)\hat{\sigma}(x,x')D(x')f(b-\beta)d^2b\,dx\,dx'$$

One may hence express the hard cross section σ_H (namely the contribution to σ_{inel} of all events with *at least* one parton collision) as

$$\boxed{\sigma_H = \sum_{N=1}^{\infty}\int d^2\beta \frac{(\sigma_S F(\beta))^N}{N!}e^{-\sigma_S F(\beta)} = \int d^2\beta\Big[1-e^{-\sigma_S F(\beta)}\Big]}$$

Differently form σ_S and σ_D, σ_H is always smaller that σ_{inel} and one may write

$$\boxed{\sigma_{inel} = \sigma_{soft} + \sigma_H}$$

While σ_S is divergent when p_t^c goes to zero, σ_H and all contributions to σ_H with a given number *N* of parton collisions are, on the contrary, finite when p_t^c goes to zero

The single and the double parton inclusive cross sections are given by the average and by the second moment of the distribution in the number of collisions

$$\langle N\rangle\sigma_H = \int d^2\beta \sum_{N=1}^{\infty} \frac{N\left[\sigma_S F(\beta)\right]^N}{N!} e^{-\sigma_S F(\beta)} = \int d^2\beta \sigma_S F(\beta) = \sigma_S$$

$$\begin{aligned} \frac{\langle N(N-1)\rangle}{2}\sigma_H &= \frac{1}{2}\int d^2\beta \sum_{N=2}^{\infty} \frac{N(N-1)\left[\sigma_S F(\beta)\right]^N}{N!} e^{-\sigma_S F(\beta)} \\ &= \frac{1}{2}\int d^2\beta \left[\sigma_S F(\beta)\right]^2 = \sigma_D \end{aligned}$$

The relations

$$\boxed{\langle N\rangle\sigma_H = \sigma_S \quad \text{and} \quad \frac{1}{2}\langle N(N-1)\rangle\sigma_H = \sigma_D}$$

are not specific of the Poissonian case and can be derived on much more general grounds

The different interactions sum up incoherent and the resulting physical picture is purely probabilistic

As a consequence one may argue that, while in the previous expressions each final state parton is integrated over the whole phase space with a cutoff in p_t, the cross section, in a different rapidity window and with a different cutoff, would have precisely the same poissonian expression. The only difference being in the value of σ_S

Notice that all contributions to σ_H, with a given number of collisions N and σ_H itself, are all well behaved for small values of the cutoff

One should also point out that all contributions to $\sigma_{H,}$, with a given number of collisions N and σ_H itself, are all quantities well defined experimentally

The physical picture can be generalized including correlations in the multi-parton distribution functions

Multi-parton correlations

One may introduce the *exclusive* n-body parton distributions

$$W_n(u_1 \ldots u_n) \quad , \quad u_i \equiv (\mathbf{b}_i, x_i)$$

and the multi-parton generating functional

$$\mathcal{Z}[J] = \sum_n \frac{1}{n!} \int J(u_1) \ldots J(u_n) W_n(u_1 \ldots u_n) du_1 \ldots du_n,$$

Probability conservation imposes the normalization condition $\mathcal{Z}[1] = 1.$

The many-body densities, i.e. the *inclusive* distributions $D_n(u_1 \ldots u_n)$ are readily obtained:

$$\begin{aligned}
D_1(u) &= \left.\frac{\delta \mathcal{Z}}{\delta J(u)}\right|_{J=1}, \\
D_2(u_1, u_2) &= \left.\frac{\delta^2 \mathcal{Z}}{\delta J(u_1) \delta J(u_2)}\right|_{J=1}, \\
&\ldots
\end{aligned}$$

One may introduce the logarithm of the generating functional $\mathcal{F}[J] = \ln(\mathcal{Z}[J])$

and, by expanding in the vicinity of *J*=1, one obtains the many-body parton correlations

$$\mathcal{F}[J] = \int D(u)[J(u)-1]du + \sum_{n=2}^{\infty} \frac{1}{n!} \int C_n(u_1 \dots u_n)\left[J(u_1)-1\right] \dots \left[J(u_n)-1\right] du_1 \dots du_n$$

A rather general expression for $\sigma_H = \int d^2\beta \sigma_H(\beta)$ is the following

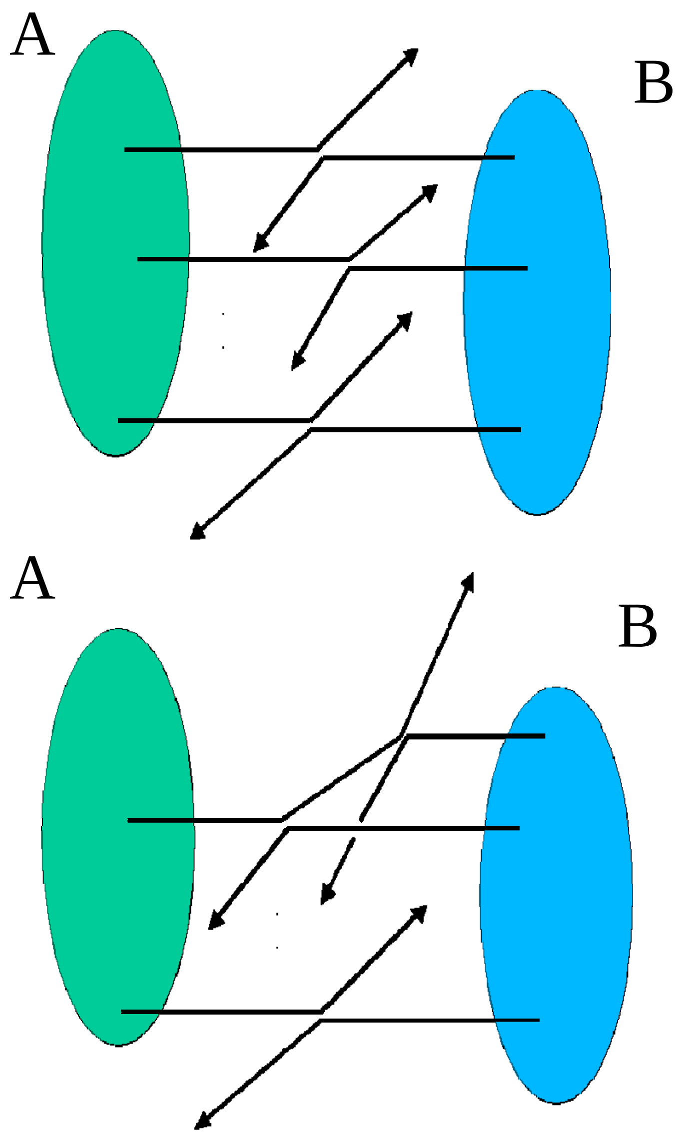
Prob. to find the hadron B in a configuration with *m* partons

$$\sigma_H(\beta) = \int \sum_n \frac{1}{n!} \frac{\delta}{\delta J(u_1)} \cdots \frac{\delta}{\delta J(u_n)} \mathcal{Z}_A[J] \times \sum_m \frac{1}{m!} \frac{\delta}{\delta J'(u'_1-\beta)} \cdots \frac{\delta}{\delta J'(u'_m-\beta)} \mathcal{Z}_B[J'] \times \left\{1 - \prod_{i=1}^{n}\prod_{j=1}^{m}[1-\hat{\sigma}_{i,j}(u,u')]\right\} \prod du du' \Big|_{J=J'=0}$$

Prob. to find the hadron A in a configuration with *n* partons

Prob. to have at least one interaction between the two configurations

This expression for σ_H includes all possible interaction (with on-shell intermediate states) between any configuration with *n* partons of hadron A and any configuration with *m* partons of hadron B. The cross section complies with the AGK cutting rules.



Both disconnected collisions

and rescatterings

are included in σ_H

A simpler expression is obtained when keeping only disconneced collisions into account, which amounts removing all addenda with repeated indices in the interaction probability

$$\left\{1 - \prod_{i,j}^{n,m} [1 - \hat{\sigma}_{ij}]\right\} \Rightarrow \sum_{ij} \hat{\sigma}_{ij} - \frac{1}{2!} \sum_{ij} \sum_{k \neq i, l \neq j} \hat{\sigma}_{ij} \hat{\sigma}_{kl} + \ldots$$

Because of the symmetry of the derivative operators, the expression may be replaced by

$$nm\hat{\sigma}_{11} - \frac{1}{2!} n(n-1)m(m-1)\hat{\sigma}_{11}\hat{\sigma}_{22} + \ldots$$

The sums can be performed and the cross section is expressed in a closed analytic form

$$\boxed{\sigma_H(\beta) = \Big[1 - \exp\big(-\delta \cdot \hat{\sigma} \cdot \delta'\big)\Big] \mathcal{Z}_A[J+1] \mathcal{Z}_B[J'+1] \Big|_{J=J'=0}}$$

where all disconnected collisions are included, while multi-parton correlations are kept into account at all orders.

The average number of collisions and the second moment of the distribution are obtained when replacing the interaction probability with

$nm\hat{\sigma}_{11}$ and with $\frac{1}{2!}n(n-1)m(m-1)\hat{\sigma}_{11}\hat{\sigma}_{22}$ respectively

The average number of collisions at fixed impact parameter is hence given by

$$\begin{aligned}\langle N(\beta)\rangle = \int \sum_n \frac{1}{n!}\frac{\delta}{\delta J(u_1)}\cdots\frac{\delta}{\delta J(u_n)}\mathcal{Z}_A[J]\Big|_{J=0} \times n \\ \times \sum_m \frac{1}{m!}\frac{\delta}{\delta J'(u'_1-\beta)}\cdots\frac{\delta}{\delta J'(u'_m-\beta)}\mathcal{Z}_B[J']\Big|_{J'=0} \times m \\ \times \hat{\sigma}(u_1,u'_1)\prod dudu'\end{aligned}$$

Summing over *m* and *n* one obtains

$$\langle N(\beta)\rangle = \int \frac{\delta}{\delta J(u_1)}\mathcal{Z}_A[J+1]\Big|_{J=0}\frac{\delta}{\delta J'(u'_1-\beta)}\mathcal{Z}_B[J'+1]\Big|_{J'=0}\hat{\sigma}(u_1,u'_1)du_1du'_1$$

which is precisely the QCD-parton model single scattering expression of the inclusive cross section

$$\boxed{\langle N(\beta)\rangle = \int D_A^{(1)}(u)D_B^{(1)}(u')\hat{\sigma}(u,u')dudu'}$$

An analogous argument holds for the second moment of the distribution in the number of collisions.

The single and double scattering expressions of the QCD parton model are hence meaningful also in the case of large numbers of parton collisions and represent the first two moments of the distribution in the number of collisions

The result holds at all orders in the multi-parton correlations

Notice that, in the case of disconnected multi-parton scatterings, the moments of the distribution in the number of collisions correspond to quantities directly measured in an inclusive experiment. e.g. the single jet inclusive cross section measurement performs an average in the number of parton collisions (since the experimental apparatus counts each event with the multiplicity of interactions).

When rescatterings are taken into account, the relation between multiple scatterings and the moments of the distribution in the number of collisions is not changed.

However, while in the case of disconneced parton collisions the average number of interactions represents precisely the quantity measured in the inclusive experiment, there is no simple relation between the two quantities when rescatterings are taken into account

In fact, in such a case, rescatterings contribute to the shadowing corrections of the parton distribution functions

Keeping into account disconnected multi-parton scatterings only and limiting correlations to the two-body case, one writes a gaussian functional integral for the cross section, which allows obtaining a closed expression for σ_H

$$\mathcal{F}_{A,B}[J+1] = \int D_{A,B}(u)J(u)du + \frac{1}{2}\int C_{A,B}(u,v)J(u)J(v)dudv$$

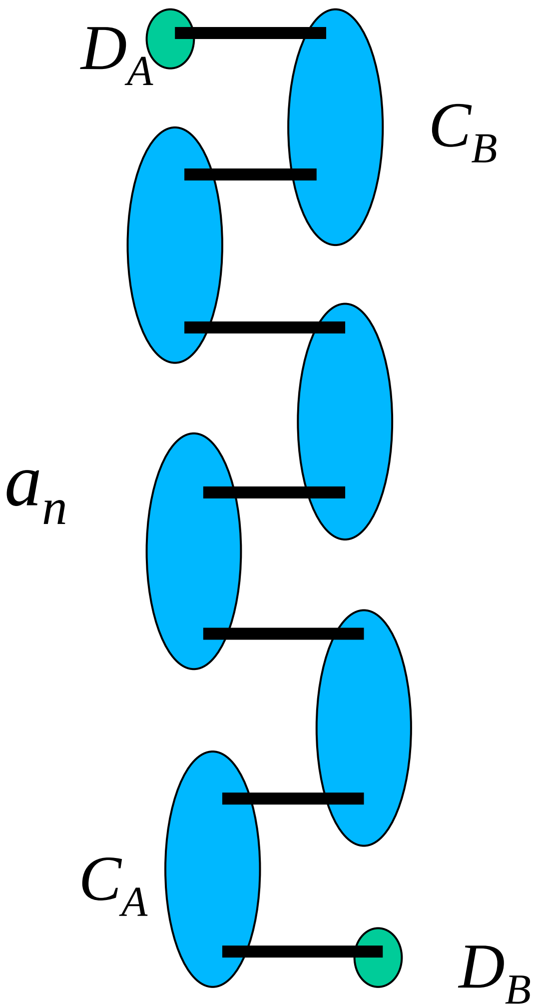
D(u) is the average number of partons and *C(u,v)* the two-body correlation. The final expression obtained for σ_H is :

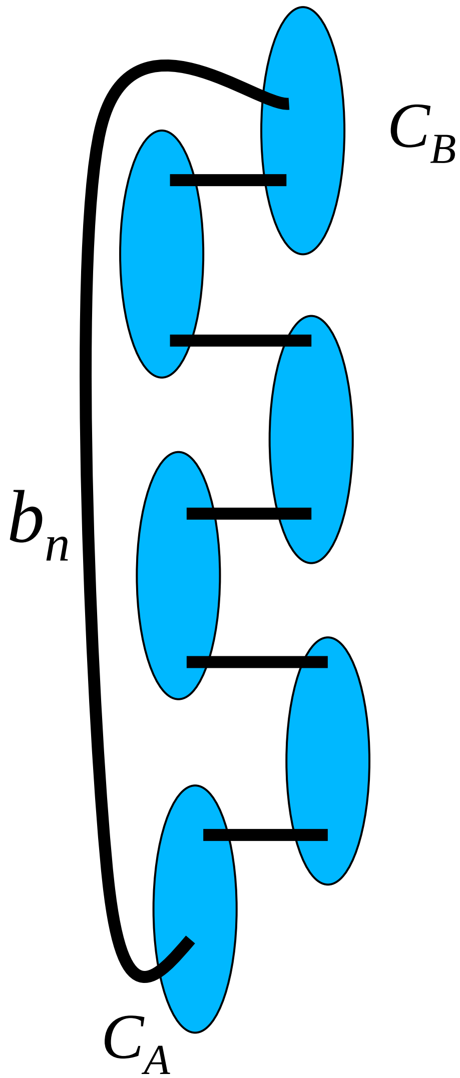
$$\boxed{\sigma_H(\beta) = 1 - \exp\left[-\tfrac{1}{2}\sum_n a_n - \tfrac{1}{2}\sum_n b_n/n\right]}$$

$$a_n = \int D_A(u_1)\hat{\sigma}(u_1,u_1')C_B(u_1'-\beta,u_2'-\beta)\hat{\sigma}(u_2',u_2)C_A(u_2,u_3)\ldots$$
$$\ldots D_B(u_n'-\beta)\prod du_i du_i'$$

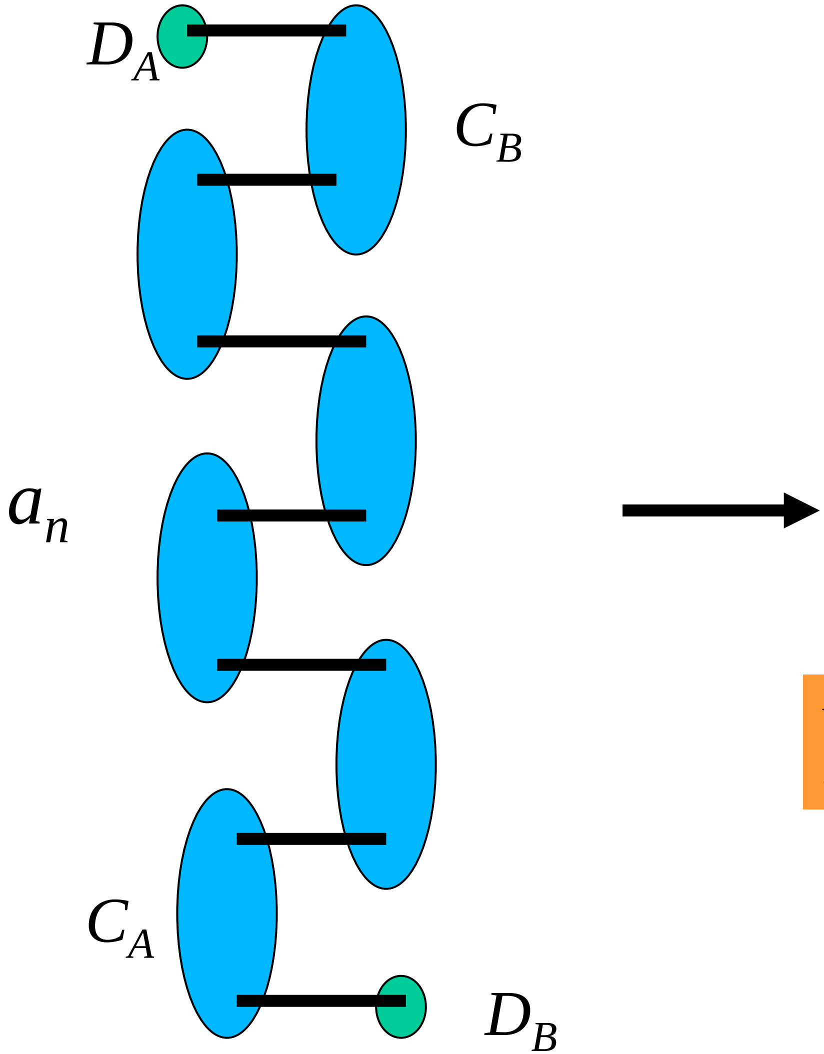
$$b_n = \int C_A(u_n,u_1)\hat{\sigma}(u_1,u_1')C_B(u_1'-\beta,u_2'-\beta)\hat{\sigma}(u_2',u_2)\ldots$$
$$\ldots C_B(u_{n-1}'-\beta,u_n'-\beta)\hat{\sigma}(u_n',u_n)\prod du_i du_i'$$

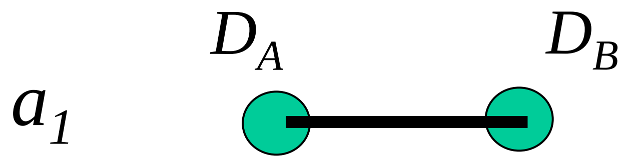
The structures a_n and b_n allow a graphic representation





The correlation term integrates to zero, which allows a further simplification when the correlation lenght is small in comparison to the hadron size. In that case one may neglect all terms a_n with $n > 1$





which is the average number of collisions at fixed impact parameter

$$\langle N(\beta)\rangle = \int D_A^{(1)}(u) D_B^{(1)}(u') \hat{\sigma}(u,u') du du'$$

What to be measured ?

All relations above are easily adapted to different choices of final state cuts

The *inclusive cross sections* are the moments of the distribution in the number of collisions. Hence e.g. triple, quadruple etc. parton interactions contribute to the inclusive double parton scattering cross section σ_D, once taken into account with the proper multiplicity factor.

One may obtain further information on the interaction dynamics by measuring, in addition to the inclusive cross sections, also the *exclusive cross sections* (including σ_H itself) in different phase space intervals.

All exclusive cross sections are well defined experimental quantities (corresponding to events with a given number of parton interactions only) and are well behaved in the infrared region

Explicit expressions, including all effects of the two-body parton correlations, are available both for the *inclusive* and the *exclusive cross sections*