FFT corrections for tune measurements

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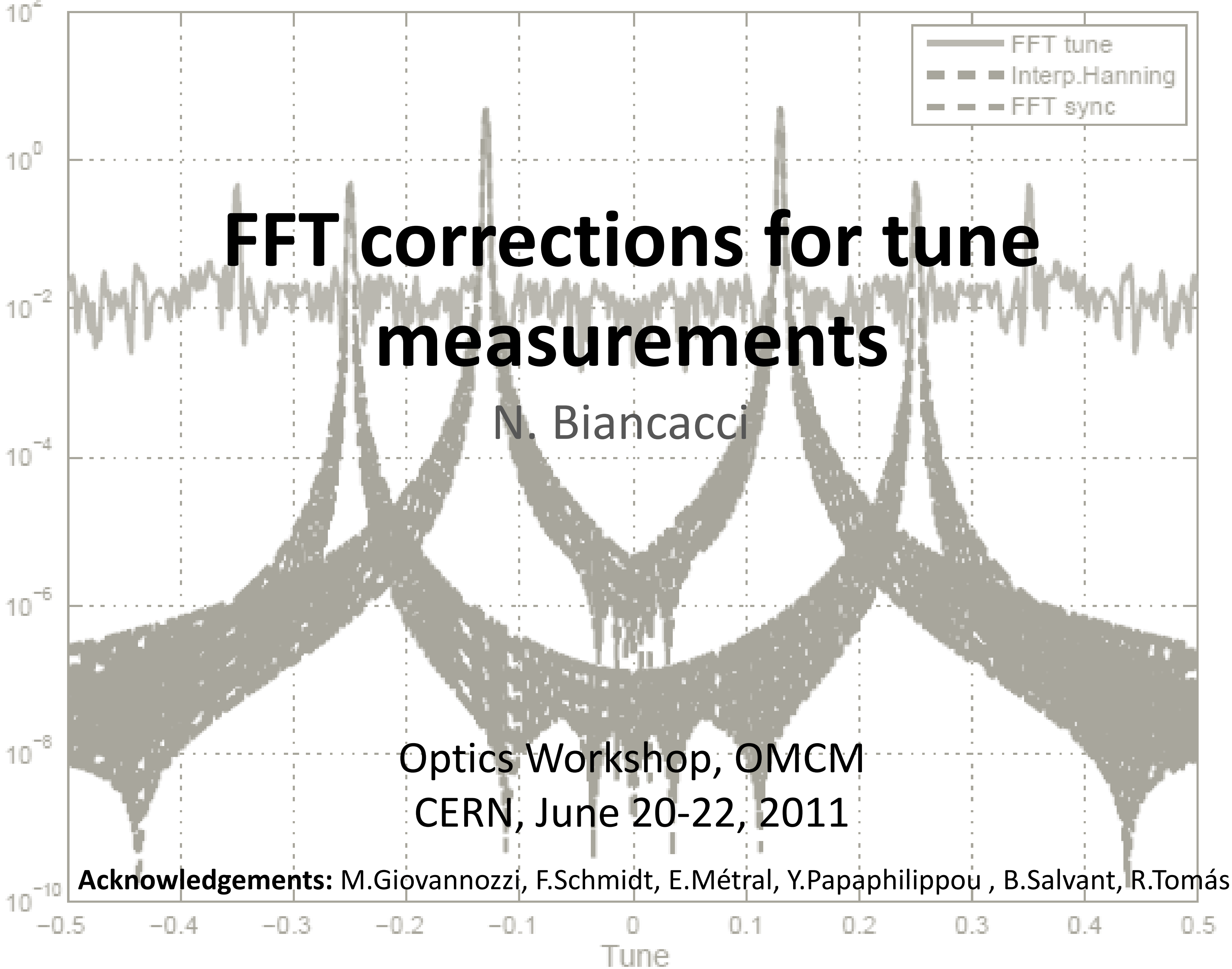
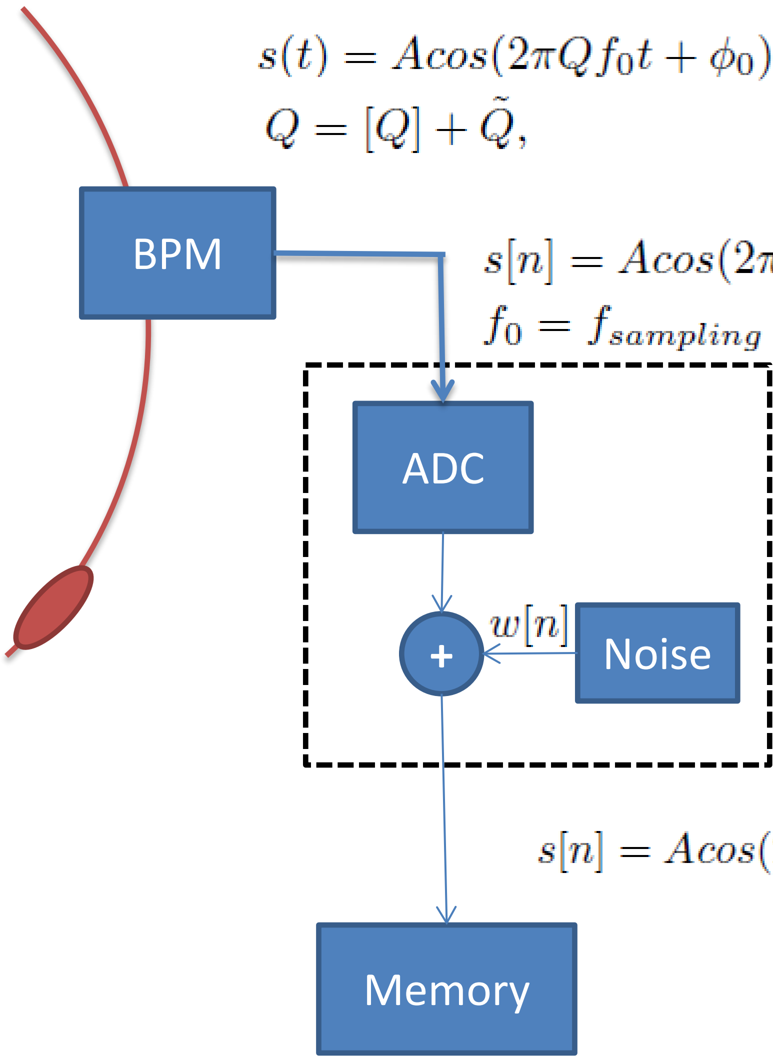


Outline

1. Discrete Fourier Transform (DFT) on harmonic signals
2. FFT-ApFFT-FFTc-NAFF-SUSSIX
3. Comparisons between different methods to achieve higher resolutions in amplitude, frequency and phase advance.
4. Conclusions and outlook

DFT on harmonic signals



$$s(t) = A cos(2\pi Q f_0 t + \phi_0),$$
$$Q = [Q] + \tilde{Q},$$

$$s[n] = A cos(2\pi \tilde{Q} n + \phi_0), \;\; n = [0..N]$$
$$f_0 = f_{sampling} = 1/T_0$$

$$s[n] = A cos(2\pi \tilde{Q} n + \phi_0) + w[n]$$

DFT on harmonic signals

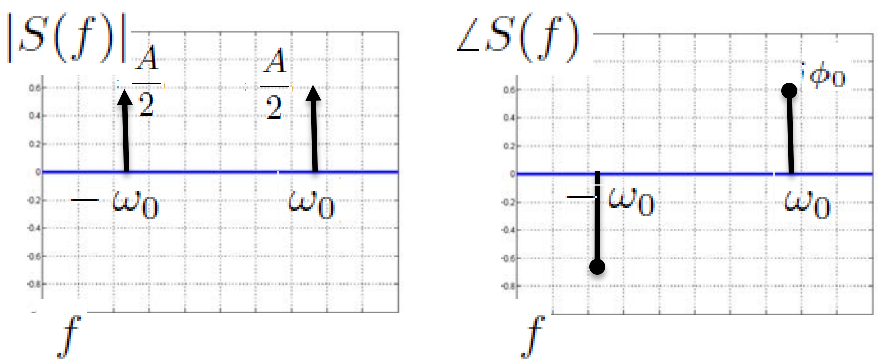
Continuum case: $s(t) = Acos(2\pi Q f_0 t + \phi_0) \qquad t \in (-\infty, +\infty)$

$$\downarrow$$

$$\boxed{S(f) = \int_{-\infty}^{+\infty} s(t) e^{-j\omega t}\, \mathrm{d}t,}$$

$$\downarrow$$

$$\boxed{S(f) = \frac{A}{2} e^{j\phi_0} \delta(\omega - \omega_0) + \frac{A}{2} e^{-j\phi_0} \delta(\omega + \omega_0)}$$



DFT on harmonic signals

Discrete case: $$x[n] = Acos(2\pi\tilde{Q}n + \phi_0), \quad n \in [-\infty, +\infty]$$

Having a finite number of samples corresponds to *windowing* with a "rect" function r[n] the sampled signal.

$$s[n] = x[n] \cdot r[n] \qquad r[n] = rect[n] = \begin{cases} 1 & \text{if } n \in [0, N], \\ 0 & \text{elsewhere} \end{cases}$$

This operation is a *convolution* in the *discrete frequency domain.*

$$S[k] = X[k] * R[k],$$

Applying the DFT formula on r[n] and applying translation property* from s[n], we get:

$$\boxed{\mathcal{D}\{x[n]\} = \sum_{k=0}^{N-1} x[n]e^{j\omega_k n}} \qquad \boxed{\omega_k = 2\pi\frac{k}{N},}$$

$$R[k] = \frac{sin(k\pi)}{sin(k\pi/N)}e^{-j\pi k\frac{N-1}{N}} \underset{N\to\infty}{\simeq} N sinc(k\pi)e^{-j\pi k},$$

$$S[k] = N\frac{A}{2}e^{j\phi_0}e^{j\pi(k-\tilde{Q}N)}sinc((k-\tilde{Q}N)\pi)+$$
$$+ N\frac{A}{2}e^{-j\phi_0}e^{j\pi(k+\tilde{Q}N)}sinc((k+\tilde{Q}N)\pi),$$

** DFT translation property* $\mathcal{D}(\{x_n \cdot e^{j\frac{2\pi m}{N}n}\}) = X[k-m]$

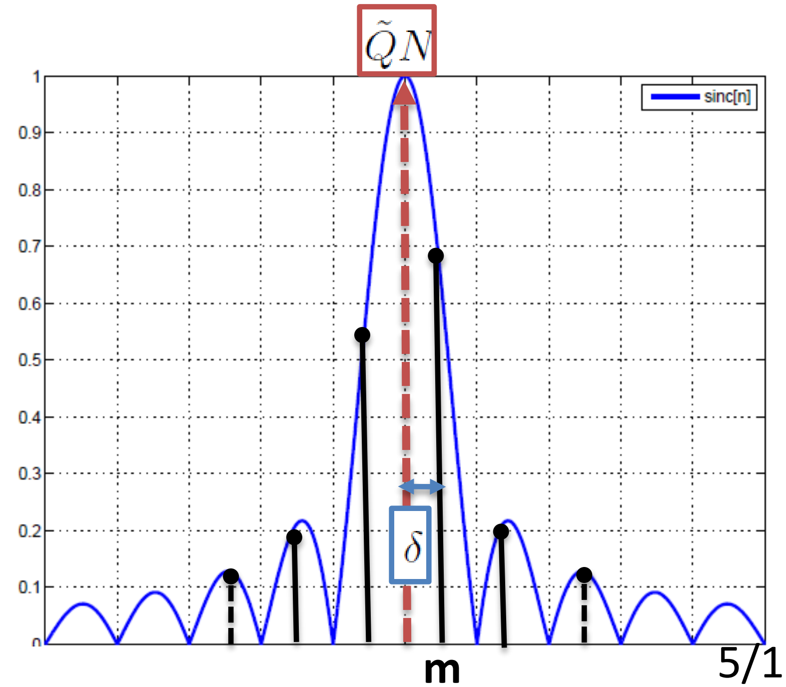
DFT on harmonic signals

Observations: $$S[k] = N\frac{A}{2}e^{j\phi_0}\underbrace{e^{j\pi(k-\tilde{Q}N)}}_{\text{Phase shift}}\underbrace{sinc((k-\tilde{Q}N)\pi)}_{\text{Frequency shift, Amplitude distortion}} + \dots$$

Naming $\delta = m - \tilde{Q}N$, where m is the index corresponding to the highest peak in amplitude spectrum, one can see that $\delta \in [-0.5, 0.5]$,

Amplitude, phase and frequency resolutions depend on the error δ in localizing the spectrum maximum.

1. Resolution in *frequency* is related to the length of samples but also to the value of δ (correspond to the real value if δ=0).
2. Without correction, $\Delta\phi \in [-\pi/2, +\pi/2]$ making the phase basically useless.
3. Resolution in *amplitude* is dependent from the resolution in frequency and the window shape.



DFT on harmonic signals

To overcome these limits and recover the real parameters for the harmonic signals different methods were proposed*:

FFTc → 'Corrected FFT' Used to correct the amplitude, phase and frequency. Based on analytical properties of windowing functions.

ApFFT → 'All phase - FFT' Used to improve the phase resolution**.** Based on properly preprocessing the measured signal in order to cancel phase errors.

Sussix → Used to improve frequency resolution. Based on iterative routines to get the frequency of the tune.

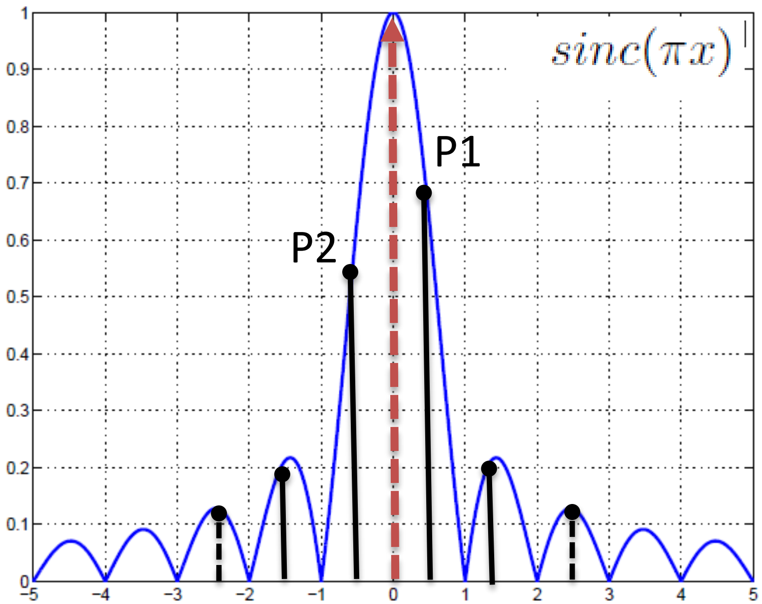
NAFF → Used to improve frequency, amplitude and phase resolution. Based on iterative routines to get the tune frequency.

In the following we present comparisons between these methods.

** All references at the end of the presentation*

Corrected FFT

Using a *"rect"* window we get a *"sinc"* shape in frequency domain. This window has a nice property if one computes the baricentre for two points in the main lobe of the sinc.



1. Take the continuum function $sinc(\pi x)$
2. Take two points P1(x1,y1) and P2(x2,y2) in the main lobe.
3. The baricentre is given by:

$$x_c = \frac{x_1 y_1 + x_2 y_2}{y_1 + y_2}$$

4. If $|x_2 - x_1| = 1$ then:

$$x_2 y_2 = (x_1 + 1)\frac{sin(\pi(x_1+1))}{\pi(x_1+1)} = \frac{-sin(\pi x_1)}{\pi}\frac{x_1}{x_1} = -x_1 y_1$$

5. It follows that $x_c = 0$

Since in spectral analysis, k-samples are spaced by unity, taking the shifted-maximum and the second maximum, and computing the baricentre Xc we get the exact position corresponding to the real maximum: $\delta = |x_c - m|$

Corrected FFT

Applying the baricentre method we can get the correction formula both for “rect” and Hanning window:

Rect window

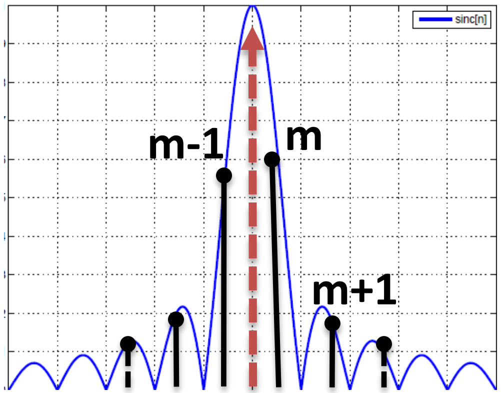
$$x_c = m + \Delta m$$

$$\Delta m = \begin{cases} \frac{Y_{m+1}}{Y_{m+1}+Y_m} & \text{if } Y_{m+1} \geq Y_{m-1}, \\ \frac{-Y_{m-1}}{Y_{m-1}+Y_m} & \text{if } Y_{m+1} < Y_{m-1}. \end{cases}$$

$$f = (m + \Delta m)\frac{f_{sampling}}{N},$$

$$A = \frac{2\pi \Delta m Y_m}{N sin(\pi \Delta m)},$$

$$\Phi = atan(Im_m/Re_m) - \Delta m \pi$$



Hanning window

$$Hann(n) = 0.5(1 - cos(\frac{2\pi n}{N}))$$

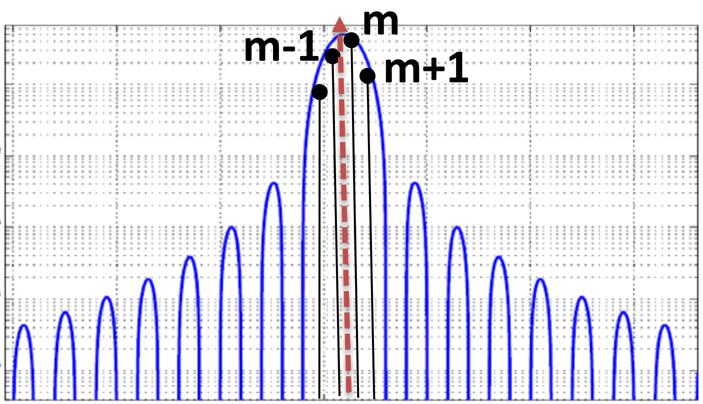
$$x_c = m + \Delta m$$

$$\Delta m = \begin{cases} \frac{2Y_{m+1}-Y_m}{Y_{m+1}+Y_m} & \text{if } Y_{m+1} \geq Y_{m-1}, \\ \frac{Y_m-2Y_{m-1}}{Y_{m-1}+Y_m} & \text{if } Y_{m+1} < Y_{m-1}. \end{cases}$$

$$f = (m + \Delta m)\frac{f_{sampling}}{N},$$

$$A = \frac{2\pi \Delta m Y_m}{N sin(\pi \Delta m)}(1 - \Delta m^2),$$

$$\Phi = atan(Im_m/Re_m) - \Delta m \pi$$



ApFFT

This is a preprocessing algorithm developed to correct ***the phase*** got out from the DFT operation. The computed phase is the ***central phase***, i.e. the one corresponding to the central sample in case of an odd number of samples.

1. Multiply the signal with a triangular window:

$$s[n] = x[n] \cdot t[n] \qquad \begin{cases} t[n] = trig[n] = N - |n| \\ x[n] = Acos(2\pi\tilde{Q}n + \phi_0) \end{cases} \qquad n \in [-N, +N]$$

2. Fold the windowed signal:

$$s_{Ap}[n] = s[0 \ldots N] \cup s[-N \cdots - 1]$$

3. Proceed to FFT evaluation and extract the phase:

$$S_{Ap}[k] = e^{j\phi_0} \frac{sin^2(\pi(\tilde{Q}N - k))}{sin^2(\pi(\tilde{Q}N - k)/N)}$$

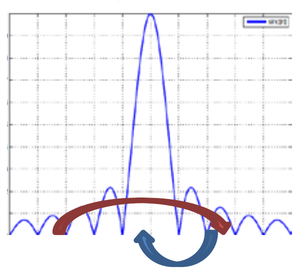
The phase is no more dependent on δ, i.e. doesn’t need to be corrected.

NAFF-Sussix

The **NAFF** code was developed by J.Laskar et al. in the Nineties as a method to analyze chaotic dynamical system by Numerical Analysis of the Fundamental Frequencies in the sampled signal.

Signal: $f = \sum_{k=1}^{N} a_k e^{j\omega_k t}$

FFT



$f - f_k$

The **FFT is used as a "first guess seed"** for the quadratic interpolation routine.

The **interpolation routine** finds the peak of $\Phi(\sigma)$ moving around the expected peak.

$$\Phi(\sigma) = < f, e^{j\sigma t} > = \frac{1}{2T}\int_{-T}^{+T} f(t)e^{-j\sigma t}\,\mathrm{d}t$$

At the maximum we get the complex amplitude a_k

After a Gram-Schmidt normalization the detected harmonic is subtracted by the original signal.

The method in **SUSSIX** and NAFF are similar. In SUSSIX it is based on the *TUNEWT* algorithm developed by A.Bazzani (University of Bologna) in the mid-Nineties and further developed and tested at CERN.

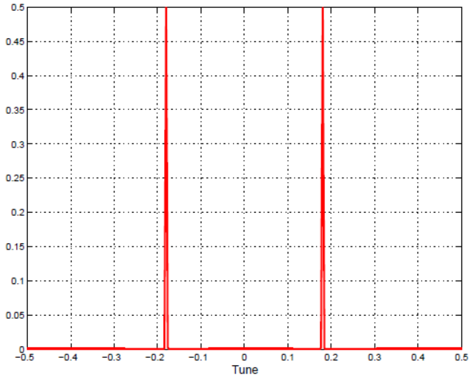
Comparisons

Different simulations were set up on detecting phase, amplitude and frequency using the different methods presented previously.

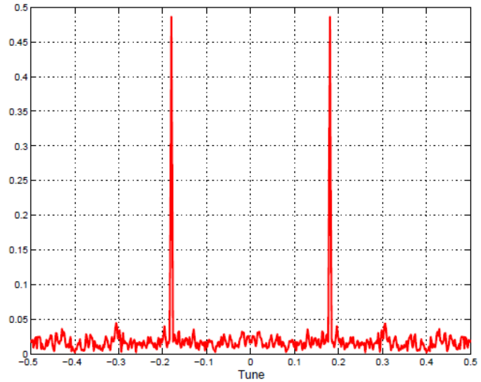
In order to properly compare the phase accuracy, the phase difference between two signals at the same frequency will be analyzed. In real life the absolute phase has no sense cause to the decoherence in the signal.

All the simulations are set in presence of noise. The level is set in terms of 1-sigma of noise.

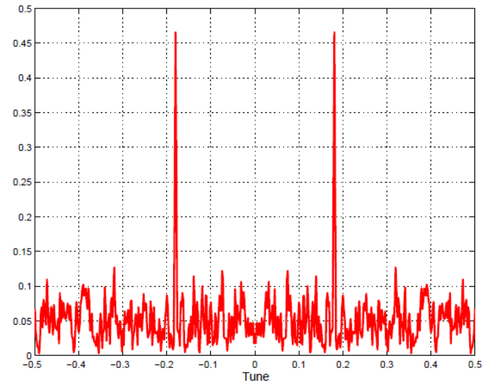
We consider the case of absence of noise (simulation case) and 1% and 10% additive gaussian noise.



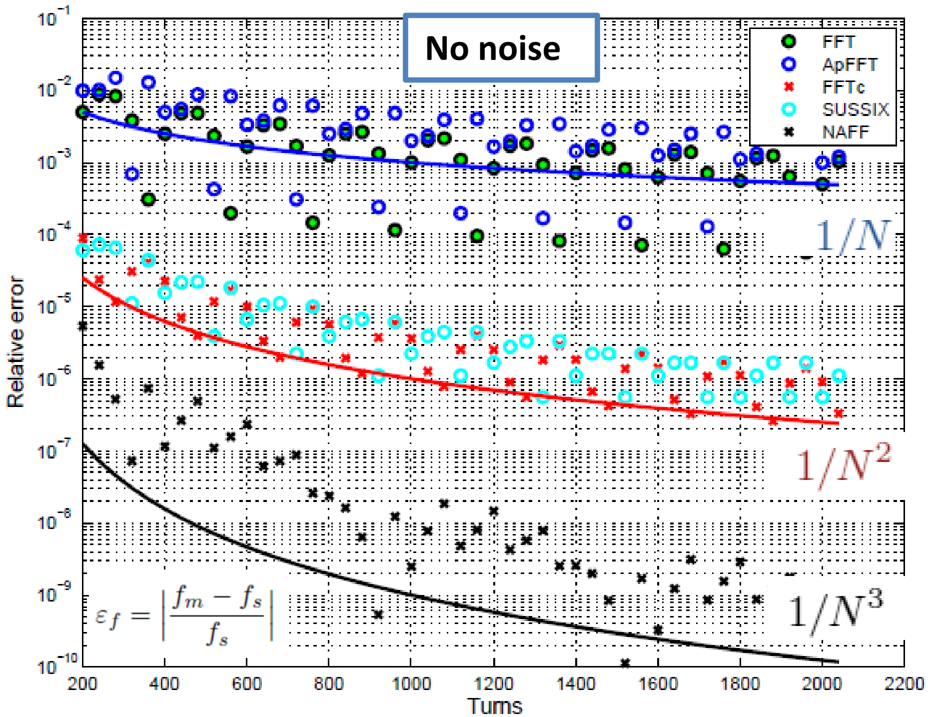
No noise



1% noise

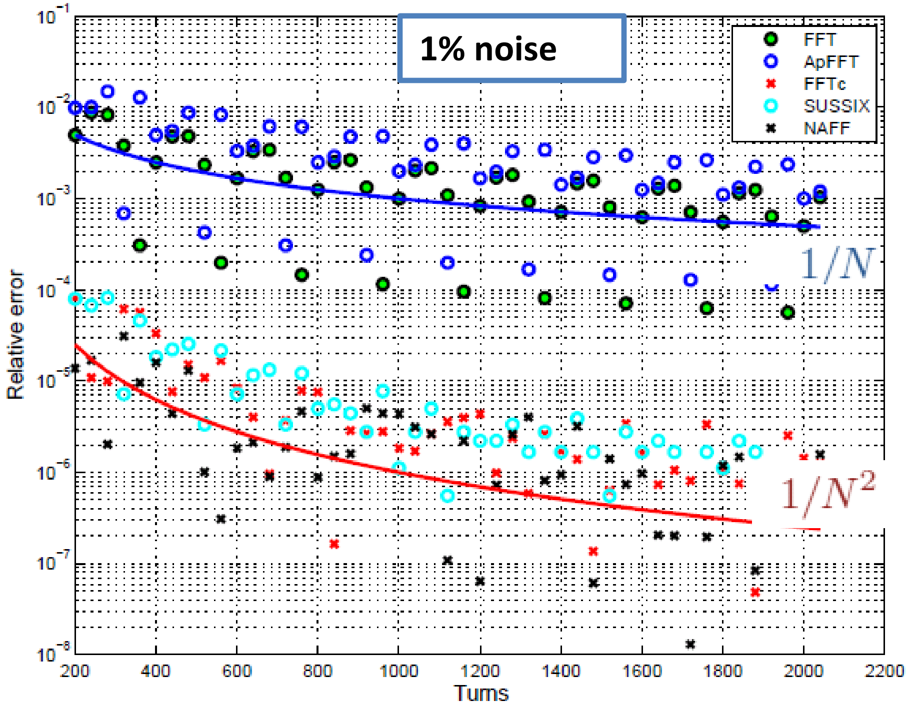


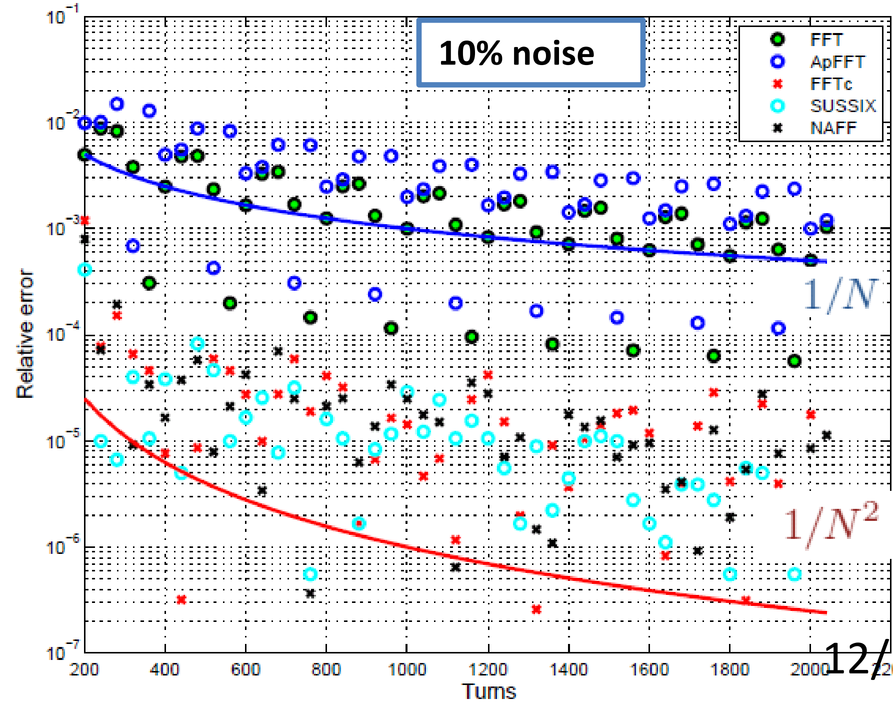
10% noise

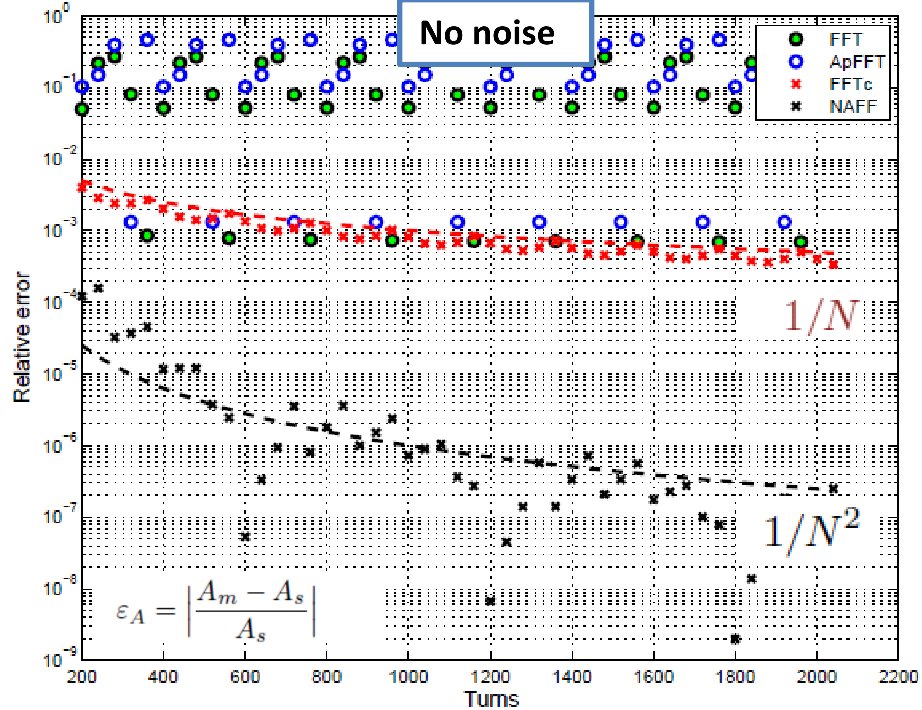


Frequency VS Turns

- Both FFTc and NAFF have Hanning window on the signal.
- The FFT is corrected with the formulae seen before.
- In case of no noise, NAFF is the most accurate (1/N^3) followed by FFTc and SUSSIX (1/N^2)
- In case of noise the resolution is still better than a normal FFT, but equivalent for all FFTc, SUSSIX and NAFF.

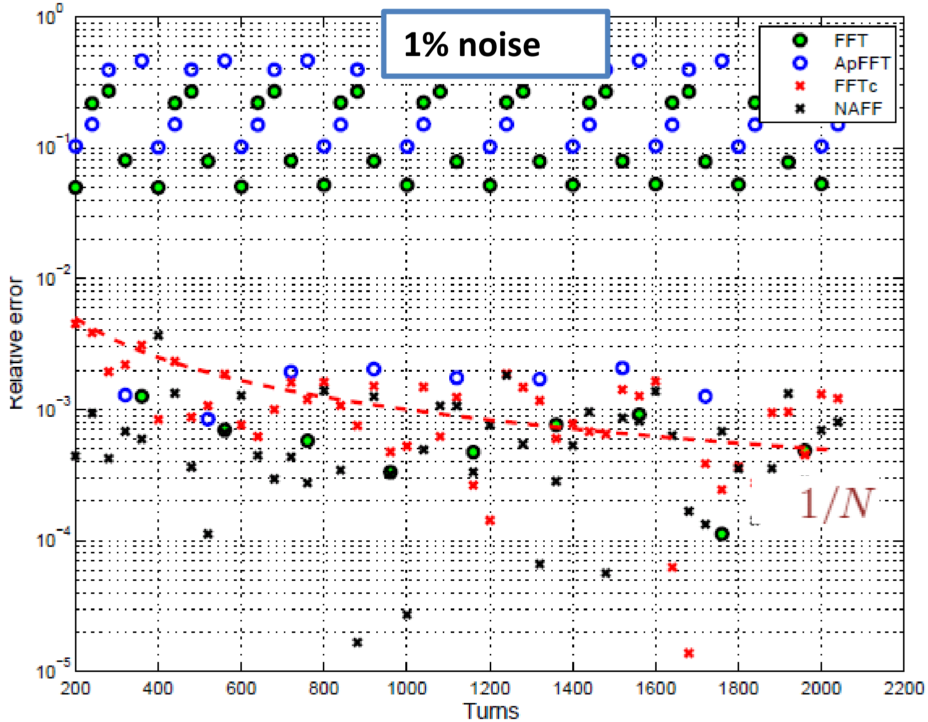


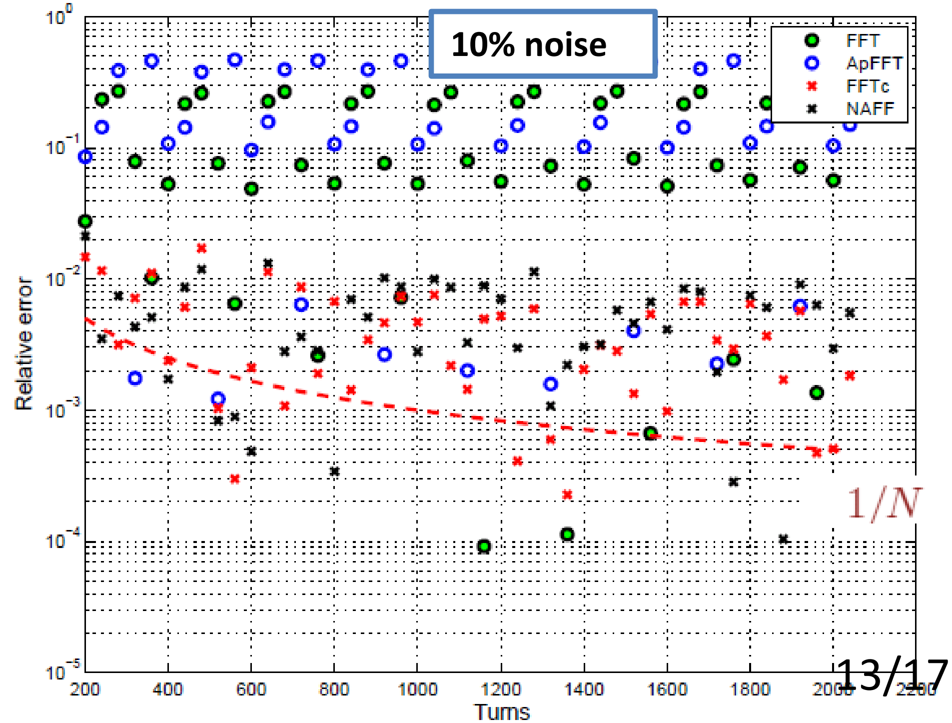




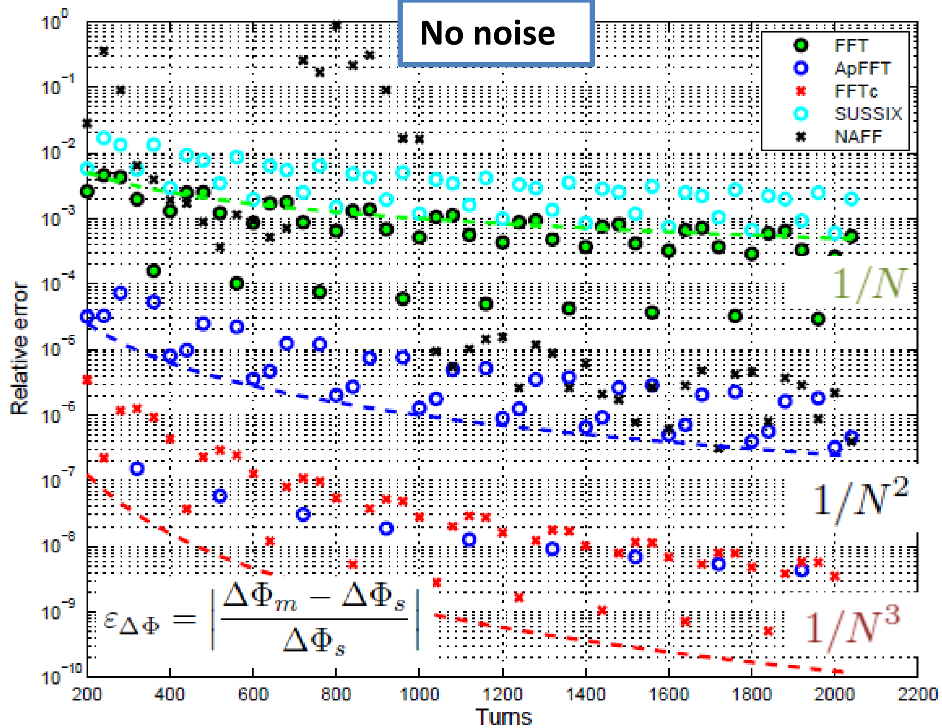
Amplitude Vs Turns

- ApFFT (as a normal FFT) doesn't implement any amplitude correction.
- The FFTc is corrected with the formulae seen before.
- Without noise FFTc goes like 1/N, NAFF like 1/N^2.
- In presence of noise, NAFF and FFTc presents similar behaviour going like 1/N.
- Sussix is not reported since is not thought for absolute amplitude detections.

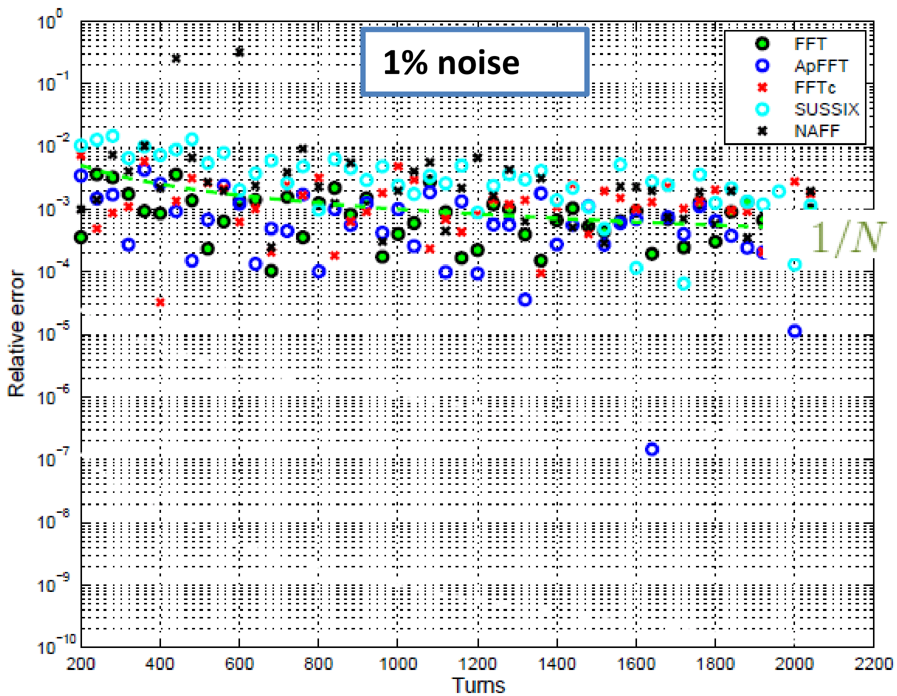


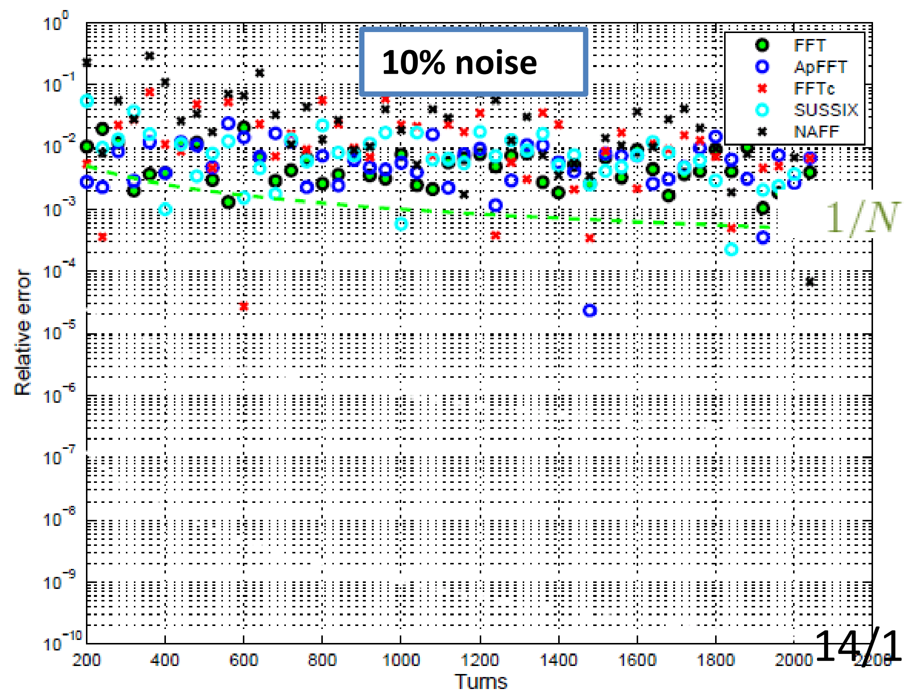


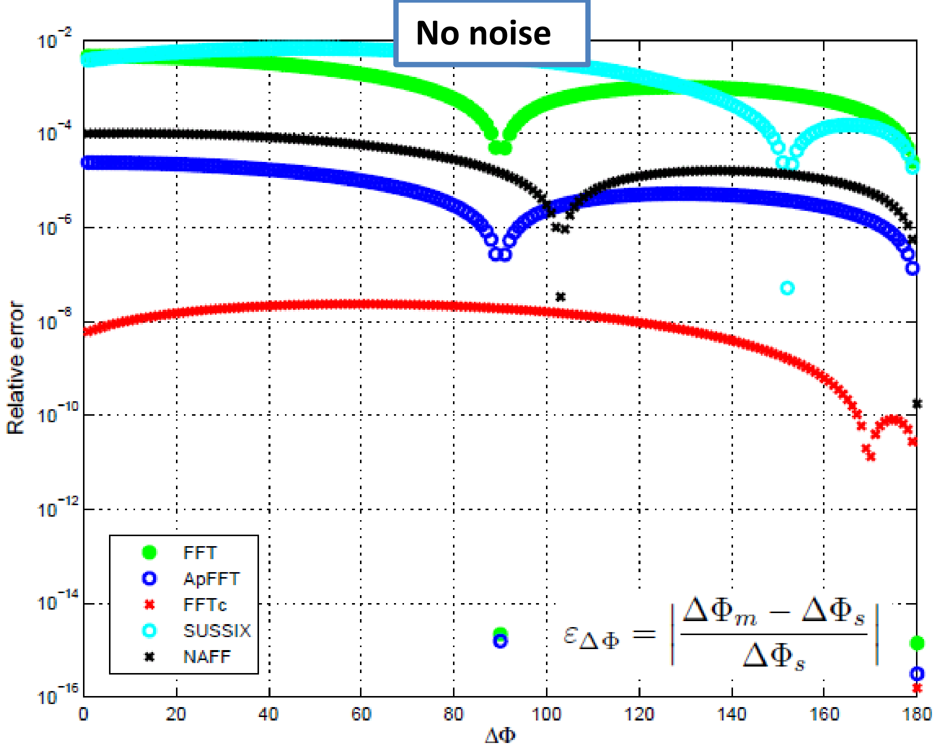
Phase Vs Turns



- This simulation is set up with a $\Delta\phi_s$=60°.
- The FFTc is corrected with the formulae seen before.
- Without noise, SUSSIX and a normal FFT have a similar behaviour (1/N).
- ApFFT goes like NAF as 1/N^2.
- FFTc reaches higher resolutions (1/N^3).
- In presence of noise all the approaches give almost the same resolution (1/N).

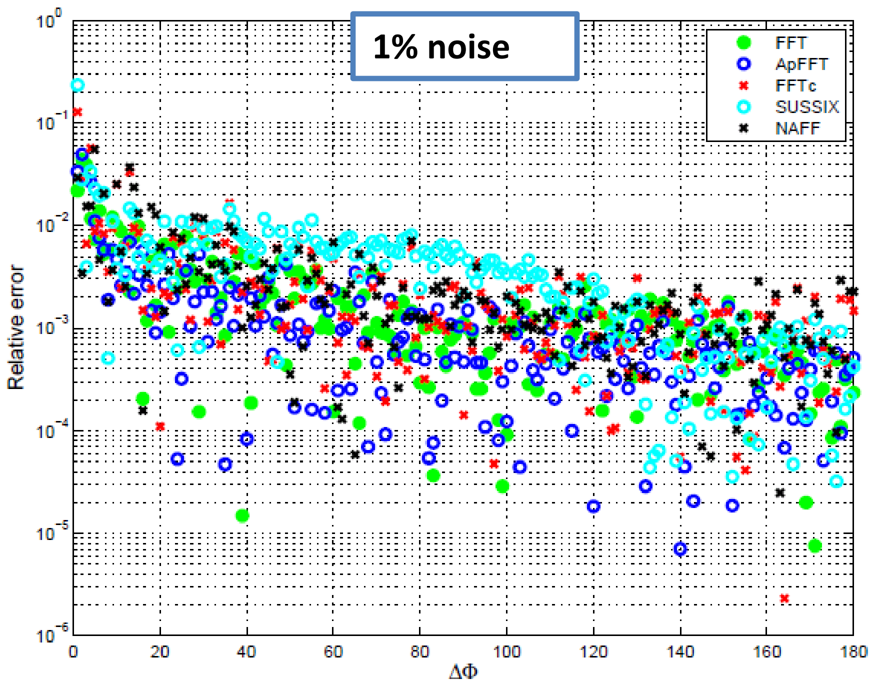


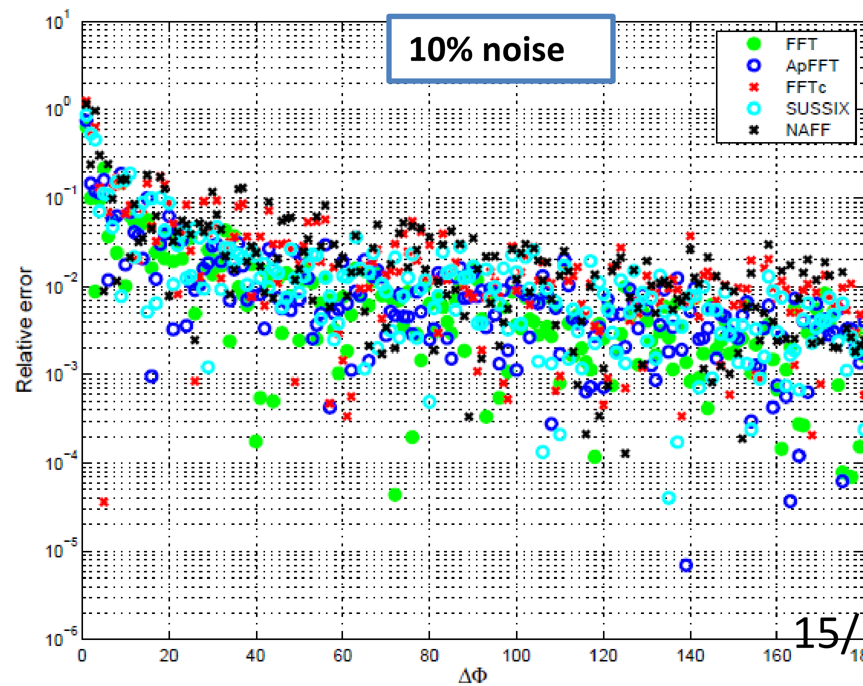




Phase Vs Phase advance

- This simulation is set up fixing N to 512 Turns and spacing the $\Delta\phi_s$ from $\Delta\phi_s = 0^o$ to $\Delta\phi_s = 180^o$.
- In case of no noise ApFFT and the corrected FFT give higher resolution.
- In case of noise, the resolution is around 10^-3 for 1% noise and 10^-2 for 10% noise.

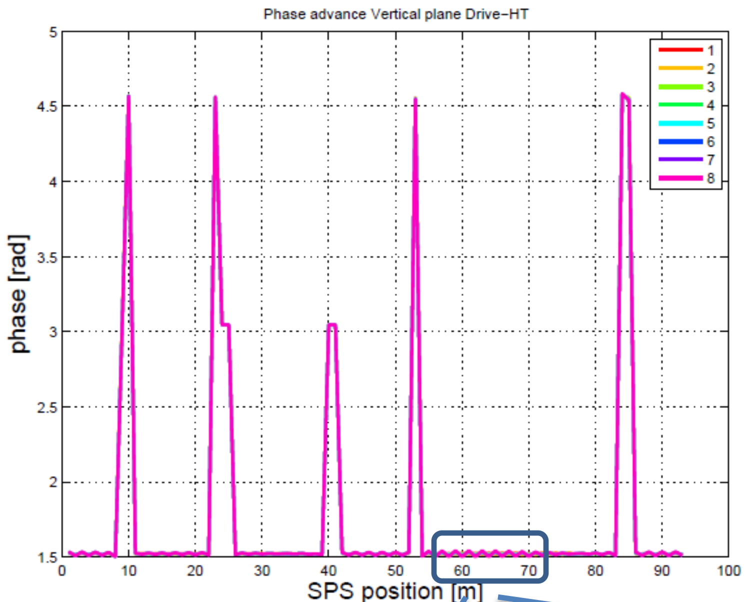




Impedance Localization

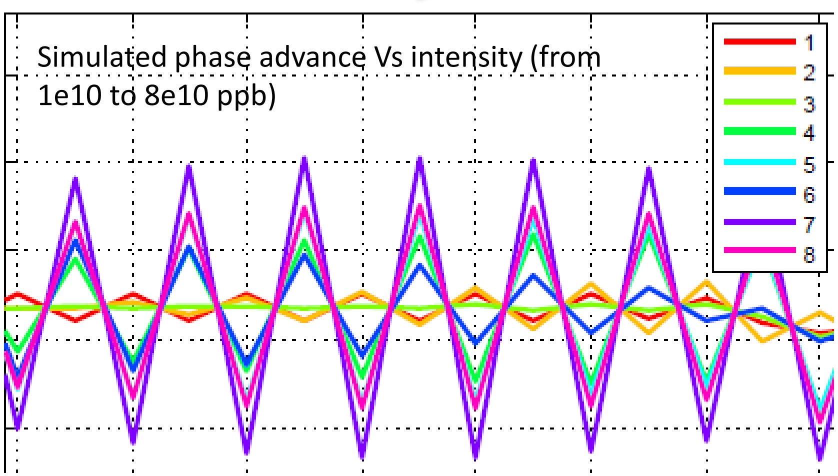
• Localizing coupling impedance sources is possible looking to the phase advances between different BPMs positions.
• The strength of the impedance is proportional to the variation of phase advance with intensity. The relation is linear.
• It is important to estimate the phase advances being over the resolution limit of the FFT used.

SPS phase advance V-plane

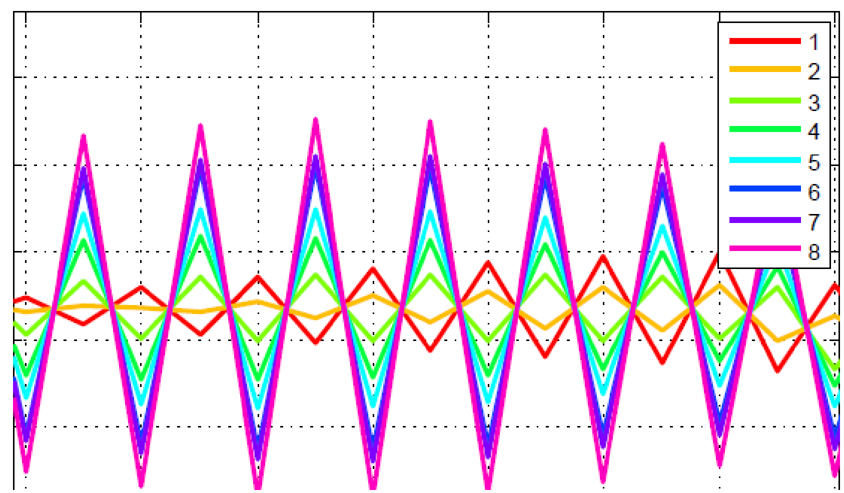


FFT

FFTc



Standard FFT



Corrected FFT

CONCLUSIONS

• The problem of getting correct phase, amplitude and frequency out of a harmonic signal was analyzed: Corrected FFT, ApFFT, SUSSIX and NAFF algorithms were presented in both a theoretical and practical point of view .

• For *frequency* measurements NAFF has the higher resolution followed by SUSSIX and FFTc. All of them become comparable in presence of low noise.
• For *amplitude* measurements NAFF has the higher resolution followed by FFTc. Still, in presence of noise they behave the same.
• For *phase* measurements FFTc has the higher resolution. All the methods behave the same in presence of low noise.

• All the algorithms break down in presence of noise. If needed, a stronger effort in order to reduce or “clean” it should be done.

OUTLOOK

• Since SUSSIX is widely used, we can improve its speed adding the analytical corrections implemented in FFTc (single shot corrections VS iterative loops).

• One can improve the corrected FFT in order to detect subsequent harmonics in the betatron signal as done in the NAFF algorithm.

References

About NAFF

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Thank you!!