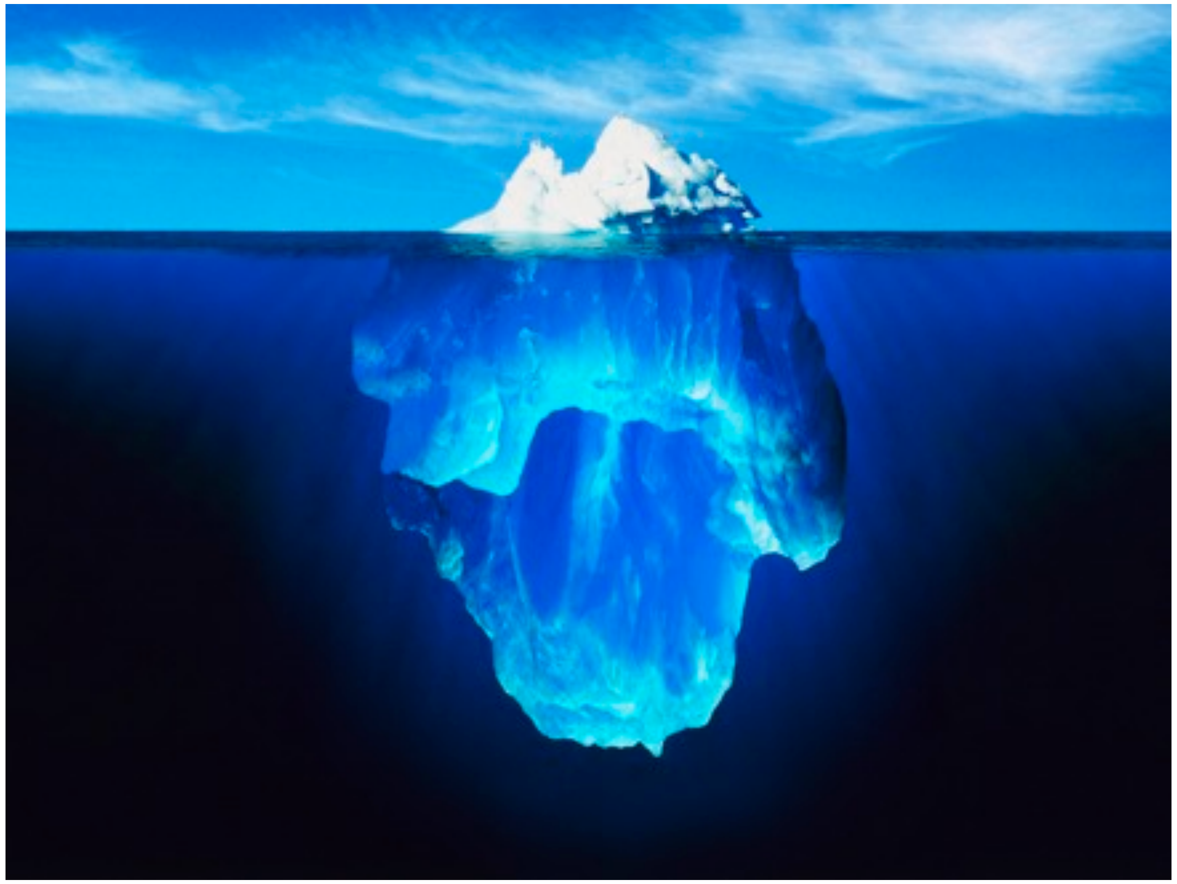
# QFT and the EW Standard Model



Based on lecture notes written with M.A. Vázquez-Mozo

Much left underneath...

# Apologies

Never underestimate the pleasure people get when they listen to something they already know

E. Fermi

# Outline

- Why Quantum Field Theory?
- Quantisation
- Kinematical symmetries
- Global symmetries
- Local symmetries
- Discrete symmetries
- Broken symmetries
- Scale symmetries, renormalisation
- Standard Model symmetries
- Amusing examples throughout time permitting

# Do we really need it?

The Schrödinger equation, plus many body physics constructions are very successful in atomic, molecular and solid state physics. The theory of bands, electrical conductivity, atomic bonding, orbitals... are adequately explained in this scheme

$$
\begin{aligned}
i\,\frac{\partial}{\partial t}\Psi(\mathbf{r_1},\mathbf{r_2},...,\mathbf{r_N},t) &= \left(\sum_i \frac{(\mathbf{p}_i - e_i\mathbf{A}_i)}{2m_i} + e_i\Phi_i + V(\mathbf{r}_i)\right)\Psi(\mathbf{r}_j,t) \\
P(\mathbf{r_1},\mathbf{r_2},...,\mathbf{r_N},t) &= |\Psi(\mathbf{r_1},\mathbf{r_2},...,\mathbf{r_N},t)|^2, \qquad \int\prod_{i=1}^{N} d^3\mathbf{r}_i\,P(\mathbf{r_j},t) = 1 \;\; \forall\, t
\end{aligned}
$$

A note on conventions

$$\hbar = c = 1, \qquad \eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1) \qquad \mathbf{F} = \frac{1}{4\pi}\frac{qq'}{r^3}\mathbf{r} \qquad \alpha = \frac{e^2}{4\pi\hbar c} \qquad e \approx .303$$

# Einstein and Heisenberg complicate our lives

Useful basic formulae. A reminder. Just this once, we reintroduce h and c

$$p^2 = \left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = m^2c^2$$

$$E = \pm\sqrt{\mathbf{p}^2c^2 + m^2c^4} \approx \pm(mc^2 + \frac{\mathbf{p}^2}{2m} + \ldots)$$

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

$$\lambda = \frac{h}{mc} \qquad \text{Compton wavelength}$$

$$E = \frac{mc^2}{\sqrt{1-\mathbf{v}^2/c^2}} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\mathbf{v}^2/c^2}}$$

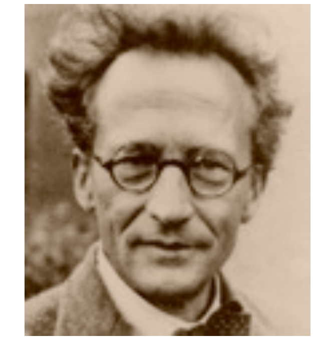
$$\Delta p \geq mc \qquad \Delta E \geq mc^2$$

$$(\Delta x)_{\min} \geq \frac{1}{2}\left(\frac{\hbar}{mc}\right)$$

When the uncertainly in momentum is bigger than mc, the uncertainty in energy is larger than mc^2, hence there is enough energy to produce another particle of the same type. In Relativity mass and energy are interchangeable. Hence we cannot localise a particle below its Compton wavelength. If we do, we will not find a single particle, but rather a fairly complicated quantum state with no well-defined number of particles.

Particle production by physical processes should be a central part of the theory.

# Klein paradoxes...

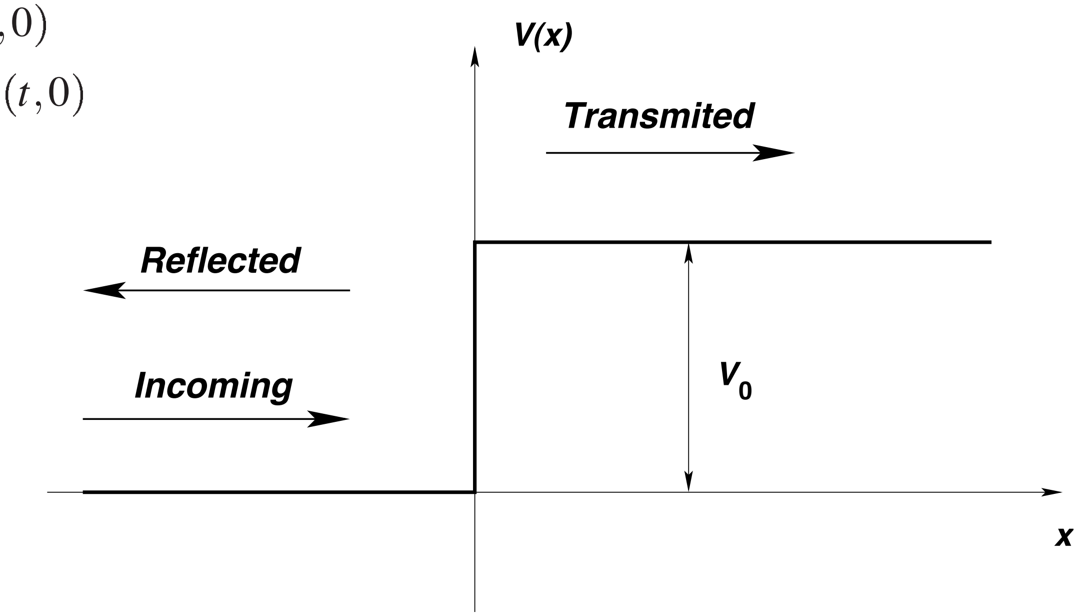



Another way to see the same problem is to consider a particle in a potential barrier in the simplest relativistic generalisation of the Schrödinger equation, the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(t,\mathbf{x}) = 0$$

$$\psi_I(t,x) = e^{-iEt+ip_1x} + Re^{-iEt-ip_1x},$$
$$\psi_{II}(t,x) = Te^{-iEt+p_2x},$$

$$\psi_I(t,0) = \psi_{II}(t,0)$$
$$\partial_x\psi_I(t,0) = \partial_x\psi_{II}(t,0)$$




$$p_1 = \sqrt{E^2 - m^2}, \qquad p_2 = \sqrt{(E-V_0)^2 - m^2}$$

$$T = \frac{2p_1}{p_1+p_2}, \qquad R = \frac{p_1-p_2}{p_1+p_2}$$

Three cases to consider

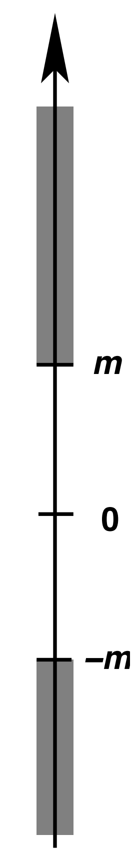
$$1)\ E - m > V_0 \qquad 2)\ E - m < V_0 \qquad 3)\ V_0 > 2m \qquad V_0 - 2m < E - m < V_0$$

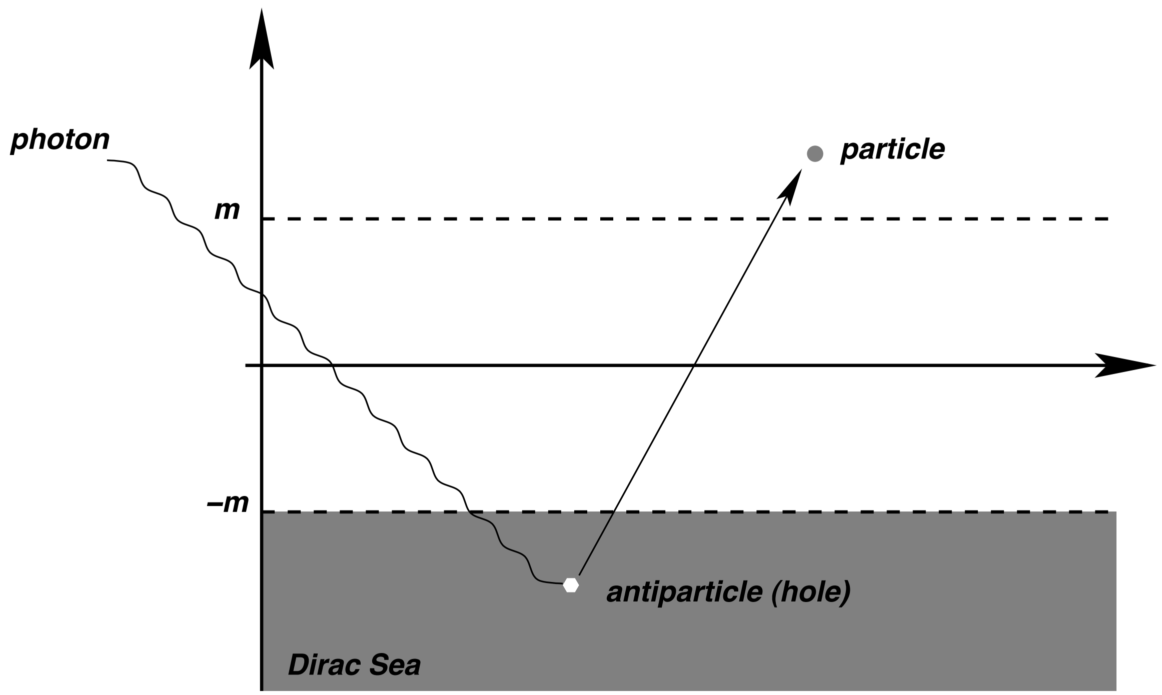
In the third case we have the strange situation that we have transmitted wave with negative kinetic energy

$$E - m - V_0$$

# .... Dirac seas

In the equation that bears his name, Dirac also found the problem with negative energy states. In his case however he found a rather ingenious way to solve the problem. Since he was describing electrons, he decided to simply fill all the negative energy states, this way Pauli's principle would guarantee stability. His equation also predicted the existence of anti-particles, although at the beginning he was reluctant to accept it. With the Dirac sea we have a simple way to understand anti-electrons = positrons (more later)




An energetic photon can make a hole. The absence of a negative energy state with negative charge manifests itself as a particle of positive energy and positive charge:

the positron

- We still have a multi-particle theory after all
- This does not work for bosons…
- We should give up the wave equation approach

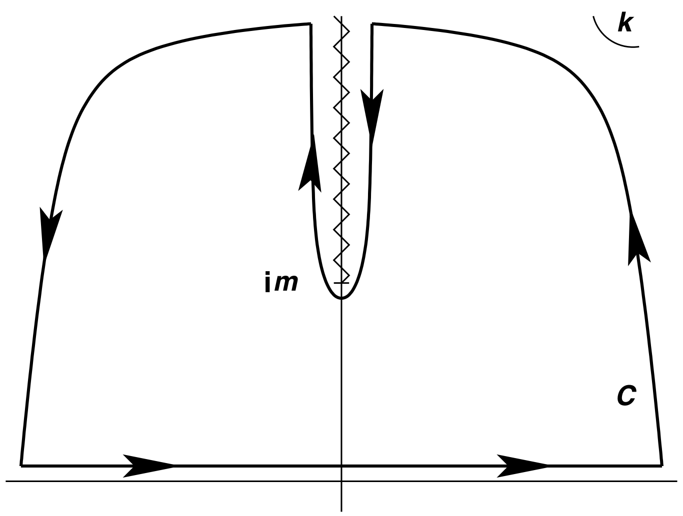
If we still insist against all odds, and decide to violate locality, but to eliminate once and for all the negative energy states by choosing our free Hamiltonian as follows:

$$H = \sqrt{-\nabla^2 + m^2}$$

$$\psi(0,\mathbf{x}) = \delta(\mathbf{x})$$

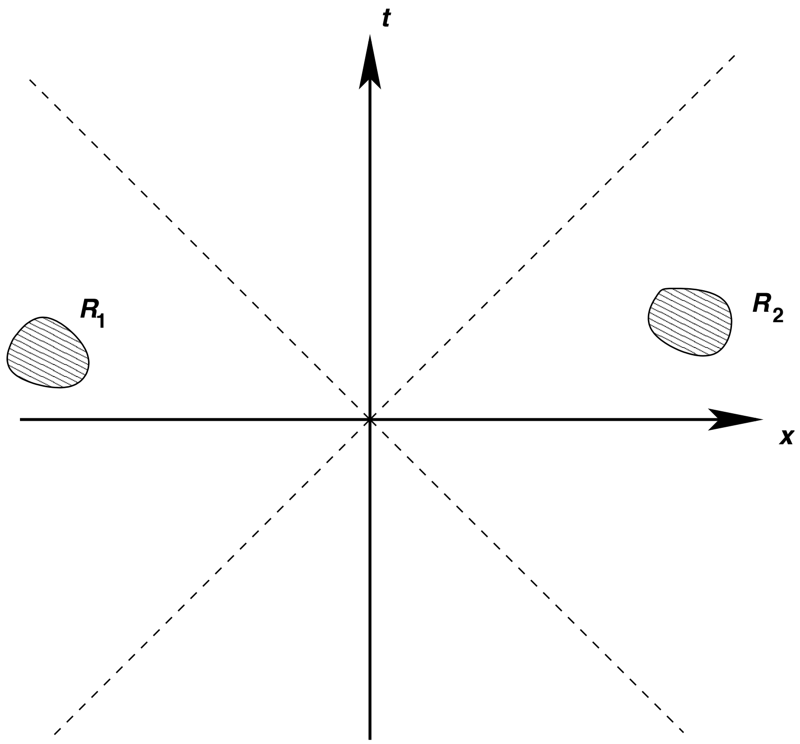
$$\psi(t,\mathbf{x}) = e^{-it\sqrt{-\nabla^2+m^2}}\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}-it\sqrt{k^2+m^2}}.$$

$$\psi(t,\mathbf{x}) = \frac{1}{2\pi^2|\mathbf{x}|}\int_{-\infty}^{\infty} k\,dk\,e^{ik|\mathbf{x}|}\,e^{-it\sqrt{k^2+m^2}}.$$




**Oops!!** we have violated causality! For any t>0 and any |x|, this wave function does not vanish!...

# Relativistic causality




Microscopic causality, Locality in Special Relativity imposes important constraints into what are observables. The light-cone decrees the causal structure of space-time. Physical measurements should be compatible with it

$$[\mathscr{O}(x), \mathscr{O}(y)] = 0, \qquad \text{if } (x-y)^2 < 0.$$

- The world is Quantum
- Particle Wave Duality
- Special Relativity
- Microscopic Causality

$\Longrightarrow$ LQFT

# From classical to quantum fields

In scattering experiments we observe asymptotic free particles characterised by their energy-momentum charge and other quantum numbers. Consider just E,p. In the NR-case we describe the one-particle states by kets carrying a unitary rep. of the rotation group.

$$|\mathbf{p}\rangle \in \mathscr{H}_1, \qquad \langle \mathbf{p}|\mathbf{p}'\rangle = \delta(\mathbf{p}-\mathbf{p}') \qquad \int d^3p\, |\mathbf{p}\rangle\langle\mathbf{p}| = \mathbf{1}. \qquad \mathscr{U}(R)|\mathbf{p}\rangle = |R\mathbf{p}\rangle \qquad \widehat{P}^i = \int d^3p\, |\mathbf{p}\rangle p^i \langle \mathbf{p}|$$

To deal with multi-particle states it is convenient to introduce creation and annihilation operators, this leads to the Fock space of states, built out of the vacuum by acting with creation operators:

$$|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle, \qquad a(\mathbf{p})|0\rangle = 0 \qquad \langle 0|0\rangle = 1$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta(\mathbf{p}-\mathbf{p}'), \qquad [a(\mathbf{p}), a(\mathbf{p}')] = [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] = 0.$$

We need relativistic invariance, hence we need to find ways to count states in an invariant way. This is necessary also when we deal with decay rates and cross sections. We need to count final states in a way consistent with Lorentz invariance. We can easily construct such an invariant phase space volume:

$$\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\,\theta(p^0)\,f(p)$$ to integrate over p0, we use a nice identity:

$$\delta[g(x)] = \sum_{x_i = \text{zeros of } g} \frac{1}{|g'(x_i)|}\delta(x-x_i) \qquad \delta(p^2-m^2) = \frac{1}{2p^0}\delta\left(p^0 - \sqrt{\mathbf{p}^2+m^2}\right) + \frac{1}{2p^0}\delta\left(p^0 + \sqrt{\mathbf{p}^2+m^2}\right)$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}$$ with $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2+m^2}$ and $(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}')$ are invariant

Now proceed by imitation of the NR case, with the non-trivial result that we have a unitary representation of the Lorentz group

$$|p\rangle = (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle \qquad \langle p|p'\rangle = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}') \qquad \widehat{P}^\mu = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|p\rangle p^\mu \langle p| \qquad \mathscr{U}(\Lambda)|p\rangle = |\Lambda^\mu_{\ \nu}p^\nu\rangle \equiv |\Lambda p\rangle$$

$$\langle 0|0\rangle = 1$$

$$\alpha(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = (2\pi)^3(2E_{\mathbf{p}})\delta(\mathbf{p}-\mathbf{p}'),$$

$$\alpha^\dagger(\mathbf{p}) \equiv (2\pi)^{\frac{3}{2}}\sqrt{2E_{\mathbf{p}}}a^\dagger(\mathbf{p}) \qquad [\alpha(\mathbf{p}),\alpha(\mathbf{p}')] = [\alpha^\dagger(\mathbf{p}),\alpha^\dagger(\mathbf{p}')] = 0.$$

$$|f\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} f(\mathbf{p})\alpha^\dagger(\mathbf{p})|0\rangle$$

Let us construct some observable in this theory. It will be an operator depending on space time, and satisfying some simple conditions:

- Hermiticity $\quad \phi(x)^\dagger = \phi(x).$
- Microcausality $\quad [\phi(x),\phi(y)] = 0, \qquad (x-y)^2 < 0.$
- Translational invariance $\quad e^{i\widehat{P}\cdot a}\phi(x)e^{-i\widehat{P}\cdot a} = \phi(x-a)$
- Lorentz invariance $\quad \mathscr{U}(\Lambda)^\dagger\phi(x)\mathscr{U}(\Lambda) = \phi(\Lambda^{-1}x).$
- Linearity $\quad \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[f(\mathbf{p},x)\alpha(\mathbf{p}) + g(\mathbf{p},x)\alpha^\dagger(\mathbf{p})\right].$

We have obtained from first principles the quantisation of the Klein-Gordon field. There are more straightforward ways, but the procedure shows how to implement the basis principles of the theory, Lorentz invariance, locality and positivity of the spectrum

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\alpha(\mathbf{p}) + e^{iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\alpha^\dagger(\mathbf{p})\right]$$

+ve energy

-ve energy

# Some important properties

$$[\phi(t,\mathbf{x}), \partial_t \phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

$$[\phi(x), \phi(x')] = i\Delta(x-x')$$

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$$

$$\begin{aligned} i\Delta(x-y) &= -\mathrm{Im} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-iE_{\mathbf{p}}(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \\ &= \int \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2-m^2)\varepsilon(p^0) e^{-ip\cdot(x-x')} \end{aligned}$$

$$\Delta(x-y) = 0 \qquad \text{for } (x-y)^2 < 0$$

The construction is free of paradoxes. It satisfies the KG equation because the +ve and -ve energy plane waves satisfy it. Of course with a free field we do not go very far…

We should design more powerful techniques leading to similar properties by for more general theories where interactions can take place.

There are two general approaches: the canonical-formalism, and the Feynman path integral. We will briefly introduce the first, just as a reminder.

Remember: PHYSICS is where the ACTION is!

Proceed by analogy with ordinary QM

$$S[x,\dot{x}] = \int dt\, L(x,\dot{x})$$

$$L = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i^2 - V(\mathbf{x})$$

$$S[\phi(x)] \equiv \int d^4x \mathscr{L}(\phi, \partial_\mu \phi) = \int d^4x \left( \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \right)$$

$$\mathbf{x}_a,\ \dot{\mathbf{x}}_a \longleftrightarrow \phi(\mathbf{x},0),\ \dot{\phi}(\mathbf{x},0)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\partial_\mu \left[ \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right] - \frac{\partial \mathscr{L}}{\partial \phi} = 0 \qquad \Longrightarrow \qquad (\partial_\mu \partial^\mu + m^2)\phi = 0$$

canonical momenta

$$p = \frac{\partial L}{\partial \dot{x}}$$

$$\pi(x) \equiv \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi)} = \frac{\partial \phi}{\partial t}$$

$$H = \sum_i p_i \dot{x}^i - L$$

$$H \equiv \int d^3x \left( \pi \frac{\partial \phi}{\partial t} - \mathscr{L} \right) = \frac{1}{2} \int d^3x \left[ \pi^2 + (\nabla \phi)^2 + m^2 \right].$$

$$[q^i, p_j] = i\hbar$$

$$[\phi(t,\mathbf{x}), \partial_t \phi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}).$$

Expanding in solutions to the KG equations and performing the canonical quantisation, we recover the algebra of creation and annihilation operator we had before, but we get a surprise

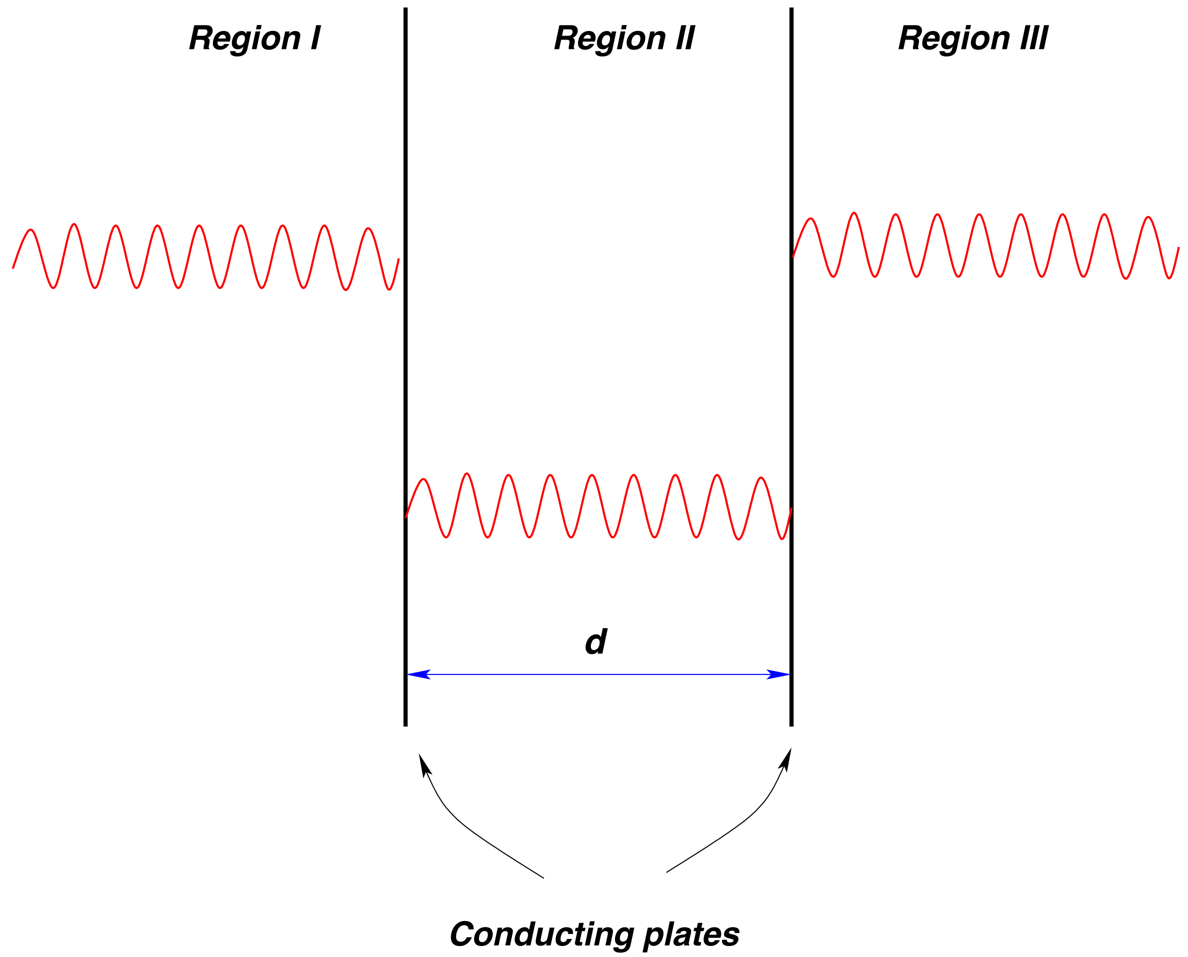
# Casimir effect

Writing the products of creation and ann. operators in NORMAL ORDERING i,e, annihilation operators to the right, we get rid of the sum of the zero point energy of the infinite number of oscillators in the field. In infinite space we subtract it, or simply normal order. When we do not have translational invariance, something interesting happens

$$\hat{H} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[\hat{\alpha}^\dagger(\mathbf{p})\hat{\alpha}(\mathbf{p}) + (2\pi)^3 E_\mathbf{p}\,\delta(\mathbf{0})\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}} E_\mathbf{p}\hat{\alpha}^\dagger(\mathbf{p})\hat{\alpha}(\mathbf{p}) + \frac{1}{2}\int d^3p\, E_\mathbf{p}\delta(\mathbf{0})$$

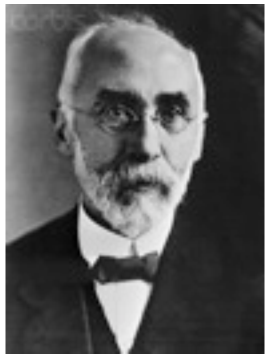
$$E(d)_{\text{reg}} = E(d)_{\text{vac}} - E(\infty)_{\text{vac}}$$




The force per unit area is the derivative of this quantity with respect to d divided by the area of the plates. The result is finite and attractive, the Casimir force! Which has been measured (of course for the electromagnetic field)

$$P_{\text{Casimir}} = -\frac{\pi^2}{240}\frac{1}{d^4}$$

# Lorentz and Poincaré Groups



In trying to systematically construct viable QFTs it is useful to understand the representations of the Lorentz (and Poincaré) groups.

The Hilbert space of states has to carry a unitary representation of the Lorentz group, so that quantum amplitudes are consistent with Unitarity and Relativistic Invariance. The fields themselves however, transform under finite dimensional representations. They are much easier to study. Just recall the usual rotation group SU(2). The Lorentz group, also known as SO(3,1) preserves the Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu \qquad \mu,\nu = 0,1,2,3$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad\qquad \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

$$\det\Lambda = \pm 1 \qquad\qquad (\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^i_0)^2 = 1$$

- $\mathfrak{L}_+^\uparrow$: proper, orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}_-^\uparrow$: improper, orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \geqslant 1$.
- $\mathfrak{L}_-^\downarrow$: improper, non-orthochronous transformations with $\det\Lambda = -1, \Lambda^0{}_0 \leqslant -1$.
- $\mathfrak{L}_+^\downarrow$: proper, non-orthochronous transformations with $\det\Lambda = 1, \Lambda^0{}_0 \leqslant -1$.

$$\mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{P}} \mathfrak{L}_-^\uparrow, \qquad \mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{T}} \mathfrak{L}_-^\downarrow, \qquad \mathfrak{L}_+^\uparrow \xrightarrow{\mathscr{P}\mathscr{T}} \mathfrak{L}_+^\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

# Lorentz and Poincaré Groups

$$R(\mathbf{e},\varphi) = e^{-i\varphi\,\mathbf{e}\cdot\mathbf{J}}$$

$$B(\mathbf{u},\lambda) = e^{-i\lambda\,\mathbf{u}\cdot\mathbf{M}}$$

Rotations and boosts generate Lorentz transformation, hence six parameter and six generators of infinitesimal transformations. The algebra is easy to obtain and "diagonalise"

$$\begin{aligned}[J_i,J_j] &= i\varepsilon_{ijk}J_k,\\ [J_i,M_k] &= i\varepsilon_{ijk}M_k,\\ [M_i,M_j] &= -i\varepsilon_{ijk}J_k\end{aligned}$$

$$J_k^{\pm} = \frac{1}{2}(J_k \pm iM_k)$$

$$\begin{aligned}[J_i^{\pm},J_j^{\pm}] &= i\varepsilon_{ijk}J_k^{\pm},\\ [J_i^{+},J_j^{-}] &= 0.\end{aligned}$$

The representations of each SU(2) are labelled by a single integer or half integer "angular" momentum s=0, 1/2, 1, 3/2, … Under parity

$$(\mathbf{s}_+,\mathbf{s}_-)$$

| Representation | Type of field |
|---|---|
| $(\mathbf{0},\mathbf{0})$ | Scalar |
| $(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0})$ | Right-handed spinor |
| $(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}})$ | Left-handed spinor |
| $(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$ | Vector |
| $(\mathbf{1},\mathbf{0})$ | Selfdual antisymmetric 2-tensor |
| $(\mathbf{0},\mathbf{1})$ | Anti-selfdual antisymmetric 2-tensor |

$$\begin{aligned}\mathbf{J} &\xrightarrow{P} \mathbf{J}\\ \mathbf{M} &\rightarrow -\mathbf{M}\\ \mathbf{J}^{\pm} &\rightarrow \mathbf{J}^{\mp}\\ (\mathbf{s_1},\mathbf{s_2}) &\rightarrow (\mathbf{s_2},\mathbf{s_1})\end{aligned}$$

$$\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^-$$

$$(\mathbf{s}_+,\mathbf{s}_-) = \sum_{\mathbf{j}=|\mathbf{s}_+-\mathbf{s}_-|}^{\mathbf{s}_++\mathbf{s}_-} \mathbf{j}$$

# Weyl spinors

The simplest representations have fundamental physical importance, they are called Weyl spinors. Clearly they are representations of the connected component of SO(3,1), but not of parity, since parity interchanges the representations

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_i^+ = \frac{1}{2}\sigma_i, \qquad J_i^- = 0 \qquad \text{for} \qquad (\tfrac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}),$$

$$J_i^+ = 0, \qquad J_i^- = \frac{1}{2}\sigma_i \qquad \text{for} \qquad (\mathbf{0}, \tfrac{\mathbf{1}}{\mathbf{2}}).$$

$$u_\pm \longrightarrow e^{-\frac{i}{2}(\theta\mathbf{n} \mp i\beta)\cdot\sigma} u_\pm \qquad\qquad u_\pm \longrightarrow e^{i\theta} u_\pm$$

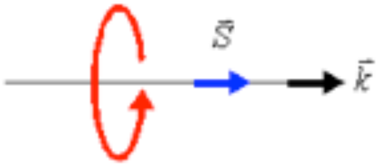
Consider for simplicity this global symmetry: fermion number

$$\sigma_\pm^\mu = (\mathbf{1}, \pm\sigma_i) \qquad \begin{matrix} u_+^\dagger \sigma_+^\mu u_+ \\ u_-^\dagger \sigma_-^\mu u_- \end{matrix}$$

$$\mathscr{L}_{\text{Weyl}}^\pm = i u_\pm^\dagger (\partial_t \pm \sigma \cdot \nabla) u_\pm = i u_\pm^\dagger \sigma_\pm^\mu \partial_\mu u_\pm$$

$$(\partial_0 \pm \sigma\cdot\nabla) u_\pm = 0 \qquad \begin{matrix} u_\pm(x) = u_\pm(k) e^{-ik\cdot x} \\ k^2 = k_0^2 - \mathbf{k}^2 = 0 \end{matrix} \qquad (|\mathbf{k}| \mp \mathbf{k}\cdot\sigma) u_\pm = 0$$

$$u_+ : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = 1,$$




positive helicity, right handed antineutrinos

$$u_- : \qquad \frac{\sigma\cdot\mathbf{k}}{|\mathbf{k}|} = -1$$


negative helicity, left handed, neutrinos

# Charge conjugation and Majorana masses

We know that under parity, the L,R Weyl spinors are exchanged. Another way to exchange them is via complex conjugation, later to be related to charge conjugation

$$\begin{aligned} M_L &= e^{-\frac{i}{2}\theta\cdot\sigma-\frac{1}{2}\beta\cdot\sigma} \\ M_R &= e^{-\frac{i}{2}\theta\cdot\sigma+\frac{1}{2}\beta\cdot\sigma} \end{aligned} \qquad \begin{aligned} \det M_L &= 1 \\ \det M_R &= 1 \end{aligned} \qquad \begin{aligned} \det M &= \epsilon_{ab} M_{a1}\, M_{b2} \\ \det M \epsilon_{ab} &= \epsilon_{cd} M_{ca}\, M_{db} \end{aligned} \qquad \epsilon = i\,\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Using $\sigma^* = -\sigma_2\,\sigma\,\sigma_2$

$$\begin{aligned} \psi_L^c &= \sigma_2\,\psi_L^* \quad \text{transforms like} \quad \psi_R \\ \psi_R^c &= \sigma_2\,\psi_R^* \quad \text{transforms like} \quad \psi_L \end{aligned}$$

- We can express any theory fully in terms of L or R fermions.
- Charge conjugation and parity exchange L and R
- A parity invariant theory requires L,R spinors at the same time
- We can construct a mass for pure L spinors if we ignore fermion number
- Fermions are anticommuting

$$\mathscr{L}^{\pm}_{\text{Weyl}} = iu^{\dagger}_{\pm}\sigma^{\mu}_{\pm}\partial_{\mu}u_{\pm} + \frac{m}{2}\left(\varepsilon_{ab}u^a_{\pm}u^b_{\pm} + \text{h.c.}\right)$$

$$\epsilon_{ab}u^a\,u^b = u^1\,u^2 - u^2\,u^1$$

Most general Majorana mass, Takagi factorisation

$$\frac{1}{2}\left(M_{IJ}\,\epsilon_{ab}\,u^{a,I}\,u^{b,J} + \text{h.c.}\right),$$
$$I, J = 1, \dots N_F, \quad M_{IJ} = M_{JI} \text{ complex}$$

$$M = U \begin{pmatrix} m_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & m_{N_F} \end{pmatrix} U^T$$

$m_i$ are positive square roots of $MM^\dagger$

This is the most general fermion mass matrix!!! It includes CKM, in fact it is more general

# Weyl + parity: Dirac

$(\tfrac{1}{2},\mathbf{0})\oplus(\mathbf{0},\tfrac{1}{2})$

$$P: u_\pm \longrightarrow u_\mp \qquad \psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \qquad \left.\begin{array}{l} i\sigma_+^\mu \partial_\mu u_+ = m u_- \\ i\sigma_-^\mu \partial_\mu u_- = m u_+ \end{array}\right\} \Longrightarrow \quad i\begin{pmatrix} \sigma_+^\mu & 0 \\ 0 & \sigma_-^\mu \end{pmatrix}\partial_\mu \psi = m\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\psi$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \qquad \overline{\psi} \equiv \psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \qquad \mathscr{L}_{\text{Dirac}} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

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$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

$$P_+\psi = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} \qquad P_-\psi = \begin{pmatrix} 0 \\ u_- \end{pmatrix}$$

$$\begin{aligned} \operatorname{Tr}\gamma^\mu\gamma^\nu &= 4\,\eta^{\mu\nu} \\ \operatorname{Tr}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta &= 4\,\eta^{\mu\nu}\eta^{\alpha\beta} - 4\,\eta^{\mu\alpha}\eta^{\beta\nu} + 4\,\eta^{\mu\beta}\eta^{\alpha\nu} \\ \operatorname{Tr}\gamma_5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu &= 4\,i\,\epsilon^{\alpha\beta\mu\nu} \end{aligned}$$

We look for +ve and -ve energy solutions as usual

$$u(k,s)e^{-ik\cdot x}$$

$$(\not{k} - m)u(k,s) = 0$$

$$v(k,s)e^{ik\cdot x}$$

$$(\not{k} + m)v(k,s) = 0$$

$$k^2 = m^2$$

$$\begin{aligned} \overline{u}(\mathbf{k},s)u(\mathbf{k},s) &= 2m, \\ \overline{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s) &= 2k^\mu, \\ \sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\overline{u}_\beta(\mathbf{k},s) &= (\not{k} + m)_{\alpha\beta} \end{aligned}$$

$$\begin{aligned} \overline{v}(\mathbf{k},s)v(\mathbf{k},s) &= -2m, \\ \overline{v}(\mathbf{k},s)\gamma^\mu v(\mathbf{k},s) &= 2k^\mu, \\ \sum_{s=\pm\frac{1}{2}} v_\alpha(\mathbf{k},s)\overline{v}_\beta(\mathbf{k},s) &= (\not{k} - m)_{\alpha\beta} \end{aligned}$$

We repeat the bosonic arguments, except for the fact that we have now anti-commutation relations between electron and positron creation-annihilation operators

$$\widehat{\psi}_\alpha(t,\vec{x}) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[u_\alpha(\vec{k},s)\,\widehat{b}(\vec{k},s)e^{-i\omega_{\vec{k}}t+i\vec{k}\cdot\vec{x}} + v_\alpha(\vec{k},s)\,\widehat{d}^{\,\dagger}(\vec{k},s)e^{i\omega_{\vec{k}}t-i\vec{k}\cdot\vec{x}}\right].$$

$$\{\widehat{\psi}_\alpha(t,\mathbf{x}),\widehat{\psi}^\dagger_\beta(t,\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$$

$$\begin{aligned}\{b(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},\\ \{b(\mathbf{k},s),b(\mathbf{k}',s')\} &= \{b^\dagger(\mathbf{k},s),b^\dagger(\mathbf{k}',s')\}=0.\\ \{d(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\} &= (2\pi)^3(2\omega_{\mathbf{k}})\delta(\mathbf{k}-\mathbf{k}')\delta_{ss'},\\ \{d(\mathbf{k},s),d(\mathbf{k}',s')\} &= \{d^\dagger(\mathbf{k},s),d^\dagger(\mathbf{k}',s')\}=0.\end{aligned}$$

$$\hat{H} = \frac{1}{2}\sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\left[b^\dagger(\mathbf{k},s)b(\mathbf{k},s)-d(\mathbf{k},s)d^\dagger(\mathbf{k},s)\right].$$

$$\widehat{H} = \sum_{s=\pm\frac{1}{2}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left[\omega_{\vec{k}}b^\dagger(\vec{k},s)b(\vec{k},s)+\omega_{\vec{k}}d^\dagger(\vec{k},s)d(\vec{k},s)\right] - \;2\int d^3k\,\omega_{\vec{k}}\delta(\vec{0}).$$

We have a conserved charge and current

$$j^\mu = \overline{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0 \qquad Q = e\int d^3x\, j^0$$

The two-point function or Feynman propagator is:

$$S_{\alpha\beta}(x_1,x_2) = \langle 0|T\left[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right]|0\rangle$$

$$T\left[\psi_\alpha(x)\overline{\psi}_\beta(y)\right] = \theta(x^0-y^0)\psi_\alpha(x)\overline{\psi}_\beta(y)-\theta(y^0-x^0)\overline{\psi}_\beta(y)\psi_\alpha(x).$$

# Introducing gauge fields

The canonical gauge field is the electromagnetic field. The first one that was understood as a gauge field. For some time this symmetry sounded like a luxury. In fact the classical theory can be formulated exclusively in terms of the E,B field that are manifestly gauge invariant. This is not so in the quantum theory, where we need to use the vector and scalar potentials. There are new, non-local observables. They are responsible for the Bohm-Aharonov effect and the quantisation of electric charge (if there is a single monopole in the Universe, (Dirac)).

What we have learned is that all fundamental interactions known to us are mediated by suitable generalisations of the EM field. They are gauge theories. In fact it seems as though Nature abhors global symmetries. It appears that all the known global symmetries are just low-energy accidents. All symmetries in the UV should be local.

We do not know why this should be so. String Theory is the only theory where this fact finds an explanation. Unfortunately there is no evidence for it at this moment...

# E&M in Quantum Mechanics

Classical EM

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t}\mathbf{E}$$

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}.$$

$$\partial_\mu F^{\mu\nu} = j^\mu \qquad j^\mu = (\rho, \mathbf{j})$$
$$\varepsilon^{\mu\nu\sigma\eta}\partial_\nu F_{\sigma\eta} = 0, \qquad A^\mu = (\varphi, \mathbf{A})$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Classical EM in relativistic form

Coupling to QM requires the gauge potentials and a non-trivial transformation of the wave function, this gives subtle consequences to gauge symmetry

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi$$
$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x})$$
$$\varphi(t,\mathbf{x}) \to \varphi(t,\mathbf{x}) + \frac{\partial}{\partial t}\varepsilon(t,\mathbf{x}), \qquad \mathbf{A}(t,\mathbf{x}) \to \mathbf{A}(t,\mathbf{x}) + \nabla\varepsilon(t,\mathbf{x})$$
$$A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

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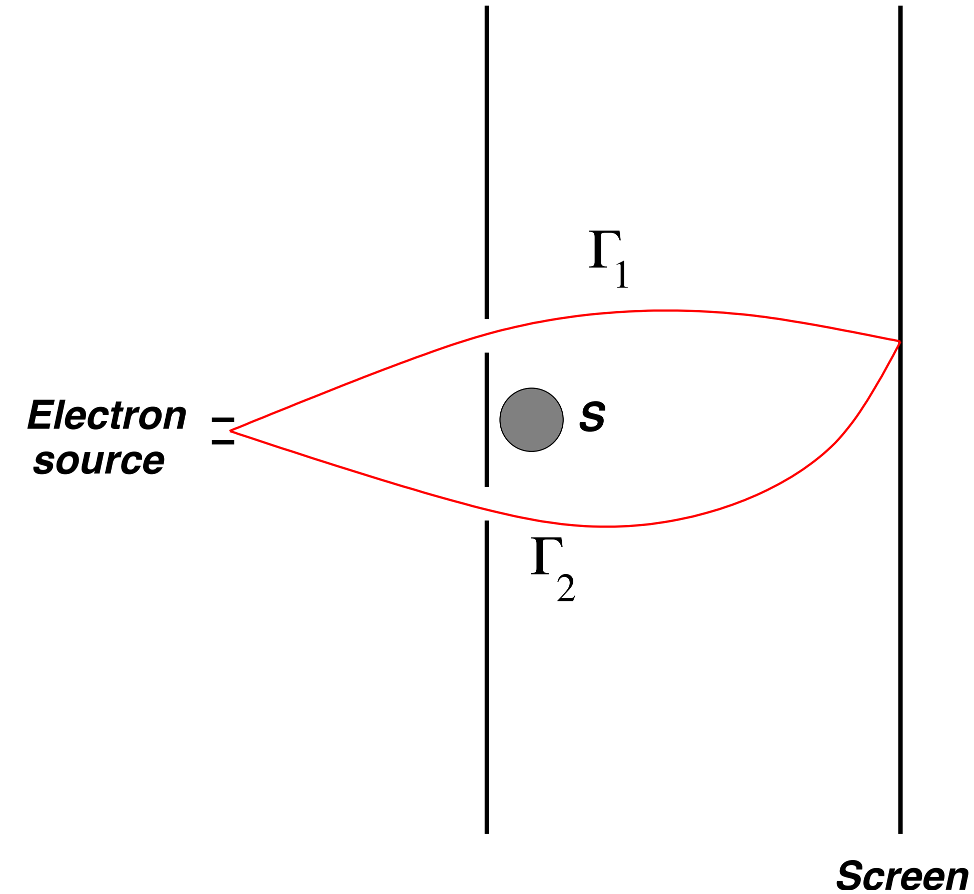
# Non-local observables

$$\Psi = e^{ie\int_{\Gamma_1} \mathbf{A}\cdot d\mathbf{x}}\Psi_1^{(0)} + e^{ie\int_{\Gamma_2} \mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}$$
$$= e^{ie\int_{\Gamma_1} \mathbf{A}\cdot d\mathbf{x}}\left[\Psi_1^{(0)} + e^{ie\oint_{\Gamma} \mathbf{A}\cdot d\mathbf{x}}\Psi_2^{(0)}\right]$$




$$U = \exp\left[ie\oint_{\Gamma} \mathbf{A}\cdot d\mathbf{x}\right]$$

This is the Aharonov-Bohm effect. The phase factor, and its non-abelian generalisation are known as “Wilson loops” or holonomies of the gauge field. Note that classically there would be no effect. The Lorentz force equation only involves E,B hence the electrons would not see the solenoid at all!!

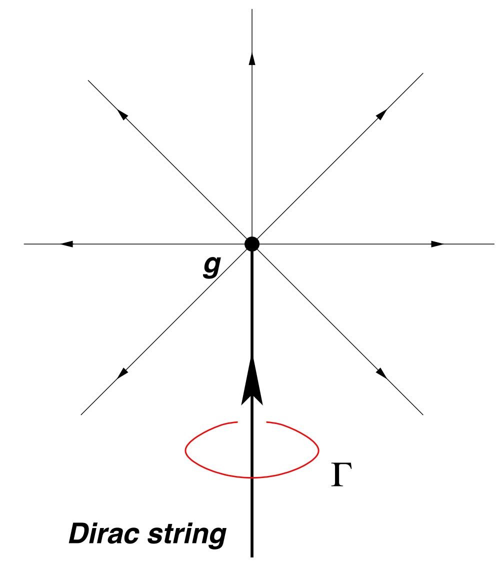
$$m\frac{du^{\mu}}{d\tau} = eF^{\mu\nu}u_{\nu}$$

# Magnetic monopoles: Dirac and charge quantisation

$$\nabla\cdot\mathbf{E}=0$$
$$\nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B}$$
$$\nabla\times\mathbf{B}=\frac{\partial}{\partial t}\mathbf{E}$$

$$\mathbf{E}-i\mathbf{B}\longrightarrow e^{i\theta}(\mathbf{E}-i\mathbf{B})$$

For angle = 90 E and B get exchanged




The symmetry extend to matter if we have magnetic sources:

$$\rho-i\rho_m\longrightarrow e^{i\theta}(\rho-i\rho_m),\qquad \mathbf{j}-i\mathbf{j}_m\longrightarrow e^{i\theta}(\mathbf{j}-i\mathbf{j}_m).$$

Consider a magnetic pole:

$$\nabla\cdot\mathbf{B}=g\,\delta(\mathbf{x}).\qquad B_r=\frac{1}{4\pi}\frac{g}{|\mathbf{x}|^2},\qquad B_\varphi=B_\theta=0$$

$$A_\varphi=\frac{1}{4\pi}\frac{g}{|\mathbf{x}|}\tan\frac{\theta}{2},\qquad A_r=A_\theta=0.$$

The Dirac string can be changed by gauge transformations, in doing QM it has to be unobservable. Then we can do a "A-B" like argument (Dirac did it 20 years earlier). We should not forget the fact that there is a factor of $\hbar c$

$$e^{ieg}=1\qquad eg=2\pi n$$

$$q_1g_2-q_2g_1=2\pi n,$$

# Electromagnetic Fields and Photons

Ignoring sources, the E&M field is a "free field"

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}.$$

The electric field is the momentum p and the vector potential the "coordinate" q

$$\mathscr{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

$$\partial_\mu F^{\mu\nu} = 0 \qquad 0 = \partial_\mu\partial^\mu A^\nu - \partial_\nu\left(\partial_\mu A^\mu\right) = \partial_\mu\partial^\mu A^\nu$$

To be able to invert, we need to fix the gauge: $\partial_\mu A^\mu = 0.$

As usual, we look for plane wave solutions
Residual gauge transformation used to fully fix the gauge

$$\varepsilon_\mu(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}}$$

$$k^\mu\varepsilon_\mu(\mathbf{k},\lambda) = 0$$

$$\epsilon_\mu(\mathbf{k},\lambda) \to \epsilon_\mu(\mathbf{k},\lambda) + k_\mu\,\chi(\mathbf{k}),\ \ k^2 = 0$$

$$k^2 = k_\mu k^\mu = (k^0)^2 - \mathbf{k}^2 = 0$$

Now, as usual we expand the field in oscillator and apply CCR. After fully fixing the gauge there are only two physical polarisations. Gauge invariance seems more a redundancy rather than a symmetry in the description of the theory

$$\widehat{A}_\mu(t,\mathbf{x}) = \sum_{\lambda=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2|\mathbf{k}|}\Big[\varepsilon_\mu(\mathbf{k},\lambda)\widehat{a}(\mathbf{k},\lambda)e^{-i|\mathbf{k}|t+i\mathbf{k}\cdot\mathbf{x}} + \varepsilon_\mu(\mathbf{k},\lambda)^*\,\widehat{a}^\dagger(\mathbf{k},\lambda)e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}\Big].$$

$$[\widehat{a}(\mathbf{k},\lambda),\widehat{a}^\dagger(\mathbf{k}',\lambda')] = (2\pi)^3(2|\mathbf{k}|)\delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'}$$

If we keep all four polarisation by partial gauge fixing, then we get negative probabilities (Gupta-Bleuler, BRST)

$$\delta_{\lambda,\lambda'} \to -\eta_{\lambda,\lambda'}$$

# Coupling matter

We imitate the coupling in the Schrödinger equation, this is what used to be called minimal coupling. We make derivatives covariant with respect to space-time dependent changes of phases in the wave-function

$$i\frac{\partial}{\partial t}\Psi = \left[-\frac{1}{2m}(\nabla - ie\mathbf{A})^2 + e\varphi\right]\Psi$$

$$\Psi(t,\mathbf{x}) \longrightarrow e^{-ie\,\varepsilon(t,\mathbf{x})}\Psi(t,\mathbf{x})$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu \varepsilon$$

$$D_\mu\left[e^{ie\varepsilon(x)}\psi\right] = e^{ie\varepsilon(x)}D_\mu\psi.$$

$$D_\mu = \partial_\mu - ieA_\mu.$$

The rigid phase rotation invariance of the Dirac Lagrangian for electrons is transformed into local phase rotations, a physically more satisfactory concept. This defines the coupling of the electron to the E&M field:

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi, \qquad \mathscr{L}^{(\text{int})}_{\text{QED}} = -eA_\mu\overline{\psi}\gamma^\mu\psi.$$

$$\psi \longrightarrow e^{ie\varepsilon(x)}\psi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\varepsilon(x).$$

This is QED, the best tested theory in the history of science, an example is the gyromagnetic ratio of the electron,

$$g\frac{e}{8m}[\gamma^\mu,\gamma^\nu]\,F_{\mu\nu}$$

$$g/2 = 1.00115965218085(76)$$

$$\alpha^{-1} = 137.035999070(98)$$

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\vec{s}, \qquad \underbrace{g_\mu = 2}_{\text{Dirac}}(1 + a_\mu)$$

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For the SM all group we will need are:

$$G: \quad U(1),\ SU(2),\ SU(3) \qquad [T^a, T^b] = if^{abc}T^c$$

$$G_{SM} = SU(3)\times SU(2)\times U(1)$$

$$g \in G \qquad g = e^{i\epsilon^a T^a} \qquad \mathrm{tr}(T^aT^b) = T_2(R)\,\delta^{ab}$$

$$\det g = 1 \Rightarrow \mathrm{tr}\,T^a = 0$$ (for SU(2), SU(3) not for U(1) of course)

U(1) is of course the simplest, just phase multiplication, i.e. as in QED

SU(2): angular momentum, isospin, and also weak isospin

$$[T^a, T^b] = i\varepsilon^{abc}T^c. \qquad T^{\pm} = \frac{1}{\sqrt{2}}(T^1 \pm iT^2), \qquad T^3$$

$$[T^3, T^{\pm}] = \pm T^{\pm}.$$
$$[T^+, T^-] = T^3$$

$$T^a = \frac{1}{2}\sigma^a$$ For spin ½ $$\mathrm{tr}\frac{\sigma^a}{2}\frac{\sigma^a}{2} = \frac{1}{2}\delta^{ab} \qquad a, b = 1, 2, 3$$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ For spin 1

For SU(3) the generators are the eight Gell-Mann 3x3 traceless hermitean matrices chosen to satisfy:

$$\mathrm{tr}\frac{\lambda^a}{2}\frac{\lambda^a}{2} = \frac{1}{2}\delta^{ab};\ a, b = 1, \ldots, 8$$

SU(3) of color, an exact gauge symmetry, also flavor SU(3), which is global (see later)

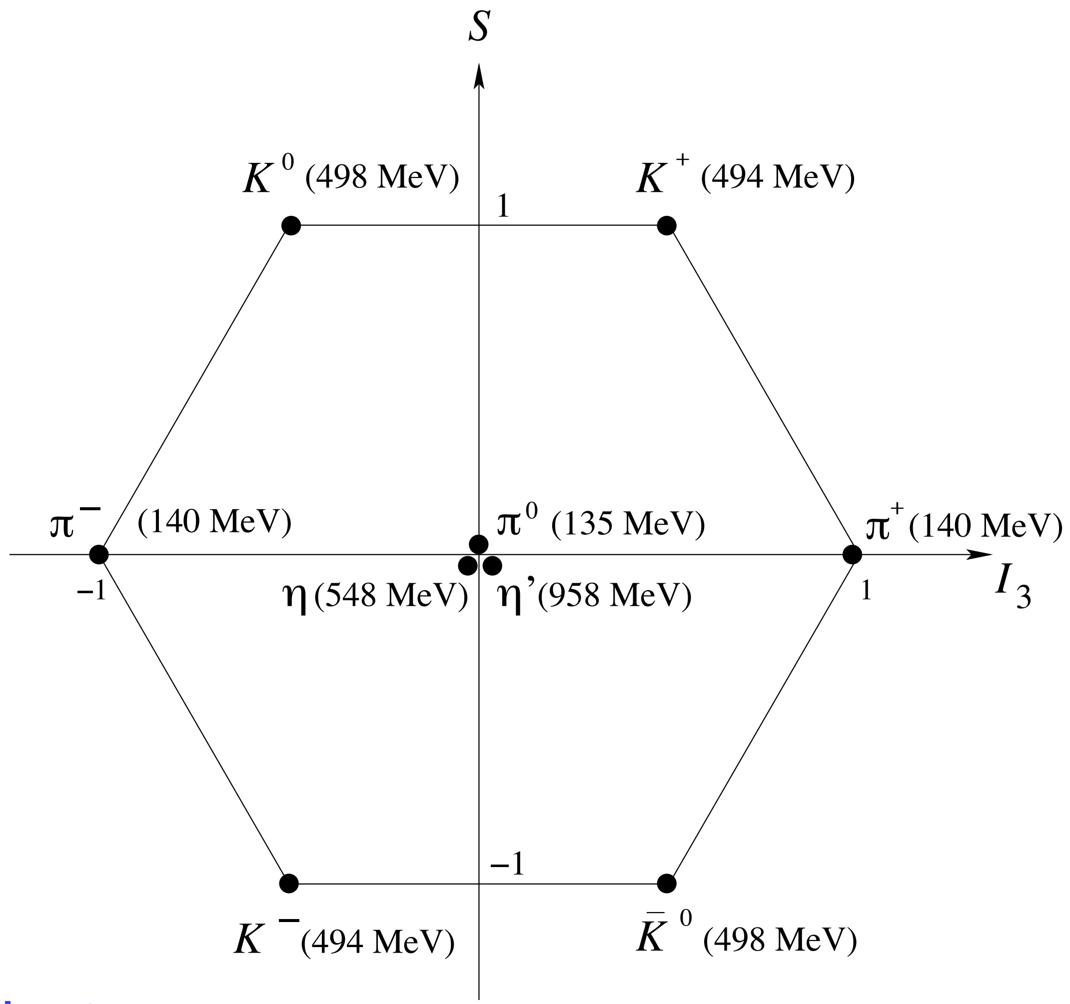
There are very few representations we will need for color SU(3):

$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8}$$

quarks antiquarks gluons

For flavor SU(3) more needed: mesons, baryons

$$\mathbf{3},\ \bar{\mathbf{3}},\ \mathbf{8},\ \mathbf{10},\ \overline{\mathbf{10}},\ \mathbf{27}\ldots$$




pseudo-scalar meson octet

A remarkable fact about the SM and QCD in particular is the fact that once we write the most general Lagrangian compatible with color gauge symmetry, flavor appears as an approximate global symmetry of the problem, although it was theorised earlier.

$$Q = I_3 + \frac{B+S}{2},$$

$$|\Delta^{++}; s_z = \tfrac{3}{2}\rangle = |uuu\rangle \otimes |\uparrow\uparrow\uparrow\rangle \equiv |u\uparrow, u\uparrow, u\uparrow\rangle.$$

$$|uud\rangle_S = \frac{1}{\sqrt{6}}\Big(|uud\rangle + |udu\rangle - 2|duu\rangle\Big),$$

$$|uud\rangle_A = \frac{1}{\sqrt{2}}\Big(|uud\rangle - |udu\rangle\Big).$$

$$|\Uparrow\rangle_S = \frac{1}{\sqrt{6}}\Big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\Big),$$

$$|\Uparrow\rangle_A = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\Big).$$

$$|p\uparrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Uparrow\rangle_A + |uud\rangle_A \otimes |\Uparrow\rangle_S\Big),$$

$$|p\downarrow\rangle = \frac{1}{\sqrt{2}}\Big(|uud\rangle_S \otimes |\Downarrow\rangle_A + |uud\rangle_A \otimes |\Downarrow\rangle_S\Big).$$

# Gauge theories and their quantisation

Imagine we have a theory with a global symmetry

$$\psi \to g\,\psi \qquad \bar{\psi} \to \bar{\psi} g^{\dagger} \qquad \mathcal{L} = \bar{\psi} i \partial\!\!\!/ \,\psi$$

Imitating electromagnetism:

$$\partial_\mu \to D_\mu \psi = \left(\partial_\mu + ieA_\mu^a T^a\right)\psi \equiv (\partial_\mu + ieA_\mu)\psi \qquad D_\mu\psi \to gD_\mu\psi$$

We can read off the gauge field transformations

$$A_\mu \to \frac{1}{ie} g\,\partial_\mu g^{-1} + gA_\mu\, g^{-1}$$

$$g \approx 1 + \epsilon \qquad A_\mu \to A_\mu + \frac{1}{ie} D_\mu \epsilon \qquad D_\mu\epsilon + ie[A_\mu, \epsilon]$$

$$[D_\mu, D_\nu] = ieT^a F^a_{\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ef^{abc} A^b_\mu A^c_\nu$$

$$F_{\mu\nu} \equiv T^a F^a_{\mu\nu} \to gF_{\mu\nu}\, g^{-1}$$

Nonabelian gauge fields have self-couplings unlike photons. This is responsible for confinement, among other things

# General Gauge Theory

General gauge theory Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + i\overline{\psi}\not{D}\psi + (D_\mu\phi)^\dagger D^\mu\phi$$
$$- \overline{\psi}\left[M_1(\phi) + i\gamma_5 M_2(\phi)\right]\psi - V(\phi).$$

We need to provide the gauge group and the matter representations for bosons and fermions and off we go

Quantising a gauge theory is no joke. There are plenty of subtleties. We give you just a taste

We can define chromoelectric and magnetic fields as in QED

$$\begin{aligned} F^a_{0i} &= \partial_0 A^a_i - \partial_i A^a_0 - ief^{abc}A^b_0 A^c_i \equiv E^a_i \\ F^a_{ij} &= \epsilon_{ijk}B^a_k, \qquad F^a_{0i} = \partial_0 A^a_0 - D_i A^a_0 \end{aligned}$$

The canonical variables are

$$\mathbf{A}^a, \mathbf{E}^a$$

$$\mathcal{L} = \mathbf{E}^a\,\partial_0\mathbf{A}^a - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - A^a_0\,(\mathbf{D}\cdot\mathbf{E})^a$$

$A^a_0$ implements a constraint

We can read off the Hamiltonian density

# General Gauge Theory

$$H = \int d^3x \left( \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + A_0^a \, (\mathbf{D}\cdot\mathbf{E})^a \right)$$

$$[A_i^a(\mathbf{x},0), E_j^b(\mathbf{y},0)] \,=\, i\,\delta_{ij}\,\delta^{ab}\delta(\mathbf{x}-\mathbf{y})$$

We can fix the gauge A_0=0 so that we only have time-independent gauge transformations in the Hamiltonian theory, but we are missing one of the equations of motion, Gauss' law that has to be implemented as a constraint.

$$(\mathbf{D}\cdot\mathbf{E})^a = 0$$

Cannot be implemented at the operator level. It generates gauge transformations

$$[Q(\epsilon), A_i^a] \,=\, i(D\epsilon)^a \qquad U(\epsilon) = \exp(i\int d^3x\,\epsilon^a(\mathbf{x})\,(\mathbf{D}\cdot\mathbf{E})^a), \qquad U\,H\,U^{-1} = H$$

Gauss' law becomes a condition on the physical states:

$$U(\epsilon)|\mathrm{phys}\rangle = |\mathrm{phys}\rangle$$

$$\mathbf{D}\cdot\mathbf{E}\,|\mathrm{phys}\rangle = 0$$

Each gauge configuration sits in an orbit and we need choose only one element, this is done by "fixing" the gauge for the t-independent gauge transf.

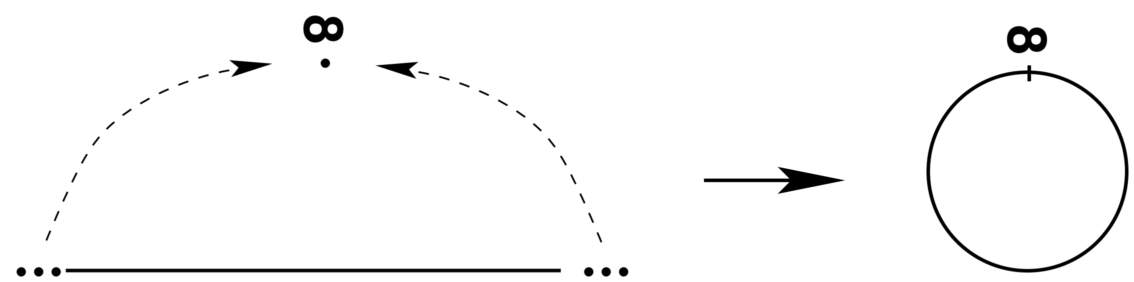
WE HAVE 2-DIM G PHYSICAL DEGREES OF FREEDOM

# Some remarks

❖ Gauge symmetry is more a redundant description of the d.o.f.

❖ Gauss' law implements gauge invariance under gauge t. connected to the identity. Consider finite-E configurations

$$g(\mathbf{x}) = e^{i\alpha(\mathbf{x})} \to 1 \qquad |\mathbf{x}| \to \infty$$
$$\alpha(\mathbf{x}) \to 0 \qquad |\mathbf{x}| \to \infty$$




There are others, and Gauss' law cannot impose invariance

$$g(\mathbf{x}) : S^3 \to G, \quad g(\infty) = 1 \qquad \pi_3(G) = Z \text{ the integers}$$

$$g : S^1 \longrightarrow U(1), \qquad g(x) = e^{i\alpha(x)}$$

$$\alpha(2\pi) = \alpha(0) + 2\pi n$$

$$\oint_{S^1} g(x)^{-1} dg(x) = 2\pi n$$

$$n = \frac{1}{24\pi^2} \int_{S^3} d^3x\, \varepsilon_{ijk} \mathrm{Tr}\left[ (g^{-1}\partial_i g)\,(g^{-1}\partial_i g)\,(g^{-1}\partial_i g) \right]$$

You cannot comb a sphere

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# A surprise: CP violation

❖ Gauge invariance only requires that under non-trivial transformations, a phase is generated. This is a vacuum angle! In fact it violates CP.

❖ It can be measured by looking for an edm of the neutron. So far no result:

❖ The strong CP problem, axions, invisible axions, axion cosmology, dark matter...

$g_1 \in \mathcal{G}/\mathcal{G}_0$ the generator

$$\mathscr{U}(g_1)|\text{phys}\rangle = e^{i\theta}|\text{phys}\rangle.$$

$$S = -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{\mu\nu\, a} - \frac{\theta g^2_{\text{YM}}}{32\pi^2}\int d^4x F^a_{\mu\nu}\widetilde{F}^{\mu\nu\, a}$$

$$\widetilde{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}F^{\sigma\lambda\, a} \qquad F^a_{\mu\nu}\widetilde{F}^{\mu\nu\, a} = 4\,\mathbf{E}^a\cdot\mathbf{B}^a$$

$$\frac{g^2_{\text{YM}}}{32\pi^2}\int d^4x F^a_{\mu\nu}\widetilde{F}^{\mu\nu\, a} = \frac{1}{24\pi^2}\int d^3x\, \varepsilon_{ijk}\text{Tr}\Big[(g\partial_i g^{-1})(g_+\partial_j g^{-1})(g_+\partial_k g^{-1})\Big].$$

# Computational tools

❖ There are two general procedures to obtain computational rules in QFT: The canonical formalism and the Path Integral formulation.

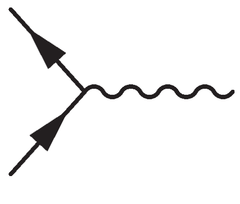
❖ You may recall that one used the Interaction Representation, Wick's theorem, T-products, Gaussian integrations…

❖ In the end we get a collection of well-defined rules that allow us to compute the probability amplitude associates to a given scattering process, out of which we can evaluate the decay width, differential and total cross section and many other quantities that can be observed for instance in collider experiments. The next few pages provide simply a reminder

# QED Feynman rules

$\alpha$  $\beta$ $\Longrightarrow$ $\left(\dfrac{i}{\not{p}-m+i\varepsilon}\right)_{\beta\alpha}$

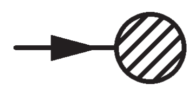
$\mu$  $\nu$ $\Longrightarrow$ $\dfrac{-i\eta_{\mu\nu}}{p^2+i\varepsilon}$

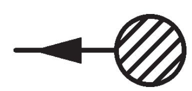


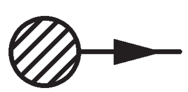
$\Longrightarrow$ $-ie\gamma^{\mu}_{\beta\alpha}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3).$

Integrate over loop momenta

$$\int \frac{d^d p}{(2\pi)^4};$$

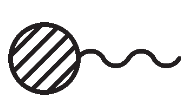
Incoming fermion: $\alpha$  $\Longrightarrow$ $u_\alpha(\mathbf{p},s)$

Incoming antifermion: $\alpha$  $\Longrightarrow$ $\overline{v}_\alpha(\mathbf{p},s)$

Outgoing fermion:  $\alpha$ $\Longrightarrow$ $\overline{u}_\alpha(\mathbf{p},s)$

Outgoing antifermion:  $\alpha$ $\Longrightarrow$ $v_\alpha(\mathbf{p},s)$

Incoming photon: $\mu$  $\Longrightarrow$ $\varepsilon_\mu(\mathbf{p})$

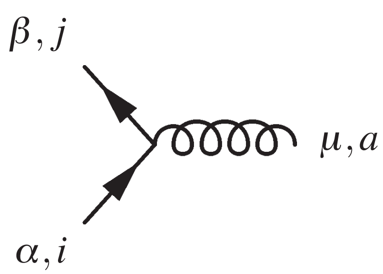
Outgoing photon:  $\mu$ $\Longrightarrow$ $\varepsilon_\mu(\mathbf{p})^*$

A minus sign has to be included for every fermion loop and for every positron line that goes from the initial to the final state. With some extra effort we can derive the Feynman rules for QCD-like theories. They appear in the next page. The quark and anti-quark factors are similar to the electron positron ones, except that we need to include color quantum numbers. The real difference comes with the gluon or non-abelian vector bosons interactions, the are quite involved and contain a large amount of interesting physics perturbatively and specially non-perturbatively.

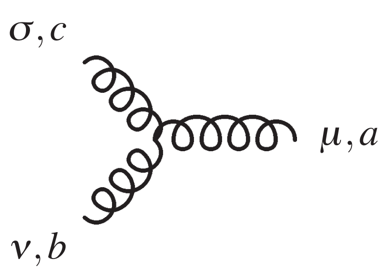
# Standard Model Feynman rules


$$\Longrightarrow \left(\frac{i}{\not p - m + i\varepsilon}\right)_{\beta\alpha} \delta_{ij}$$

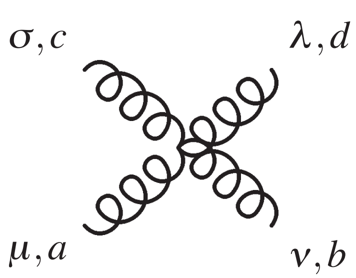

$$\Longrightarrow \frac{-i\eta_{\mu\nu}}{p^2 + i\varepsilon} \delta^{ab}$$




$$\Longrightarrow -ig\gamma^{\mu}_{\beta\alpha} t^a_{ij}$$




$$\Longrightarrow g f^{abc}\left[\eta^{\mu\nu}(p_1^{\sigma} - p_2^{\sigma})\text{permutations}\right]$$




$$\Longrightarrow -ig^2\left[f^{abe} f^{cde}\left(\eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\sigma}\right) + \text{permutations}\right]$$

Although the rules seem to be those for QCD, notice that we could always include in the group theory factors t^a-{ij} chiral projectors and make the group not simple but semi-simple as in the case of the SM: SU(3) xSU(2)xU(1). If we work in nice renormalizable gauges, the only difference is that we have to include the Feynman rules for the couplings of the scalar sector. Something we will do later.

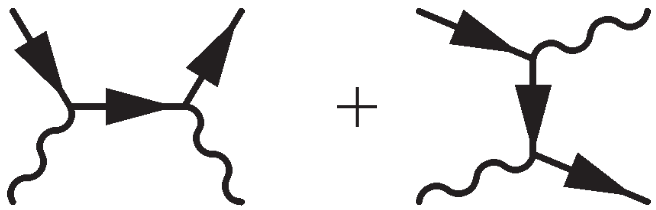
$$t^a_{ij} \to t^a_{ij}\,\frac{1}{2}(1 \pm \gamma_5)$$

With this simple trick the hard part, which is the coupling of the W,Z, and photons can be read simply from the rules in the LHS

# One example: Thomson Scattering

$$\gamma(k,\varepsilon)+e^-(p,s)\longrightarrow\gamma(k',\varepsilon')+e^-(p',s')$$

We work in the NR approximation for simplicity but keeping explicitly the dependence on the photon polarisations. We can guess that the answer has to be a pure number times the classical electron radius



$$= (ie)^2\overline{u}(\mathbf{p}',s')\not\varepsilon'(\mathbf{k}')^*\frac{\not p+\not k+m_e}{(p+k)^2-m_e^2}\not\varepsilon(\mathbf{k})u(\mathbf{p},s)$$

$$+ (ie)^2\overline{u}(\mathbf{p}',s')\not\varepsilon(\mathbf{k})\frac{\not p-\not k'+m_e}{(p-k')^2-m_e^2}\not\varepsilon'(\mathbf{k}')^*u(\mathbf{p},s)$$

$$p^2=m_e^2=p'^2$$

$$k^2=0=k'^2$$

$$|\mathbf{p}|,|\mathbf{k}|,|\mathbf{p}'|,|\mathbf{k}'|\ll m_e$$

$$\not a\not b=-\not b\not a+2(a\cdot b)\mathbf{1}$$

$$(p+k)^2-m_e^2\approx 2m_e|\mathbf{k}|,\qquad (p-k')^2-m_e^2\approx -2m_e|\mathbf{k}'|$$

$$(\not k-m)u(k,s)=0.$$

$$\overline{u}(\mathbf{k},s)\gamma^\mu u(\mathbf{k},s)=2k^\mu$$

$$\langle f|\hat{S}|i\rangle = \langle f|i\rangle + (2\pi)^4\delta^{(4)}\left(\sum_{\text{final}} p'_i - \sum_{\text{initial}} p_j\right) i\mathscr{M}_{i\to f}$$

$$d\sigma = \frac{|\mathscr{M}_{i\to f}|^2}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|}(2\pi)^4\,\delta^{(4)}\left(p_1+p_2-\sum_{j=1}^{n} p'_j\right) d\Phi_k$$

$$\begin{aligned} F_{\text{coll}} &= 4E_1E_2|\mathbf{v}_1-\mathbf{v}_2| = 4E_1E_2\left|\frac{\mathbf{p}_1}{E_1}-\frac{\mathbf{p}_2}{E_2}\right| \\ &= 4|E_2\mathbf{p}_1 - E_1\mathbf{p}_2| = 4\Big(E_2|\mathbf{p}_1| + E_1|\mathbf{p}_2|\Big) \\ &= 4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}. \end{aligned}$$

Square the amplitude, sum over final electron polarisations, and sum over the initial ones. We will consider unpolarised incoming photons and study how the outgoing photons can gain some degree of polarisation

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(\mathbf{k},s)\bar{u}_\beta(\mathbf{k},s) = (\not{k}+m)_{\alpha\beta}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2m_e^2}\overline{|i\mathscr{M}_{i\to f}|^2} = \left(\frac{e^2}{4\pi m_e}\right)^2\left|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*\right|^2$$

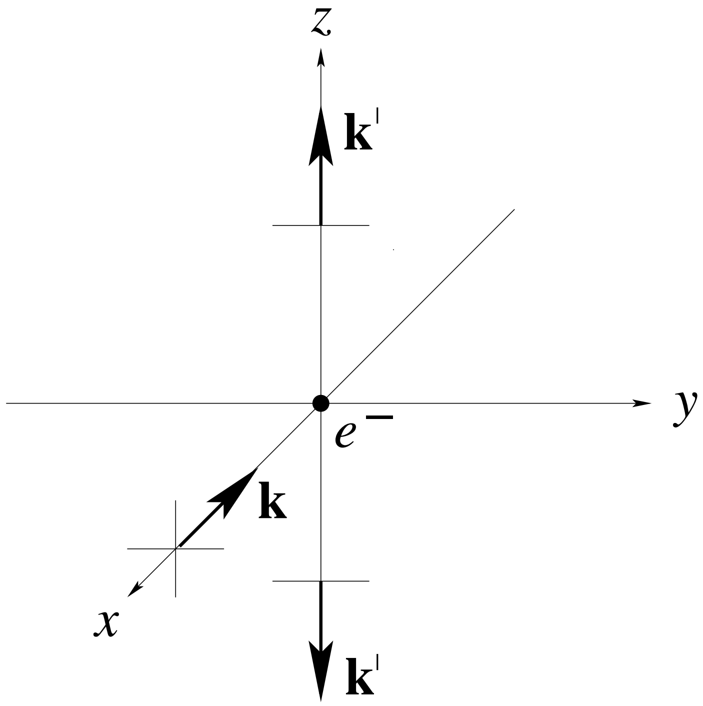
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}\sigma_T\left|\varepsilon(\mathbf{k})\cdot\varepsilon'(\mathbf{k}')^*\right|^2$$

$$\sigma_T = \frac{e^4}{6\pi m_e^2} = \frac{8\pi}{3}r_{\text{cl}}^2$$

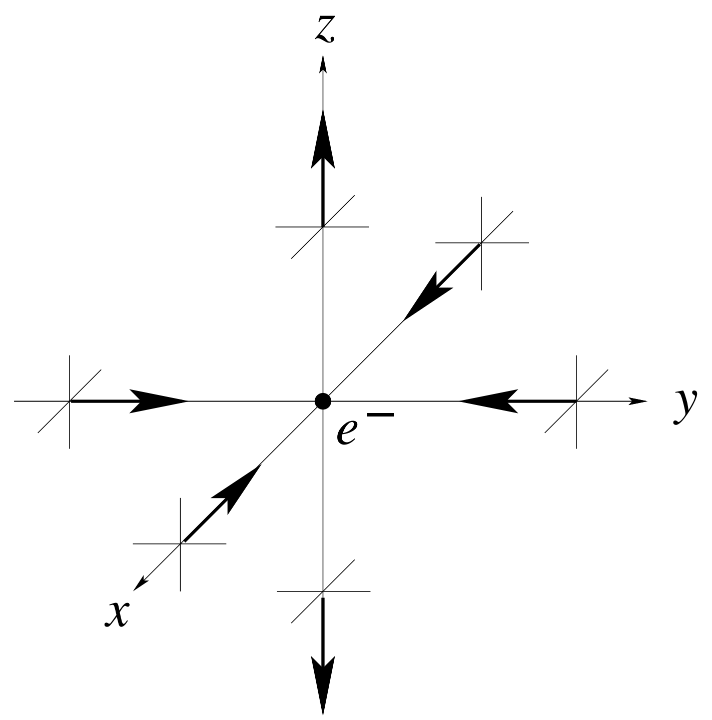
$$\begin{aligned} \frac{1}{2}\sum_{a=1,2}\left|\varepsilon(\mathbf{k},a)\cdot\varepsilon'(\mathbf{k}')^*\right|^2 &= \frac{1}{2}\left(\delta_{ij} - \frac{k_ik_j}{|\mathbf{k}|^2}\right)\varepsilon'_j(\mathbf{k}')\varepsilon'_i(\mathbf{k}')^* \\ &= \frac{1}{2}\left[1 - |\hat{\mathbf{k}}\cdot\varepsilon'(\mathbf{k}')|^2\right], \end{aligned}$$

We want to monitor the polarisation of the outgoing photons even when the incoming ones are not polarised

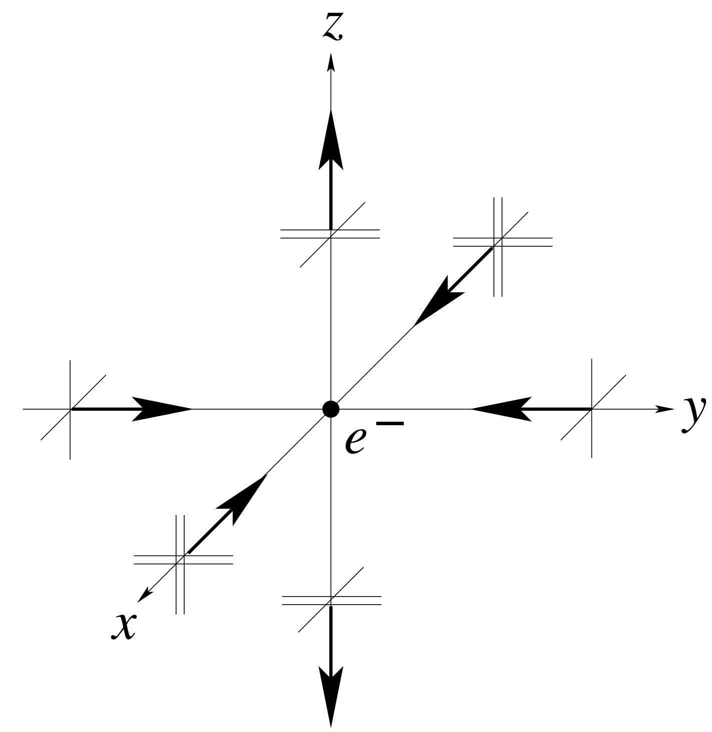
# Thomson and CMB Polarisation



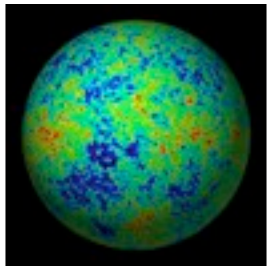

How we can get polarised light




An isotropic incoming distribution of light does not generate polarisation




A incoming light with a quadrupole perturbation generates net polarisation

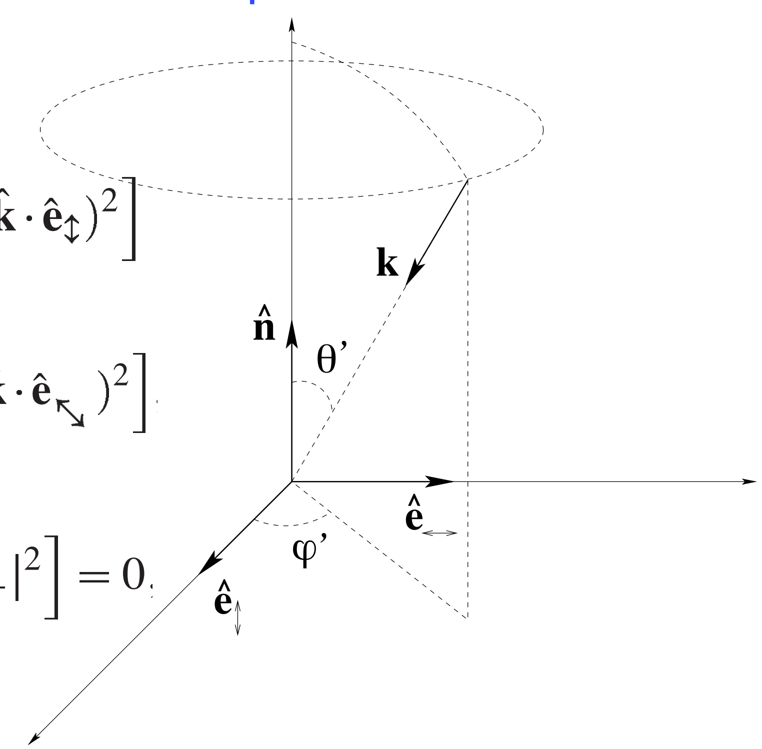


Stokes parameters:

$$Q(\hat{\mathbf{n}}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \Big[ |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\leftrightarrow}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\updownarrow}|^2 \Big] \qquad -\frac{1}{2}\int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \Big[ (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\leftrightarrow})^2 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\updownarrow})^2 \Big]$$

$$U(\mathbf{u}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \Big[ |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\swarrow\!\nearrow}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{\nwarrow\!\searrow}|^2 \Big] = -\frac{1}{2}\int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\hat{\mathbf{n}}) \Big[ (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\swarrow\!\nearrow})^2 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\nwarrow\!\searrow})^2 \Big].$$

$$V(\hat{\mathbf{u}}) \sim \sum_{a=1,2} \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \Big[ |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{+}|^2 - |\varepsilon(\mathbf{k},a)\cdot\hat{\mathbf{e}}_{-}|^2 \Big] = \int d\Omega(\hat{\mathbf{k}})\, f(\hat{\mathbf{k}},\mathbf{u}) \Big[ |\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{+}|^2 - |\hat{\mathbf{k}}\cdot\hat{\mathbf{e}}_{-}|^2 \Big] = 0.$$




$$\hat{\mathbf{e}}_{\pm} = -\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\varphi} \pm i\,\hat{\mathbf{e}}_{\theta}\right)$$

Finally we reach the punch line. No circular polarisation is generated by Thomson scattering, and we can write the combination:

$$Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}}) \sim -\int d\Omega(\theta', \varphi') f(\theta', \varphi'; \hat{\mathbf{n}}) \sin^2\theta' e^{\pm 2i\varphi'}$$

$$Y_2^{\pm 2}(\theta', \varphi') = 3\sqrt{\frac{5}{96\pi}} \sin^2\theta' e^{\pm 2i\varphi'}$$

One of the obvious generators of quadrupole anisotropies are gravitational waves. Inflation predicts primordial gravitation waves, the measurement of polarisation in the CMB offers an amazing window to obtain this information. The simple computation of Thomson scattering has unexpected consequences

$$Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}}) = -\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(E_{\ell m} \pm iB_{\ell m}\right) {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}})$$

$$\langle E^*_{\ell m} E_{\ell' m'} \rangle = C_\ell^{EE} \delta_{\ell\ell'} \delta_{mm'}, \qquad \langle B^*_{\ell m} B_{\ell' m'} \rangle = C_\ell^{BB} \delta_{\ell\ell'} \delta_{mm'}$$

# Noether's Theorem

Quantum mechanical realisation of Symmetries (Wigner's theorem). In a QM theory physical symmetries are maps among states that preserve probability amplitudes (their modulus). They can be unitary or anti-unitary

$$|\alpha\rangle \longrightarrow |\alpha'\rangle, \qquad |\beta\rangle \longrightarrow |\beta'\rangle$$

$$|\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|.$$

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle$$ unitary

$$\langle \mathscr{U}\alpha|\mathscr{U}\beta\rangle = \langle\alpha|\beta\rangle^*$$ anti-unitary T-reversal, CPT

For continuous symmetries we have Noether's celebrated theorem: If under infinitesimal transformations, AND WITHOUT USING THE EQUATIONS OF MOTION you can show that:

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$$

$$\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu$$

then there is a conserved current in the theory

In formulas:

$$\begin{aligned}\delta_\varepsilon \mathscr{L} &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\partial_\mu \delta_\varepsilon \phi + \frac{\partial \mathscr{L}}{\partial \phi}\delta_\varepsilon \phi \\ &= \partial_\mu \left( \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi \right) + \left[ \frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta_\varepsilon \phi \\ &= \partial_\mu K^\mu .\end{aligned}$$

$$\partial_\mu J^\mu = 0$$

$$\boxed{J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi - K^\mu}$$

With a conserved charge that generates the symmetry:

$$Q \equiv \int d^3x J^0(t,\mathbf{x})$$

$$\frac{dQ}{dt} = \int d^3x\, \partial_0 J^0(t,\mathbf{x}) = -\int d^3x\, \partial_i J^i(t,\mathbf{x}) = 0,$$

$$\delta\phi = i[\phi, Q].$$

Space-time translations -- Energy-Momentum
Lorentz transformation-- Angular momentum and CM motion
Phase rotation -- abelian and non-abelian charges

# Useful examples

Massive Dirac fermions:

$$\mathscr{L} = i\overline{\psi}_j \partial\!\!\!/ \psi_j - m\overline{\psi}_j \psi_j \qquad \psi_i \longrightarrow U_{ij}\psi_j \qquad U \in U(N)\ \ N \text{the number of fermions}$$

$$U = \exp\left(i\alpha^a T^a\right), \qquad (T^a)^\dagger = T^a$$

$$j^{\mu\, a} = \overline{\psi}_i T^a_{ij} \gamma^\mu \psi_j \qquad \partial_\mu j^\mu = 0 \qquad Q^a = \int d^3x\, \psi_i^\dagger T^a_{ij} \psi_j$$

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha) = e^{i\alpha^a Q^a}.$$

When U is the identity, we have fermion number, or charge

In the m=0 we have more symmetry: CHIRAL SYMMETRY, rotate L,R fermions independently

$$\mathcal{L} = i\bar{\psi}_{jL}\, \partial\!\!\!/\, \psi_{Lj} + i\bar{\psi}_{jR}\, \partial\!\!\!/\, \psi_{Rj}$$

$$\psi_{L,R} \to U_{L,R}\, \psi_{L,R} \qquad U(N)_L \times U(N)_R$$

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# Wigner-Weyl mode

Imagine we have a symmetry that is a symmetry of the ground state

$$[Q^a, H] = 0. \qquad \mathscr{U}(\alpha)|0\rangle = |0\rangle \qquad Q^a|0\rangle \;=\; 0$$

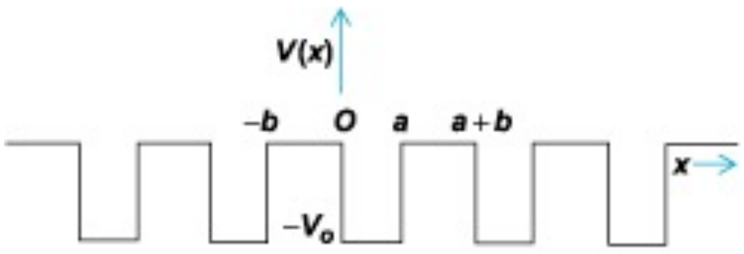
Then the states of the theory fall into multiplets of the symmetry group

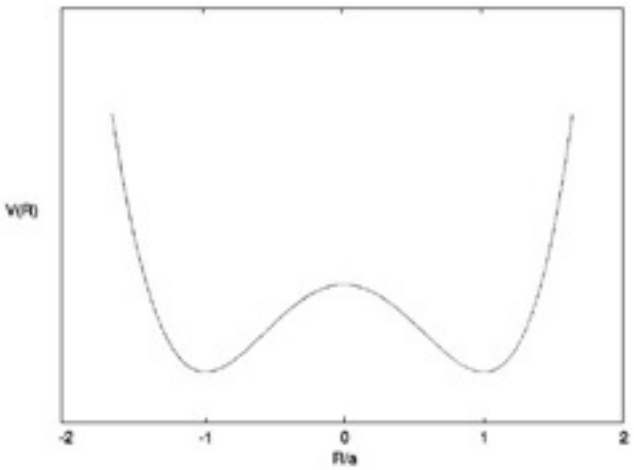
$$\mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1} = U_{ij}(\alpha)\phi_j$$

$$|i\rangle = \phi_i|0\rangle$$

$$\mathscr{U}(\alpha)|i\rangle = \mathscr{U}(\alpha)\phi_i\mathscr{U}(\alpha)^{-1}\mathscr{U}(\alpha)|0\rangle = U_{ij}(\alpha)\phi_j|0\rangle = U_{ij}(\alpha)|j\rangle$$

The spectrum of the theory is classified in terms of multiplets of the symmetry group. This is the case of the Hydrogen atom. The Hamiltonian is rotational invariant, the ground state is an s-wave state, hence all excited states fall into degenerate representations of the rotation group: 1s, 2s, 2p, 2s, 3p, 3d,… In QM (finite number of d.o.f.) this is always the case (tunnelling, band theory in solids)



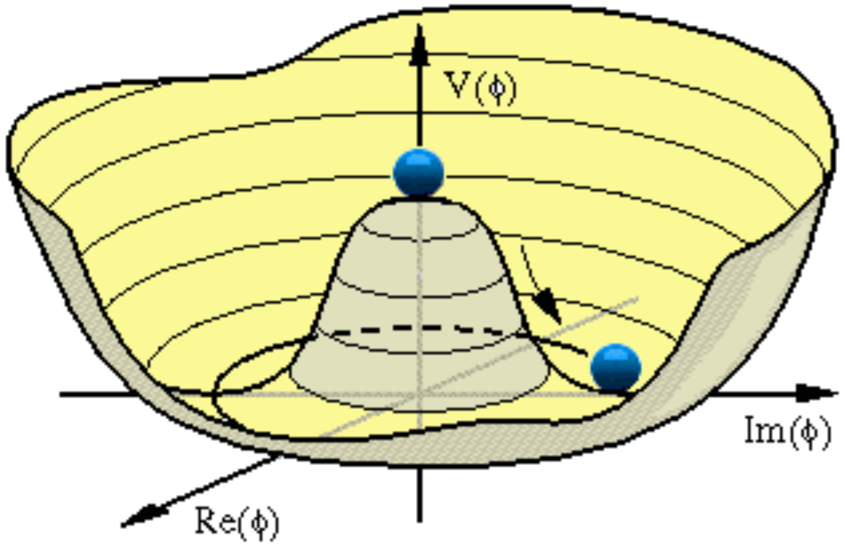



# Nambu-Goldstone mode

Sometimes also called hidden symmetry. The symmetry is spontaneously broken by the vacuum

$$[Q^a, H] = 0. \quad Q^a|0\rangle \neq 0.$$

Consider a collection of N scalar fields with a global symmetry group G




$$\mathscr{L} = \frac{1}{2}\partial_\mu \varphi^i \partial^\mu \varphi^i - V(\varphi^i) \qquad \delta\varphi^i = \varepsilon^a (T^a)^i_j \varphi^j$$

$$H[\pi^i, \varphi^i] = \int d^3x \left[\frac{1}{2}\pi^i\pi^i + \frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right]$$

$$\mathscr{V}[\varphi^i] = \int d^3x \left[\frac{1}{2}\nabla\varphi^i \cdot \nabla\varphi^i + V(\varphi^i)\right]$$

The minima satisfy

$$\langle\varphi^i\rangle \qquad V(\langle\varphi^i\rangle) = 0, \qquad \nabla\varphi = \mathbf{0} \qquad \left.\frac{\partial V}{\partial\varphi^i}\right|_{\varphi^i = \langle\varphi^i\rangle} = 0$$

$$T^a = \{H^\alpha, K^A\} \qquad (H^\alpha)^i_j\langle\varphi^j\rangle = 0 \qquad (K^A)^i_j\langle\varphi^j\rangle \neq 0.$$

unbroken broken

# Nambu-Goldstone mode

The masses are given by the second derivatives of the potential (assuming canonical normalisation)

$$M^2_{ij} \equiv \left. \frac{\partial^2 V}{\partial\varphi^i \partial\varphi^j} \right|_{\varphi=\langle\varphi\rangle}$$

Invariance

$$\delta V(\varphi) = \varepsilon^a \frac{\partial V}{\partial \varphi^i} (T^a)^i_j \varphi^j = 0 \qquad \frac{\partial^2 V}{\partial\varphi^i\partial\varphi^k}(T^a)^i_j\varphi^j + \frac{\partial V}{\partial\varphi^i}(T^a)^i_k = 0$$

$$M^2_{ik}(T^a)^i_j\langle\varphi^j\rangle = 0. \qquad M^2_{ik}(K^A)^i_j\langle\varphi^j\rangle = 0$$

For every broken generator there is a massless scalar field

The argument works at the full quantum level

The fields acquiring a VEV need not be elementary

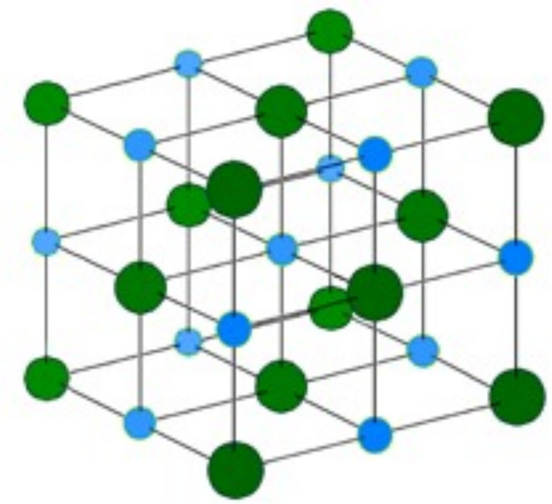
Simplest example:

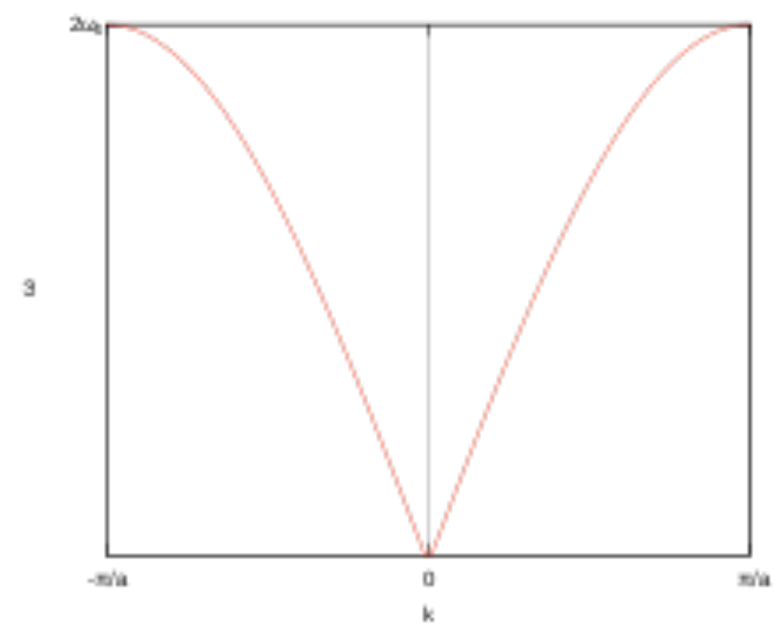
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

$$\phi \to \phi + c$$

Its own NG-boson

# Phonons are NG bosons






A liquid is translationally invariant

The crystal after solidification has discrete translational symmetry

The low energy excitation of the lattice contain acoustic phonons

Their dispersion relation is as for NG bosons

They propagate at the speed of sound

$$\omega(k) \;=\; 2\omega|\sin(ka/2)|$$

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# Order parameters

❖ The notion of symmetry breaking is intimately connected with the theory of phase transitions in CMP

❖ It is quite frequent that in going from one phase to another the symmetry of the ground state (vacuum) changes

❖ In real physical systems this is what we see with magnetic domains in magnetic material below the Curie point

❖ In going from one phase to the other, some parameters change in a noticeable way. These are the order parameters.

❖ In liquid-solid transition it is the density

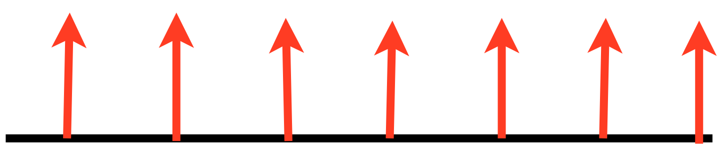
❖ In magnetic materials it is the magnetisation

❖ In the Ginsburg-Landau theory of superconductivity, the Cooper pairs acquire a VEV. This breaks U(1) inside the superconductor and thus explains among other things the Meissner effect. The Cooper pairs are pairs of electrons bound by the lattice vibrations. In ordinary superconductors their size is several hundred Angstroms.

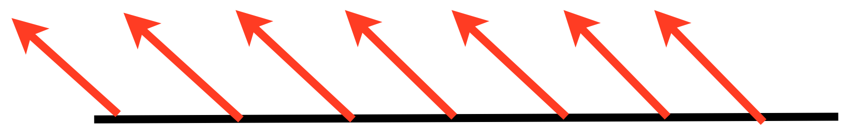
❖ The order parameters need not be elementary fields...

# Misconceptions, vacuum degeneracy

By abuse of language we often hear, or say that in theories with SSB there is vacuum degeneracy. This is fact is not the case, at least in LQFT. In understanding this we will also understand why there are massless states in theories with SSB. N is the volume in the example. The Heisenberg model of magnetism. H is rotational invariant above the critical temperature, and magnetised below it



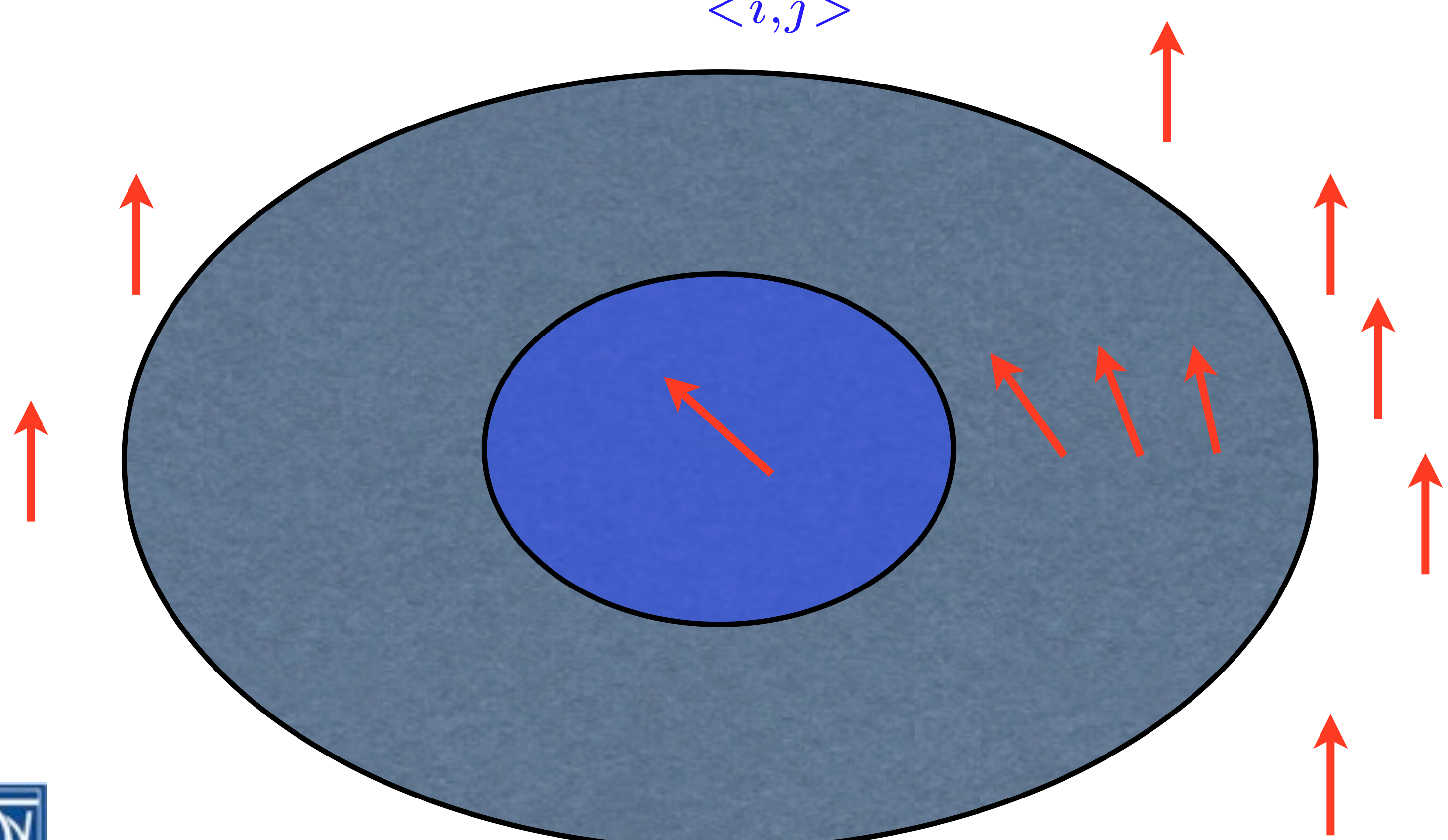
$|0\rangle$



$|\theta\rangle$

$$H = -J \sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j$$

$$\begin{aligned} \langle 0|\theta\rangle &= (\cos(\theta/2))^N \\ &\to 0 \qquad N \to \infty \end{aligned}$$



By making the transitions very slowly we can manage to make this configuration to have as small an energy as we wish. Hence we have a continuum spectrum above zero. This is the sign of a massless particle, the NG-boson

# No Goldstone bosons in finite volume

This simple example contains the ingredients of the general case. Consider a theory in a box of side L and PBCs, the plane waves solutions are easy to write down

$$\Phi = (\phi_1, \phi_2) \qquad \zeta(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)] \equiv \frac{1}{\sqrt{2}}[a + h(x)]\, e^{i\theta(x)}.$$

$$\mathscr{L} = \frac{1}{2}\partial_\mu \Phi \cdot \partial^\mu \Phi - \frac{\lambda}{4}\left(\Phi^2 - a^2\right)^2$$

$$= \partial_\mu \zeta^* \partial^\mu \zeta - \lambda\left(|\zeta|^2 - \frac{a^2}{2}\right)^2 = \frac{a^2}{2}\partial_\mu\theta\partial^\mu\theta + \cdots, \qquad \partial_\mu \partial^\mu \theta = 0 \qquad \partial_\mu \partial^\mu \phi = 0$$

$$\varphi_{\mathbf{k}}(t, \mathbf{x}) = \frac{1}{\sqrt{V}} e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n} \qquad \varphi(t, \mathbf{x}) = \varphi_0 + \pi_0 t + \sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{\sqrt{2V|\mathbf{k}|}}\left[\alpha(\mathbf{k})e^{-i|\mathbf{k}|t + i\mathbf{k}\cdot\mathbf{x}} + \alpha^\dagger(\mathbf{k})e^{i|\mathbf{k}|t - i\mathbf{k}\cdot\mathbf{x}}\right].$$

$$[\varphi(t, \mathbf{x}_1), \dot\varphi(t, \mathbf{x}_2)] = i\delta(\mathbf{x}_1 - \mathbf{x}_2) = \frac{i}{V} + \frac{i}{V}\sum_{\mathbf{k}\neq\mathbf{0}} e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}$$

$$[\varphi_0, \pi_0] = \frac{i}{V}. \qquad a = \frac{1}{\sqrt{2}}\left(\varphi_0 + i V^{\frac{1}{3}}\pi_0\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\varphi_0 - i V^{\frac{1}{3}}\pi_0\right), \qquad :H:= \frac{V}{2}\pi_0^2 + \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}).$$

$$[a, a^\dagger] = V^{-\frac{2}{3}}. \qquad Q = \int d^3x\, \partial_0\varphi = V\pi_0 = \frac{V^{\frac{2}{3}}}{i\sqrt{2}}\left(a - a^\dagger\right). \qquad e^{-i\xi Q}\varphi(x)e^{i\xi Q} = \varphi(x) + \xi,$$

$$|\xi\rangle \sim e^{i\xi Q}|0\rangle = e^{-\frac{1}{\sqrt{2}}\xi V^{\frac{2}{3}}(a^\dagger - a)}|0\rangle. \qquad \boxed{\langle 0|\xi\rangle = e^{-\frac{1}{4}\xi^2 V^{\frac{2}{3}}}\langle 0|0\rangle.}$$

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http://carlossicoli.free.fr/A/Alvarez-Gaume_L.,_Vazquez-Mozo_M.A.-An_Invitation_to_Quantum_Field_Theory_-Springer(2011).pdf

# Pions

In HEP they provide the only observed NG bosons

The order parameter is not an elementary field

To find other NG bosons in the SM we have to go to the Higgs sector, and there they are "eaten" to provide masses for the W and Z vector bosons

In QCD there are no fundamental scalars. Consider just two flavors u,d. We have chiral symmetry

$$\begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix} \longrightarrow M_{L,R} \begin{pmatrix} u_{L,R} \\ d_{L,R} \end{pmatrix}$$

$$G = \underbrace{SU(2)_L \times SU(2)_R}_{SU(2)_V} \times U(1)_B \times U(1)_A$$

$$q^f_\alpha \qquad f = u, d, \;\; \alpha = 1, 2, 3$$

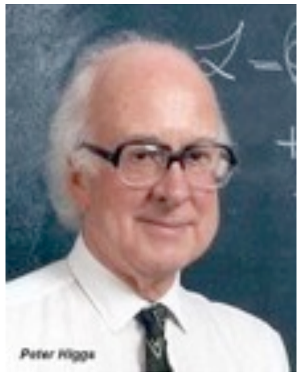
$$\langle \bar{q}^f \cdot q^{f'} \rangle \; = \; \Lambda^3_{\chi SB} \; \delta^{f f'}$$

$$\bar{q}^f \cdot q^{f'} \; \simeq \; \Lambda^3_{\chi SB} \; e^{i\pi^a \sigma^a / f_\pi}$$

These are the pions.
This is an IR property of QCD, not accessible to Pert.Th.
Low-E pion theorems, chiral Lagrangians….

# The BEH mechanism





Notice we say the mechanism, not necessary the particle! In gauge theories one cannot just add a mass for the gauge bosons. This badly destroys the gauge symmetry and the theory is inconsistent.

BEH showed that in gauge theories with SSB the NG bosons are "eaten" by the gauge bosons to become massive but preserving the basic properties of the gauge symmetry. Ex. Abelian Higgs model

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) - \frac{\lambda}{4}\left(\varphi^\dagger\varphi - \mu^2\right)^2 \qquad \varphi \longrightarrow e^{i\alpha(x)}\varphi, \qquad A_\mu \longrightarrow A_\mu + \partial_\mu\alpha(x)$$

$$\langle\varphi\rangle = \mu e^{i\vartheta_0} \longrightarrow \mu e^{i\vartheta_0 + i\alpha(x)} \qquad \varphi(x) = \left[\mu + \frac{1}{\sqrt{2}}\sigma(x)\right]e^{i\vartheta(x)}$$

Take the unitary gauge

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\mu^2 A_\mu A^\mu + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}\lambda\mu^2\sigma^2 \\ &- \lambda\mu\sigma^3 - \frac{\lambda}{4}\sigma^4 + e^2\mu A_\mu A^\mu\sigma + e^2 A_\mu A^\mu\sigma^2.\end{aligned}$$

$$m_\gamma^2 = 2e^2\mu^2$$

The simplest example is the GL and BCS theory of superconductivity, in this case the "Higgs" particle is composite, it is an object of charge made of two bound electrons that get a "VEV" (Cooper pairs) that get a VEV in the superconducting state. The photon is massive in this state. This explains among other things the Meissner effect.

# Gauge couplings: colour

There are three gauge groups in the theory, the colour group SU(3) and the electroweak group SU (2)xU(1) of weak isospin and hypercharge. Y and T3 mix to generate electric charge and the photon

$$SU(3)_c \times SU(2) \times U(1)_Y \rightarrow SU(3) \times U(1)_Q$$

QCD by itself is a perfect theory in many ways

$$\mathscr{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{f=1}^{6} \overline{Q}^f \left( i\not{D} - m_f \right) Q^f . \qquad Q^f_i \longrightarrow U(g)_{ij} Q^f_j , \qquad \text{with} \quad g \in \mathrm{SU}(3)$$

Isospin as an approximate symmetry:

$$\mathscr{L} = (\overline{u}, \overline{d}) \begin{pmatrix} i\not{D} - \frac{m_u + m_d}{2} & 0 \\ 0 & i\not{D} - \frac{m_u + m_d}{2} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} - \frac{m_u - m_d}{2} (\overline{u}, \overline{d}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

Once the electroweak sector is included the story of the masses is far more complicated (see later)

The physics that led to the SM is the combined results of many people over more than a hundred years, some of the them were awarded the Nobel Prize in Physics. They appear in the pages that follow. We can think of the beginning of the SM Odyssey with the discovery of the electron by Thomson in 1897.

Try putting names to the faces as well as the associated contributions.

This is an amusing exercise in HEP history

# Who is who in the Standard Model

If you do it more carefully (Werner Riegler), what you find is:

**87 Nobel Prices related to the Development of the Standard Model**

**31 for Standard Model Experiments**
**13 for Standard Model Instrumentation and Experiments**
**3 for Standard Model Instrumentation**
**21 for Standard Model Theory**
**9 for Quantum Mechanics Theory**
**9 for Quantum Mechanics Experiments**
**1 for Relativity**

See at the end of the lectures, the list prepared by Werner last year. Unfortunately this year we could not add Englert, Higgs,…

# Fermion quantum numbers

The fundamental fermions come in three families with the same quantum numbers with respect to the gauge group

| Leptons | | | | | |
|---|---|---|---|---|---|
| $i$ (generation) | 1 | 2 | 3 | $T^3$ | $Y$ |
| $\mathbf{L}^i$ | $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$ | $\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L$ | $\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $-\frac{1}{2}$ |
| $\ell_R^i$ | $e_R^-$ | $\mu_R^-$ | $\tau_R^-$ | 0 | $-1$ |

| Quarks | | | | | |
|---|---|---|---|---|---|
| $i$ (generation) | 1 | 2 | 3 | $T^3$ | $Y$ |
| $\mathbf{Q}^i$ | $\begin{pmatrix} u \\ d \end{pmatrix}_L$ | $\begin{pmatrix} c \\ s \end{pmatrix}_L$ | $\begin{pmatrix} t \\ b \end{pmatrix}_L$ | $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ | $\frac{1}{6}$ |
| $U_R^i$ | $u_R$ | $c_R$ | $t_R$ | 0 | $\frac{2}{3}$ |
| $D_R^i$ | $d_R$ | $s_R$ | $b_R$ | 0 | $-\frac{1}{3}$ |

In principle one could add sterile neutrinos, right handed neutrinos who are singlets under the gauge group. They would generate Dirac masses for the known neutrinos

# Gauge couplings: EW sector

## The EW group has four generators

$$\mathbf{W}_\mu = W_\mu^+ T^- + W_\mu^- T^+ + W_\mu^3 T^3, \qquad \mathbf{B}_\mu = B_\mu Y.$$

$$Q = T^3 + Y.$$

$$[Q, T^\pm] = \pm T^\pm, \qquad [Q, T^3] = [Q, Y] = 0.$$

$$\begin{aligned} A_\mu &= B_\mu \cos\theta_w + W_\mu^3 \sin\theta_w, \\ Z_\mu &= -B_\mu \sin\theta_w + W_\mu^3 \cos\theta_w \end{aligned}$$

$$\begin{aligned} D_\mu &= \partial_\mu - ig\mathbf{W}_\mu - ig'\mathbf{B}_\mu \\ &= \partial_\mu - igW_\mu^+ T_R^- - igW_\mu^- T_R^+ - igW_\mu^3 T_R^3 - ig' B_\mu Y_R, \end{aligned}$$

$$e = g\sin\theta_w = g'\cos\theta_w.$$

$$\begin{aligned} D_\mu = {} & \partial_\mu - igW_\mu^+ T_R^- - igW_\mu^- T_R^+ - iA_\mu (g\sin\theta_w T_R^3 + g'\cos\theta_w Y_R) \\ & - iZ_\mu (gT_R^3 \cos\theta_w - g' Y_R \sin\theta_w). \end{aligned}$$

$$\begin{aligned} \mathscr{L}_{\text{gauge}} = {} & -\frac{1}{2} W_{\mu\nu}^+ W^{-\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{ig}{2} \cos\theta_w W_\mu^+ W_\nu^- Z^{\mu\nu} \\ & + \frac{ie}{2} W_\mu^+ W_\nu^- F^{\mu\nu} - \frac{g^2}{2} \left[ (W_\mu^+ W^{+\mu})(W_\mu^- W^{-\mu}) - (W_\mu^+ W^{-\mu})^2 \right] \end{aligned}$$

$$\begin{aligned} \mathscr{L}_{\text{matter}} = \sum_{i=1}^{3} \Big( & i\overline{\mathbf{L}}^j \not{D} \mathbf{L}^j + i\overline{\ell}_R^j \not{D} \ell_R^j \\ & + i\overline{\mathbf{Q}}^j \not{D} \mathbf{Q}^j + i\overline{U}_R^j \not{D} U_R^j + i\overline{D}_R^j \not{D} D_R^j \Big) \end{aligned}$$

$$W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm \mp ie\left( W_\mu^\pm A_\nu - W_\nu^\pm A_\mu \right) \mp ig\cos\theta_w \left( W_\mu^\pm Z_\nu - W_\nu^\pm Z_\mu \right)$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$\frac{g}{2\cos\theta_w} Z_\mu \overline{\nu}_\ell \gamma^\mu \nu_\ell, \qquad \frac{g}{\cos\theta_w} \left( -\frac{1}{2} + \sin^2\theta_w \right) Z_\mu \overline{\ell}_L \gamma^\mu \ell_L.$$

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The Higgs couplings responsible for the masses of the leptons and the current algebra masses of the quarks are:

$$\mathbf{H} = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \qquad \tilde{\mathbf{H}} \equiv i\sigma^2 \mathbf{H}^* = \begin{pmatrix} H^{0*} \\ H^{+*} \end{pmatrix} \qquad \mathscr{L}_{\text{Higgs}} = (D_\mu \mathbf{H})^\dagger D^\mu \mathbf{H} - V(\mathbf{H}, \mathbf{H}^\dagger), \qquad V(\mathbf{H}, \mathbf{H}^\dagger) = \frac{\lambda}{4}\left(\mathbf{H}^\dagger \mathbf{H} - \mu^2\right)^2$$

$$Y(\mathbf{H}) = \tfrac{1}{2}$$

$$\mathscr{L}^{(\ell)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(\ell)}_{ij}\,\overline{\mathbf{L}}^i\,\mathbf{H}\,\ell^j_R + C^{(\ell)*}_{ji}\,\overline{\ell}^i_R \mathbf{H}^\dagger\,\mathbf{L}^j\right)$$

$$\mathscr{L}^{(q)}_{\text{Yukawa}} = -\sum_{i,j=1}^{3}\left(C^{(q)}_{ij}\,\overline{\mathbf{Q}}^i\,\mathbf{H}\,D^j_R + C^{(q)*}_{ji}\overline{D}^i_R\,\mathbf{H}^\dagger\,\mathbf{Q}^j\right) - \sum_{i,j=1}^{3}\left(\tilde{C}^{(q)}_{ij}\,\overline{\mathbf{Q}}^i\,\tilde{\mathbf{H}}\,U^j_R + \tilde{C}^{(q)*}_{ji}\,\overline{U}^i_R\,\tilde{\mathbf{H}}^\dagger\,\mathbf{Q}^j\right).$$

The most general Lagrangian compatible with the gauge symmetry and up to dimension 4, so that the theory is renormalisable. One H gets its VEV the masses are generated from the Yukawa couplings. Use unitary gauge. The gauge fields get masses from the kinetic term

$$\mathbf{H}(x) = e^{i\mathbf{a}(x)\cdot\frac{\sigma}{2}}\begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}}h(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \mu + \frac{1}{\sqrt{2}}h(x) \end{pmatrix}$$

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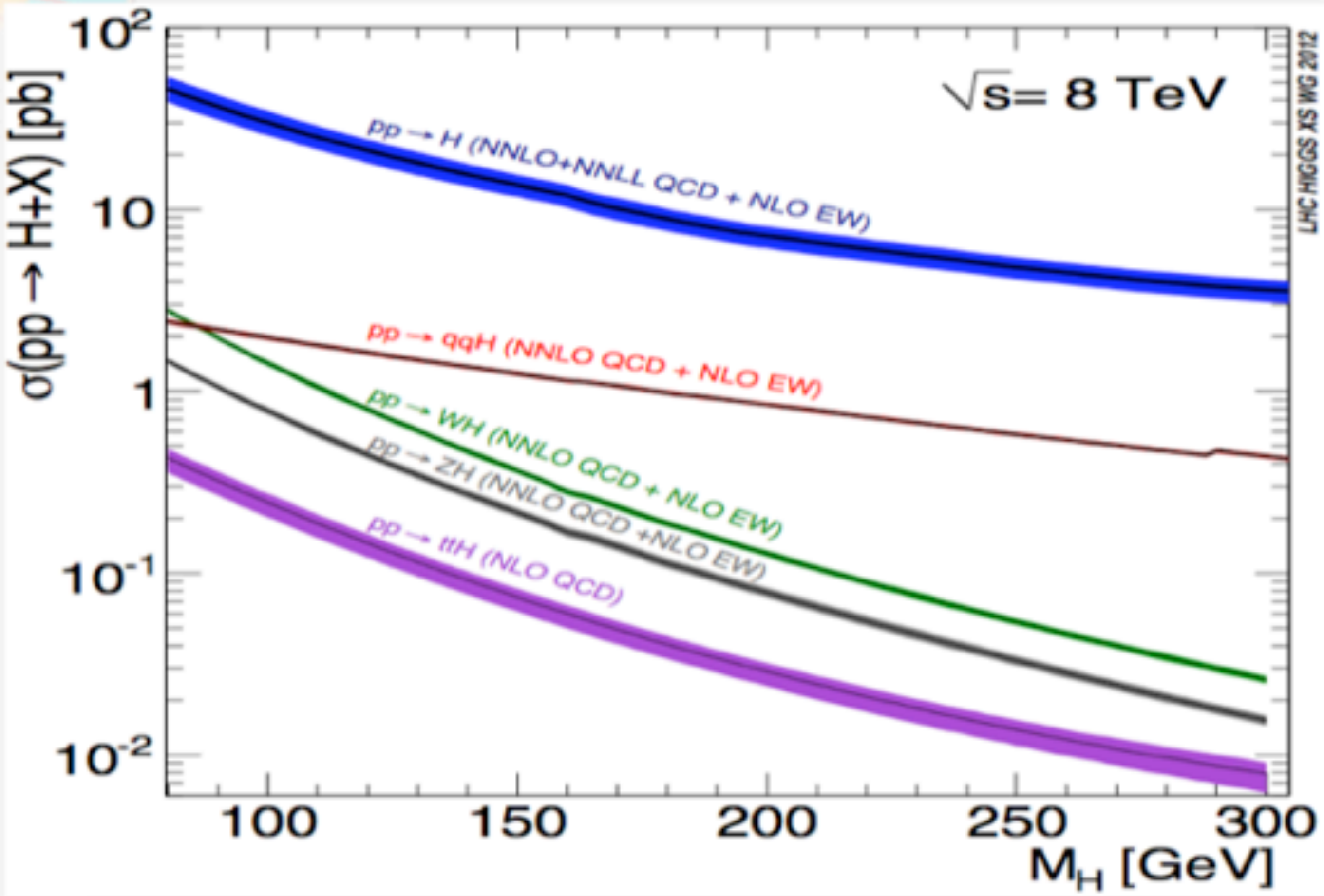
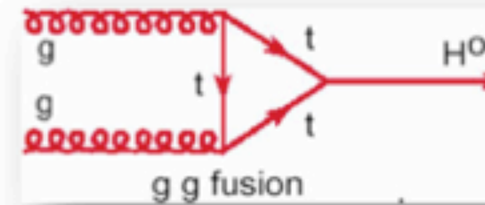
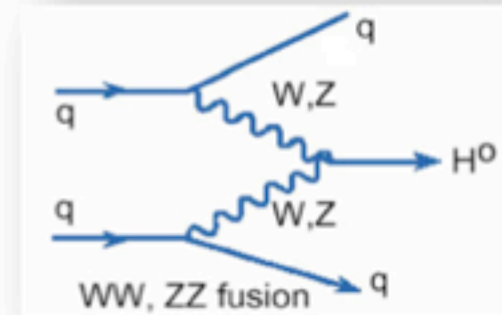
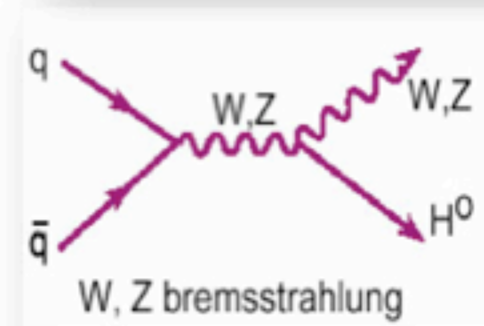
# Higgs boson production

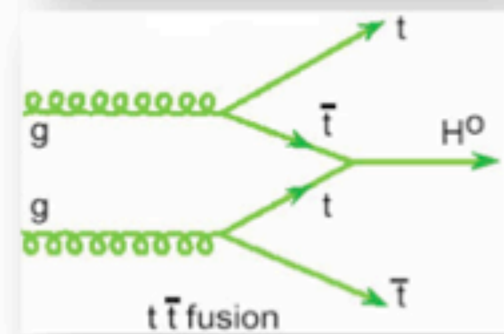
July 4th 2012 The Status of the Higgs Search J. Incandela for the CMS COLLABORATION



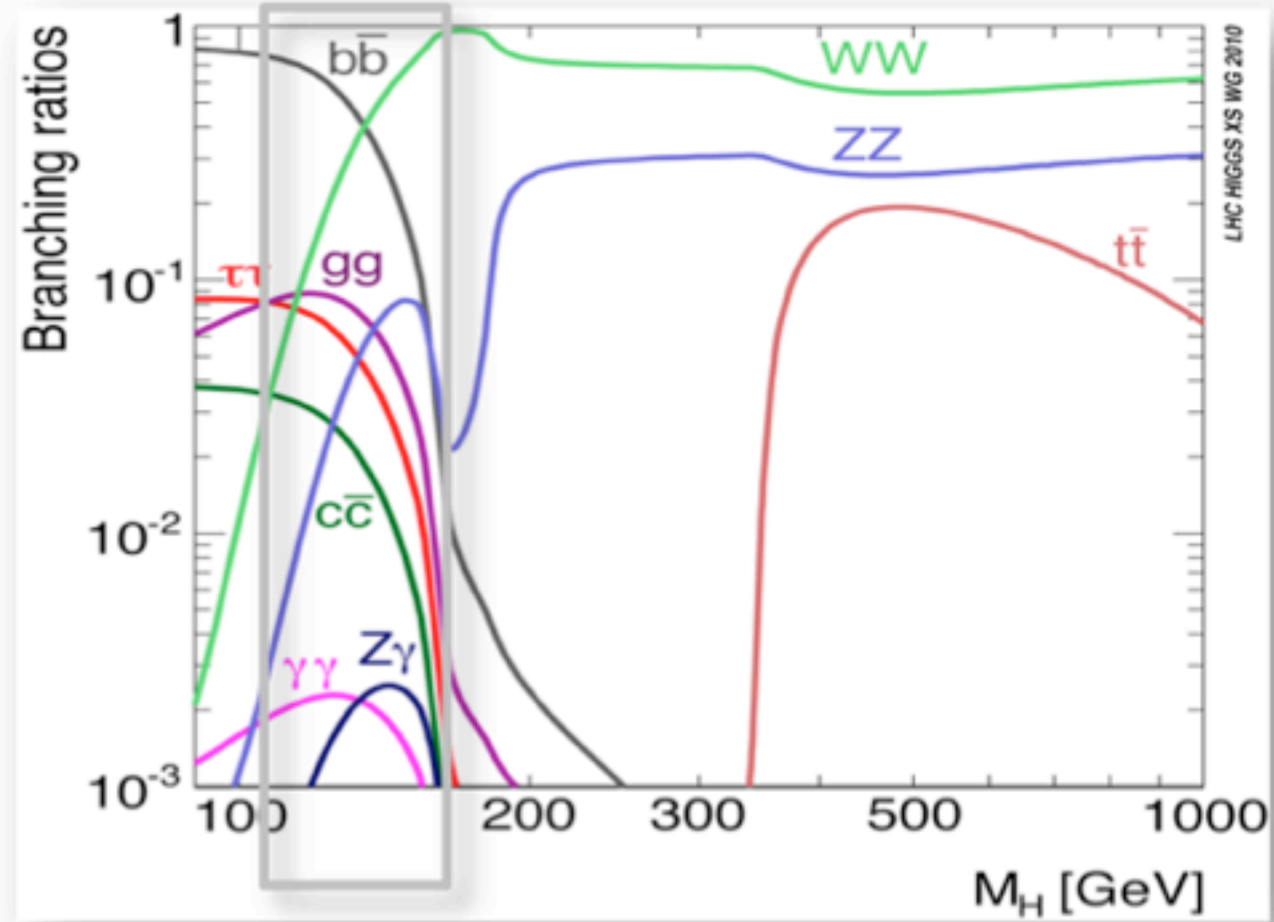

- $\sqrt{s}$=8 TeV: 25-30% higher σ than $\sqrt{s}$=7 TeV at low $m_H$
- All production modes to be exploited
  - gg VBF VH ttH
  - Latter 3 have smaller cross sections but better S/B in many cases

Courtesy J. Incancella

# Higgs boson decays

5 decay modes exploited

- High mass: WW, ZZ
- Low mass: $b\bar{b}$, $\tau\tau$, WW, ZZ, $\gamma\gamma$
- Low mass region is very rich but also very challenging:
  main decay modes ($b\bar{b}$, $\tau\tau$) are hard to identify in the huge background
- Very good mass resolution (1%): $H \rightarrow \gamma\gamma$ and $H \rightarrow ZZ \rightarrow 4l$




Courtesy J. Incancella

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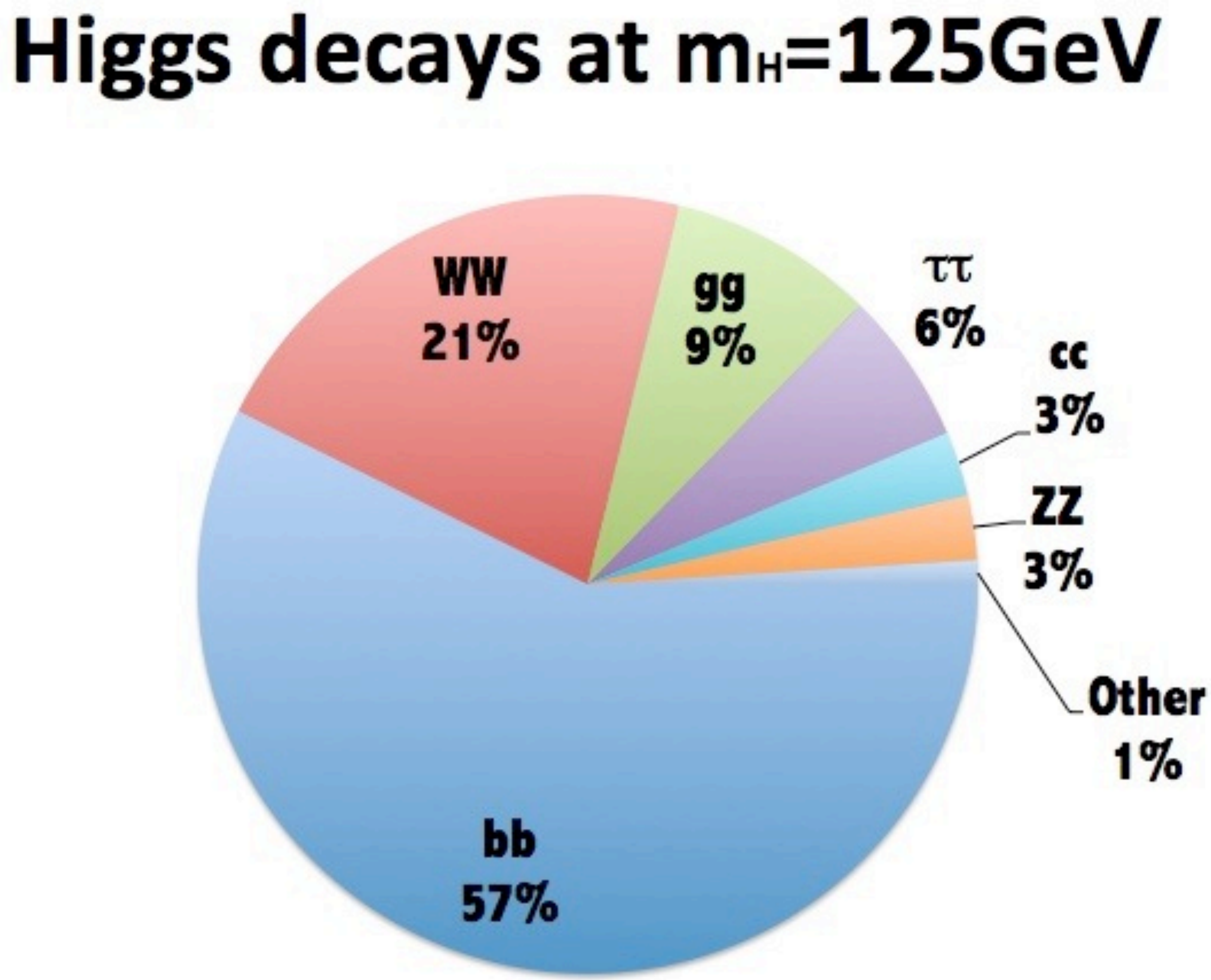

Higgs decays at mH=125GeV
WW
21%
gg
9%
ττ
6%
cc
3%
ZZ
3%
Other
1%
bb
57%

# CKM matrix

$$\mathscr{L}^{(\ell)}_{\rm mass} = -(\overline{e}_L, \overline{\mu}_L, \overline{\tau}_L)\, M^{(\ell)} \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} + \text{h.c.} \qquad \mathscr{L}^{(q)}_{\rm mass} = -(\overline{d}_L, \overline{s}_L, \overline{b}_L)\, M^{(q)} \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} - (\overline{u}_L, \overline{c}_L, \overline{t}_L)\, \widetilde{M}^{(q)} \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix} + \text{h.c.}$$

$$M^{(\ell,q)}_{ij} = \mu C^{(\ell,q)}_{ij}, \qquad \widetilde{M}^{(q)}_{ij} = \mu \widetilde{C}^{(q)}_{ij}.$$

$$V_L^{(\ell)\dagger} M^{(\ell)} V_R^{(\ell)} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \qquad V_L^{(q)\dagger} M^{(q)} V_R^{(q)} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \qquad \widetilde{V}_L^{(q)\dagger} \widetilde{M}^{(q)} \widetilde{V}_R^{(q)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

In the quark sector going from to mass eigenstates leaves a matrix of phases in the charged currents, the CKM matrix. Not for neutral currents GIM

$$j^\mu_+ = (\overline{u}_L, \overline{c}_L, \overline{t}_L)\gamma^\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = (\overline{u}'_L, \overline{c}'_L, \overline{t}'_L)\gamma^\mu \widetilde{V}_L^{(q)\dagger} V_L^{(q)} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} \qquad V \equiv \widetilde{V}_L^{(q)\dagger} V_L^{(q)}$$

# Discrete symmetries

❖ In the classical world, we have invariance uncer P,C,T. All we had was E&M and gravity.

❖ In QFT they are not guaranteed in fact P,C,T, CP are broken symmetries. The only one that survives so far is CPT. It has several important consequences. CP violation is fundamental in the generation of matter. In the SM we need at least three families

❖ The existence of antiparticles with the same mass and decay rate

❖ The connection between spin and statistics

❖T-reversal and CPT are the only ones implemented by anti-unitary operators

$$\begin{tikzcd}
\mathbf{q}_0, \mathbf{p}_0 \arrow[r, "\mathrm{T}"] \arrow[d, "t"] & \mathbf{q}_0, -\mathbf{p}_0 \\
\mathbf{q}(t), \mathbf{p}(t) \arrow[r, "\mathrm{T}"] & \mathbf{q}(t), -\mathbf{p}(t) \arrow[u, "t"']
\end{tikzcd}$$

# Anomalous Symmetries

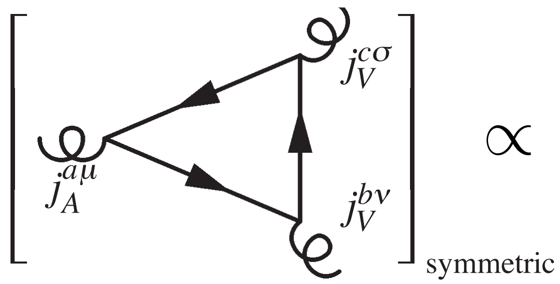
Sometimes symmetries of the classical Lagrangian do not survive quantisation. There are three examples we can cite:

❖Global chiral symmetries, responsible for the electromagnetic decay of the neutral pion

❖Gauged chiral symmetries. This happens when left and right multiplets have different representations of the gauge group. At the one-loop level we find a non-trivial conditions among the quantum numbers necessary to maintain gauge invariance. It suffices to satisfy this condition at the one-loop level

❖Scale invariance. The behaviour of the theory under scale transformation. Rather how physics depends on scales is far more interesting that just dimensional analysis.

$$\langle 0|T\left[j_A^{a\mu}(x)\,j_V^{b\nu}(x')\,j_V^{c\sigma}(0)\right]|0\rangle =$$




$$\propto \pm \ \mathrm{tr}\left[\tau^a_{i,\pm}\{\tau^b_{i,\pm},\tau^c_{i,\pm}\}\right]$$

$$\sum_{i=1}^{N_+}\mathrm{tr}\left[\tau^a_{i,+}\{\tau^b_{i,+},\tau^c_{i,+}\}\right]-\sum_{j=1}^{N_-}\mathrm{tr}\left[\tau^a_{j,-}\{\tau^b_{j,-},\tau^c_{j,-}\}\right]=0.$$

Anomaly cancellation condition, it has highly non-trivial implications for the family structure

# Anomalous Symmetries

quarks: $\begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_{L,\frac{1}{6}}$ $\quad u^\alpha_{R,\frac{2}{3}}$ $\quad d^\alpha_{R,\frac{2}{3}}$

leptons: $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L,-\frac{1}{2}}$ $\quad e_{R,-1}$

$(3,2)^L_{\frac{1}{6}}$ $\quad (1,2)^L_{-\frac{1}{2}}$

$(3,1)^R_{\frac{2}{3}}$ $\quad (3,1)^R_{-\frac{1}{3}}$ $\quad (1,1)^R_{-1}$

Anomalies cancel generation by generation. In fact the hypercharge assignments is completely determined if we also impose the traceless-ness of any U(1)

$$\sum_{\text{left}} Y_+^3 - \sum_{\text{right}} Y_-^3 = 3\times 2\times\left(\frac{1}{6}\right)^3 + 2\times\left(-\frac{1}{2}\right)^3 - 3\times\left(\frac{2}{3}\right)^3 - 3\times\left(-\frac{1}{3}\right)^3 - (-1)^3 = \left(-\frac{3}{4}\right) + \left(\frac{3}{4}\right) = 0.$$

| | | |
|---|---|---|
| $SU(3)^3$ | $SU(2)^3$ | $U(1)^3$ |
| $SU(3)^2\, SU(2)$ | $SU(2)\, U(1)$ | |
| $SU(3)^2\, U(1)$ | $SU(2)\, U(1)^2$ | |
| $SU(3)\, SU(2)^2$ | | |
| $SU(3)\, SU(2)\, U(1)$ | | |
| $SU(3)\, U(1)^2$ | | |

# Deriving quantum numbers

$$SU(N)_c \times SU(2) \times U(1) \qquad \text{S.M. anomaly}$$

$$\begin{array}{ccc} (N,2)^L_{q_L} & \oplus & (1,2)^L_{l_L} \\ (N,1)^R_{u_R} & \oplus & (N,1)^R_{d_R} \oplus (1,1)^R_{e_R} \end{array}$$

Single family

Anomaly conditions, we will normalize e_R=-1 as in the SM

$$U(1)SU(2)^2 \qquad 2N\, q_L + 2\, l_L = 0$$

$$U(1)SU(N)^2 \qquad 2\, q_L - (u_R + d_R) = 0$$

$$U(1)^3 \qquad 2\, N\, q_L^3 + 2\, l_L^3 - N\, u_R^3 - N\, d_R^3 - e_R^3 \;=\; 0$$

$$U(1) \qquad 2\; q_L + 2\, l_L - N(u_R + d_R) - e_R \;= 0$$

A simple computation now yields a (nearly) unique solution:

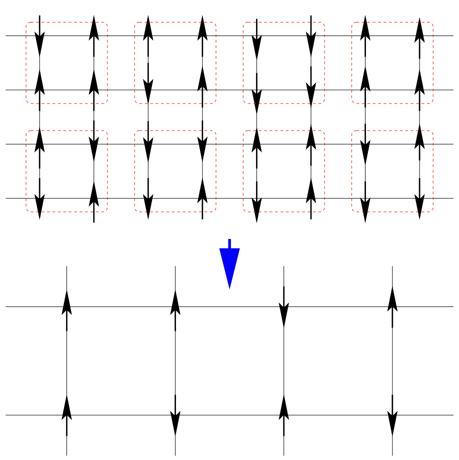
$$q_L \;=\; \frac{1}{2\,N} \qquad l_L \;=\; -\frac{1}{2} \qquad e_R \;=\; -1$$

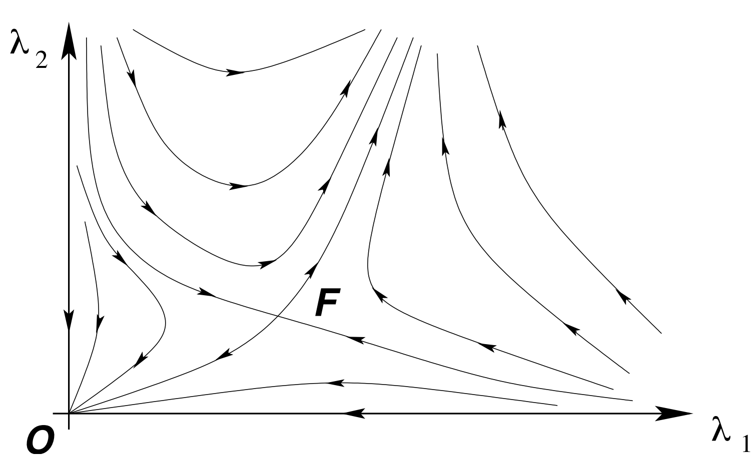
$$\begin{aligned} u_R + d_R &= 1/N \\ u_R\, d_R &= -\frac{1}{4}\left(1 - \frac{1}{N^2}\right) \end{aligned} \qquad \begin{aligned} u_R &= \frac{N+1}{2N} \\ d_R &= -\frac{N-1}{2N} \end{aligned}$$

For N=3 we obtain the hypercharges of the SM!!

# Scale invariance, renormalisation

$\times \lambda^{-1}$






Renormalisation deals with the scale dependence of the the physics even if the original theory is scale invariant.

Virtual phenomena can get more complicated or simplify as we move to larger and shorter distances

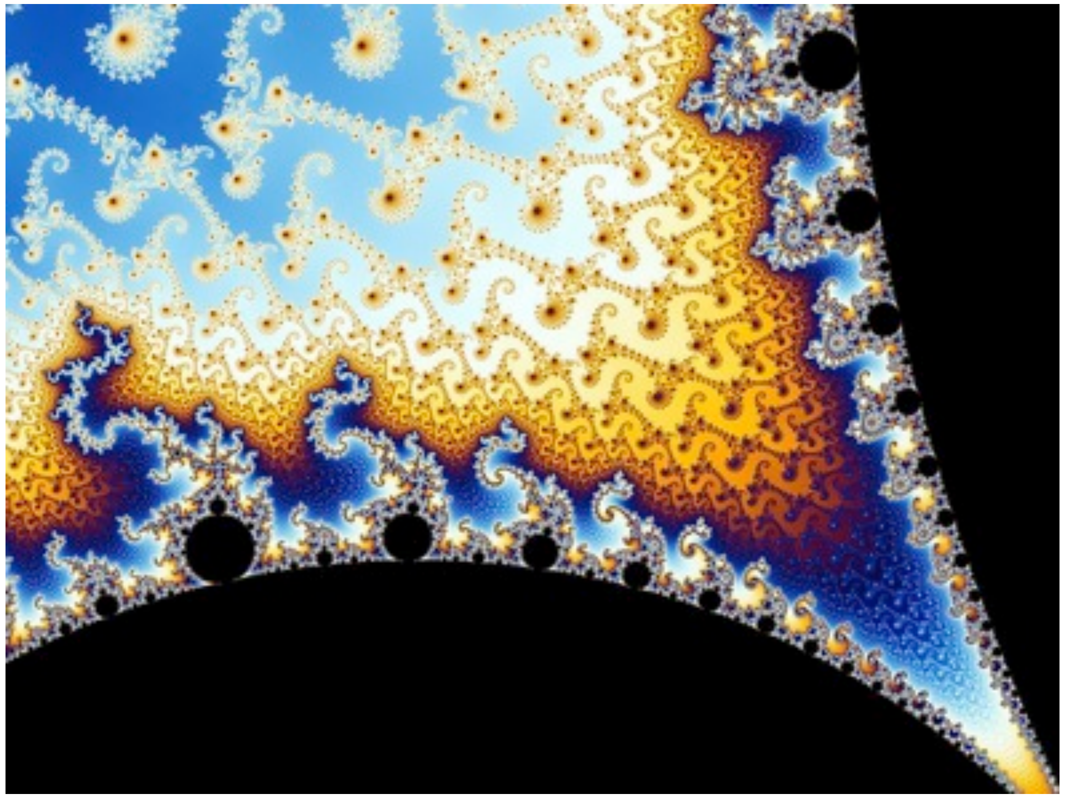
$$x^\mu \longrightarrow \lambda x^\mu, \qquad \phi(x) \longrightarrow \lambda^{-\Delta}\phi(\lambda^{-1}x),$$

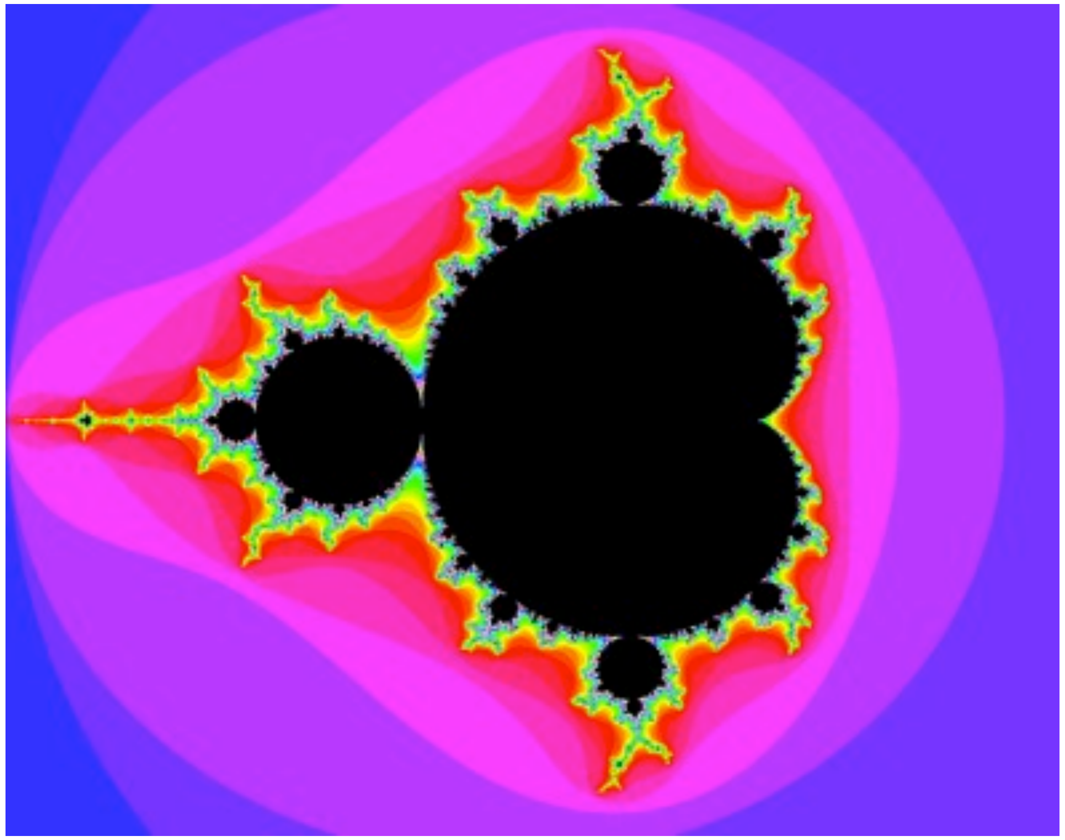
$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{g}{4!}\phi^4, \qquad \mathscr{L} \longrightarrow \lambda^{-4}\mathscr{L}[\phi]$$

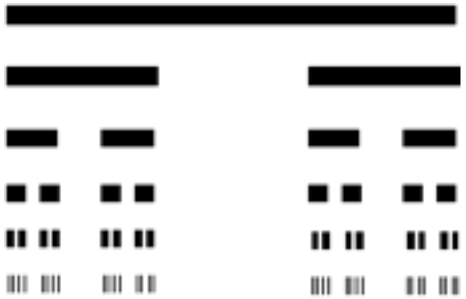
$$H \xrightarrow{\mathscr{R}} H^{(1)} \xrightarrow{\mathscr{R}} H^{(2)} \xrightarrow{\mathscr{R}} \ldots \xrightarrow{\mathscr{R}} H_\star.$$

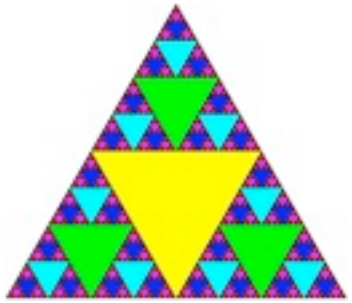
In relativistic QFT we seem to get only fixed points, no limit cycles nor strange attractors

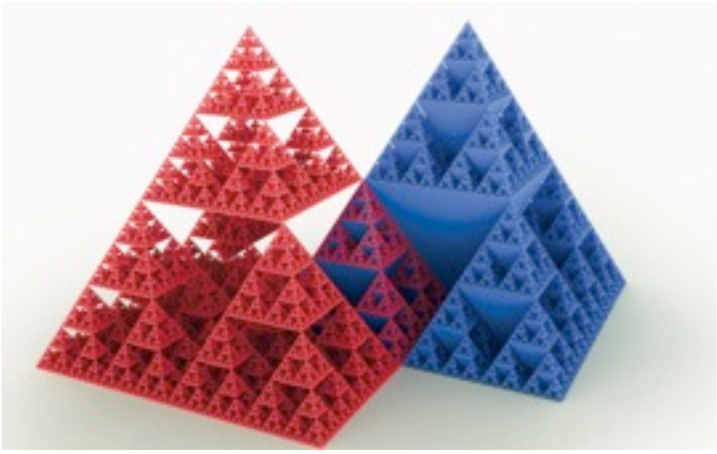
# Scale invariance, fractals





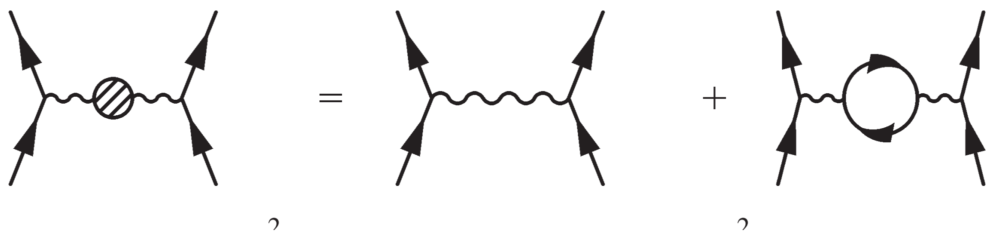






Why not in QFT? It would be rather remarkable if in a field theory we found strange attractors at high or low energies. Lorentz or Poincaré invariance play an important role in determining the possible limit structures. Only fixed points?

# Fixed points beta functions



$$\eta_{\alpha\beta}\left(\bar{v}_e\gamma^\alpha u_e\right)\left\{\frac{e^2}{4\pi q^2}\left[1+\frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\Lambda^2}\right)\right]\right\}\left(\bar{v}_\mu\gamma^\beta u_\mu\right)$$

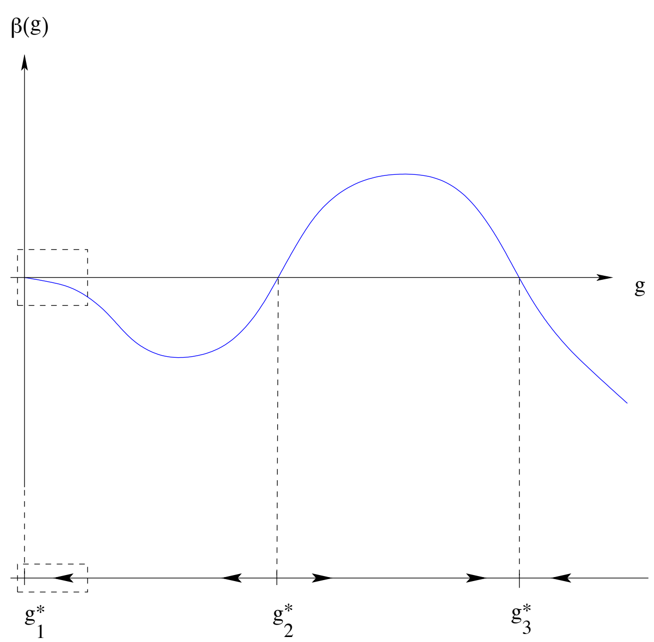
$$e(\mu)^2 = e(\Lambda)^2_{\text{bare}}\left[1+\frac{e(\Lambda)^2_{\text{bare}}}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right]$$

$$\beta(g) = \mu\frac{dg}{d\mu}.$$

$$\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2}.$$

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right).$$

At one loop




$$\beta(g^*) = 0.$$

IR free (QED)

$$\beta'(g)|_{g^*} > 0 \quad , \qquad \mu\frac{dg}{d\mu} = \beta'(g-g^*)+\dots$$

$$\mu\uparrow \quad , \qquad g\uparrow$$

UV free (QCD)

$$\beta'(g)|_{g^*} < 0 \quad , \qquad \mu\frac{dg}{d\mu} = \beta'(g-g^*)+\dots$$

$$\mu\uparrow \quad , \qquad g\downarrow$$

There is a dynamically generated scale responsible for most of the mass of the nucleons

$$\langle \mathbf{p}^2\rangle = \Lambda^2_{\text{QCD}}. \qquad \Lambda_{\text{QCD}} \gg m_u, m_d$$

# Farewell

- QFT is a vast and complex subject
- SM is a big achievement
- It summarises our knowledge of the fundamental laws of Nature
- But also our ignorance
- Many puzzles and unanswered questions remain
- We may be at the end of a cycle. Perhaps the symmetry paradigm has been exhausted.
- Naturalness, a red herring? Higgs or not Higgs
- Gravity into the picture finally?
- Hopefully we are entering a golden decade

Thank you