Gauge invariant cosmological perturbations

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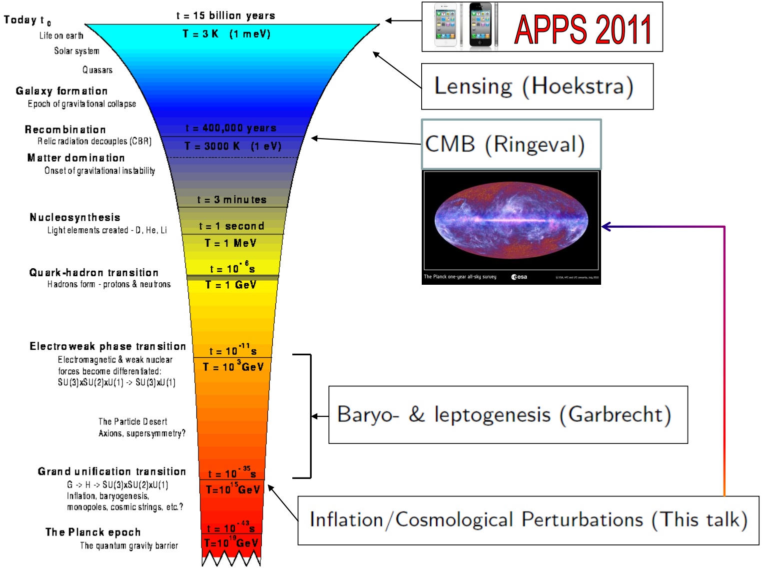
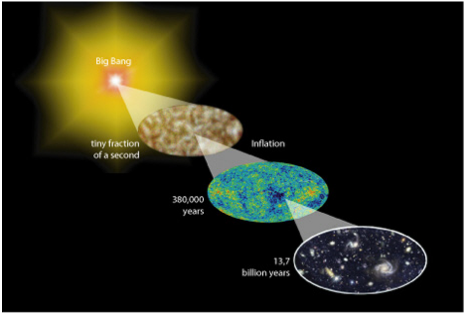
History of the universe



Physics of the very early universe: inflation

Cosmological inflation (Guth 1981)

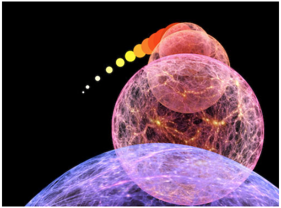
- is an accelerated expansion of the early universe ($10^{-37} - 10^{-32}$ s)
- solves horizon and flatness problems
- can amplify quantum fluctuations in the early universe



How do we describe quantum fluctuations during inflation?

Quantum fluctuations in the very early universe

Inflation can be described by a scalar field rolling down a potential in an expanding universe



Action:

$$S = \int d^4x \sqrt{-g} \left\{ -R - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) \right\}$$

with background FLRW metric $g^{(0)}_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$.

Density perturbations are associated with perturbations of the matter fields, *i.e.* $\Phi = \phi + \varphi$

But in very early universe, fluctuations of spacetime are equally important, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$

However, issue of *gauge dependence* for metric fluctuations..

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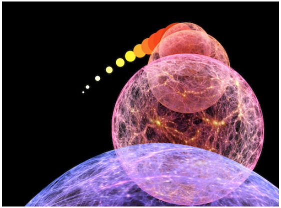

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Gauge dependence in general relativity

The action

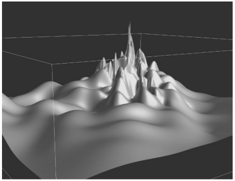
$$S = \int d^4x\sqrt{-g}\left\{-R - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)\right\}$$

is *covariant*: it is invariant under coordinate transformations. Metric tensor transforms as:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{dx^\alpha}{d\tilde{x}^\mu}\frac{dx^\beta}{d\tilde{x}^\nu}g_{\mu\nu}(x).$$

Now:

- split the metric field in a *fixed* background $g^{(0)}_{\mu\nu}$ and a fluctuation $\delta g_{\mu\nu}$
- this breaks the general covariance for the fluctuations



$\longrightarrow$ Fluctuations on a fixed background become dependent on coordinate transformations!

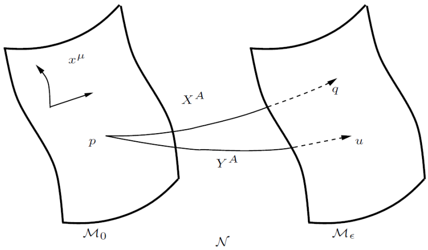
Gauge freedom in general relativity

A general perturbation of a quantity Q is defined as

$$\text{Perturbation } \delta Q = (Q \text{ in perturbed spacetime) - (} Q^{(0)} \text{ in background spacetime)}$$

In order to compare Q and $Q^{(0)}$, one has to choose a mapping between the perturbed and background spacetime $\Longrightarrow$ this is a *gauge choice*

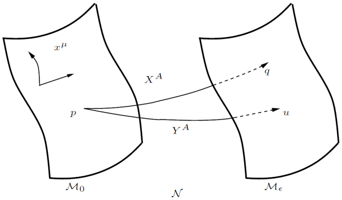
The freedom in choosing a mapping is called a *gauge freedom*



Gauge transformations in general relativity

Under an infinitesimal coordinate transformation

$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$, gauge transformations of the fields:



$$\delta g_{\mu\nu}(x) \quad \to \quad \delta g_{\mu\nu}(\tilde{x}) = \delta g_{\mu\nu}(x) - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$$

$$\varphi(x) \quad \to \quad \varphi(\tilde{x}) = \varphi(x) + \xi^0 \partial_0 \phi(t)$$

(compare to QED: $A_\mu \to A_\mu + \partial_\mu \Lambda$)

Problems with gauge freedom: calculating physical quantities (similar to QED).

Two solutions:

- Fix the gauge (in QED: Coulomb gauge, Lorenz gauge)
- Construct gauge invariant variables (in QED: electric field E_i)

  (For GR: Bardeen 1980)

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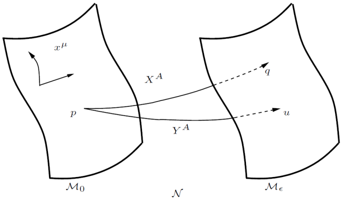

$$\begin{aligned} \delta g_{\mu\nu}(x) &\to \delta g_{\mu\nu}(\tilde{x}) = \delta g_{\mu\nu}(x) - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \\ \varphi(x) &\to \varphi(\tilde{x}) = \varphi(x) + \xi^0 \partial_0 \phi(t) \end{aligned}$$

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Dynamical variables in the action

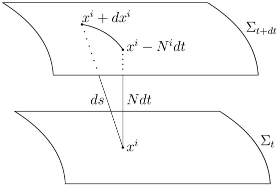
$$S = \int d^4x\sqrt{-g}\left\{-R - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)\right\}.$$

What are the dynamical fields in this action?

"Decompose spacetime into space and time"

ADM line element (Arnowitt, Deser, Misner 1959)

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$



- 7 dynamical fields: spatial metric g_{ij} and scalar field Φ
- 4 constraint fields: lapse function N, shift function N^i (compare to A_0 for QED)
- 7-4=3 dynamical degrees of freedom!

Perturbations

Now insert linear fluctuations of all fields around FLRW background ($ds^2 = -dt^2 + a^2 dx_i dx^i$):

$$g_{ij} = a(t)^2 \left(\delta_{ij} + h_{ij}(t, \mathbf{x})\right)$$

$$\Phi = \phi(t) + \varphi(t, \mathbf{x})$$

Scalar-vector-tensor decomposition:

$$h_{ij} = \frac{\delta_{ij}}{3} h + \left(\partial_i \partial_j - \frac{\delta_{ij}}{3} \nabla^2\right) \tilde{h} + \partial_{(i} h^T_{j)} + h^{TT}_{ij},$$

with

$$\partial^i h^T_i = 0, \qquad \partial^i h^{TT}_{ij} = 0 = \partial^j h^{TT}_{ij}.$$

Substitute these in action, solve constraint equations to eliminate unphysical degrees of freedom..

Final result

Finally, the action $S = \int d^4x\sqrt{-g}\left\{-R - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)\right\}$ up to second order is

$$S^{(2)} = \int d^3x dt a^3 \left\{ \frac{\dot{\phi}^2}{36H^2}\left[\frac{1}{2}\dot{\mathcal{R}}^2 - \frac{1}{2}\left(\frac{\partial_i \mathcal{R}}{a}\right)^2\right] + \frac{1}{4}\left[(\dot{h}_{ij}^{TT})^2 - \left(\frac{\partial h_{ij}^{TT}}{a}\right)^2\right]\right\}$$

Mukhanov 1981

- 1 dynamical scalar: comoving curvature perturbation:

$$\mathcal{R} = (h - \nabla^2 \tilde{h}) - 6\frac{H}{\dot{\phi}}\varphi$$

  Combination of scalar metric and inflaton fluctations

  Together with inflation forms the primordial power spectrum of the CMB

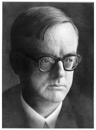
- 1 dynamical tensor: graviton h_{ij}^{TT}
- All fields are gauge invariant!

$\mathcal{R}$ and h_{ij}^{TT} are *gauge invariant cosmological perturbations*

Einstein and Jordan frame

Einstein frame action

$$S = \int d^4x\sqrt{-g}\left\{-R - \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)\right\}.$$



Jordan frame action (important for Higgs inflation)

$$S_J = \int d^4x\sqrt{-g}\left\{-R_J F(\Phi_J) - \frac{1}{2}g_J^{\mu\nu}\partial_\mu\Phi_J\partial_\nu\Phi_J - V_J(\Phi_J)\right\}.$$

Frames related through field redefinitions

$$g_{\mu\nu} = \omega^2 g_{\mu\nu,J}, \qquad \omega^2 = F(\Phi_J), \qquad d\Phi = \frac{d\Phi}{d\Phi_J}d\Phi_j$$

Just field redefinitions, therefore *frames are physically equivalent*

Are the Jordan and Einstein frame physically equivalent at the quantum level?

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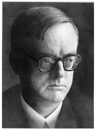

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The quantum equivalence of frames

Free Jordan frame action (JW, Prokopec 2010):

$$S_J^{(2)} = \int d^3x \bar{N}_J dt a_J^3 \left\{ z_J^2 \left[\frac{1}{2} \dot{\mathcal{R}}_J{}^2 - \frac{1}{2} \left(\frac{\partial_i \mathcal{R}_J}{a_J} \right)^2 \right] + \frac{F}{4} \left[(\dot{h}_{ij,J}^{TT})^2 - \left(\frac{\partial h_{ij,J}^{TT}}{a_J} \right)^2 \right] \right\}$$

Under the field redefinitions $g_{\mu\nu} = \omega^2 g_{\mu\nu,J}$, $d\Phi = \frac{d\Phi}{d\Phi_J} d\Phi_j$:

$$S_J^{(2)} \rightarrow S^{(2)}!$$

Important observation:

$$\mathcal{R} = \mathcal{R}_J \quad \text{and} \quad h_{ij}^{TT} = h_{ij,J}^{TT}.$$

Comoving curvature perturbation $\mathcal{R}$ and graviton h_{ij}^{TT} are gauge invariant **and** invariant under a conformal transformation $\longrightarrow$ observables!

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Current project: gauge invariance for higher order perturbations

Calculate the gauge invariant action up to 3rd order in perturbations

- Maldacena (2003) used gauge fixing to obtain 3rd order action for single scalar field
- We want to do this in a completely gauge-invariant way in the Jordan frame

  (with T. Prokopec and G. Rigopoulos)
- Challenging: Gauge-invariant variables are nonlinear

Why is this useful?

- Possible to calculate quantum corrections to power spectrum
- Possible to calculate non-Gaussianities: bispectrum
- Show the quantum equivalence of Jordan and Einstein frames at 3rd order

Summary and outlook

Summary:

- Quantum fluctuations in the early universe can be amplified during inflation
- Issue of gauge dependence: gauge invariant cosmological perturbations
- Cosmological perturbations + inflation predicts primordial power spectrum for CMB

for a review, see Mukhanov, Feldman, Brandenberger 1992

Outlook:

- Calculate gauge invarian action up to 3rd order (non-linear GI variables)
- (Dis)prove quantum equivalence of Jordan and Einstein frames
- Quantum corrections, non-gaussianity