ICFA Beam–Beam Workshop CERN 2013

Analytical and Numerical Tools for Beam–Beam Studies

Mathias Vogt (DESY–MFL)

- **Intro**
- **Weak–Strong Beam–Beam (WSBB)**
- **A little bit on WSBB codes**
- **Strong–Strong Beam–Beam (SSBB)**
- **A little bit on SSBB codes**

. . . not necessarily in that strict order!

FLASH
Free−Electron Laser
in Hamburg

Beam Beam Models (Basics)

Immanent symmetry: "beam" $\leftrightarrow$ "other beam" $\Rightarrow$ **"other beam"** $=:$ **"beam*"**

We don't need the $\star$ to indicate IP–properties: **"at-the-IP" is the default for beam–beam–stuff!!**

- **Phase space:** $\vec{z} \in \mathbb{R}^{2n}$, $n = 1, 2, 3$
  $\{z_i\}_{i=1,\ldots,6} \rightarrow$
  $\rightarrow x, (a := p_x/p_0), y, (b := p_y/p_0), \tau, \delta$
- **Indep. var.** $\theta := 2\pi s/C$
- **Hamiltonian:**
  $H = H_0 + \sum_{i=1}^{N_{\rm IP}} a_{2\pi}(\theta - \theta_i) H_i^{\rm bb}$
- $a_{2\pi}(\theta) = a_{2\pi}(\theta + 2\pi) =$
  $\begin{cases} \delta_{2\pi}(\theta) & : \sigma_\tau \ll \beta_{x,y} \\ \text{loc. hump around } 0 & : \text{otherwise} \end{cases}$
- $a_{2\pi} \rightarrow \delta_{2\pi} \Rightarrow$
  $H_i^{\rm bb} \rightarrow U_i^{\rm bb}$ **(kick–potential)**
- extended $a_{2\pi}$: $H_i^{\rm bb} = T^{\rm free-space} + U_i^{\rm bb}$
  $\leftarrow$ **beam–waist**
  $\rightarrow$ **Hourglass–Effect**

- include **long. phase space (τ, δ)** $\Rightarrow$ potential **crossing angle**

... and more fun with **beam–waists!**

- Note of course : **Hamiltonian*:**
  $H^\star = H_0{}^\star + \sum_{i=1}^{N_{\rm IP}} a_{2\pi}{}^\star(\theta - \theta_i) H_i^{\rm bb\star}$
- $H_i^{\rm bb}$ can be **head–on** or **long–range** (a.k.a. "parasitic")
- $H_i^{\rm bb}$ can be **weak–strong** (beam* **fixed** from turn-to-turn)
- $H_i^{\rm bb}$ can be **strong–strong** (beam* **changes** from turn-to-turn due to beam)
- Some collision schemes (RHIC, Tevatron, **LHC!**) need to consider more than 1 bunch per beam!

Beam Beam Models ("Time"–Continuous)

For the moment : only one short bunch per beam and **head–on w/o crossing angle**, only one IP.

- **Phase space densities :** $\Psi(\vec{z},\theta)$ & $\Psi^\star(\vec{z},\theta)$
- **SSBB (the real thing!) :** dependence of H ($H^\star$) on $\Psi^\star$ (Ψ) :
  $H[\Psi^\star] = H_0 + U^{\mathrm{ss}}[\Psi^\star]$
  $H^\star[\Psi] = {H_0}^\star + U^{\mathrm{ss}\star}[\Psi]$
- via $\rho(\vec{q},\theta) := \int \Psi(\vec{q},\vec{p},\theta)\, d^n p$ & $\rho^\star(\vec{q},\theta) := \int \Psi^\star(\vec{q},\vec{p},\theta)\, d^n p$
- $U^{\mathrm{ss}}[\Psi^\star](\vec{q}) \propto \int G(\vec{q}-\vec{q}\,')\rho^\star(\vec{q}\,')d^n q'$, G : Green's function

$\Rightarrow$ Evolution of trajectories $\vec{z}(\theta)$, $\vec{z}^\star(\theta)$ needs **up to date** densities Ψ, $\Psi^\star$ (both!) : ($\underline{J}$: symplectic structure)

$$\boxed{\begin{aligned} \tfrac{d}{d\theta}\vec{z} &= \underline{J}\,\partial_{\vec{z}}\, H[\Psi^\star](\vec{z},\theta) \\ \tfrac{d}{d\theta}\vec{z}^\star &= \underline{J}\,\partial_{\vec{z}}\, H^\star[\Psi](\vec{z}^\star,\theta) \end{aligned}}$$

$\rightarrow$ so, why not skip the trajectories ?!

$$\boxed{\begin{aligned} \partial_t\Psi &= \{H[\Psi^\star],\Psi\} &\equiv (\partial_{\vec{z}}\Psi)^{\mathrm{T}}\underline{J}\,(\partial_{\vec{z}}H[\Psi^\star]) \\ \partial_t\Psi^\star &= \{H[\Psi],\Psi^\star\} &\equiv (\partial_{\vec{z}}\Psi^\star)^{\mathrm{T}}\underline{J}\,(\partial_{\vec{z}}H[\Psi]) \end{aligned}}$$

$\rightarrow$ SSBB coupled Vlasov–Poisson eq's

$\rightarrow$ **coupled system of 2 non–linear 1-st order PIDEs**

$\rightarrow$ Can treat coherent (and incoherent) motion and **collective** interactions

- **WSBB :** $\Psi^\star$ given & **fixed** $\forall$ turns
  $\rightarrow$ study only $\vec{z}(\theta)$ (and/or $\Psi(\vec{z},\theta)$)
  $\rightarrow$ $U^{\mathrm{ws}}(q) \equiv U^{\mathrm{ss}}[{\Psi^\star}_{\mathrm{fixed}}](q)$
- $\frac{d}{d\theta}\vec{z} = \underline{J}\,\partial_{\vec{z}}\,H^{\mathrm{ws}}(\vec{z},\theta)$ $\leftarrow$ Can. eq's
- $\partial_t\Psi = \{H^{\mathrm{ws}},\Psi\}$ $\leftarrow$ Liouville eq.
  $\rightarrow$ **linear 1-st order PDE**

$\rightarrow$ Can **NOT** treat collective effects.

Beam Beam Models (“Time”–Discrete WSBB)

- **WSBB :**
- $\frac{d}{d\theta}\vec{z} = \underline{J}\,\partial_{\vec{z}} H(\vec{z},\theta)$
  $\leftarrow$ **Hamiltonian Vectorfield**

$\Rightarrow$ $\vec{z}(\theta_i) \mapsto \vec{z}(\theta_f) \equiv \vec{M}_{\theta_f,\theta_i}(\vec{z}(\theta_i))$
$\leftarrow$ **Symplectic Flow**
$\underline{M}(\vec{z}_0) := \partial\vec{M}_{\theta_f,\theta_i}(\vec{z}_0) \in \mathbf{Sp}(2n) \forall \vec{z}_0 \in \mathbb{R}^{2n}$
$\vec{M}_{\theta,\theta} = \vec{\mathrm{Id}}$ (identity)

$\Rightarrow$ **Measure Preserving Flow** :
$\mu_\Psi(\mathcal{A}) = \mu_\Psi(\vec{M}(\mathcal{A}))\ \forall \mathcal{A} \in \mathbb{B}^{2n}$
i.a.w.: $\boxed{\Psi = \mathrm{const.}\text{ along trajectories}}$

$\leftarrow$ this is why **Liouville eq.** holds!

$\rightarrow$ Meth. o. Characteristics / P.F.–Meth.
$\Psi(\vec{z},\theta)$ at point $\vec{z}$ and “time” θ is given by $\Psi(\vec{M}^{-1}_{\theta,\theta_0}(\vec{z}),\theta_0)$ at an earlier “time” θ_0 and the **backward tracked** point $\vec{M}^{-1}_{\theta,\theta_0}(\vec{z}) \equiv \vec{M}_{\theta_0,\theta}(\vec{z})$

$\rightarrow$ linear(!) Perron–Frobenius Operator
$\mathcal{M} : \Psi \mapsto \Psi \circ \vec{M}^{-1}$

- **Discrete “time” maps :**
  restrict θ to discrete set $\{\theta_j\}_{j=1,\ldots}$
  $\boxed{\vec{z}_j := \vec{z}(\theta_j),\ \Psi_j(\vec{z}) := \Psi(\vec{z},\theta_j)}$
  $\boxed{\vec{M}_{f,i}(\vec{z}) := \vec{M}_{\theta_f,\theta_i}(\vec{z})}$
  and forget about $\theta \in \mathbb{R}$ . . .
- **OneTurnMap** (OTM, monodromy map)
  $\vec{T}_j(\vec{z}) := \vec{M}_{\theta_j+2\pi,\theta_j}(\vec{z})$
- Since $\mathbf{Sp}(2n)$ is connected, all symplectic $\mathcal{C}^1$ maps are connected to $\vec{\mathrm{Id}}$ (identity) and thus can all be a flow.

$\Rightarrow$ extra freedom : **use effective maps from θ_i to θ_f w/o caring what happens in–between!**

Beam Beam Models (“Time”–Discrete SSBB)

- from WSBB:

$$\Psi_f(\vec{z}) = (\mathcal{M}_{f,i}\Psi_i)(\vec{z})$$
$$= \left(\Psi_i \circ \vec{M}_{f,i}^{-1}\right)(\vec{z}) = \Psi_i(\vec{M}_{i,f}(\vec{z}))$$

- **SSBB :**

- For every given decent ψ ($\in \mathcal{L}^1$ & normalized) $\underline{J}\partial_{\vec{z}}H[\psi]$ **is a perfectly Hamiltonian V.F.** and defines the perfectly **Symplectic Flow** $\vec{M}[\psi]$

$\Rightarrow$ Thus (at least) the following model is perfectly well defined:

- **BB–Kick & Lattice** (One IP) :

- $\vec{T}[\Psi^\star] := \vec{L} \circ \vec{K}[\Psi^\star]$

$$\vec{K}[\Psi^\star] := \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} \mapsto \begin{pmatrix} \vec{q} \\ \vec{p} - \partial_{\vec{q}} U[\rho^\star](\vec{q}) \end{pmatrix}$$

$\vec{L}$ represents the lattice w/o collective effects

$\Rightarrow$ $\vec{T}[\Psi^\star]^{-1} = \vec{K}[\Psi^\star]^{-1} \circ \vec{L}^{-1}$ (inv. OTM)

$\Rightarrow$ $\mathcal{T}[\Psi^\star] : \Psi \mapsto \Psi \circ \vec{T}[\Psi^\star]^{-1}$ (P.F.)

$\Rightarrow$ Evolution from n-th turn to $(n+1)$-st :

$$\boxed{\begin{aligned} \Psi_{n+1}(\vec{z}) &= \Psi_n\left(\vec{K}[\Psi^\star_n]^{-1}\left(\vec{L}^{-1}(\vec{z})\right)\right) \\ \Psi^\star_{n+1}(\vec{z}) &= \Psi^\star_n\left(\vec{K}[\Psi_n]^{-1}\left(\vec{L}^{-1}(\vec{z})\right)\right) \end{aligned}}$$

- Extension to more IPs straight forward!

- Example : HERA with “hadronic leptons”

  $\rightarrow$ needs only one bunch per beam

  2×2 arcs: $\vec{L}_e^{\,W}$, $\vec{L}_e^{\,E}$, $\vec{L}_p^{\,W}$, $\vec{L}_p^{\,E}$

  2×2 bb–kicks:

  $\vec{K}_e[\Psi^{p,N}]$, $\vec{K}_e[\Psi^{p,S}]$, $\vec{K}_p[\Psi^{e,N}]$, $\vec{K}_p[\Psi^{e,S}]$

"Time"–Discrete SSBB : HERA–Example

- 2×2 arcs: $\vec{L}_e{}^W$, $\vec{L}_e{}^E$, $\vec{L}_p{}^W$, $\vec{L}_p{}^E$ $(e^{\pm}, p) \times$(West, East)
- 2×2 bb–kicks: $\vec{K}_e[\Psi^{p,N}]$, $\vec{K}_e[\Psi^{p,S}]$, $\vec{K}_p[\Psi^{e,N}]$, $\vec{K}_p[\Psi^{e,S}]$ $(e^{\pm}, p) \times$(North, South)
- Evolution of Ψ^e and Ψ^p over $2n$ **half turns**:

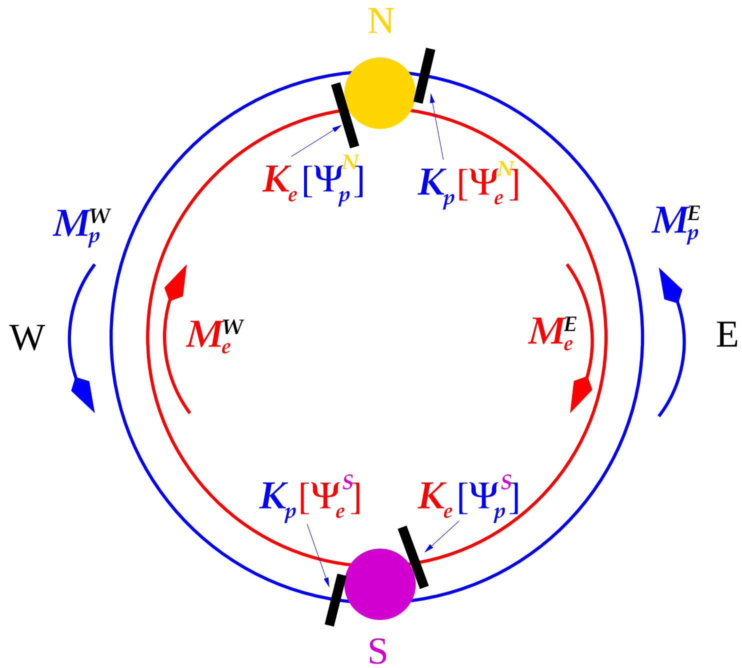
1:N→S: $\Psi^{e,S}_n = \Psi^{e,N}_n \circ \vec{K}_e{}^{-1}[\Psi^{p,N}_n] \circ \vec{L}_e{}^{O^{-1}}$ $\Psi^{p,S}_n = \Psi^{p,N}_n \circ \vec{K}_p{}^{-1}[\Psi^{e,N}_n] \circ \vec{L}_p{}^{W^{-1}}$

2:S→N: $\Psi^{e,N}_{n+1} = \Psi^{e,S}_n \circ \vec{K}_e{}^{-1}[\Psi^{p,S}_n] \circ \vec{L}_e{}^{W^{-1}}$ $\Psi^{p,N}_{n+1} = \Psi^{p,S}_n \circ \vec{K}_p{}^{-1}[\Psi^{e,S}_n] \circ \vec{L}_p{}^{O^{-1}}$

⇒ No fundamental difference between 2 IPs and 1 IP

⇒ Just more intricate dependence on the lattice parameters

- There's more complicated examples: RHIC, Tevatron, **LHC!!!**
- Also: approximate extended BB waists with (kick→drift→)^k, $k > 1$.



The Rigid Bunch Model (RBM)

. . . just for completeness: the **Rigid Bunch Model (RBM) :**

- Quick and dirty: only centroid motion
- However, well suited for **first multi $(= N)$ bunch & multi $(= M)$ IP** analysis :
- One "macro particle" $\vec{z}_i$ per bunch$_i$ and WS–like interaction potential for crossing of i–th and j–th bunch at l-th IP $U_l(\vec{q}_i - \vec{q}_j)$
- Further simplification : linearization, no long. & uncoupled, kick
  $\rightarrow$ study (x, a) and (y, b) plane separately

$\Rightarrow$ e.g. $\vec{K}_l[\vec{z}^\star](\vec{z}) = \begin{pmatrix} 1 & 0 \\ -\kappa_l & 1 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ +\kappa_l \, q^\star \end{pmatrix}$ and vice versa $(\vec{z} \leftrightarrow \vec{z}^\star)$

- Now glue together: bunches $\vec{Z} := \vec{z}_1 \oplus \vec{z}_2 \oplus \ldots \oplus \vec{z}_N$, sections of lattice $\underline{M}_l := \underline{L}_l^1 \oplus \underline{L}_l^2 \oplus \ldots \oplus \underline{L}_l{}^N$ and join with IPs $\underline{K}_l$ (bunch-to-bunch coupling)

$\rightarrow$ linear stability analysis of $2N \times 2N$ OTM $\quad \underline{T} := \underline{K}_1 \underline{M}_1 \ldots \underline{K}_M \underline{M}_M$

The Absolutely Most Famous Results from Linear WSBB :-)

- unperturbed linear OTM seen from IP ($\alpha = 0$):
- $\underline{T}_0 := \begin{pmatrix} \cos(2\pi Q_0) & \beta_0 \sin(2\pi Q_0) \\ -\sin(2\pi Q_0)/\beta_0 & \cos(2\pi Q_0) \end{pmatrix}$
- insert linear (focusing) WSBB kick $\underline{K} := \begin{pmatrix} 1 & 0 \\ -\kappa & 1 \end{pmatrix}$ before IP
- with κ from $\boxed{\kappa_{x,y} = \frac{2N^\star r_p}{\gamma} (\sigma^\star{}_{x,y}(\sigma^\star{}_x + \sigma^\star{}_y))^{-1}}$

$\Rightarrow \underline{T} := \underline{T}_0 \, \underline{K} = \begin{pmatrix} \cos(2\pi Q_0) - \beta_0 \sin(2\pi Q_0)\kappa & \beta_0 \sin(2\pi Q_0) \\ -\sin(2\pi Q_0)/\beta_0 - \cos(2\pi Q_0)\kappa & \cos(2\pi Q_0) \end{pmatrix}$

$\Rightarrow \cos(2\pi Q) = \frac{1}{2}\mathrm{trace}\underline{T} = \cos(2\pi Q_0) - \frac{\beta_0\kappa}{2}\cos(2\pi Q_0)$

$\Rightarrow$ Perturbed tune $Q = Q_0 + \frac{\beta_0\kappa}{4\pi} + O(\kappa^2)$

- **Linear Beam–Beam Tuneshift Parameter** $\boxed{\xi := \frac{\beta_0\kappa}{4\pi}}$

Famous Results from WSBB

- Purely transverse motion, head–on
- **Round Gaussian Beam:**
  $\rho(r) = \frac{1}{2\pi\sigma_r^2}\exp\left(-\frac{r^2}{2\sigma_r^2}\right)$
  $\rightarrow$ kick $\Delta r' \propto 1/r\left(1-\exp\left(-\frac{r^2}{2\sigma_r^2}\right)\right)$
- **Elliptic Gaussian Beam:**
  $\rho(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$
  $\rightarrow$ **Bassetti–Erskine!** $\rightarrow$contains complex error function $\rightarrow$ numerically slow

$\leftarrow$ both however have
$U(x,y) = U(-x,y) = U(x,-y)$

$\Rightarrow$ Only resonances $2k_xQ_x + 2k_yQ_y = k_0$ are driven by H–O collisions w/o crossing angle

- Long–range drives also odd reson.
- Crossing angle$\rightarrow$ sidebands
  $k_xQ_x + k_yQ_y + k_sQ_s = k_0,\ k_x + k_y + k_s = 2k$

- **Canonical Averaging**
  $\rightarrow$ **Tune Footprint** $\vec{Q}(\vec{J})$

$\rightarrow$ neat feature: detuning $\rightarrow$0 at infinite amplitudes

- Phase space close to h.o. resonances might be subject to action diffusion

$\rightarrow$ driven by beam beam + (any of: orbit jitter, multipoles, external noise, $\emptyset$,. . .)

$\rightarrow$ The full machinery of the canonical incoherent resonance analysis needed !

$\rightarrow$ recent paper by T.Sen (PRSTAB, **15** 101001 (2012)) on “Anomalous beam diffusion near beam-beam synchrobetatron resonances”

WSBB Tracking

- In principle every "single particle" tracking code may implement beam–beam lenses.
- However, while Round Gaussian Beams are relatively cheap, the complex error function needed for Elliptic Gaussian Beams is a major pain!
- Long beam waists can effectively be approximated by kick–drift expansions
- Crossing angle can be treated by Lorentz–boosting into the rest system of the lens (and back)
- Fairly complete 6d description is in: Leunissen, Schmidt, Ripken, PRSTAB **3** 124002 (2000)
- BB–compensation (H–O & L–R) : electron lenses & electric wires
- Typical codes are, to my recognition, MAD, sixtrack, BBsim, Lifetrack, PTC
- Leptons : include damping and stoch. excitation

Famous Results from SSBB

- SSBB coupled Vlasov–Poisson eq's = coupled system of 2 non–linear 1-st order partial integro–differential equations $\Rightarrow$ solving them analytically is quite some challenge.
- Standard procedure(s):
  Linearization about equilibrium.
  $\rightarrow$ Which equilibrium? $\rightarrow$ **averaging**
  $\rightarrow$ equilibria $\Psi_{\text{eq}}(\vec{J})$ of the averaged system give quasi–equilibria of the exact system. $\{\overline{H}[\Psi_{\text{eq}}{}^\star], \Psi_{\text{eq}}\} = 0$
- Linearize around $\Psi_{\text{eq}}(\vec{J})$:
  $\Psi_n(\vec{z}) = \Psi_{\text{eq}}(\vec{J}) + \Phi_n(\vec{z}) \Rightarrow$
  $\partial_t \Phi_n = \{\overline{H}[\Psi_{\text{eq}}{}^\star], \Phi_n\} + \{\overline{H}[\Phi^\star{}_n], \Psi_{\text{eq}}\}$
  $\partial_t \Phi^\star{}_n = \{\overline{H}[\Psi_{\text{eq}}], \Phi^\star{}_n\} + \{\overline{H}[\Phi_n], \Psi_{\text{eq}}{}^\star\}$
- Decouple by introducing Eigenmodes for 2 and/or more bunches

$\Rightarrow \partial_t f_n = \{\overline{H}[F_{\text{eq}}], f_n\} + \{\overline{H}[f_n], F_{\text{eq}}\}$

- Laplace in t and Fourier in angles $\vec{\varphi}$ (or similar)

$\rightarrow$ **Fredholm type integral equation for the harmonics**

- There's a multitude of slightly different **Linearized Averaged Vlasov Models**: see e.g. Chao, Yokoya/Koiso, Alexahin, Ellison/Sobol/Vogt, . . .

$\rightarrow$ Theory and observation suggest:
For moderate BB parameter, civilized equilibria (not unique!) the plain collective beam–beam modes are at best neutrally stable.
I.a.w.: they don't grow unless externally driven.

SSBB Tracking

- when people want all at the same time...
  high resolution for Ψ, for $U[\Psi]$, maybe in 6d with beam–beam waists and crossing angles, including multi–bunch and multi–IP schemes and lattice non–linearities and for many turns and all that in little time

... then things become a little tough ! However if one puts up with only parts of that,

1. There's some **Perron–Frobenius codes** that evolve Ψ_n, $\Psi^{\star}{}_n$ on a grid :
   (Bob Warnock's code(s), Andrey Sobol's code, and my BBPF, and probably more...)
2. There's many **Macro–Particle codes** that evolve ensembles of particles :
   (Ji Quiang's massive parallel code BeamBeam3D, Kazuhito Ohmi's code, Werner Herr et al., Y.-H. Cai's code, my BBDeMo,...)

- Every code needs an adapted, **fast** & **accurate Poisson solver**!
- Relation Perron–Frobenius $\leftrightarrow$ Macro–Particle Tracking:
  given $\Psi_f(\vec{z}) = \Psi_i(\vec{M}^{-1}(\vec{z}))$, compute expectation values = integrals :
  $E_f[g] := \int g(\vec{z})\, \Psi_f(\vec{z})\, d^{2n}z = \int g(\vec{z})\, \Psi_i(\vec{M}^{-1}(\vec{z}))\, d^{2n}z = \int g(\vec{M}(\vec{z}))\, \Psi_i(\vec{z})\, d^{2n}z$
- Leptons : try operator splitting : Perron–Frobenious for Vlasov and finite–difference for Fokker–Planck ($\rightarrow$ R.L.Wanock, M.–P.Zorzano)

Summary

- The growing hunger of the experiments for Luminosity assures beam beam theory & simulation will be hot topics as long as colliders are built/operated!

← BB can drive resonances and action diffusion and thus severely degrade beam- & luminosity–lifetime, and background conditions at the experiments.

← It can however, also help provide (incoherent) tune spread and Landau damping.

← Coherent, collectively driven beam–beam modes have been predicted by theory and simulation and have been observed in real machines.

- It appears however, that in many cases they are not by-themselves unstable, i.e. growing.
- Instead they often tend to be either Landau damped or neutrally stable.
- Collective BB–modes are an active interesting field.
- Progress in parallel computing will strongly enhance the simulations in the strong–strong regime.