Proton structure functions at small x:

recent results using NLO BFKL and DGLAP evolution

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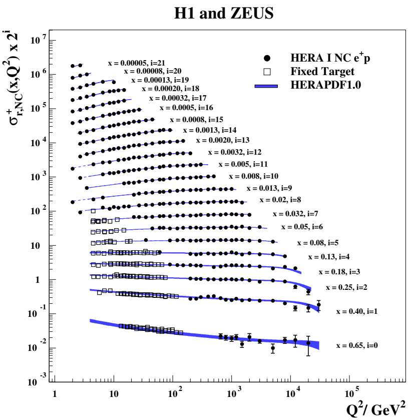
Based on:

- **arXiv:1209.1353**
  [PRL 110 (2013) 041601]
- **arXiv:1301.5283**
- **in preparation**

Outline



DeepInelasticScattering at small x

collinear factorization: excellent description

$$F_{2,L}(x, Q^2) = \text{coeff. funct.} \otimes \text{pdfs}$$

Q^2 dependence: DGLAP evolution

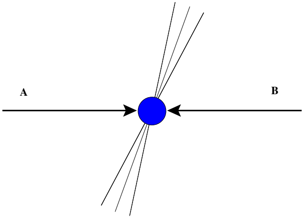
rise at small x

- DGLAP splitting functions
- intrinsic (non-pert.?) x-dependence of initial conditions

Question: Can we understand this x-dependence from first principles?

Perturbative analysis of high energy scattering

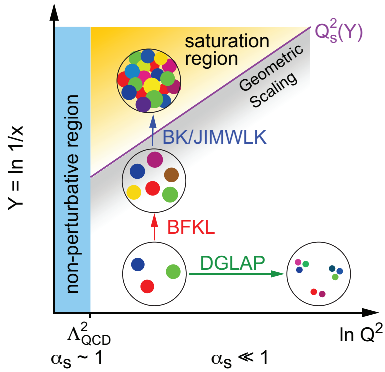
- multi scale regime $s \gg Q^2 \gg \Lambda_{\mathsf{QCD}}^2$
- $\alpha_s \ln(s/s_0) \sim 1$
  resummation $\rightarrow$ BFKL equation



- LL $\sum_n \left(\alpha_s \ln s\right)^n$ [Fadin, Kuraev, Lipatov (1977)], [Balitsky, Lipatov (1978)]
- NLL $\sum_n \alpha_s \left(\alpha_s \ln s\right)^n$ [Fadin, Lipatov (1998)], [Ciafaloni, Gamici (1998)]

DIS: $\ln \frac{s}{Q^2} \equiv \ln \frac{1}{x}$

Life beyond BFKL: high density region

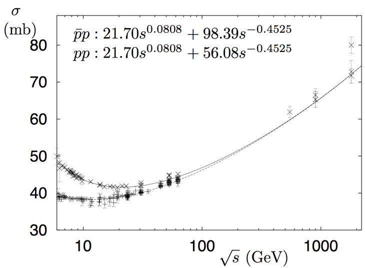


[EIC White Paper, arXiv:1212.1701]

here:

- restrict to BFKL at NLO
  no saturation effects considered
- breakdown at ultra small $x \equiv$ evidence for saturation/CGC
- expect deviation for DIS on nuclear target
- higher sensitivity of Linear small x evolution to non-linear effects than *e.g.* DGLAP

BFKL $\equiv$ interesting physics by itself



[Landshoff (2007)]

strong int$^{a.}$ before QCD: soft Pomeron

$$\sigma(s) \sim s^{\alpha}$$

$\alpha \simeq .08$ soft Pomeron intercept

perturbative realization: BFKL

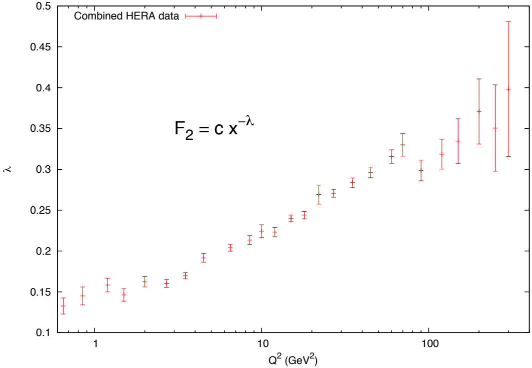
DIS: effective intercept

$$\lambda(Q^2) = \left\langle \frac{d\ln F_2}{d\ln 1/x} \right\rangle_x$$

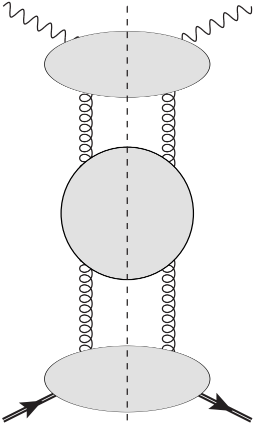
- transition between soft and hard 'Pomeron'
- natural description in pert. QCD?
- NLO BFKL study [Ellis, Kowalski, Ross (2008)]; [Kowalski, Lipatov, Ross, Watt (2010)]

  'Discrete Pomeron Solution';

  successful description of combined HERA data



high energy factorization (up to NLL) of F_2



photon impact factor

$$F_2 = \int \frac{d\boldsymbol{q}^2}{\boldsymbol{q}^2} \frac{d\boldsymbol{p}^2}{\boldsymbol{p}^2} \Phi_{\gamma^*}\left(\frac{\boldsymbol{q}^2}{Q^2}, \mu^2, s_0\right) f_{\text{BFKL}}\left(\frac{s}{s_0}, \boldsymbol{q}^2, \boldsymbol{p}^2, \mu^2\right) \times \Phi_{\text{proton}}\left(\frac{\boldsymbol{p}^2}{Q_0^2}, s_0\right)$$

gluon Green's function

proton impact factor

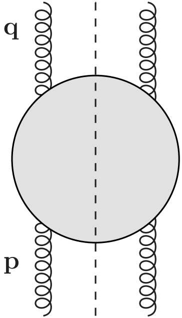
convolution in k_T space

s_0: reggeization scale

μ: renormalization scale

The BFKL gluon Green's function - scale choice

$$f_{\mathrm{BFKL}}\left(\frac{s}{s_0}, \boldsymbol{q}^2, \boldsymbol{p}^2, \mu^2\right) = \int \frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega} f_\omega\left(\boldsymbol{q}^2, \boldsymbol{p}^2, \mu^2\right)$$



symmetric kinematic $\boldsymbol{q}^2 \sim \boldsymbol{p}^2$

$\ln(s/s_0)$ ➡ $\Delta y \simeq \ln \frac{s}{|\boldsymbol{q}||\boldsymbol{p}|}$

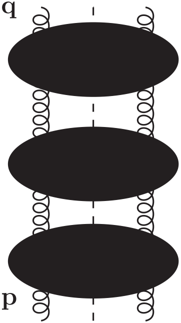
asymmetric DIS kinematic $\boldsymbol{q}^2 \gg \boldsymbol{p}^2$

DGLAP suggests

$\ln(s/s_0)$ ➡ $\ln 1/x_g \simeq \ln s/\boldsymbol{q}^2$

The BFKL gluon Green's function - BFKL equation

$$f_{\mathrm{BFKL}}\left(\frac{s}{s_0}, \boldsymbol{q}^2, \boldsymbol{p}^2, \mu^2\right) = \int \frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega} f_\omega\left(\boldsymbol{q}^2, \boldsymbol{p}^2, \mu^2\right)$$



BFKL equation

$$\omega f_\omega\left(\boldsymbol{q}^2, \boldsymbol{p}^2\right) = \delta^{(2)}(\boldsymbol{q}^2 - \boldsymbol{p}^2) + \int \frac{d^2\boldsymbol{k}}{\pi} K_{\mathrm{BFKL}}(\boldsymbol{q}, \boldsymbol{k}) f_\omega\left(\boldsymbol{k}^2, \boldsymbol{p}^2\right)$$

BFKL Kernel

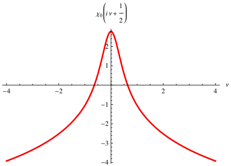
$$K_{\mathrm{BFKL}}(\boldsymbol{q}, \boldsymbol{k}) = \left(\frac{\alpha_s N_c}{\pi}\right) K_{\mathrm{LO}}(\boldsymbol{q}, \boldsymbol{k}) + \left(\frac{\alpha_s N_c}{\pi}\right)^2 K_{\mathrm{NLO}}(\boldsymbol{q}, \boldsymbol{k})$$

LO diagonalized by scale invariant eigenfunctions

$$\int \frac{d^2\boldsymbol{q}}{\pi} K_{\mathrm{LO}}(\boldsymbol{q}, \boldsymbol{p})(\boldsymbol{p}^2)^{\gamma-1} = \frac{\alpha_s N_c}{\pi}\chi_0(\gamma)(\boldsymbol{q}^2)^{\gamma-1}$$

Leading order solution

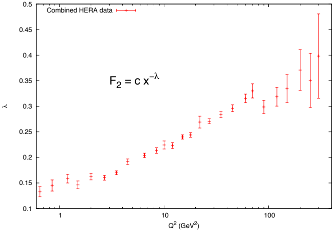
$$f^{\text{LO}}_{\text{BFKL}}\left(\frac{1}{x}, \boldsymbol{q}^2, \boldsymbol{p}^2, \mu^2\right) = \frac{1}{\boldsymbol{q}^2}\int_{1/2-i\infty}^{1/2+i\infty} \frac{d\gamma}{2\pi^2 i}\left(\frac{\boldsymbol{q}^2}{\boldsymbol{p}^2}\right)^{\gamma} x^{-\bar{\alpha}_s\chi_0(\gamma)}$$



$$\gamma = \frac{1}{2} + i\nu$$

intercept at saddle point $\nu = 0$: $\lambda^{\text{LO}}_{s.p.} = \frac{\alpha_s N_c}{\pi} 4\ln 2 \simeq .53$ for $\alpha_s = 0.2$

$\simeq .40$ for $\alpha_s = 0.15$



- order of magnitude correct
- adding naive running of the coupling $\alpha_s(Q^2)$ entirely contradicts data

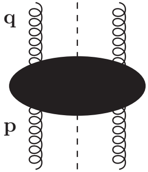
NLO corrections: scale invariant terms

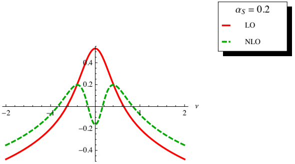
$$\int \frac{d^2\boldsymbol{p}}{\pi}\left(\frac{\boldsymbol{p}^2}{\boldsymbol{q}^2}\right)^{\gamma-1} K^{\text{sc.inv.}}(\boldsymbol{q},\boldsymbol{p}) = \chi^{\text{NLO}}(\gamma,\mu^2) =$$

$$= \bar{\alpha}_s(\mu^2)\left[\chi_0(\gamma) + \bar{\alpha}_s(\mu^2)\left(\chi_1(\gamma) - \frac{1}{2}\chi_0(\gamma)\chi_0'(\gamma)\right)\right]$$

$\uparrow$

NLO correction due to asymmetric s_0



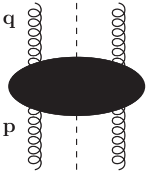


- negative NLO intercept in DIS kinematics
  $\lambda^{\text{NLO}}_{s.p.} \simeq -0.16$ for $\alpha_s = 0.2$
  $\lambda^{\text{NLO}}_{s.p.} \simeq .01$ for $\alpha_s = 0.15$
- numerical instability
- reason: large (double) logs for (anti-) collinear limit of external momoenta
- γ rep.: (triple/double) poles at $\gamma = 0, 1$

(anti-)collinear poles of the NLO eigenvalue

$$\chi^{\text{NLO}}(\gamma) \simeq \frac{\bar{\alpha}_s(1+\alpha_s a)}{\gamma} + \frac{\bar{\alpha}_s^2 b}{\gamma^2} + \frac{\bar{\alpha}_s(1+\alpha_s a)}{1-\gamma} + \frac{\bar{\alpha}_s^2 b}{(1-\gamma)^2} - \frac{\bar{\alpha}_s^2}{(1-\gamma)^3}$$

treatment: collinear factorization



- $\frac{1}{(1-\gamma)^3} \quad \Leftrightarrow \quad \boldsymbol{p}^2 \gg \boldsymbol{q}^2$
- choice $s/s_0 = 1/x_g$ dictated by DGLAP for $\boldsymbol{q}^2 \gg \boldsymbol{p}^2$

  $\boldsymbol{p}^2 \gg \boldsymbol{q}^2$ DGLAP suggests: $\left(1/x_g\right)^\omega \left(\boldsymbol{q}^2/(\boldsymbol{p}^2)\right)^\omega$,

fix at LL : replace $\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$

$$\rightarrow \quad \tilde{\chi}_0(\gamma,\omega) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma+\omega)$$

[Andersson, Gustafson, Samuelsson (1996)]

expand $\omega = \bar{\alpha}_s \tilde{\chi}_0(\gamma,\omega) = \bar{\alpha}_s \chi_0(\gamma) - \frac{\bar{\alpha}_s^2}{(1-\gamma)^3} + \mathcal{O}(\bar{\alpha}_s^3)$

(anti-)collinear poles of the NLO eigenvalue

$$\chi^{\text{NLO}}(\gamma) \simeq \frac{\bar{\alpha}_s(1+\alpha_s a)}{\gamma} + \frac{\bar{\alpha}_s^2 b}{\gamma^2} + \frac{\bar{\alpha}_s(1+\alpha_s a)}{1-\gamma} + \frac{\bar{\alpha}_s^2 b}{(1-\gamma)^2} - \frac{\bar{\alpha}_s^2}{(1-\gamma)^3}$$

coefficients a, b closely related to high energy limit of (N)LO DGLAP splitting kernels
➡ resummation [Salam (1998)]

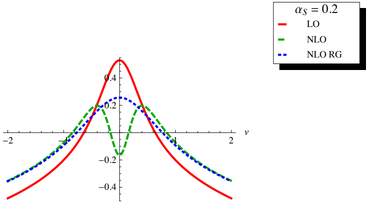
$$\begin{aligned}\omega &= \bar{\alpha}_s(1+A\bar{\alpha}_s)\left(2\psi(1) - \psi\left(\gamma + B\bar{\alpha}_s\right) - \psi\left(1-\gamma+\omega+B\bar{\alpha}_s\right)\right) \\ &= \bar{\alpha}_s(1+A\bar{\alpha}_s)\sum_{m=0}^{\infty}\left(\frac{1}{\gamma+m+B\bar{\alpha}_s} + \frac{1}{1-\gamma+m+\omega+B\bar{\alpha}_s} - \frac{2}{m+1}\right).\end{aligned}$$

disentangle complicated ω-dependence through ‘all-pole-approximation’ [Sabio Vera (2005)], [MH, Salas, Sabio Vera (2012)] ➡ resummed NLO eigenvalue

Note: do not add new information, only resum NLO BFKL divergenceses (in contrast to [Ciafaloni, Colferai, Salam, Stasto (2003), (2004), (2006), (2007)], [Altarelli, Ball, Forte (2003), (2004), (2005), (2008)])

(anti-)collinear poles of the NLO eigenvalue

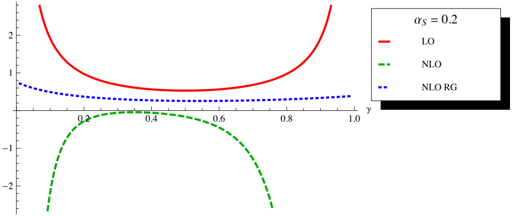
$$\chi(\gamma) = \bar{\alpha}_s \chi_0(\gamma) + \bar{\alpha}_s^2 \chi_1(\gamma) - \frac{1}{2}\bar{\alpha}_s^2 {\chi_0}'(\gamma)\chi_0(\gamma) + \chi_{\mathsf{RG}}(\bar{\alpha}_s, \gamma, a, b),$$

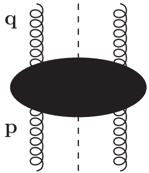


with

$$\chi_{\mathsf{RG}}(\bar{\alpha}_s, \gamma, a, b) = \bar{\alpha}_s(1 + a\bar{\alpha}_s)\left(\psi(\gamma) - \psi(\gamma - b\bar{\alpha}_s)\right) - \frac{\bar{\alpha}_s^2}{2}\psi''(1-\gamma) - b\bar{\alpha}_s^2 \frac{\pi^2}{\sin^2(\pi\gamma)}$$
$$+ \frac{1}{2}\sum_{m=0}^{\infty}\left(\gamma - 1 - m + b\bar{\alpha}_s - \frac{2\bar{\alpha}_s(1+a\bar{\alpha}_s)}{1-\gamma+m} + \sqrt{(\gamma-1-m+b\bar{\alpha}_s)^2 + 4\bar{\alpha}_s(1+a\bar{\alpha}_s)}\right).$$

Weak k_T ordering at NLO





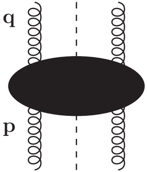
asymmetric scale choice s/s_0 ➡ $K(\boldsymbol{q}, \boldsymbol{q}) = \int \frac{d\gamma}{2\pi i} \frac{1}{\boldsymbol{q}^2} \left(\frac{\boldsymbol{q}^2}{\boldsymbol{p}^2}\right)^{\gamma} \chi(\gamma)$:

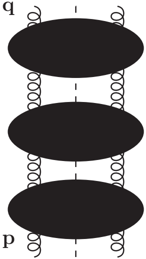
- breaks at NLO the symmetry $\boldsymbol{q} \leftrightarrow \boldsymbol{p}$
- $\boldsymbol{q}^2 > \boldsymbol{p}^2$ in average ➡ weak ordering ➡ IR stability

running coupling corrections with external scale dependence

$$\int \frac{d\gamma}{2\pi i} \frac{1}{\boldsymbol{q}^2} \left(\frac{\boldsymbol{q}^2}{\boldsymbol{p}^2}\right)^{\gamma} \bar{\alpha}_s \chi_0(\gamma) \left(-\frac{\alpha_s \beta_0}{4\pi} \ln \frac{|\boldsymbol{q}||\boldsymbol{p}|}{\mu^2}\right)$$

do not exponeniate correction → treated as pure NLO

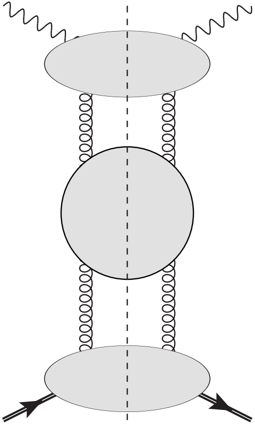




NLO resummed Green's function:

$$f_{\mathsf{BFKL}}\left(\frac{1}{x_g}, \boldsymbol{q}^2, \boldsymbol{p}^2\right) = \int \frac{d\gamma}{2\pi^2 i} \frac{1}{\boldsymbol{q}^2} \left(\frac{\boldsymbol{q}^2}{\boldsymbol{p}^2}\right)^{\gamma} x_g^{-\chi(\gamma)} \left[1 - \ln\left(\frac{1}{x_g}\right) \frac{\bar{\alpha}_s^2 \beta_0 \chi_0(\gamma)}{4N_c} \ln \frac{|\boldsymbol{q}||\boldsymbol{p}|}{\mu^2}\right]$$

high energy factorization (up to NLL) of F_2



photon impact factor

$\downarrow$

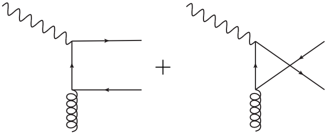
$$F_2 = \int \frac{d\boldsymbol{q}^2}{\boldsymbol{q}^2}\frac{d\boldsymbol{p}^2}{\boldsymbol{p}^2}\Phi_{\gamma^*}\left(\frac{\boldsymbol{q}^2}{Q^2},\mu^2\right) f_{\mathsf{BFKL}}\left(\frac{1}{x_g},\boldsymbol{q}^2,\boldsymbol{p}^2,\mu^2\right) \quad\times\quad \Phi_{\mathsf{proton}}\left(\frac{\boldsymbol{p}^2}{Q_0^2}\right)$$

$\uparrow$ gluon Green's function

$\uparrow$ proton impact factor

photon impact factor

photon at LO



strict LO

- $\frac{\alpha_s(\mu^2)}{2\pi}\Phi_{\gamma^*}(\gamma)$
- $\lim_{\gamma\to 0}\Phi_{\gamma^*}(\gamma) = \frac{1}{\gamma^2}\gamma_{qg}(N=1) + \ldots,$

improved kinematics

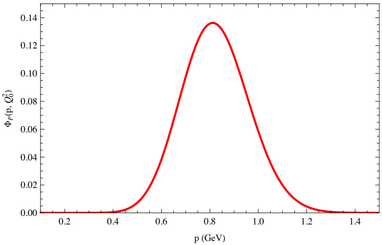
- $\frac{\alpha_s(\mu^2)}{2\pi}\Phi_{\gamma^*}(\gamma, \omega = \chi(\gamma)),$
  [Kwiecinski, Martin, Stasto (1997)], [Bialas, Navelet, Peschanski (2001)]
- $\lim_{\gamma\to 0}\Phi_{\gamma^*}(\gamma, \omega) = \frac{1}{\gamma^2}\gamma_{qg}(N = 1+\omega) + \ldots$

proton impact factors

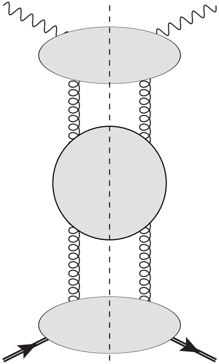
proton: Poisson distribution

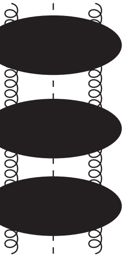
$$\Phi_P\left(p, Q_0^2\right) = \frac{\mathcal{C}}{\Gamma(\delta)}\left(\frac{p^2}{Q_0^2}\right)^{\delta} e^{-\frac{p^2}{Q_0^2}}$$

Normalization $\mathcal{C}$ and parameters Q_0^2, δ to be determined from fit to data



Setting the renormalization scale





Several choices possible: ...

here: $\mu^2 = Q \cdot Q_0$

symmetric choice: both hard and soft scale

believe: takes into account best t-channel virtualities

Green's function:

$$f_{\mathsf{BFKL}}\left(\frac{1}{x_g}, \boldsymbol{q}^2, \boldsymbol{p}^2\right) = \int \frac{d\gamma}{2\pi^2 i} \frac{1}{\boldsymbol{q}^2} \left(\frac{\boldsymbol{q}^2}{\boldsymbol{p}^2}\right)^{\gamma} x_g^{-\chi(\gamma)}$$

$$\left[1 - \ln\left(\frac{1}{x_g}\right) \frac{\bar{\alpha}_s^2 \beta_0 \chi_0(\gamma)}{4N_c} \ln \frac{|\boldsymbol{q}||\boldsymbol{p}|}{\mu^2}\right]$$

Running coupling corrections

- Observation 1: can obtain description of data above $Q^2 = 5$ GeV^2 possible, but requires extreme proton scales $Q_0^2 \simeq 1$ GeV^2 and above (applies also to $\mu^2 = Q^2 etc.$)

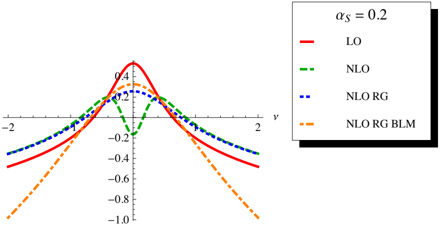
- Observation 2: more natural description, if one follows [Brodsky, Fadin, Kim, Lipatov, Pivovarov (1999)]:

  - non-Abelian physical renormalization scheme ('MOM-scheme') in 'Yennie-gauge', $\xi = 3$ - absorb entire (and sizeable) β_0 dependent terms of $\chi_{\rm NLO}(\gamma)$ into running coupling

$$\alpha_s\left(QQ_0\right) \to \alpha_s\left(QQ_0, \gamma\right) = \frac{4N_c}{\beta_0\left[\log\left(\frac{QQ_0}{\Lambda^2}\right) + \frac{1}{2}\chi_0(\gamma) - \frac{5}{3} + 2\left(1 + \frac{2}{3}Y\right)\right]}$$

modified $\chi_1(\gamma)$ and coefficents of RG term

$\chi_1(\gamma) \to \tilde{\chi_1}(\gamma) \qquad a, b \to \tilde{a}, \tilde{b}$



parametrization in the infra-red

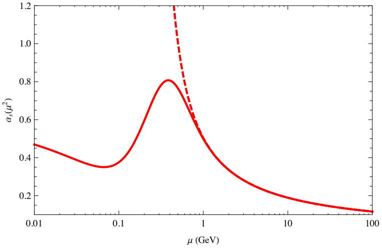
Useful, but not essential:

parametrization of the running coupling [Webber (1998)]

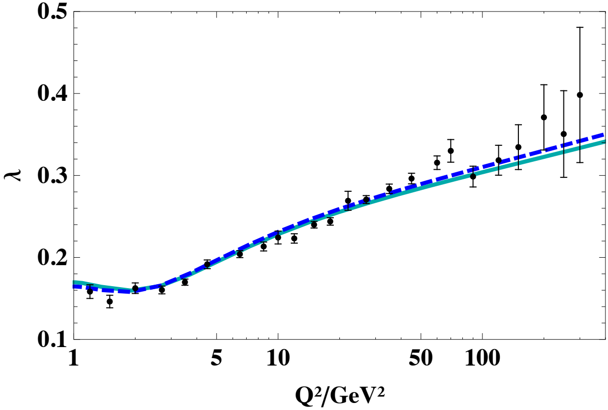
- compatible with power corrections to jet observables, preferable over simple freezing
- for the data: helps for $Q^2 < 4$ GeV^2

$$\alpha_s\left(\mu^2\right) = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}} + f\left(\frac{\mu^2}{\Lambda^2}\right)$$

$$f\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{4\pi}{\beta_0}\;\frac{125\left(1+4\frac{\mu^2}{\Lambda^2}\right)}{\left(1-\frac{\mu^2}{\Lambda^2}\right)\left(4+\frac{\mu^2}{\Lambda^2}\right)^4}.$$

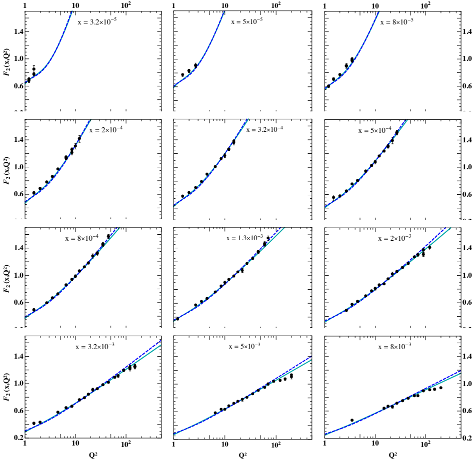


Comparison with HERA data - Pomeron intercept

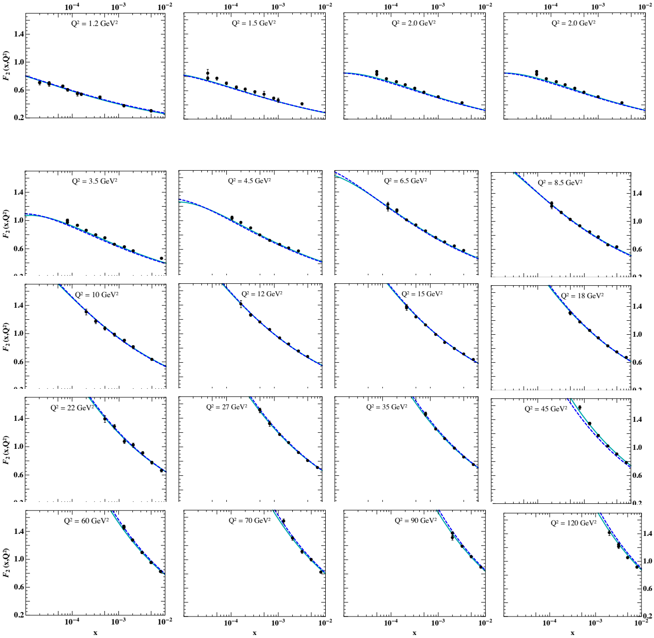


- MOM scheme with gauge parameter $\xi = 3$ ('Yennie-gauge'), $\Lambda_{\rm QCD}^{\rm MOM} = 0.21$GeV
- LO photon (**straight**): $\delta = 8.4$, $Q_0 = 0.28$ GeV
- photon with improved kinematics: (**dashed**): $\delta = 6.5$, $Q_0 = 0.28$ GeV

Comparison with HERA data - F_2

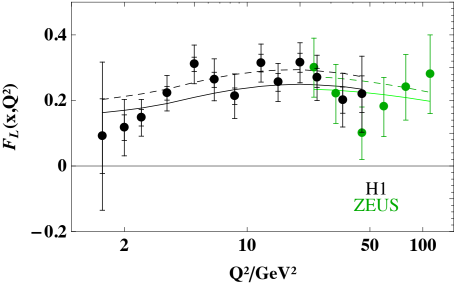


- very accurate description of data for both LO photon (**straight**) [$\delta = 8.4$, $Q_0 = 0.28$ GeV, $\mathcal{C} = 1.50$] and photon with improved kinematics (**dashed**) [$\delta = 6.5$, $Q_0 = 0.28$ GeV, $\mathcal{C} = 2.39$]
- find expected deviations at large x and large Q^2



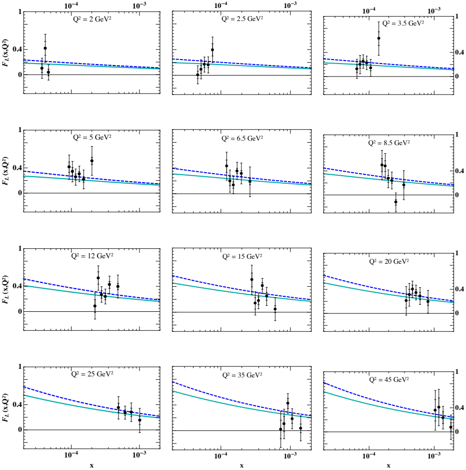
Q² = 1.2 GeV²
Q² = 1.5 GeV²
Q² = 2.0 GeV²
Q² = 2.0 GeV²
Q² = 3.5 GeV²
Q² = 4.5 GeV²
Q² = 6.5 GeV²
Q² = 8.5 GeV²
Q² = 10 GeV²
Q² = 12 GeV²
Q² = 15 GeV²
Q² = 18 GeV²
Q² = 22 GeV²
Q² = 27 GeV²
Q² = 35 GeV²
Q² = 45 GeV²
Q² = 60 GeV²
Q² = 70 GeV²
Q² = 90 GeV²
Q² = 120 GeV²
F₂ (x,Q²)
x

Comparison with HERA data - F_L



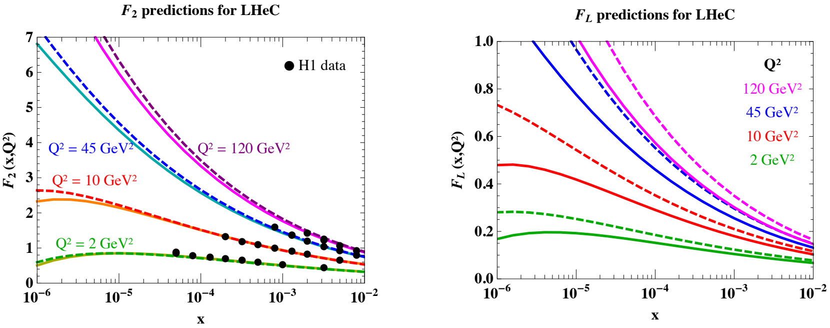
- accuracy of the data considerably diminished w.r.t. F_2
- Inclusive observable: $\langle F_L \rangle_x$
- only change in setup: photon imact factor (quark loop) for long. polarized photon
- LO photon (**straight**) and photon with improved kinematics (**dashed**)

F_L unaveraged



Predictions for an *e.g.* LHeC

Extrapolating our result to proton structure functions at ultra-small x



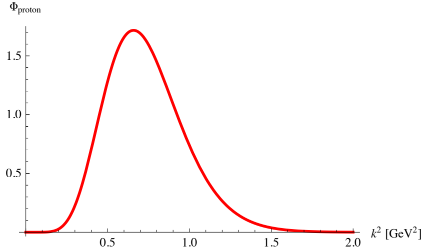
LO photon (**straight**)

LO photon with improved kinematics (**dashed**)

Short intermediate summary

- NLO BFKL describes combined HERA data in the kinematic regime associated with the BFKL framework
  - small $x < x_0$, $x_0 \sim 0.5 \cdot 10^{-2} - 1.0 \cdot 10^{-2}$
  - perturbative QCD $Q^2 > 2\text{GeV}^2$
  - $Q^2 > 150\text{GeV}^2$ and $x > .5 \cdot 10^{-2}$ require probably more DGLAP dynamics

  → NLO corrections to photon
- both collinear resummations and BLM like treating of the running coupling needed
- (resummed) not-running coupling NLO corrections still sizeable and important; roughly a 30% correction over the full range of Q^2
- proton impact factor at 'natural' values
  - $Q_0 = 0.28\text{GeV} \sim \Lambda_{\text{QCD}}$
  - average $k_T^2 = 0.66\text{GeV}^2$
  - $\mathcal{O}(1)$ normalization comparable to pdfs →



Part II
DGLAP evolution and physical splitting functions
–
a first numerical implementation

DGLAP evolution (at small x)

collinear factorization $F_{2,L}(Q^2) = C\left(\frac{Q^2}{\mu_f^2}\right) \otimes f(\mu_f^2)$

DGLAP $\partial_{\ln \mu_f^2} f = P \otimes f$ evolution w.r.t. factorization scale μ_f^2

factorization ➡ universality of $f = q, g$ (we like that!)

rise of F_2 at small x

- perturvative: DGLAP splitting kernels P (share DL with BFKL)
- non-perturbative: intrinsic distribution from fit ($g \sim x^{0.1}$)

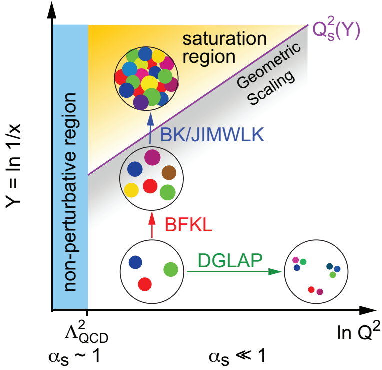
drawback of factorization:

- q, g given by theory definition (➡ factorization scheme)
- differences between schemes can be large $\sim (\alpha_s \ln 1/x)^n$

(don't like that so much ....)

don't care so much if I only want to parametrize my data ....

.... but if I want to understand physics at small x, this matters



[EIC White Paper, arXiv:1212.1701]

- what is evolution (JIMWLK, BFKL, DGLAP)?
- what is a clever scheme choice?

affects also BFKL and BK/JIMWLK evolution ...

DGLAP also many pdfs with many parameters

solution: direct evolution of structure functions

$$\text{“}\quad \partial_{\ln Q^2} F(x, Q^2) = \quad K \quad \otimes \quad F(x, Q^2) \quad \text{”}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

obs. $\rightarrow$ obs. $\leftarrow$ obs.

evolution kernels K

- physical
- no scheme ambiguity (only renormalization scale)

in principle equivalent to [Catani (1996)]

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

idea around for a while:

[Bardeen, Buras (1979)], [Floratos, Kounnas, Lacaze (1981)], [Grunberg (1984)], [Catani (1996)], [Blümlein, Ravindran, van Neerven (2000)], [Vogt, Moch, Soar, Vermaseren (2010)]

→ address for the first time numerical implementation

issue 1

DGLAP equation = matrix equation:

$$\partial_{\ln \mu_f^2} f = P \otimes f \qquad f = q, g$$

$n_f = 3$
3 quark, 1 gluon pdf → singlet = (Σ, g)
NS_1, NS_2

need 4 observables to disentangle this (F_2^p, F_2^n, F_L, F_3) → possible, but involved

small x: Σ = sea quark, NS$= 0$

$$(F_2, F_L) \Leftrightarrow (\Sigma, g)$$

Determination of physical evolution kernels

define:

$$\boldsymbol{F} = \begin{pmatrix} F_2 \\ F_L \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} C_{2q} & C_{2g} \\ C_{Lq} & C_{Lg} \end{pmatrix} \quad \boldsymbol{f} = \begin{pmatrix} \Sigma \\ g \end{pmatrix} \quad \boldsymbol{P} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}$$

requires Mellin/Moment space with $a(N) = \int_0^1 dx x^{N-1} a(x)$ ➡ $\otimes \to \cdot$

$$\begin{aligned}
\partial_{\ln Q^2} \boldsymbol{F} &= \partial_{\ln Q^2} (\boldsymbol{C} \cdot \boldsymbol{f}) \\
&= \beta \frac{d\boldsymbol{C}}{da_s} \cdot \boldsymbol{f} + \boldsymbol{C} \cdot \boldsymbol{P} \cdot \boldsymbol{f} \\
&= \underbrace{\left[\beta \frac{d\boldsymbol{C}}{da_s} \cdot \boldsymbol{C}^{-1} + \boldsymbol{C} \cdot \boldsymbol{P} \cdot \boldsymbol{C}^{-1} \right]}_{\tilde{\boldsymbol{K}}} \boldsymbol{F}
\end{aligned}$$

with $a_s \equiv \dfrac{\alpha_s}{4\pi}$

issue 2

$$\tilde{\boldsymbol{K}} = \left[\beta \frac{d\boldsymbol{C}}{da_s} \cdot \boldsymbol{C}^{-1} + \boldsymbol{C} \cdot \boldsymbol{P} \cdot \boldsymbol{C}^{-1}\right]$$

$$C_{2q} = \mathcal{O}(1) \qquad C_{Lq}, C_{Lg} = \mathcal{O}(a_s)$$

→ mix different orders of a_s in $\tilde{\boldsymbol{K}}$

→ scale dependence inside the kernels!

solution: scheme independence of $C_{Lq}^{(1)}, C_{Lg}^{(1)}$ → evolve $\tilde{\boldsymbol{F}} = (F_2, \tilde{F}_L)$

$$\tilde{F}_L^{(g)} = \frac{F_L}{a_s C_{Lg}^{(1)}}$$

[Blümlein, Ravindran, van Neerven (2000)]

or

$$\tilde{F}_L^{(q)} = \frac{F_L}{a_s C_{Lq}^{(1)}}$$

[Vogt, Moch, Soar, Vermaseren (2010)]

argued that choice matters → up to NLO identical results

→ modified coefficient matrix $\tilde{\boldsymbol{C}}$

$$\boldsymbol{K} = \left[\beta \frac{d\tilde{\boldsymbol{C}}}{da_s} \cdot \tilde{\boldsymbol{C}}^{-1} + \tilde{\boldsymbol{C}} \cdot \boldsymbol{P} \cdot \tilde{\boldsymbol{C}}^{-1}\right]$$

perturbative expansion: LO

$$\boldsymbol{K} = \begin{pmatrix} K_{22} & K_{2L} \\ K_{L2} & K_{LL} \end{pmatrix}$$

perturbative expansion

$$\boldsymbol{K} = a_s \boldsymbol{K}^{(0)} + a_s^2 \boldsymbol{K}^{(1)} \quad \boldsymbol{C} = \boldsymbol{C}^{(0)} + a_s \boldsymbol{C}^{(1)} + a_s^2 \boldsymbol{C}^{(2)} \quad \boldsymbol{P} = a_s \boldsymbol{P}^{(0)} + a_s^2 \boldsymbol{P}^{(1)}$$

LO kernels (using the quark convention)

$$K_{22}^{(0)} = P_{\mathsf{qq}}^{(0)} - \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}} \qquad K_{2L}^{(0)} = \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}}$$

$$K_{L2}^{(0)} = \frac{C_{\mathsf{Lg}}^{(1)} P_{\mathsf{gq}}^{(0)}}{C_{\mathsf{Lq}}^{(1)}} - \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}} - P_{\mathsf{gg}}^{(0)} + P_{\mathsf{qq}}^{(0)} \qquad K_{LL}^{(0)} = \frac{C_{\mathsf{Lq}}^{(1)} P_{\mathsf{qg}}^{(0)}}{C_{\mathsf{Lg}}^{(1)}} + P_{\mathsf{gg}}^{(0)}$$

LO solution

standard solution for $\partial_{\ln Q^2} \boldsymbol{F} = \boldsymbol{K} \cdot \boldsymbol{F}$:

$$\boldsymbol{F}(N, Q^2) = \boldsymbol{D}^{-1}(Q^2) \left[\left(\frac{a_s}{a_0} \right)^{\lambda_+(N)} e_+(N) + \left(\frac{a_s}{a_0} \right)^{\lambda_-(N)} e_-(N) \right] \boldsymbol{D}(Q_0^2) \cdot \boldsymbol{F}(N, Q_0^2)$$

$$a_s = a_s(Q^2) \qquad a_0 \equiv a_s(Q_0^2)$$

$\lambda_\pm(N)$ eigenvalues of the matrix $\boldsymbol{K}^{(0)}$, $e_\pm(N)$ projectors on eigenspaces
matrix $\boldsymbol{D}$ achieves rotation $\tilde{\boldsymbol{F}} = \boldsymbol{D} \cdot \boldsymbol{F}$

Observation:

- $\lambda_\pm(N)$ agree for $\boldsymbol{K}^{(0)}$ and $\boldsymbol{P}^{(0)}$
- eigenspaces differ, but final result for (F_2, F_L) agrees exactly!

→ at LO, physical and DGLAP evolution are the same

NLO Kernels obtained easily ....

for the 'quark' convention

$$K_{22}^{(1)} = -\frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{gg}^{(0)}}{C_{Lg}^{(1)}} + C_{2g}^{(1)} P_{gq}^{(0)} + \frac{\beta_0 C_{2g}^{(1)} {C_{Lq}}^{(1)}}{{C_{Lg}}^{(1)}}$$
$$- \frac{{C_{2g}}^{(1)} C_{Lq}^{(1)\,2} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}} + \frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{qq}^{(0)}}{C_{Lg}^{(1)}} - \beta_0 C_{2q}^{(1)}$$
$$+ \frac{C_{Lg}^{(2)} C_{Lq}^{(1)} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}} - \frac{C_{Lq}^{(1)} P_{qg}^{(1)}}{C_{Lg}^{(1)}} - \frac{C_{Lq}^{(2)} P_{qg}^{(0)}}{C_{Lg}^{(1)}} + P_{qq}^{(1)}$$

$$K_{2L}^{(1)} = \frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{gg}^{(0)}}{C_{Lg}^{(1)}} - \frac{\beta_0 C_{2g}^{(1)} C_{Lq}^{(1)}}{C_{Lg}^{(1)}} + \frac{C_{2g}^{(1)} C_{Lq}^{(1)\,2} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}}$$
$$- \frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{qq}^{(0)}}{C_{Lg}^{(1)}} + \frac{C_{2q}^{(1)} C_{Lq}^{(1)} P_{qg}^{(0)}}{C_{Lg}^{(1)}} - \frac{C_{Lg}^{(2)} C_{Lq}^{(1)} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}} + \frac{C_{Lq}^{(1)} P_{qg}^{(1)}}{C_{Lg}^{(1)}}$$

$$K_{L2}^{(1)} = -\frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{gg}^{(0)}}{C_{Lg}^{(1)}} + C_{2g}^{(1)} P_{gq}^{(0)} - \frac{C_{2g}^{(1)} C_{Lq}^{(1)\,2} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}} + \frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{qq}^{(0)}}{C_{Lg}^{(1)}}$$
$$+ C_{2q}^{(1)} P_{gg}^{(0)} - \frac{C_{2q}^{(1)} C_{Lg}^{(1)} P_{gq}^{(0)}}{C_{Lq}^{(1)}} + \frac{C_{2q}^{(1)} C_{Lq}^{(1)} P_{qg}^{(0)}}{C_{Lg}^{(1)}} - C_{2q}^{(1)} P_{qq}^{(0)} - \frac{C_{Lq}^{(2)} P_{gg}^{(0)}}{C_{Lq}^{(1)}}$$
$$+ \frac{C_{Lg}^{(2)} P_{gq}^{(0)}}{C_{Lq}^{(1)}} + \frac{C_{Lg}^{(1)} P_{gq}^{(1)}}{C_{Lq}^{(1)}} + \frac{\beta_0 C_{Lg}^{(2)}}{C_{Lg}^{(1)}} + \frac{C_{Lg}^{(2)} C_{Lq}^{(1)} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}} - \frac{C_{Lq}^{(1)} P_{qg}^{(1)}}{C_{Lg}^{(1)}}$$
$$- \frac{2 C_{Lq}^{(2)} P_{qg}^{(0)}}{C_{Lg}^{(1)}} - \frac{\beta_0 C_{Lq}^{(2)}}{C_{Lq}^{(1)}} + \frac{C_{Lq}^{(2)} P_{qq}^{(0)}}{C_{Lq}^{(1)}} - P_{gg}^{(1)} + P_{qq}^{(1)}$$

$$K_{LL}^{(1)} = \frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{gg}^{(0)}}{C_{Lg}^{(1)}} - C_{2g}^{(1)} P_{gq}^{(0)} + \frac{C_{2g}^{(1)} C_{Lq}^{(1)\,2} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}} - \frac{C_{2g}^{(1)} C_{Lq}^{(1)} P_{qq}^{(0)}}{C_{Lg}^{(1)}}$$
$$- \frac{\beta_0 C_{Lg}^{(2)}}{C_{Lg}^{(1)}} - \frac{C_{Lg}^{(2)} C_{Lq}^{(1)} P_{qg}^{(0)}}{C_{Lg}^{(1)\,2}} + \frac{C_{Lq}^{(1)} P_{qg}^{(1)}}{C_{Lg}^{(1)}} + \frac{C_{Lq}^{(2)} P_{qg}^{(0)}}{C_{Lg}^{(1)}} + P_{gg}^{(1)}$$

NLO solution

no exact analytic solution at NLO, use solution by Glück, Reya Vogt (1990)] , also used in PEGASUS [Vogt (2004)] :

$$\boldsymbol{F}(N,Q^2) = \boldsymbol{D}^{-1}\Bigg\{ \left(\frac{a_s}{a_0}\right)^{\lambda_+} \Bigg[e_+(N) + (a_0 - a_s)e_+ \cdot \boldsymbol{R}^{(1)} \cdot e_+ + \left[a_0 - a_s\left(\frac{a_s}{a_0}\right)^{\lambda_- - \lambda_+}\right] \frac{e_- \cdot \boldsymbol{R}^{(1)} \cdot e_-}{\lambda_+ - \lambda_- + 1}\Bigg] + (+) \leftrightarrow (-)\Bigg\} \boldsymbol{D} \cdot \boldsymbol{F}(N, Q_0^2)$$

NLO corrections inside

$$\boldsymbol{R}^{(1)} = \frac{1}{\beta_0}\boldsymbol{K}^{(1)} - \frac{\beta_1}{\beta_0^2}\boldsymbol{K}^{(0)}$$

running coupling at NLO

$$\frac{da_s}{d\ln\mu^2} = -a_s^2\beta_0 - a_s^3\beta_1$$

– solved numerically

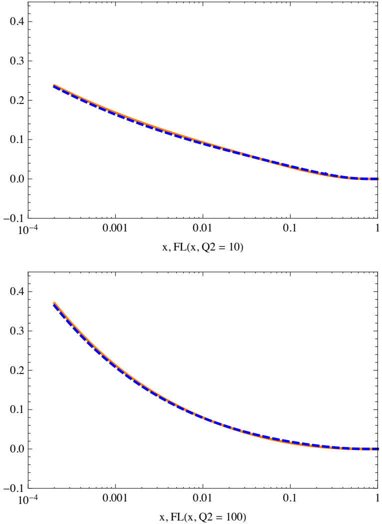
Numerical implementation

- parallel implementation for conventional 'coeff $\otimes$ pdf' and physical evolution kernels
- two independent codes (Fortran, Mathematica) + pdf evolution cross-checked with Pegasus [Vogt (2004)]
- use same input at $Q_0^2 = 2\text{GeV}^2$, given in terms of toy-pdfs (Pegasus default intial parton distributions) with $n_f = 3$ and $\alpha_s(Q_0^2) = 0.35$

$$
\begin{aligned}
xu_v(x, Q_0^2) &= 5.10722x^{0.8}(1-x)^3 \\
xd_V(x, Q_0^2) &= 3.064320x^{0.8}(1-x)^4 \\
xg(x, Q_0^2) &= 1.70000x^{-0.1}(1-x)^5 \\
x\bar{d}(x, Q_0^2) &= 0.1939875x^{-0.1}(1-x)^6 \\
x\bar{u}(x, Q_0^2) &= (1-x)x\bar{d}(x, Q_0^2) \\
xs(x, Q_0^2) &= x\bar{s}(x, Q_0^2) = 0.2(\bar{u}+\bar{d})(x, Q_0^2)
\end{aligned}
$$

- $\boldsymbol{C}^{(0)}$, $\boldsymbol{C}^{(1)}$ and $\boldsymbol{P}^{(0)}$, $\boldsymbol{P}^{(1)}$ taken from [Floratos, Kounas, Lacaze (1981)], $C_{Lq,g}^{(2)}$ in the parametrized version of [van Neerven, Vogt (1999), (2000)]

First numerical results – the (toy) structure function F_L

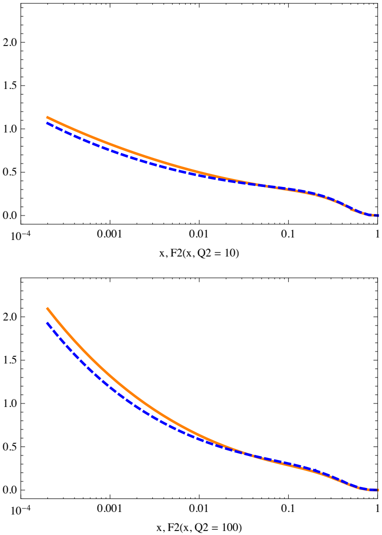


- (straight) physical kernels
- (dashed) coeff ⊗ pdf

- relative difference at large x huge :
  $\mathcal{O}(20\%)$ for $Q^2 = 10$ GeV^2
  $\mathcal{O}(30\%)$ for $Q^2 = 100$ GeV^2

- reasonable at small x:
  $\mathcal{O}(1.3\%)$ for $Q^2 = 10$ GeV^2
  $\mathcal{O}(1.6\%)$ for $Q^2 = 100$ GeV^2

First numerical results – the (toy) structure function F_2

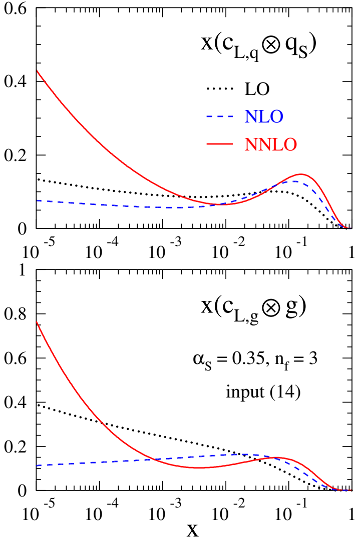


- (**straight**) physical kernels
- (**dashed**) coeff $\otimes$ pdf

- relative difference at large x less dramatic, but still ... :
  $\mathcal{O}(2.4\%)$ for $Q^2 = 10$ GeV^2
  $\mathcal{O}(7.4\%)$ for $Q^2 = 100$ GeV^2

- increased at small x:
  $\mathcal{O}(6.3\%)$ for $Q^2 = 10$ GeV^2
  $\mathcal{O}(8.8\%)$ for $Q^2 = 100$ GeV^2

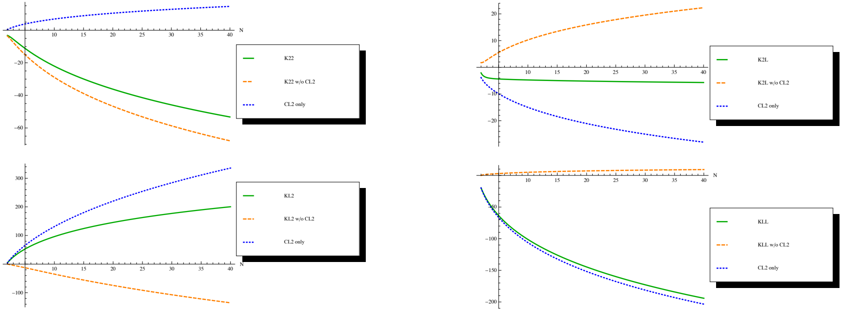
Why are difference so large?



[Moch, Vogt, Vermaseren (2004)]

- coefficents $C_{L,qg}^{(2)}$ are large corrections ➡ rotated into evolution kernels
- "coeff $\otimes$ pdf": with $a_s(Q^2)$
- physical kernels: at the initial scale $a_s(Q_0^2)$

Why is this difference so large?



- coefficents $C^{(2)}_{L,qg}$ are large corrections ➡ dominate the NLO evolution kernels
- can one control this ...? study in progress ....

Summary and Conclusions

2nd NLO BFKL fit to combined HERA data

- solution expressed through LO eigenfunctions (no discrete Pomeron solution as by KLRW)
- use RG improvements + BLM scale setting for Green's function + parametrization of running coupling in the infra-red
- excellent agreement with data in BFKL region, deviations for $Q^2 > 150$GeV^2, $x > 0.5 \cdot 10^{-2}$ and $Q^2 < 1.2$GeV^2.

possible improvements mainly for the photon

- NLO corrections [Bartels, Gieseke, Qiao (2000)], [Bartels, Gieseke, Kyrieleis (2001)], [Bartels, Colferai, Gieseke, Kyrieleis (2002)],[Bartels, Kyrieleis (2004)], [Bartles, Chachamis (2006)], [Balitsky, Chirillis (2011), (2013)], [Beuf (2012)]
- quark masses
- treatment of running coupling

Summary and Conclusions

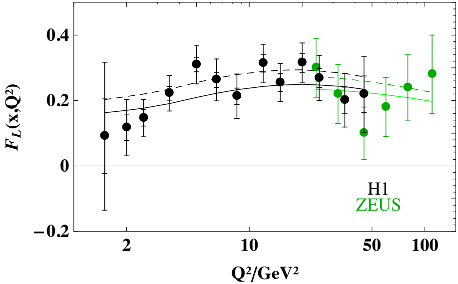
1st numerical implementation of physical anomalous dimensions

- use small x approximation ➡ (F_2, F_L)
- at LO identical to conventional DGLAP 'coeff. $\otimes$ pdf'
- at NLO, 'quark' and 'gluon' convention for $\tilde{F}_L$ give same result
- at NLO, difference between 'coeff. $\otimes$ pdf' and physical kernels sizeable

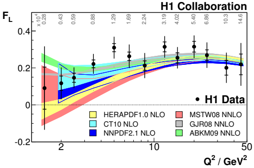
Question:

- Can this differences be reduced or simply a feature of physical evolution kernels?

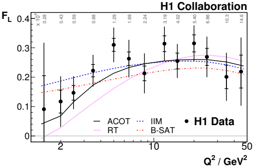
Backup



F_L(x,Q²)
0.4
0.2
0
−0.2
2
10
50
100
Q²/GeV²
H1
ZEUS



H1 Collaboration
F_L
0.4
0.2
0
-0.2
x 10⁻⁴
0.28
0.43
0.59
0.88
1.29
1.69
2.24
3.19
4.02
5.40
6.86
10.3
14.6
H1 Data
HERAPDF1.0 NLO
CT10 NLO
NNPDF2.1 NLO
MSTW08 NNLO
GJR08 NNLO
ABKM09 NNLO
2
10
50
Q² / GeV²



H1 Collaboration
F_L
0.4
0.3
0.2
0.1
0
x 10⁻⁴
0.28
0.43
0.59
0.88
1.29
1.69
2.24
3.19
4.02
5.40
6.86
10.3
14.6
ACOT
RT
IIM
B-SAT
H1 Data
2
10
50
Q² / GeV²