

# Time-like splitting functions at NNLO in QCD

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in collaboration with **A. Mitov** and **A. Vogt** on [hep-ph/0604053](#)

and with **A. Vogt** on [arXiv:0709.3899v1](#)

- ECT\* Trento workshop "*Parton fragmentation processes: in the vacuum and in the medium*", Trento, Feb 26 2008 –

# Plan

- The ECT\* Trento workshop  
*"Parton fragmentation processes: in the vacuum and in the medium"*  
coordinate system Arleo, d'Enterria '08

	vacuum	medium
theory		
experiment		

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# Introduction

## Basic concepts of perturbative QCD

- QCD theory predictions at high energies rely on few basic concepts
  - infrared safety
  - factorization
  - evolution

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## Basic concepts of perturbative QCD

- QCD theory predictions at high energies rely on few basic concepts
  - infrared safety
  - factorization
  - evolution

### Infrared safety

- Small class of cross sections at high energies directly calculable in perturbation theory
- Infrared safe quantities
  - free of long range dependencies at leading power in large momentum scale  $Q$  Kinoshita '62; Lee, Nauenberg '64
- General structure of cross section
  - large momentum scale  $Q$ , renormalization scale  $\mu$

$$Q^2 \sigma(Q^2, \mu^2, \alpha_s(\mu^2)) = \sum_n \alpha_s^n c_n(Q^2/\mu^2)$$

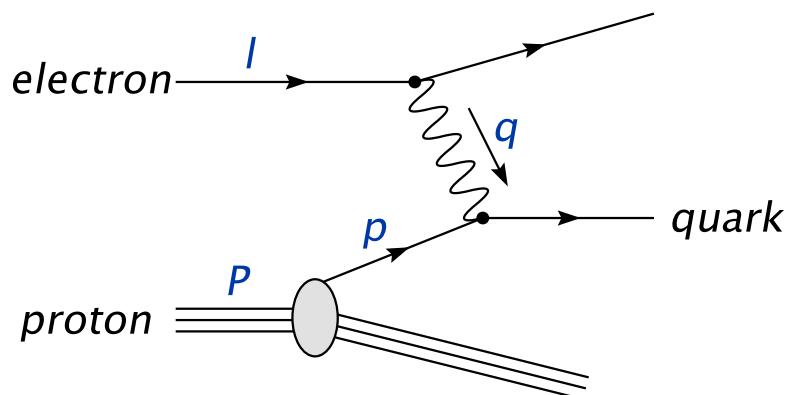
# Factorization

- Large class of hard-scattering reactions (e.g. initial state hadrons)
  - sensitive to dynamics from different scales (e.g. soft and collinear)
- Structure of factorized cross section
  - large momentum scale  $Q$ , factorization scale  $\mu$ 
$$Q^2 \sigma_{\text{phys}}(Q) = \hat{\sigma}_{\text{pt}}(Q/\mu, \alpha_s(\mu)) \otimes \phi(\mu)$$
  - convolution  $\otimes$  in suitable kinematical variables
  - generalization of operator product expansion

# Evolution

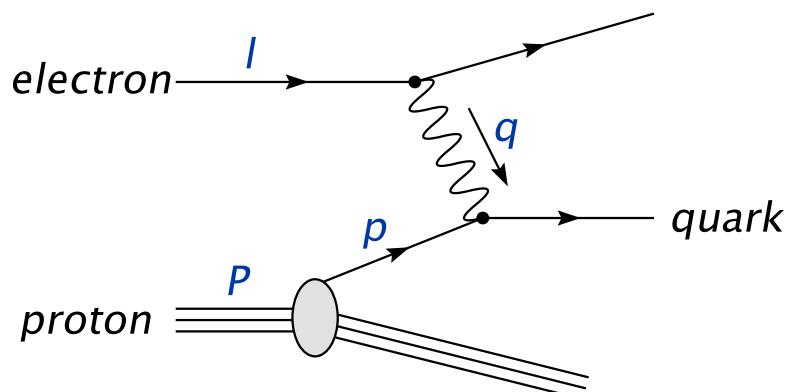
- Dependence of cross sections for observable on momentum transfer
- Physical cross section in factorization ansatz cannot depend on  $\mu$ 
$$\mu \frac{d\sigma_{\text{phys}}}{d\mu} = 0 \quad (\text{factorization scale } \mu \text{ arbitrary})$$
- Classic example: QCD corrections to deep-inelastic scattering
  - scaling violations Gross, Wilczek '73; Politzer '73
  - evolution of parton densities Altarelli, Parisi '77

# Deep-inelastic scattering



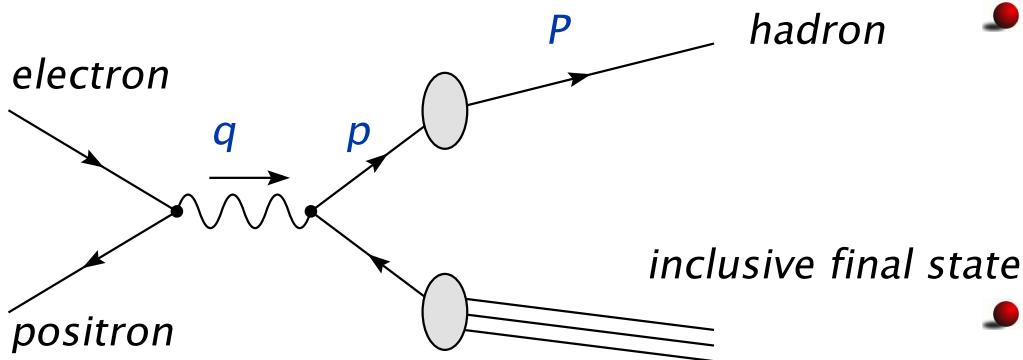
- Kinematic variables
  - momentum transfer  $Q^2 = -q^2$  (space-like)
  - Bjorken variable  $x = Q^2/(2p \cdot q)$
- Parton distributions  $PDF$ 
  - scale evolution governed by splitting functions  $P_{ij}$

# Deep-inelastic scattering



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# $e^+e^-$ annihilation



- Kinematic variables
  - momentum transfer  $Q^2 = +q^2$  (time-like)
  - scaling variable  $x = (2p \cdot q)/Q^2$
- Fragmentation functions  $D$ 
  - scale evolution governed by (time-like) splitting functions  $P_{ij}$

# Predictions in perturbative QCD

## DIS (space-like)

- LO and NLO splitting functions

$$P_{\text{ns}}^{(0)}(x) = \mathcal{C}_F (2p_{\text{qq}}(x) + 3\delta(1-x))$$

$$P_{\text{ps}}^{(0)}(x) = 0$$

$$P_{\text{qg}}^{(0)}(x) = 2n_f p_{\text{qg}}(x)$$

$$P_{\text{gg}}^{(0)}(x) = 2\mathcal{C}_F p_{\text{gg}}(x)$$

$$P_{\text{gg}}^{(0)}(x) = \mathcal{C}_A \left( 4p_{\text{gg}}(x) + \frac{11}{3}\delta(1-x) \right) - \frac{2}{3}n_f \delta(1-x)$$

- NNLO splitting functions

S.M., Vermaseren, Vogt '04

- 3 pages for nonsinglet
- 8 pages for singlet

- Coefficient functions for  $F_2$  and  $F_L$  at three loops
  - $\mathcal{O}(100)$  pages

S.M., Vermaseren, Vogt '05

# Predictions in perturbative QCD

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$$P_{\text{ns}}^{(0)}(x) = \textcolor{blue}{C}_F (2p_{\text{qq}}(x) + 3\delta(1-x))$$

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## $e^+e^-$ (time-like)

- LO and NLO splitting functions  
Curci, Furmanski, Petronzio '80;  
Floratos, Kounnas, Lacaze '81

- NNLO splitting functions

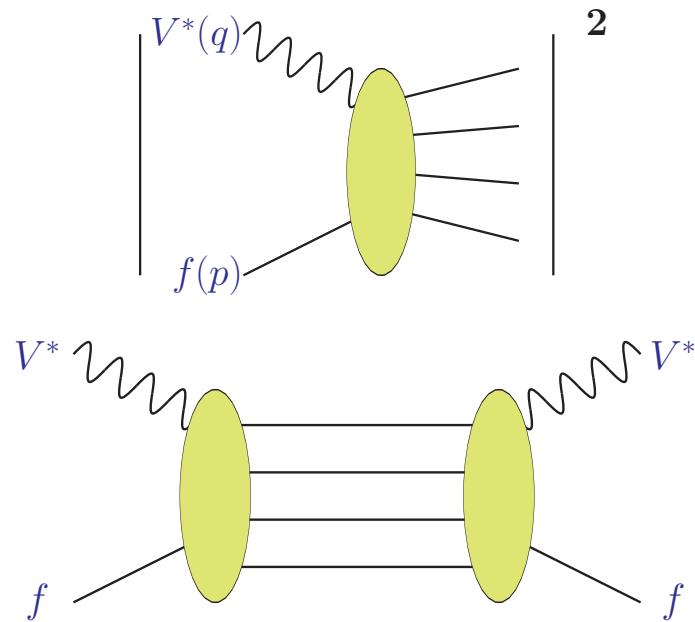
- non-singlet time-like  
Mitov, S.M., Vogt '06
- singlet (diagonal)  
S.M., Vogt '07

- Coefficient functions for  
 $e^+e^- \rightarrow V \rightarrow h + X$   
only to two loops

Rijken, van Neerven '97; Mitov, S.M. '06

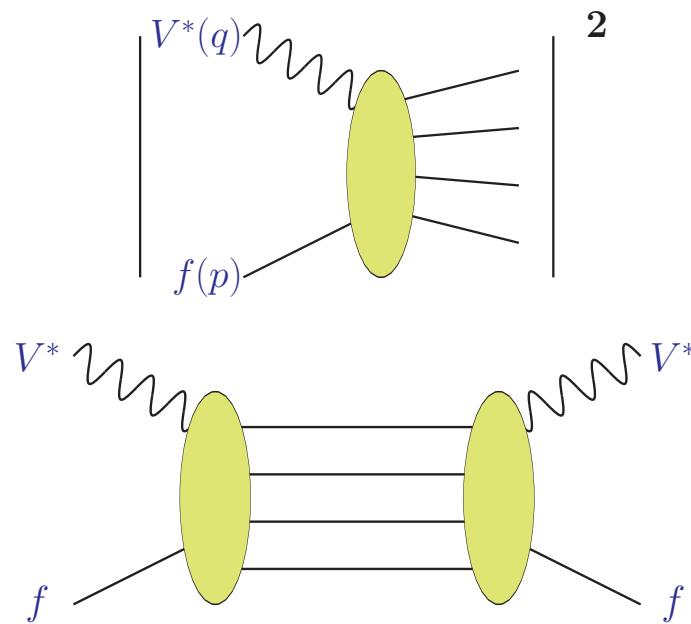
# Our calculation in deep-inelastic scattering

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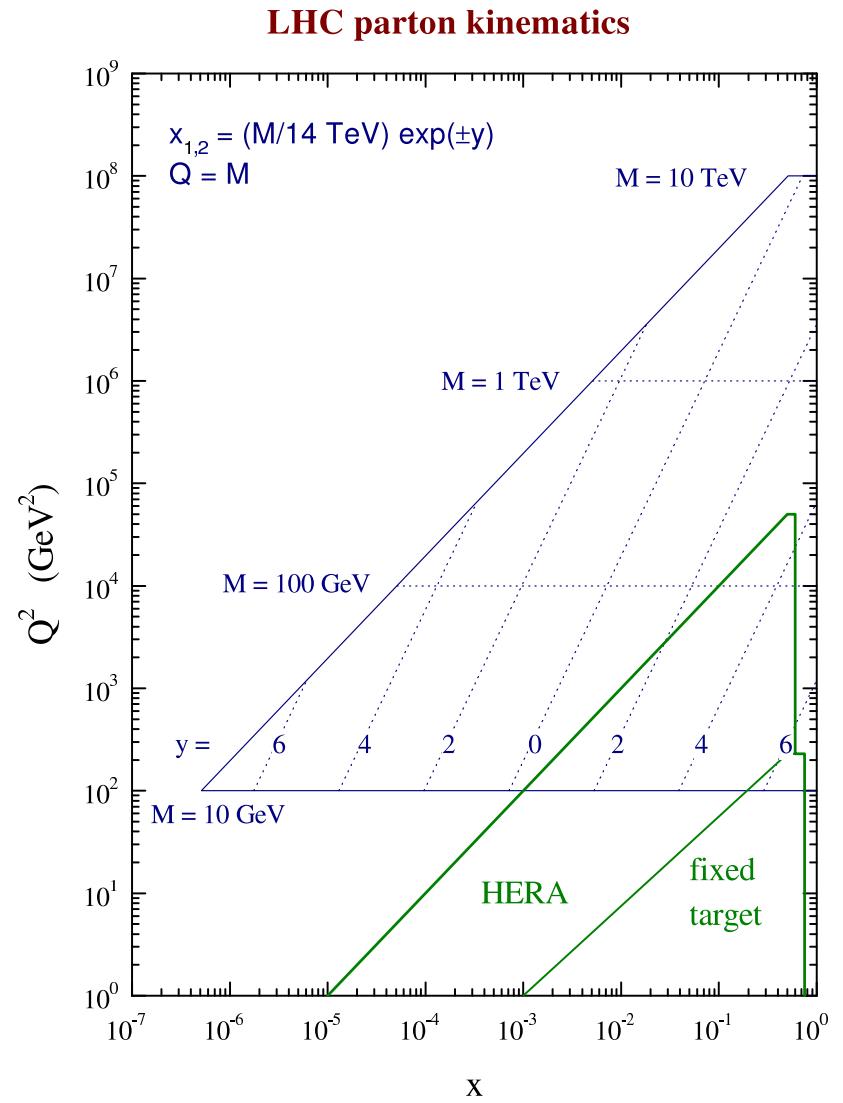
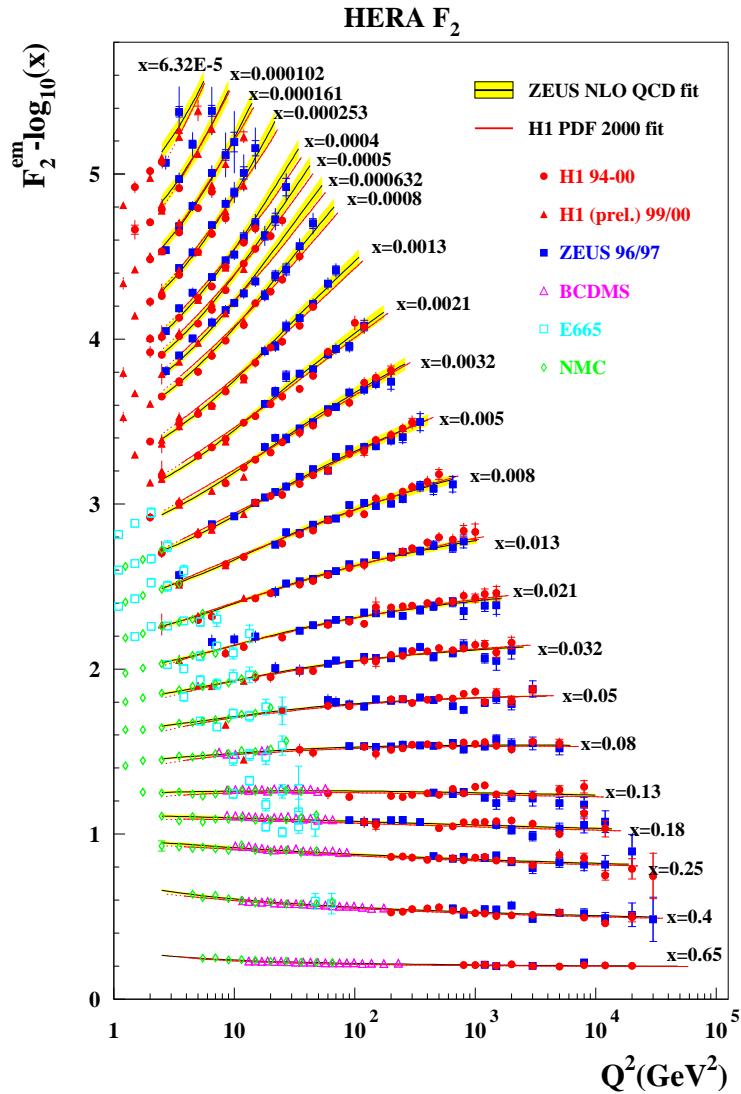
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	tree	1-loop	2-loop	3-loop
$q\gamma$	1	3	25	<b>359</b>
$g\gamma$		2	17	<b>345</b>
$h\gamma$			2	<b>56</b>
$qW$	1	3	32	<b>589</b>
$q\phi$		1	23	<b>696</b>
$g\phi$	1	8	218	<b>6378</b>
$h\phi$		1	33	<b>1184</b>
sum	3	18	350	<b>9607</b>

- more than 10 FTE years and a few CPU years
  - computer algebra updates:  $\rightarrow$  [3.1](#)  $\rightarrow$  [3.2](#)  $\rightarrow \dots$
  - $> 10^5$  tabulated symbolic integrals ( $> 3\text{GB}$ )

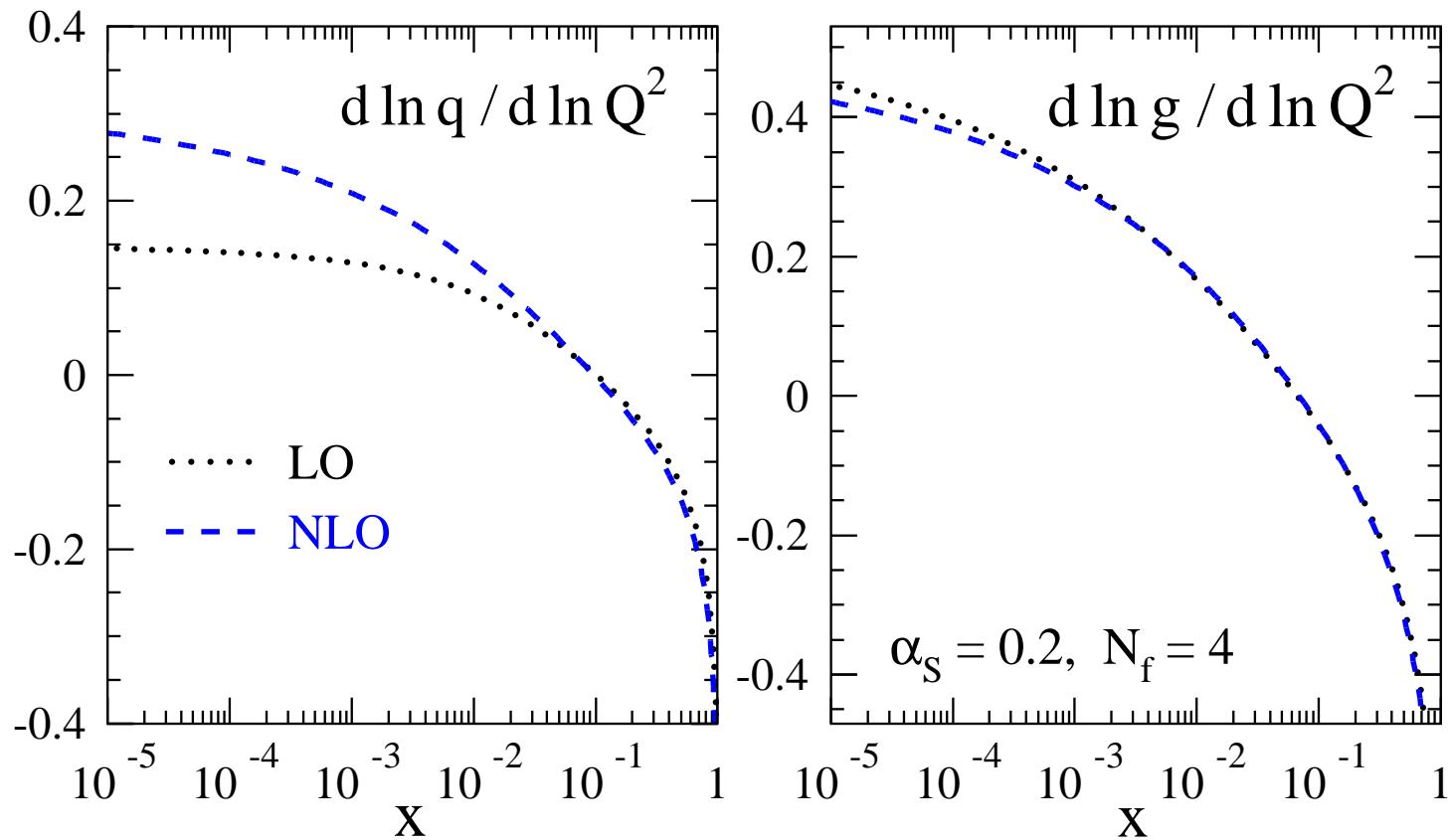
# PDFs from HERA to LHC



- **HERA → LHC:** scale evolution in  $Q^2$  over three orders of magnitude

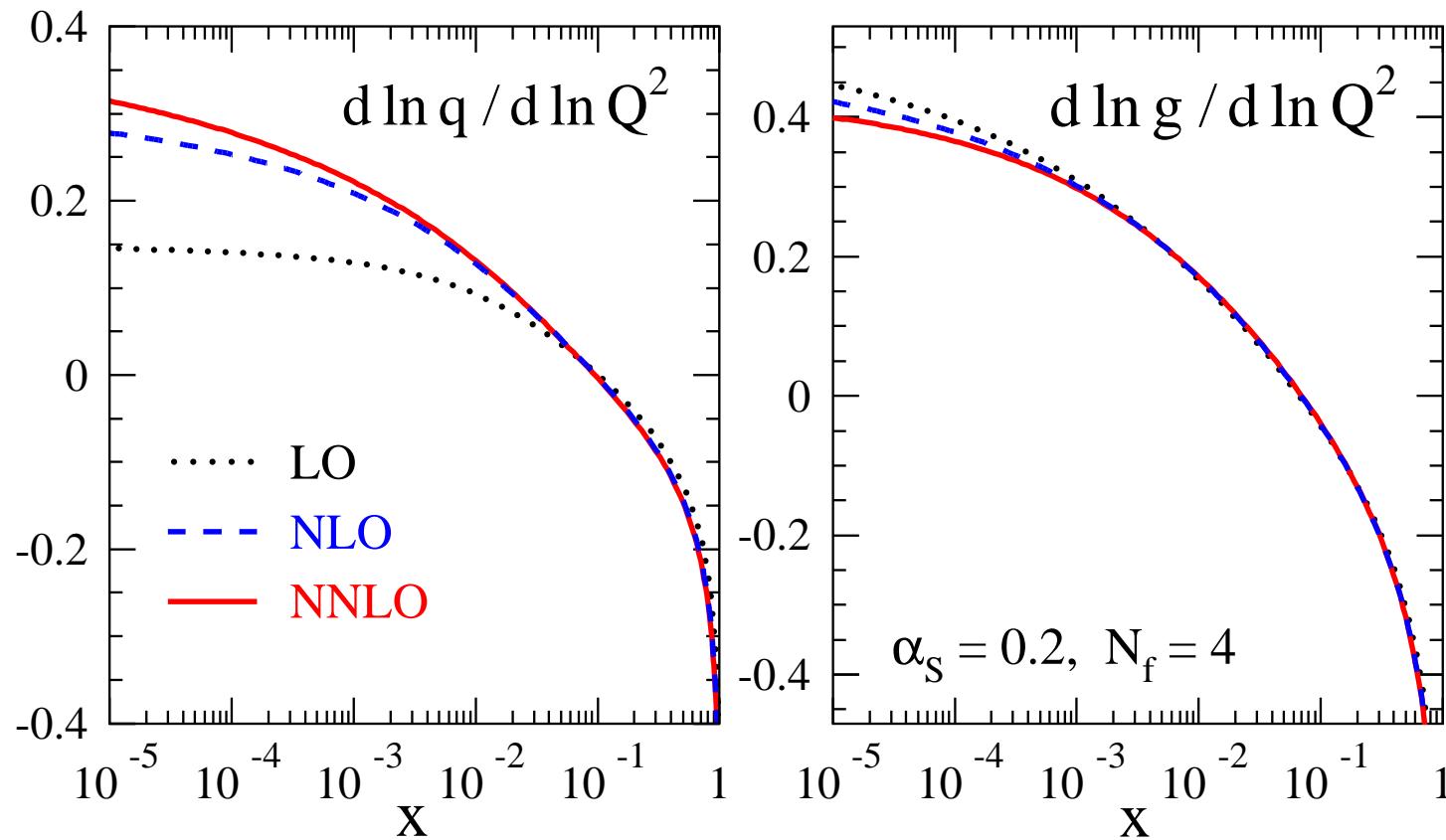
# Perturbative stability of evolution

- Scale derivatives of quark and gluon distributions at  $Q^2 \approx 30 \text{ GeV}^2$



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- Expansion very stable except for very small momenta  $x \lesssim 10^{-4}$

# Relation between space- and time-like kinematics

## Crewther relation

- From conformal and chiral invariance of leading singularity of short distance OPE simple relation between Crewther '72
  - amplitude  $\pi^0 \rightarrow \gamma\gamma$
  - polarized Bjorken sum rule  $\int_0^1 dx g_1^{ep-en}(x, Q^2)$
  - Adler function  $D_V$  (derivative of correlator  $Q^2 \frac{\partial}{\partial Q^2} \Pi_V$ )

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- Higher order radiative QCD corrections exhibit relations between
  - polarized Gross-Llewellyn Smith sum-rule  $C_{GLS}$  at  $\mathcal{O}(\alpha_s^3)$  (first Mellin moment of  $F_3^{\bar{\nu}p+\nu p}$ ) Larin, Vermaseren '91
  - Adler function  $D_V$  at  $\mathcal{O}(\alpha_s^3)$  Gorishny, Kataev, Larin '91
- $C_{GLS}$  and  $D_V$  related by running coupling ( $\beta$ -function) through  $\mathcal{O}(\alpha_s^3)$  Broadhurst, Kataev '96; Maxwell, Broadhurst, Kataev '06

# Drell-Yan-Levy relation

- Analytic continuation in energy  $-q^2 \rightarrow +q^2$   
(exploit analyticity properties)  
Curci, Furmanski, Petronzio '80; Floratos, Kounnas, Lacaze '81; Stratmann, Vogelsang '96;  
Blümlein, Ravindran, van Neerven '00; ...
- Relation between DIS structure function  $F_1^{\text{s-like}}$  and fragmentation function  $F_T^{\text{t-like}}$

$$F_T^{\text{t-like}}(x) = -xF_1^{\text{s-like}}\left(\frac{1}{x}\right)$$

- Example: leading order splitting function  $P_{qq}^{(0)}$ 
  - respects “naive” Drell-Yan-Levy relation  
(with  $\delta(1-x) \rightarrow \delta(1-x)$ )

$$P_{qq}^{(0)}(x) = 2C_F \left( \frac{2}{1-x} - 1 - x \right) + 3C_F \delta(1-x)$$

- Beware: naive version of Drell-Yan-Levy relation not valid beyond LO

# Mapping DIS to $e^+e^-$ annihilation

## Real and virtual contributions

- Partonic forward Compton amplitude  $\mathcal{T}_n$  in  $D = 4 - 2\epsilon$  combines
  - virtual corrections  $\mathcal{F}_n$  (QCD form factor,  $\mathcal{F}_n \propto \delta(1 - x)$ )
  - real-emission contributions  $\mathcal{R}_n(x)$   
(depend on harmonic polylogarithms in  $x$ )

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- $D$ -dimensional +-distributions in  $\mathcal{R}_n$  for soft/collinear region

$$[(1-x)^{-1-k\epsilon}]_+ = -\frac{1}{\epsilon k} \delta(1-x) + \sum_{i=0} \frac{(-k\epsilon)^i}{i!} \left( \frac{\ln^i(1-x)}{1-x} \right)_+$$

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- Laurent-series for  $\mathcal{T}_n$  in  $\epsilon$  at  $n^{\text{th}}$ -order
  - soft and collinear singularities in  $\mathcal{F}_n$  and  $\mathcal{R}_n$  behave as  $1/\epsilon^{2n}$
  - mass-factorization predicts  $1/\epsilon^n$
- Infrared safety ( $\rightarrow$  KLN Kinoshita '62; Lee, Nauenberg '64)
  - constructive approach to  $\mathcal{F}_n$  and  $\mathcal{R}_n$

# Mass factorization

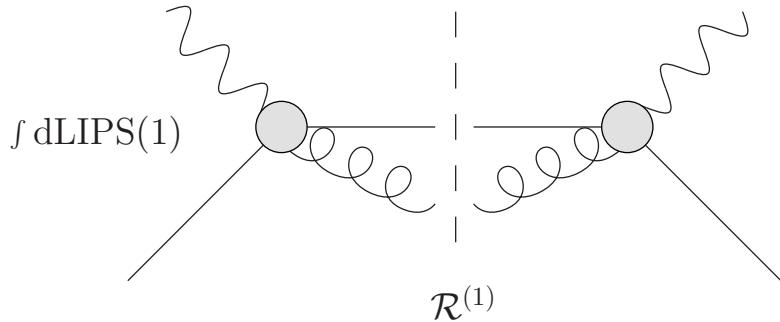
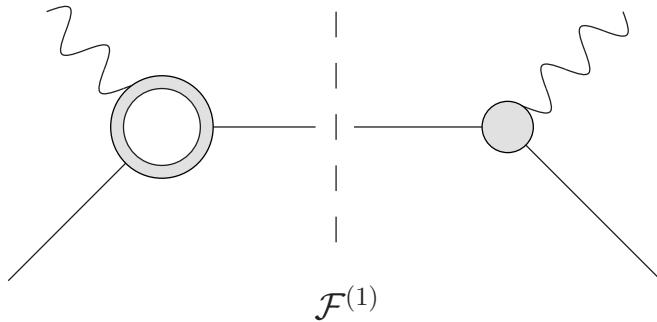
- Universal structure of bare quantity after cancellation of poles between  $\mathcal{F}_n$  and  $\mathcal{R}_n$ 
  - structure function  $F_1$  (space-like)
  - transverse fragmentation function  $F_T$  (time-like)
- Laurent-series at  $n^{\text{th}}$ -order behaves as  $\frac{1}{\epsilon^n}$

$$F^{(1)} = -\frac{1}{\epsilon} P^{(0)} + c^{(1)} + \epsilon a^{(1)} + \epsilon^2 b^{(1)} + \epsilon^3 d^{(1)} + \dots$$

$$\begin{aligned} F^{(2)} = & \frac{1}{2\epsilon^2} P^{(0)}(P^{(0)} + \beta_0) - \frac{1}{2\epsilon} [P^{(1)} + 2P^{(0)}c^{(1)}] + c^{(2)} - P^{(0)}a^{(1)} \\ & + \epsilon [a^{(2)} - P^{(0)}b^{(1)}] + \dots \end{aligned}$$

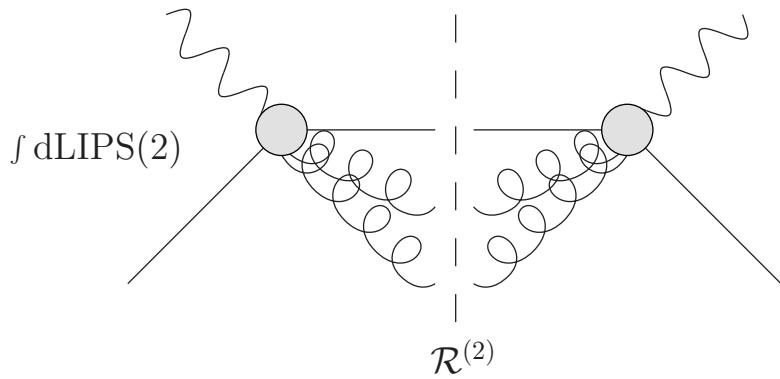
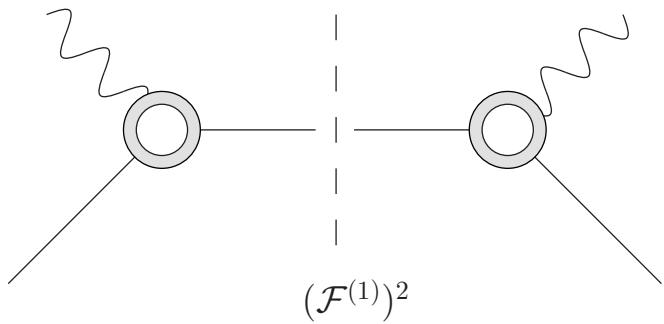
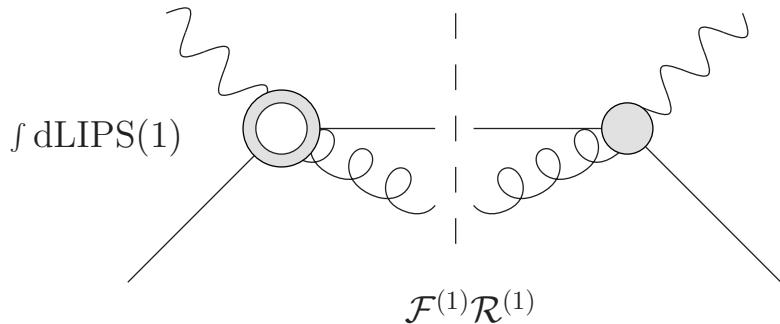
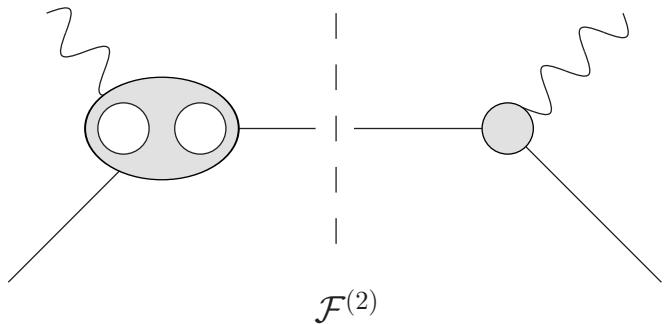
$$\begin{aligned} F^{(3)} = & -\frac{1}{6\epsilon^3} P^{(0)}(P^{(0)} + \beta_0)(P^{(0)} + 2\beta_0) \\ & + \frac{1}{6\epsilon^2} [P^{(1)}(3P^{(0)} + 2\beta_0) + P^{(0)}(3P^{(0)}c^{(1)} + 3\beta_0c^{(1)} + 2\beta_1)] \\ & - \frac{1}{6\epsilon} [2P^{(2)} + 3P^{(1)}c^{(1)} + P^{(0)}(6c^{(2)} - 3P^{(0)}a^{(1)} - 3\beta_0a^{(1)})] + \dots \end{aligned}$$

# Anatomy of DIS result (1 loop)



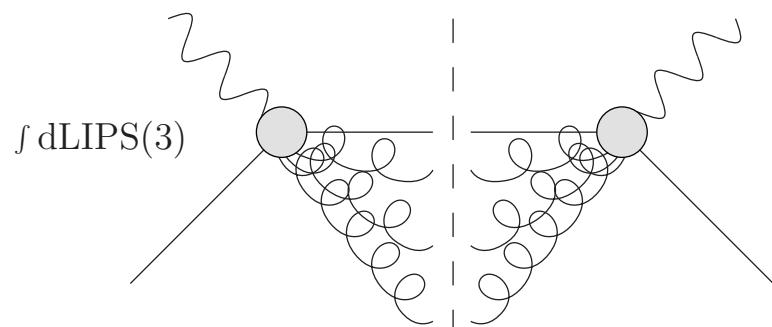
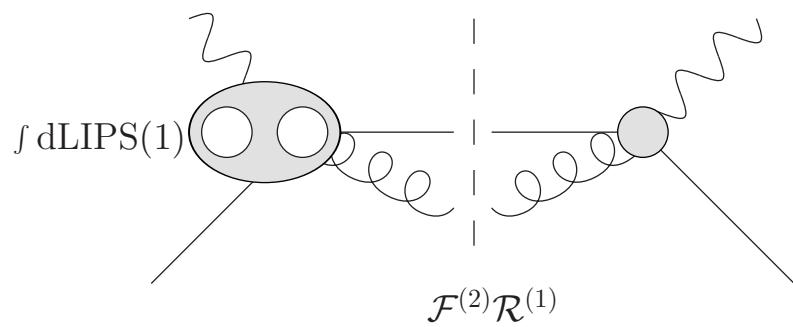
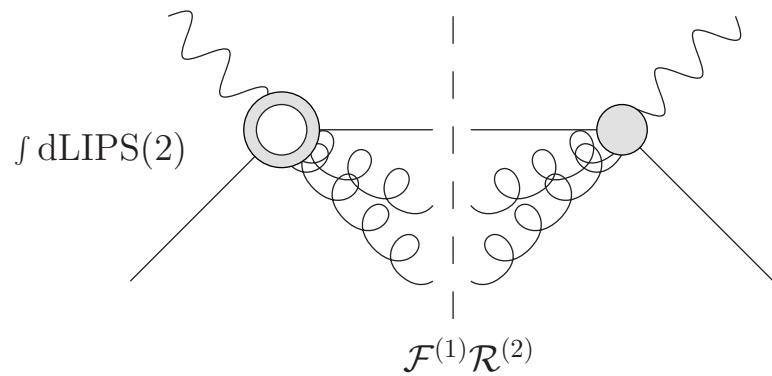
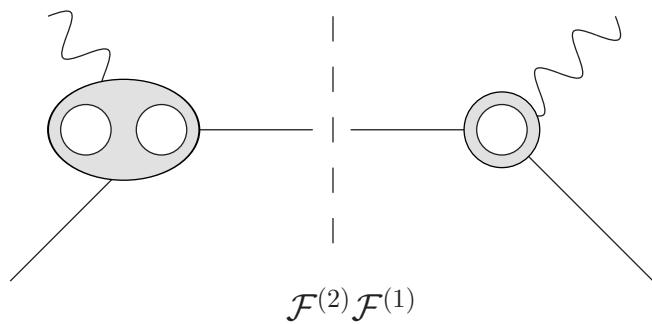
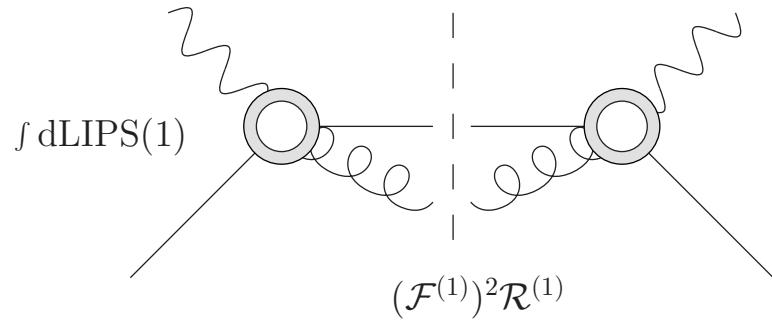
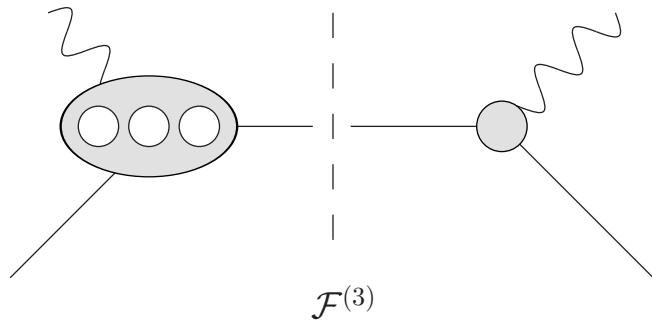
$$\mathcal{T}_1^b = 2 \operatorname{Re} \mathcal{F}_1 \delta(1-x) + \mathcal{R}_1$$

# Anatomy of DIS result (2 loops)



$$\mathcal{T}_2^b = (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \delta(1-x) + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{R}_1 + \mathcal{R}_2$$

# Anatomy of DIS result (3 loops)



$$T_3^b = (2 \operatorname{Re} \mathcal{F}_3 + 2 |\mathcal{F}_1 \mathcal{F}_2|) \delta(1-x) + (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \mathcal{R}_1 + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{R}_2 + \mathcal{R}_3$$

# Analytic continuation

- $\mathcal{R}_n$  is Laurent series in  $\epsilon$  with coefficients being harmonic polylogarithms  $H_{m_1, \dots, m_k}(x)$  or polynomials in  $x, (1-x), (1+x)$
- Analytic continuation from space-like to time-like kinematics requires  
Curci, Furmanski, Petronzio '80; Stratmann, Vogelsang '96; ...
  - mapping  $-q^2 \rightarrow +q^2$  and  $x \rightarrow \frac{1}{x}$

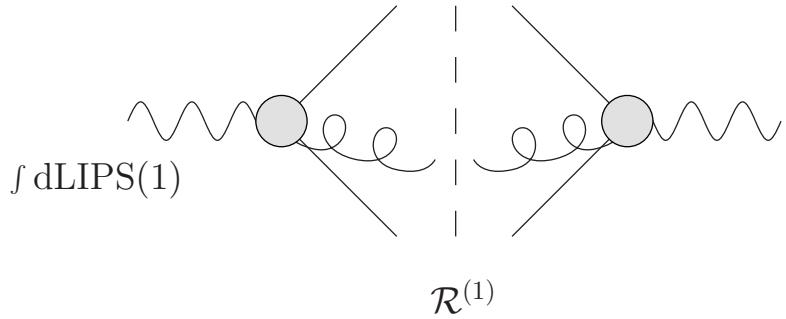
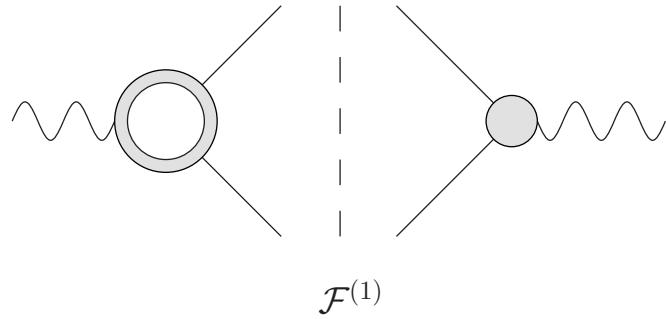
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 $H_0(x) = \ln x$  ,    $H_1(x) = -\ln(1-x)$  ,    $H_{-1}(x) = \ln(1+x)$   
Goncharov '98; Borwein, Bradley, Broadhurst, Lisonek '99; Remiddi, Vermaseren '99
  - all branch cuts from analytic continuation  $\frac{1}{x} - i\delta$  to  $x > 0$  uniquely defined through  $H_1(1/x - i\delta) = H_1(x) + H_0(x) - i\pi$
  - phase space of detected parton in  $e^+e^-$ -annihilation in  $D$ -dimensions (take phase space factor  $x^{1-2\epsilon}$ )

# Analytic continuation

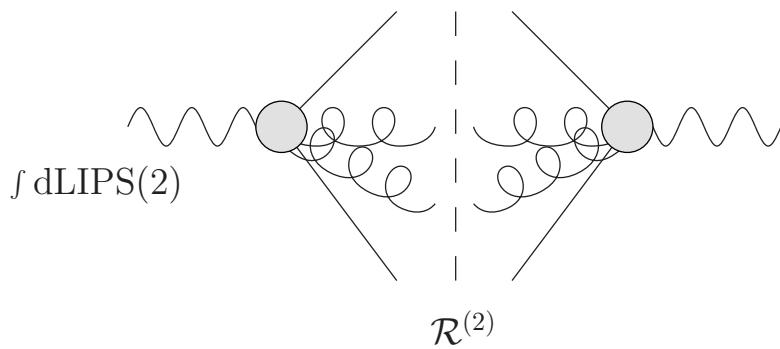
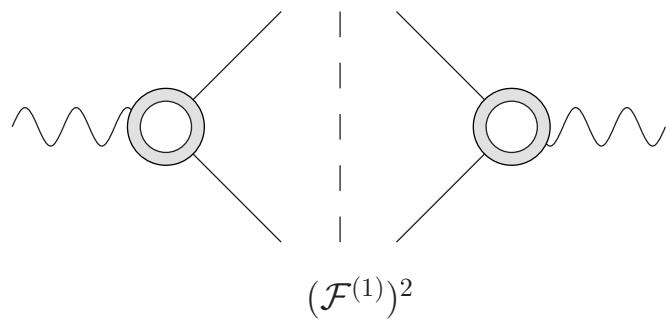
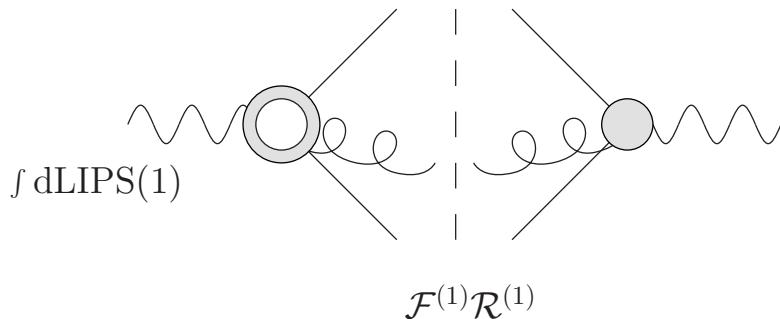
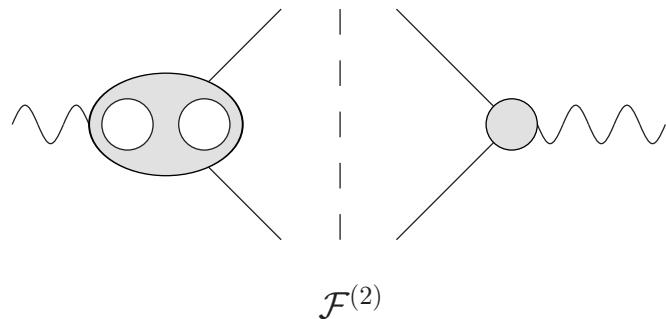
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- Constructive approach to  $\mathcal{R}_n^{\text{t-like}}$  (given we know  $\mathcal{R}_n^{\text{s-like}}$ )

# Assembly of $e^+e^-$ (1 loop)



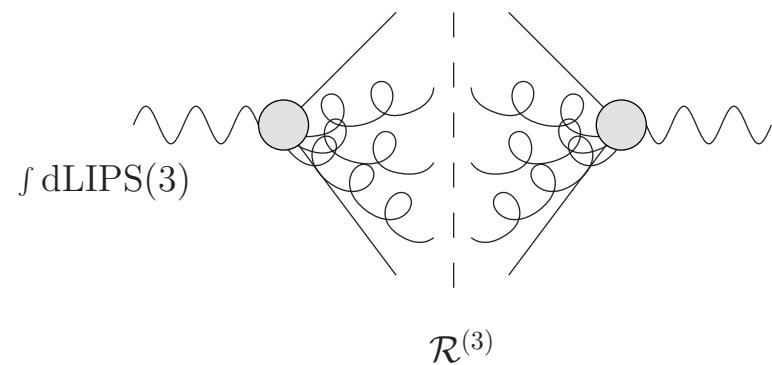
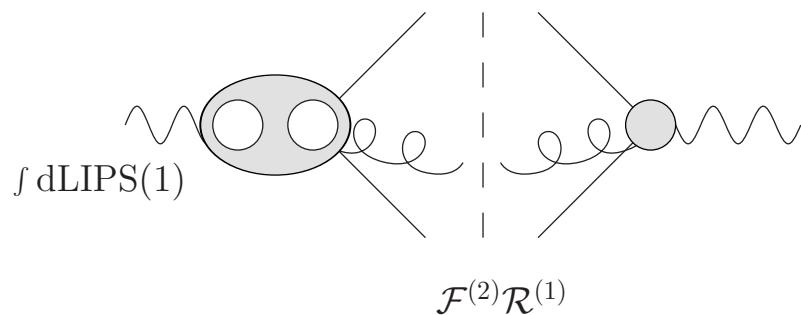
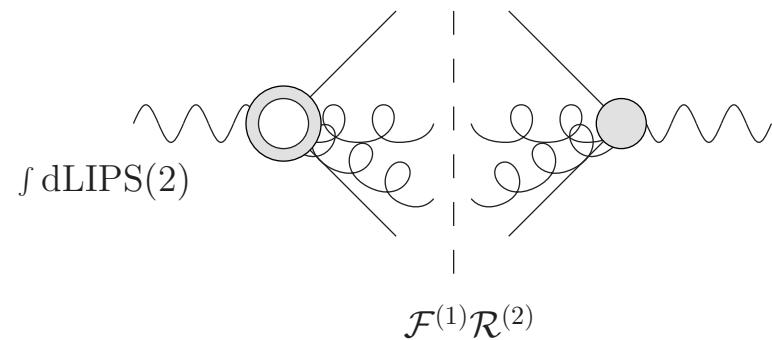
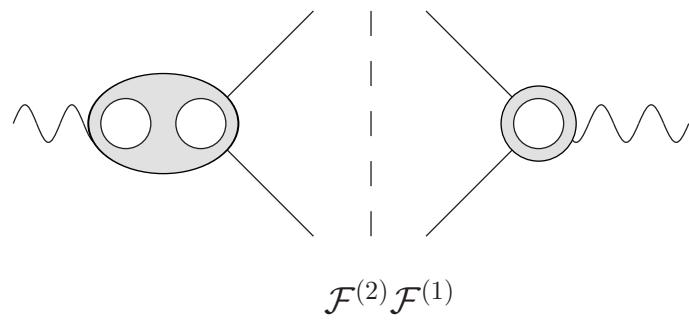
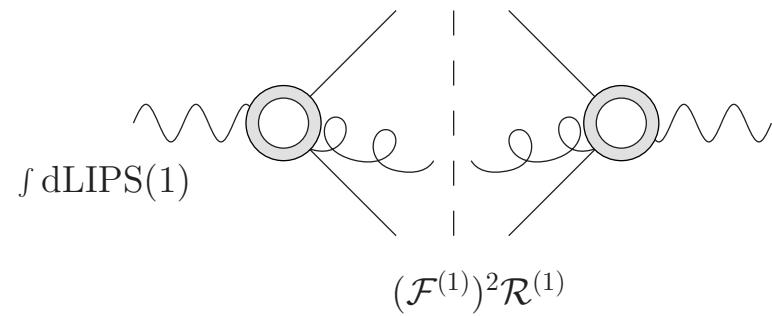
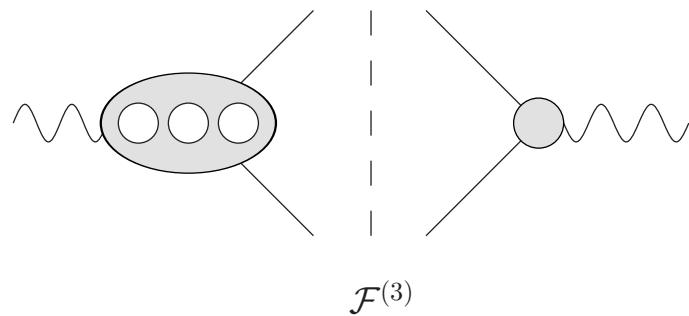
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## Assembly of $e^+e^-$ (2 loop)



$$\mathcal{T}_2^b = (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \delta(1-x) + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{R}_1 + \mathcal{R}_2$$

## Assembly of $e^+e^-$ (3 loop)



$$T_3^b = (2 \operatorname{Re} \mathcal{F}_3 + 2 |\mathcal{F}_1 \mathcal{F}_2|) \delta(1-x) + (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \mathcal{R}_1 + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{R}_2 + \mathcal{R}_3$$

## Time-like results

- Read off results in time-like kinematics from mass factorization of bare transverse fragmentation function  $F_T$

$$F^{(1)} = -\frac{1}{\epsilon} P^{(0)} + c^{(1)} + \epsilon a^{(1)} + \epsilon^2 b^{(1)} + \epsilon^3 d^{(1)} + \dots$$

$$\begin{aligned} F^{(2)} = & \frac{1}{2\epsilon^2} P^{(0)}(P^{(0)} + \beta_0) - \frac{1}{2\epsilon} \left[ P^{(1)} + 2P^{(0)}c^{(1)} \right] + c^{(2)} - P^{(0)}a^{(1)} \\ & + \epsilon \left[ a^{(2)} - P^{(0)}b^{(1)} \right] + \dots \end{aligned}$$

$$\begin{aligned} F^{(3)} = & -\frac{1}{6\epsilon^3} P^{(0)}(P^{(0)} + \beta_0)(P^{(0)} + 2\beta_0) \\ & + \frac{1}{6\epsilon^2} \left[ P^{(1)}(3P^{(0)} + 2\beta_0) + P^{(0)}(3P^{(0)}c^{(1)} + 3\beta_0c^{(1)} + 2\beta_1) \right] \\ & - \frac{1}{6\epsilon} \left[ 2P^{(2)} + 3P^{(1)}c^{(1)} + P^{(0)}(6c^{(2)} - 3P^{(0)}a^{(1)} - 3\beta_0a^{(1)}) \right] + \dots \end{aligned}$$

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- New three-loop results for  $P^{(2)}$

# Upshot

- Results for mapping space-like to time-like processes  
 $(F_1^{\text{s-like}} \leftrightarrow F_T^{\text{t-like}})$ ,  $(F_3^{\text{s-like}} \leftrightarrow F_A^{\text{t-like}})$  and  $(F_\phi^{\text{s-like}} \leftrightarrow F_\phi^{\text{t-like}})$ 
  - read off three-loop splitting function  $P_{\text{ns}}^{(2)T}$ ,  $P_{\text{ps}}^{(2)T}$  and  $P_{gg}^{(2)T}$
- Checks
  - one-loop, two-loops (even through order  $\epsilon$  with loop technology)  
Mitov, S.M. '06
  - three loops through order  $1/\epsilon$  (soft/collinear limit)
  - sum rules

$$\int_0^1 dx P_{\text{ns}}^{(2)T}(x) = 0 \quad \text{and} \quad \int_0^1 dx \left( P_{qg}^{(2)T}(x) + P_{gg}^{(2)T}(x) \right) = 0$$

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- Problems in sum rule check
  - coefficient  $\frac{1}{\epsilon} C_F^3 P_{qq}^{(0)} \zeta_2 \ln^2 x$  incorrect in  $P_{\text{ns}}^{(2)T}(x)$
  - coefficient  $\frac{1}{\epsilon} C_A^3 P_{gg}^{(0)} \zeta_2 \ln^2 x$  incorrect in  $P_{gg}^{(2)T}(x)$

# Rescue

- Alternative approach relates space- and time-like kinematics  
Dokshitzer, Marchesini, Salam '05
  - idea: universal splitting function (kinematics independent)
- Our result
  - analytic continuation with correct sum rule  
Mitov, S.M., Vogt '06; S.M. Vogt '07
  - agreement with approach based on universal splitting functions  
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- Alternative approach relates space- and time-like kinematics  
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  - agreement with approach based on universal splitting functions  
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- Additional check
  - compute three-loop coefficient functions for one-particle inclusive Higgs decay
  - second-moment combination enters the Higgs decay rate
$$(C_{\phi,q}^T + C_{\phi,g}^T)(N=2) = 1 + \alpha_s c_\phi^{(1)} + \alpha_s^2 c_\phi^{(2)} + \alpha_s^3 c_\phi^{(3)} + \dots ,$$
- agreement
  - NNLO with Chetyrkin, Kniehl, Steinhauser '97; Schreck, Steinhauser '07
  - N<sup>3</sup>LO with Baikov, Chetyrkin '06 (up to  $\zeta_2$ -terms)

# NLO time-like splitting functions (diagonal singlet)

$$\begin{aligned} \delta P_{\text{ns},+}^{(1)}(x) &\equiv P_{\text{ns},+}^{(1)T}(x) - P_{\text{ns},+}^{(1)S}(x) = \\ &4\mathcal{C}_F^2 \left( H_0(6(1-x)^{-1} - 5 - x) + H_{0,0}(-8(1-x)^{-1} + 6 + 6x) + (H_{1,0} + H_2)(-8(1-x)^{-1} \right. \\ &\left. + 4 + 4x) \right). \end{aligned}$$

$$\begin{aligned} \delta P_{\text{ps}}^{(1)}(x) &\equiv P_{\text{ps}}^{(1)T}(x) - P_{\text{ps}}^{(1)S}(x) = \\ &8\mathcal{C}_F n_f \left( -20/9x^{-1} - 3 - x + 56/9x^2 - (3 + 7x + 8/3x^2)H_0 + 2(1+x)H_{0,0} \right). \end{aligned}$$

$$\begin{aligned} \delta P_{\text{gg}}^{(1)}(x) &\equiv P_{\text{gg}}^{(1)T}(x) - P_{\text{gg}}^{(1)S}(x) = \\ &8\mathcal{C}_A^2 \left( p_{\text{gg}}(x) \left[ 11/3H_0 - 4(H_{0,0} + H_{1,0} + H_2) \right] + [6(1-x) - 22/3(x^{-1} - x^2)]H_0 \right. \\ &- 8(1+x)H_{0,0} \left. \right) - 16/3\mathcal{C}_A n_f p_{\text{gg}}(x)H_0 + 8\mathcal{C}_F n_f \left( 20/9x^{-1} + 3 + x - 56/9x^2 \right. \\ &+ [4 + 6x + 4/3(x^{-1} + x^2)]H_0 + 2(1+x)H_{0,0} \left. \right). \end{aligned}$$

# NNLO time-like splitting functions (diagonal singlet)

$$\begin{aligned} \delta P_{\text{ps}}^{(2)}(x) &\equiv P_{\text{ps}}^{(2)T}(x) - P_{\text{ps}}^{(2)S}(x) = \\ &+ 16\mathcal{C}_F^2 (P_{\text{gg}}(x) \left[ 311/24H_0 + 4/3H_0\zeta_2 - 169/9H_{0,0} + 8H_0\zeta_2 - 22H_{0,0,0} \right. \\ &- 268/9H_{1,0} - 8H_0\zeta_2 - 44/3H_{1,0,0} - 268/9H_2 + 8H_0\zeta_2 - 44/3H_{2,0} - 44/3H_3 \left. \right] \\ &+ (1+x) \left[ -4H_0\zeta_2 + 25/2H_{0,0,0} + H_2\zeta_2 + 2H_3 \right] - (1-x) \left[ 325/18H_0 + 50/3H_{1,0} \right. \\ &+ 50/3H_2 \left. \right] - (3 - 5x)H_0\zeta_2 - (173/18 - 691/18x)H_{0,0} \left. \right] \\ &+ 16\mathcal{C}_F^2 (\mathcal{C}_A - 2\mathcal{C}_F) \left( P_{\text{gg}}(x) \left[ 151/24H_0 + H_0\zeta_2 - 13/6H_0\zeta_2 - 169/18H_{0,0} + 8H_0\zeta_2 \right. \right. \\ &- 13/2H_{0,0,0} - 8H_{0,0,0,0} - 13/4H_{1,0,0} + 4H_0\zeta_2 - 22/3H_{1,0,0,0} - 6H_{1,0,0,0,0} - 134/9H_2 \\ &+ 4H_0\zeta_2 - 22/3H_{2,0,0} - 2H_{2,0,0,0} - 22/3H_3 - 2H_{3,0} - 6H_4 \left. \right] + P_{\text{gg}}(-x) \left[ -8H_{-3,0} \right. \\ &+ 8H_1\zeta_2 - 8H_{-2,0} - 10 - 4H_{-2,0,0} - 4H_{-2,0,0,0} - 4H_{-2,0,0,0,0} - 2H_{-1,0} + 16H_{-1,0,0} \\ &- 8H_0\zeta_2 - 9/2H_{0,0,0,0} + 8H_{0,0,0,0,0} + 2H_{1,0,0,0,0} \left. \right] - (1+x) \left[ 4H_{-2,0} - 8H_{-1,0,0} \right. \\ &+ (1-x) \left[ 4H_{-3,0} + 4H_{-2,0,0} - 88/9H_0 + 3H_0\zeta_2 - 28/3H_{1,0} - 28/3H_2 \right] - 4H_0\zeta_2 \\ &- (50/9 - 184/9)xH_{0,0,0} - 4xH_0\zeta_2 + (11/2 + 35/2)xH_0 + 8xH_{0,0,0} \left. \right] \\ &+ 16\mathcal{C}_F^2 n_f \left( P_{\text{gg}}(x) \left[ -11/2H_0 - 2/3H_0\zeta_2 + 11/9H_{0,0} + 2H_{0,0,0} + 20/9H_{1,0} \right. \right. \\ &+ 4/3H_{1,0,0} + 20/9H_2 + 4/3H_{2,0} + 4/3H_3 \left. \right] - (1-x)H_{0,0,0} + (1-x) \left[ 13/9H_0 \right. \\ &+ 4/3H_{1,0} + 4/3H_2 \left. \right] + (8/9 - 28/9)xH_{0,0} \left. \right] . \end{aligned}$$
  

$$\begin{aligned} \delta P_{\text{gg}}^{(2)}(x) &\equiv P_{\text{gg}}^{(2)T}(x) - P_{\text{gg}}^{(2)S}(x) = \\ &+ 8\mathcal{C}_F C_F n_f \left( 269/6x^{-1} + 14 + 113/2x - 346/3x^2 + \zeta_2(172 + 167x + 8x^2)/3 \right. \\ &- \zeta_2(12x^{-1} - 13 + 6x + 28x^2) - 2(1+x) \left[ 16\zeta_2^2 + 4H_{-1,0,0} + 9H_{1,0} + 4H_{1,1} \right. \\ &+ 10\zeta_2H_2 + 9H_{2,0,0} - 12H_{2,1,0,0} - 2H_{2,1,0,0,0} - 9H_{2,1,0} + 4H_{3,1} - H_3 \left. \right] \\ &+ 2(1-x) \left[ 5\zeta_2H_1 + 2H_{2,0,0} + 139/12H_{1,0,0} - 6H_{1,0,0,0} - H_{1,0,0,0,0} - 3H_{1,2,0} + H_{2,1} \right] \\ &+ 8/3(x^{-1} - x^2) \left[ 5\zeta_2H_1 + 5/3H_{1,0,0} - 6H_{1,0,0,0} - H_{1,0,0,0,0} - 3H_{1,2,0} + 2H_{2,0,0} + H_{2,1} \right] \\ &- (57/2 + 247/3x + 811/2x^2 + 72\zeta_2)/18H_0 + (62 - 81/2x + 208/9x^2)H_{0,0} \\ &+ (6 + 18x - x^2)H_{0,0,0} + (385/18 - 1/x + 190/3 - 143/3x - 667/18x^2)H_1 \\ &+ (28/9x^{-1} + 71 + 46x + 248/9x^2)H_2 - 4/3(4x^{-1} - 6 - 3x + 8x^2)H_3 \\ &+ 8\mathcal{C}_F n_f^2 \left( 2/9(23x - 2x^{-1} - 20 - x^2) + 2(1+x) \left[ \zeta_2 - \zeta_2H_0 - H_1 + 2H_2 + H_3 \right. \right. \\ &- H_{0,0,0} \left. \right] - (1-x)(H_1 - H_{1,0}) + 4/(x^{-1} - x^2)H_{1,0} + 2/9(3 + 18x + 10x^2)H_0 \\ &- (7 + x - 4x^2)/3H_{0,0} - (20x^{-1} - 56x^2)/9H_1 + (3 + 7 + 8/x^2)\zeta_2(H_2 - H_3) \left. \right] \\ &+ 8\mathcal{C}_F n_f \left( 217/18 + 55/3x^{-1} - 122/9x - 101/6x^2 - \zeta_2(16x^{-1} + 36 + 24x) + \zeta_2/3 \right. \\ &+ (127 - 188x + 128x^2) - 2(1+x) \left[ 16\zeta_2^2 + 5\zeta_2H_0 + 10\zeta_2H_2 + 17\zeta_2H_3 - 12H_{0,0,0} \right. \\ &- 2H_{1,0,0,0} - 6H_{2,0,0} + 9H_{0,0,0} + 4H_{1,1,0,0} - H_{1,1,0} \left. \right] - \left[ \frac{8}{3}(x^{-1} - x^2) + 2(1-x) \right] \left[ 5\zeta_2H_1 \right. \\ &+ 3H_{1,0} - 6H_{1,0,0} - 3H_{1,2,0} - H_{1,1,0,0} + H_{2,1} \left. \right] + (4x^{-1} - 283/6 + 239/2x + 739/18x^2 \\ &- 8\zeta_2 - 20\zeta_2x - 16/3\zeta_2x^2)H_0 - (18 + 97/2 + 16x^2)H_{0,0} - (6 - 6x + 8x^2)H_{0,0,0} \\ &- (385x^{-1} + 1140 - 858x - 667x^2)/18H_1 + 53/6(1 - x)H_{1,0} - (20/3x^{-1} + 45 \\ &+ 72x + 24x^2)H_2 - (32/3x^{-1} + 14 + 6x)H_{2,0} - (16/3x^{-1} - 8 - 12x)H_3 \left. \right] \\ &+ 8/9\mathcal{C}_F n_f^2 \left( 4x^{-1} + 40 - 46x + 2x - 9\zeta_2(3 + 7 + 8/x^2) - 6(1+x) \right. \\ &+ 3H_{0,0,0} - H_{2,0,0} - H_3 - (92/3x^{-1} - 6 - 48x - 32/3x^2)H_0 - (16x^{-1} + 83 \\ &+ 101x + 28x^2)H_0 + (20x^{-1} + 27 + 9x - 56x^2)H_1 + (4x^{-1} + 3 - 3x - 4x^2)H_2 \\ &+ (16x^{-1} + 39 + 51x + 8x^2)H_3 \left. \right]. \end{aligned}$$

S.M., Vogt '07  
2007

# The large $x$ -limit: $x \rightarrow 1$

- Large  $x$ -limit for diagonal splitting functions  $P_{aa}^{(2)}$ ,  $a = q, g$

$$P_{aa, \rightarrow 1}^{(2)}(x) = \frac{A_3^a}{(1-x)_+} + B_3^a \delta(1-x) + C_3^a \ln(1-x) + \mathcal{O}(1)$$

one-loop       $A_1^q = 4 C_F$

two-loop       $A_2^q = 8 C_F C_A \left( \frac{67}{18} - \zeta_2 \right) - \frac{5}{9} C_F n_f$

- $A_3^a$  important for threshold resummation in soft/collinear limit

$$A_3^q = 16 C_F C_A^2 \left( \frac{245}{24} - \frac{67}{9} \zeta_2 + \frac{11}{6} \zeta_3 + \frac{11}{5} \zeta_2^2 \right) + 16 C_F^2 n_f \left( -\frac{55}{24} + 2 \zeta_3 \right)$$

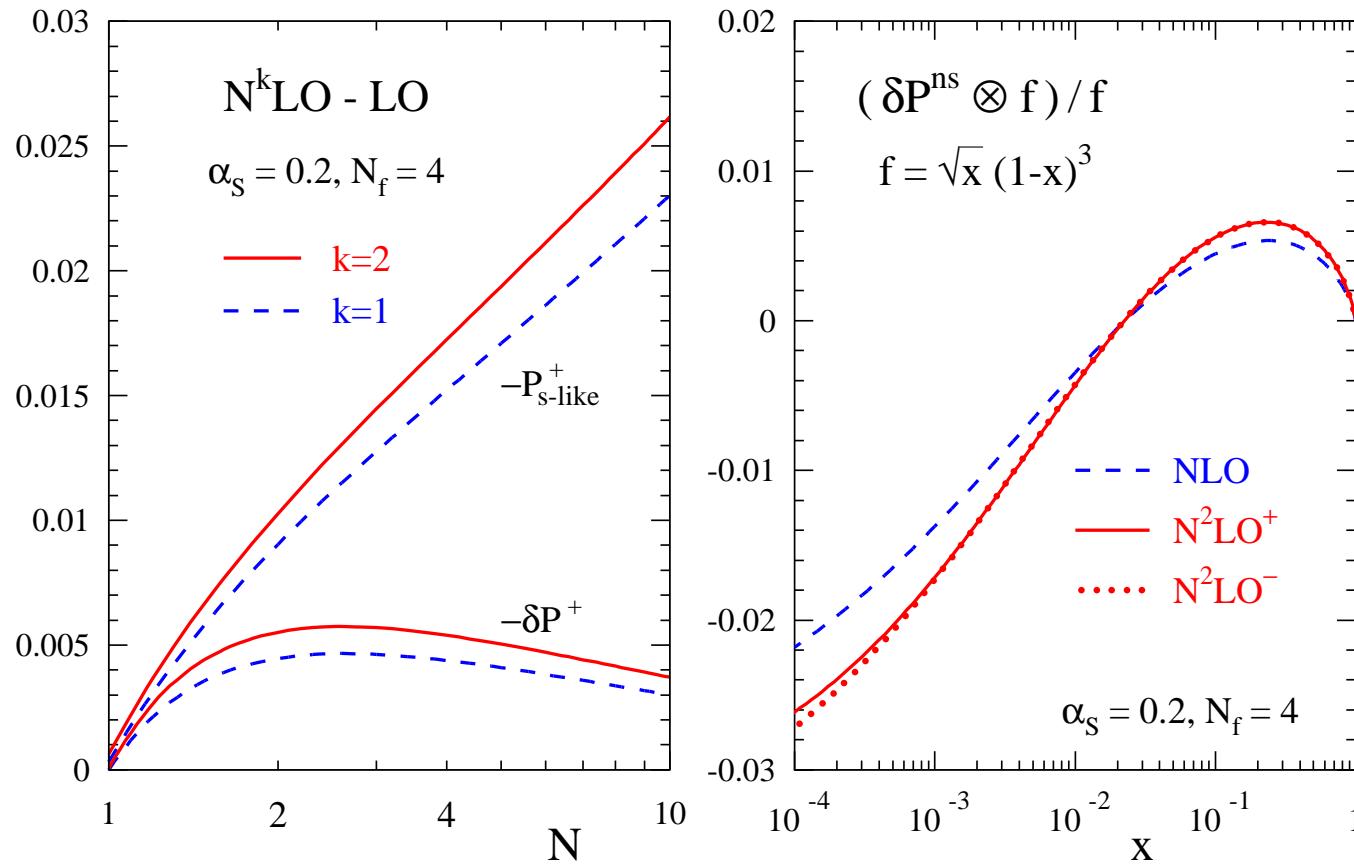
$$+ 16 C_F C_A n_f \left( -\frac{209}{108} + \frac{10}{9} \zeta_2 - \frac{7}{3} \zeta_3 \right) + 16 C_F n_f^2 \left( -\frac{1}{27} \right)$$

- Surprising relation for subleading logarithms

$$C_1^a = 0, \quad C_2^a = \pm (A_1^a)^2, \quad C_3^a = \pm 2 A_1^a A_2^a$$

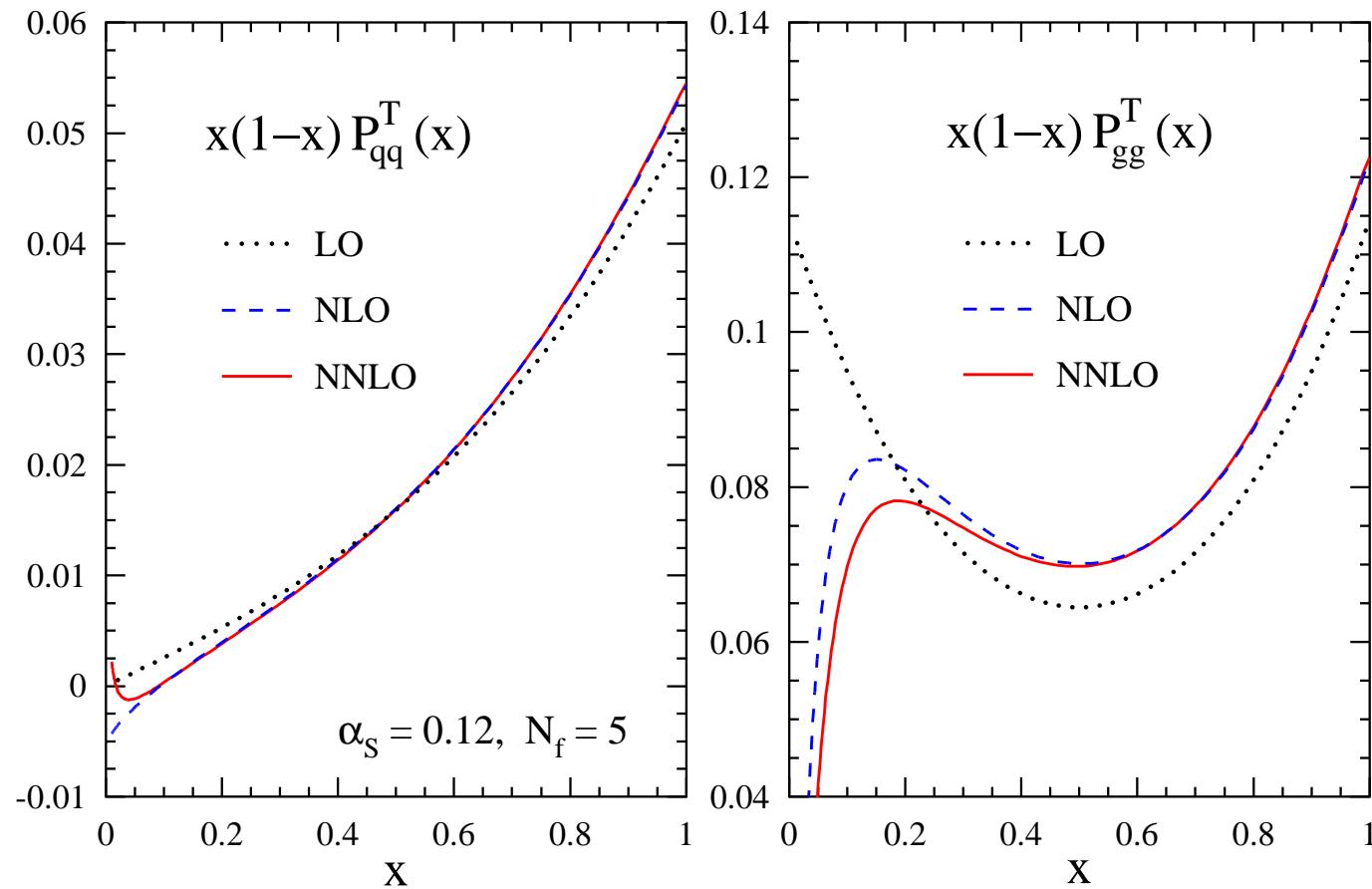
- $\pm$  → explanation see Dokshitzer, Marchesini, Salam '05

# Time-like splitting functions



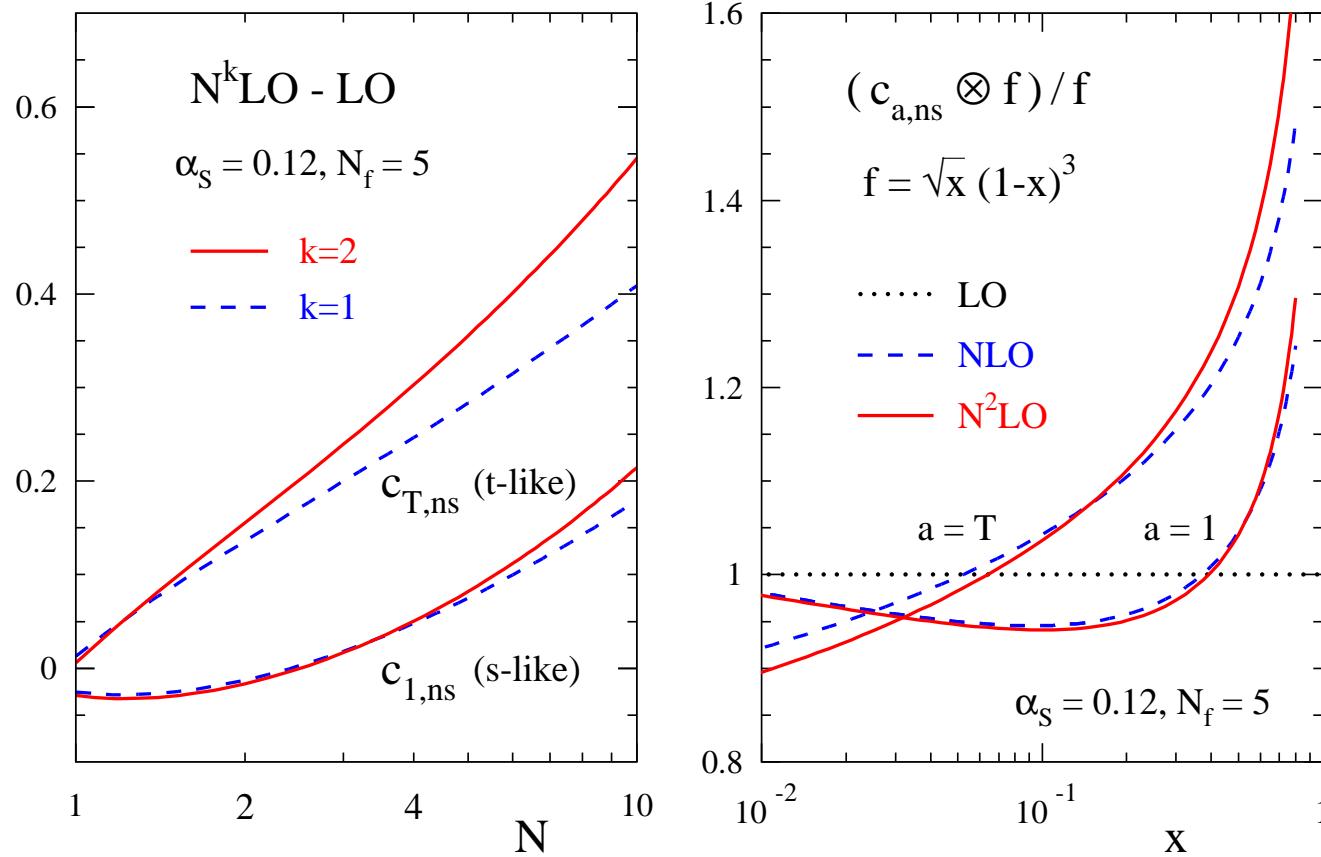
- Differences  $\delta P_{\text{ns}} = P_{\text{ns}}^T - P_{\text{ns}}^S$  between time-/space-like non-singlet splitting functions at low scale of  $\alpha_s = 0.2$
- Mellin moments (left); convolutions input shape  $f = \sqrt{x}(1-x)^3$  (right)  
Mitov, S.M., Vogt '06

# Time-like splitting functions



- Perturbative expansion of  $P_{\text{qq}}^T$  and  $P_{\text{gg}}^T$  (multiplied by  $x(1-x)$ )  
S.M., Vogt '07

# Time-like coefficient functions



- Coefficient functions for  $Q^2 \simeq M_Z^2$  Rijken, van Neerven '96; Mitov, S.M., Vogt '06
- Mellin moments (left); convolutions with input shape  $f = \sqrt{x}(1-x)^3$  (right)
  - $c_{T,ns}$  for (time-like) process  $e^+e^- \rightarrow h + X$
  - $c_{1,ns}$  in (space-like) deep-inelastic scattering

# The small $x$ -limit: $x \rightarrow 0$ (space-like)

- Structure of gluon splitting functions at small  $x$

$$P_{gg, x \rightarrow 0}^{(2)S}(x) = E_1^{gg} \frac{\ln x}{x} + E_2^{gg} \frac{1}{x} + \mathcal{O}(\ln^4 x)$$

- No logarithm  $\ln^2 x/x$  in  $P_{gg}^{(2)}$ 
  - predicted by leading logarithmic BFKL equation  
Kuraev, Lipatov, Fadin '77; Balitsky, L.N. Lipatov '78; Jaroszewicz '82
- Coefficient  $E_1^{gg}$  in agreement with prediction of next-to-leading logarithmic BFKL equation Fadin, Lipatov '98

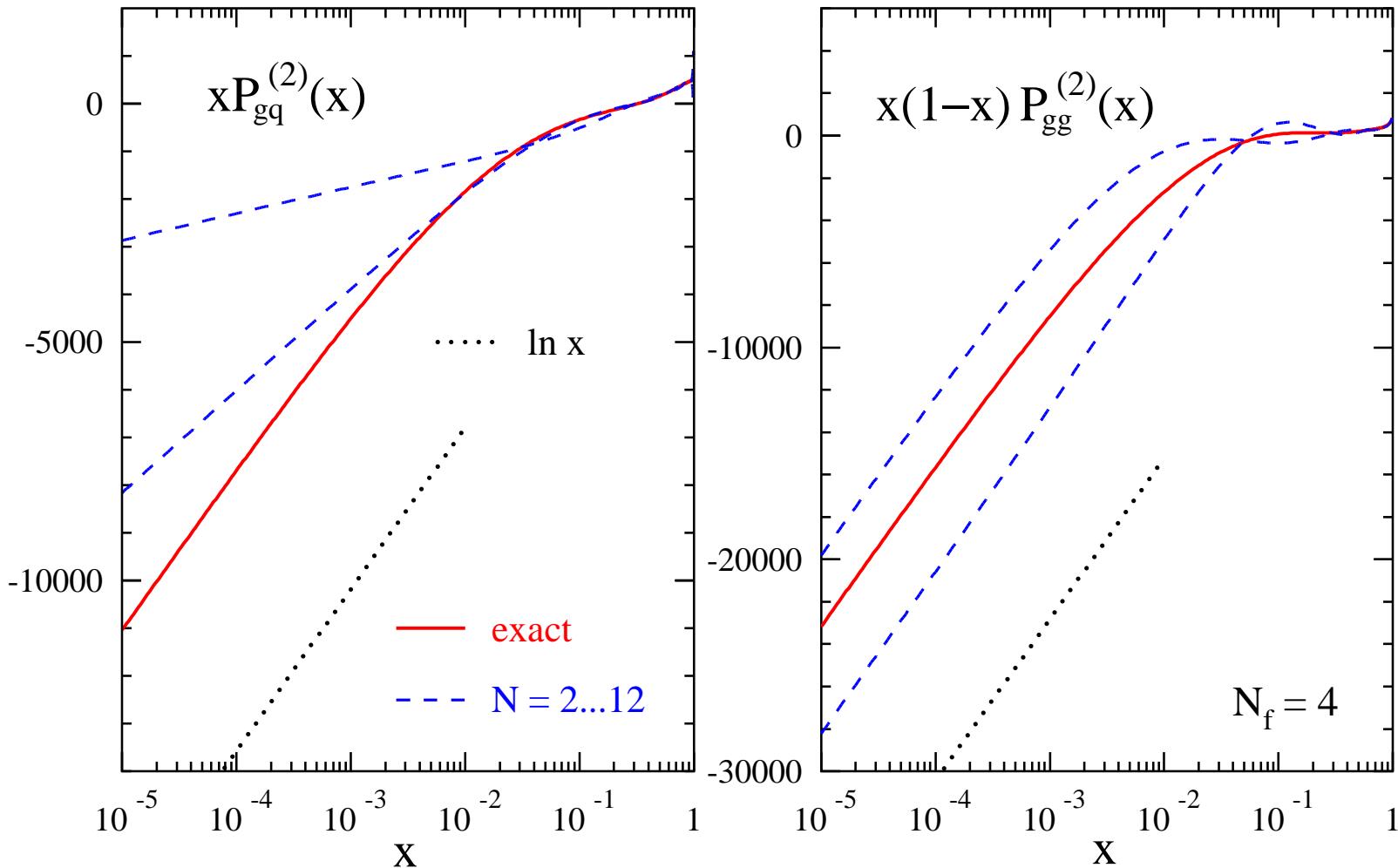
$$E_1^{gg} = \left( \frac{6320}{27} - \frac{176}{3} \zeta_2 - 32 \zeta_3 \right) C_A^3 + \left( \frac{1136}{27} - \frac{32}{3} \zeta_2 \right) C_A^2 n_f$$

$$- \left( \frac{1376}{27} - \frac{64}{3} \zeta_2 \right) C_A C_F n_f$$

$$\cong 2675.85 + 157.269 n_f$$

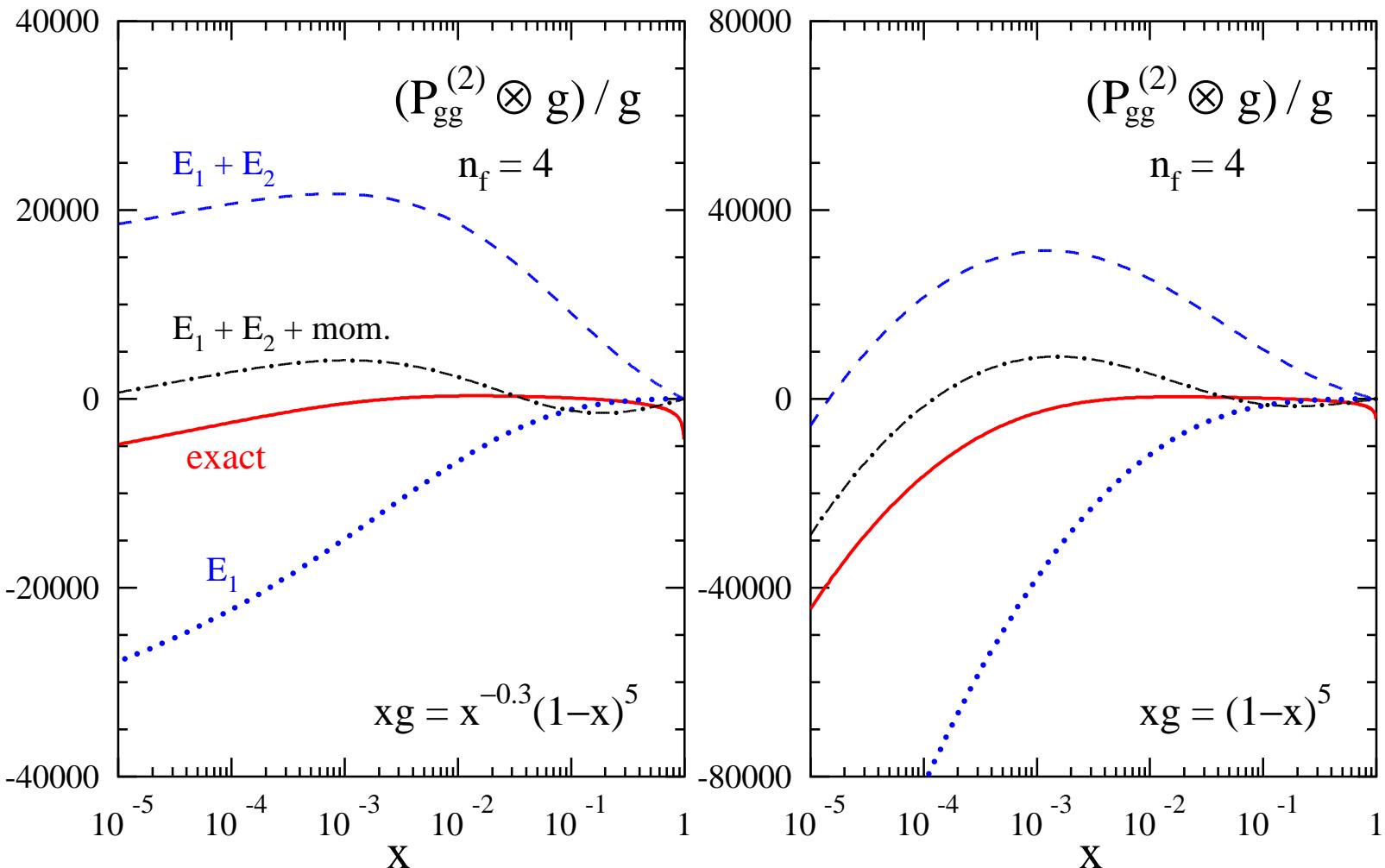
$$E_2^{gg} \cong 14214.2 + 182.958 n_f - 2.79835 n_f^2$$

# $P_{gg}^{(2)S}$ for $x \rightarrow 0$ (space-like)



- Splitting function  $P_{gq}^{(2)}$  (left) and  $P_{gg}^{(2)}$  (right)
- exact result, estimates from  $N$ -moments and leading small- $x$  term

# Convolution of small- $x$ terms with gluon PDF



- Comparison of exact result for  $P_{gg}^{(2)}$  with various approximations of small- $x$  terms, schematic ‘steep’ (left) and ‘flat’ distributions (right)

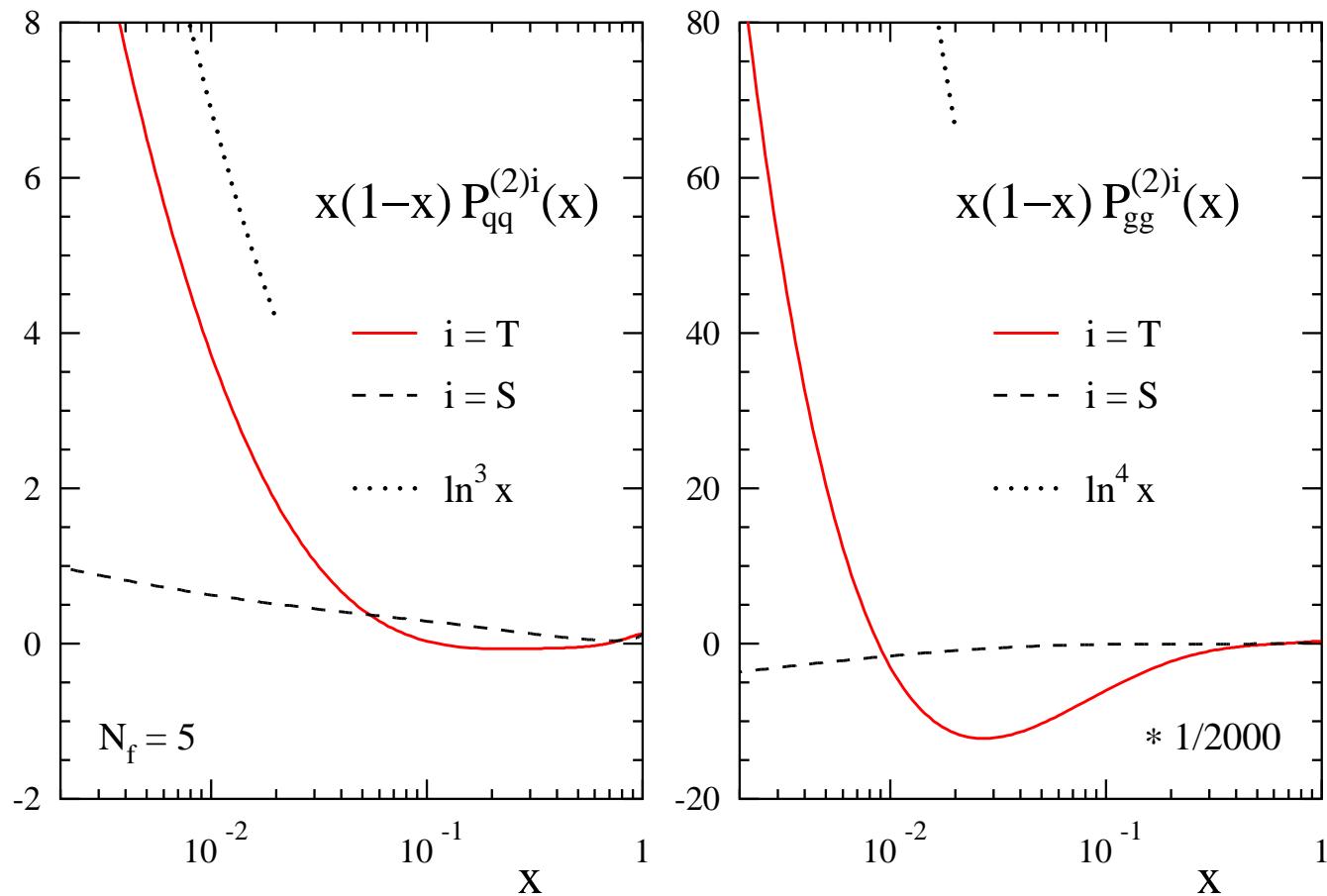
## The small $x$ -limit: $x \rightarrow 0$ (time-like)

- Structure of (diagonal) splitting functions at small  $x$ 
  - double-logarithmic contributions with very large coefficients
  - huge enhancement already at  $x \gtrsim 10^{-3}$

$$x P_{\text{qq}}^{(2)T}(x) = -\frac{32}{9} C_A C_F n_f (2 \ln^3 x + \ln^2 x) + \frac{8}{27} (155 + 72 \zeta_2) C_A C_F n_f \ln x + \mathcal{O}(1)$$

$$\begin{aligned} x P_{\text{gg}}^{(2)T}(x) = & \frac{64}{3} C_A^3 \ln^4 x + \frac{32}{9} (33 C_A^3 + 6 C_A^2 n_f - 10 C_A C_F n_f) \ln^3 x \\ & + \frac{8}{9} [(389 - 144 \zeta_2) C_A^3 + 136 C_A^2 n_f - 232 C_A C_F n_f + 4 n_f^2 (C_A - 2 C_F)] \ln^2 x \\ & + \frac{8}{27} [(4076 - 990 \zeta_2 - 972 \zeta_3) C_A^3 + (739 - 36 \zeta_2) C_A^2 n_f \\ & - (1819 - 144 \zeta_2) C_A C_F n_f + 108 C_F^2 n_f + 46 n_f^2 (C_A - 2 C_F)] \ln x + \mathcal{O}(1) \end{aligned}$$

## $P_{qq}^{(2)T}$ and $P_{gg}^{(2)T}$ for $x \rightarrow 0$ (time-like)



- Splitting function  $P_{qq}^{(2)}$  (left) and  $P_{gg}^{(2)}$  (right) S.M., Vogt '07
  - five flavours, multiplied by  $x(1-x)$ , divided by  $2000 \simeq (4\pi)^3$
  - comparison with space-like splitting functions

## Complete $N = 2$ Mellin moments

$$\begin{aligned}
P_{\text{qq}}^{(2)T}(N=2) &= -P_{\text{gq}}^{(2)T}(N=2) = \\
&- C_F^3 \left( \frac{54556}{243} - \frac{7264}{27} \zeta_2 - 320 \zeta_3 + 256 \zeta_2^2 \right) \\
&- C_F^2 C_A \left( \frac{6608}{243} - \frac{2432}{9} \zeta_2 + \frac{2464}{9} \zeta_3 - \frac{128}{3} \zeta_2^2 \right) - C_F C_A^2 \left( \frac{20920}{243} + \frac{64}{3} \zeta_3 \right) \\
&- C_F C_A n_f \left( \frac{55}{81} + \frac{296}{27} \zeta_2 - \frac{512}{9} \zeta_3 \right) - C_F^2 n_f \left( \frac{2281}{81} - \frac{32}{9} \zeta_2 + \frac{64}{9} \zeta_3 \right) \\
P_{\text{gg}}^{(2)T}(N=2) &= -P_{\text{qg}}^{(2)T}(N=2) = \\
&- C_A^2 n_f \left( \frac{6232}{243} - \frac{2132}{27} \zeta_2 - \frac{128}{9} \zeta_3 + \frac{160}{3} \zeta_2^2 \right) \\
&+ C_A n_f^2 \left( \frac{2}{27} - \frac{160}{27} \zeta_2 + \frac{64}{9} \zeta_3 \right) - C_A C_F n_f \left( \frac{2681}{243} - \frac{760}{27} \zeta_2 + \frac{56}{9} \zeta_3 \right) \\
&- C_F^2 n_f \left( \frac{10570}{243} - \frac{352}{27} \zeta_2 - \frac{32}{9} \zeta_3 \right) - C_F n_f^2 \left( \frac{41}{9} - \frac{128}{27} \zeta_2 \right)
\end{aligned}$$

- First steps towards the complete time-like singlet splitting functions

# Summary

## Deep-inelastic scattering

- QCD precision predictions for electron-proton collision
  - wealth of experimental information on proton structure
  - radiative corrections for parton evolution

## $e^+e^-$ annihilation

- Theory results transferred to time-like evolution
  - successful recycling of DIS
  - use factorization and infrared safety (KLN) of observables
- Theory predictions
  - non-singlet to NNLO Mitov, S.M., Vogt '06
  - singlet (diagonal) to NNLO S.M., Vogt '07

## Outlook

- Complete time-like singlet splitting functions ( $P_{qg}^{(2)T}$ ,  $P_{gq}^{(2)T}$ ) ...