Scattering Amplitudes and the Positive Grassmannian

Jacob L. Bourjaily
Harvard University

based on work in collaboration with
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Organization

1. *Spiritus Movens*: Using (Generalized) Unitarity at One-Loop
2. The *On-Shell* Analytic *S*-Matrix
   - Basic Building Blocks of the *S*-Matrix: On-Shell Diagrams
   - On-Shell, All-Loop Recursion Relations
   - Classification, Combinatorics, Canonical Coordinates, and Computation of Diagrams, and the manifestation of the *Yangian*
3. *Revisiting* Generalized Unitarity at One-Loop and Beyond
   - Scalar-Box Integrals and their Divergences
   - The *Dual-Conformal* Regularization of Infrared Divergences
   - The *Chiral* Box Expansion for One-Loop *Integrands*
4. General Lessons Learned from On-Shell Physics

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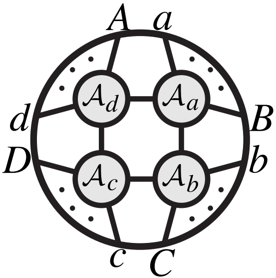

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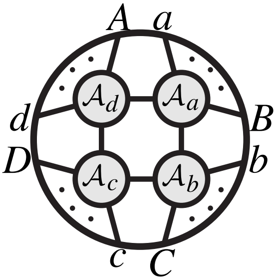
$f_{a,b,c,d} \equiv$  $I_{a,b,c,d}$

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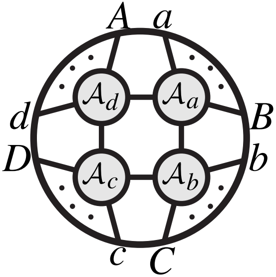
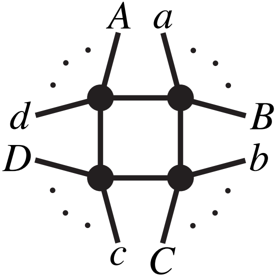
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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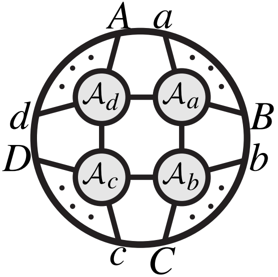
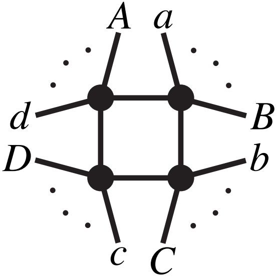
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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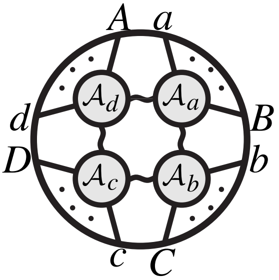

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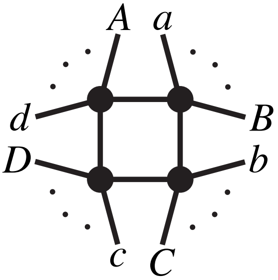

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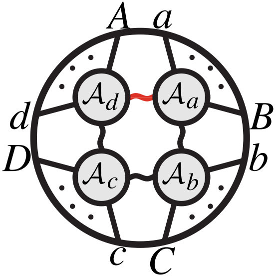
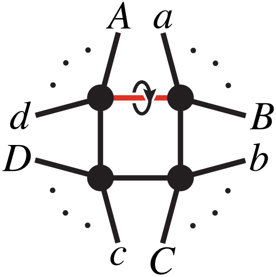
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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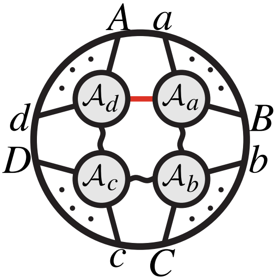

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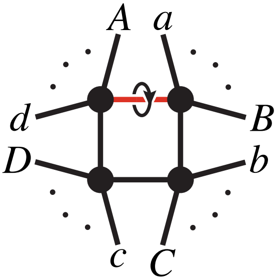

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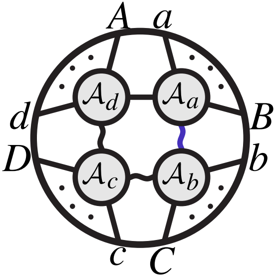
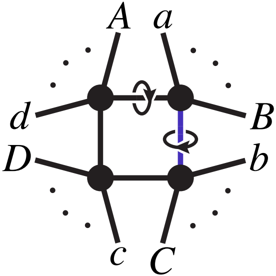
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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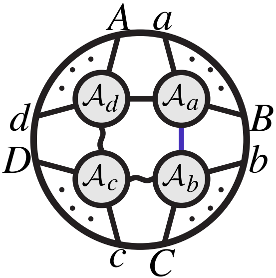
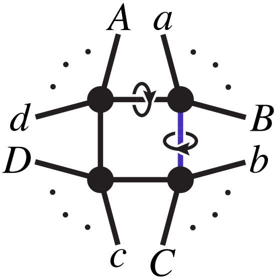
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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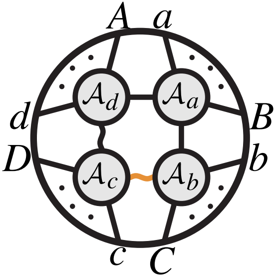
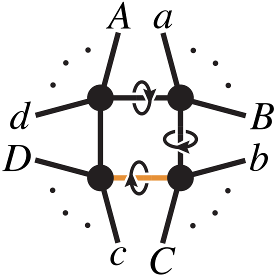
$f_{a,b,c,d} \equiv$  $I_{a,b,c,d} \equiv$ 

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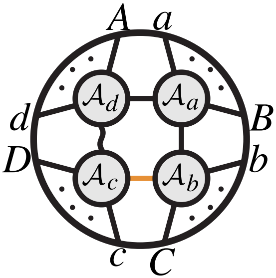

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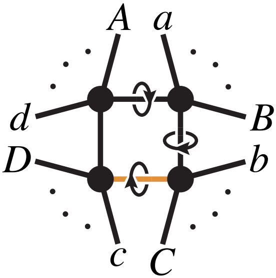

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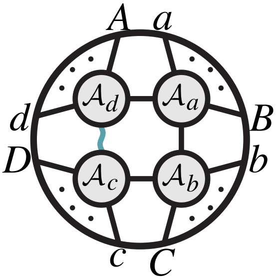

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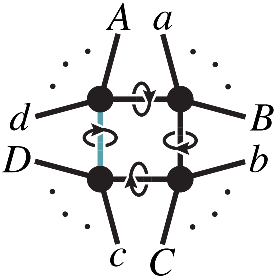

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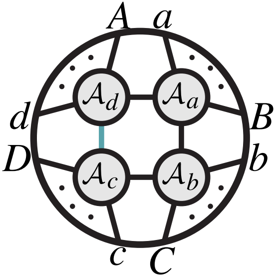
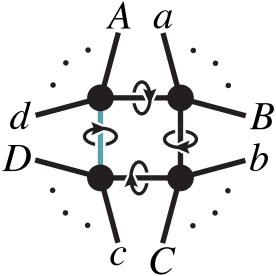
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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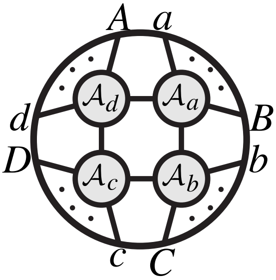
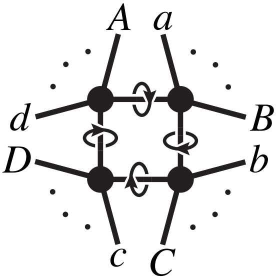
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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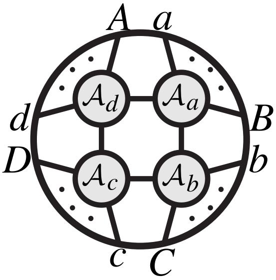

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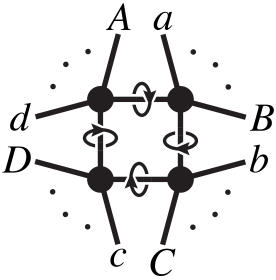

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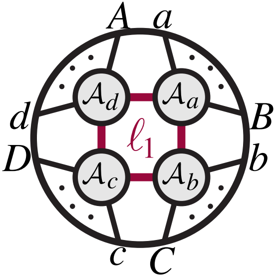
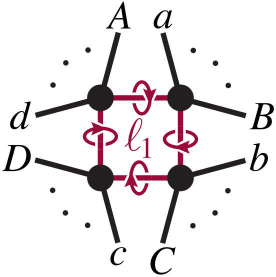
$f^1_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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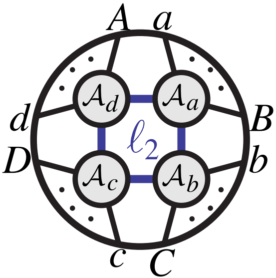
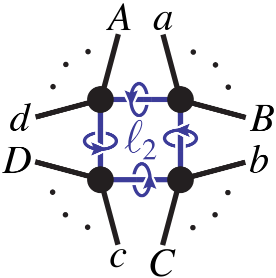
$f^2_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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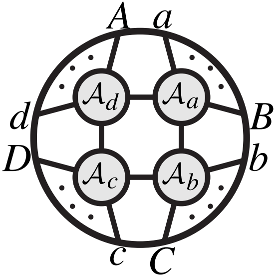
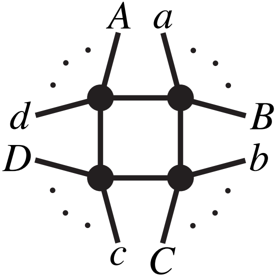
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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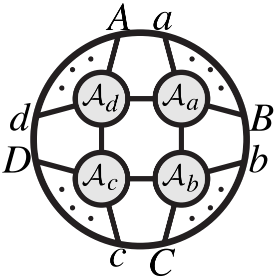

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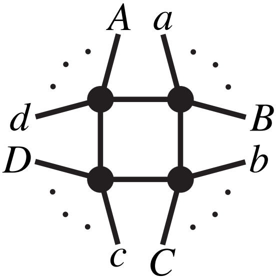

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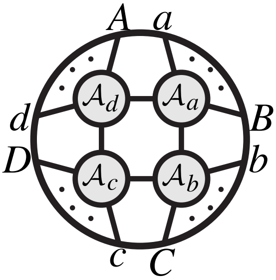
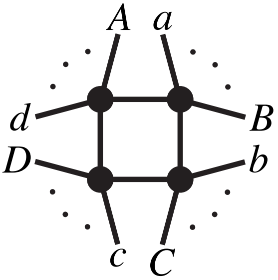
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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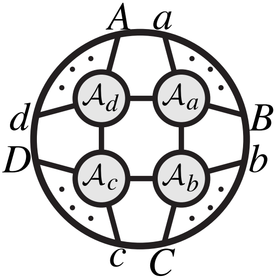

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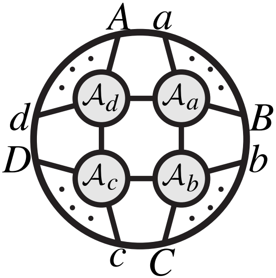

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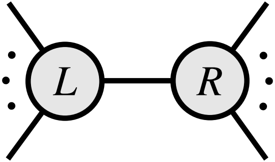

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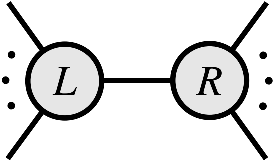

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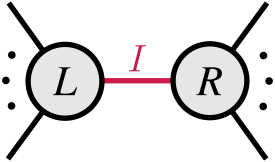

Internal Particles:

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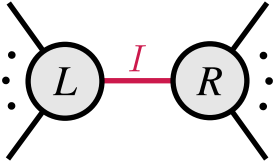

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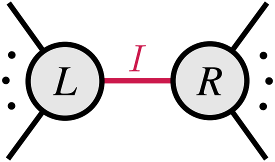

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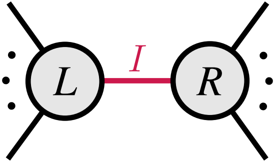

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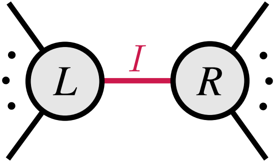

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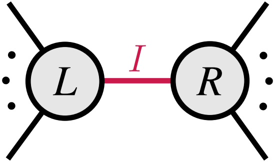

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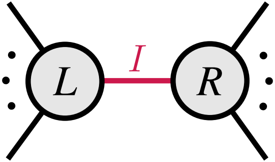

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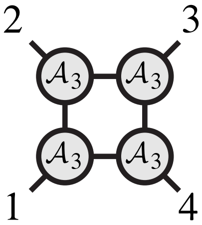

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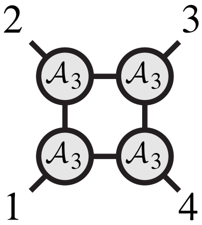

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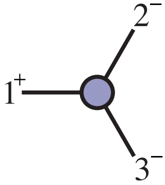

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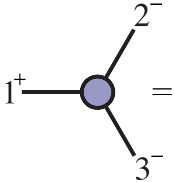

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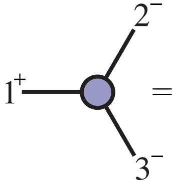

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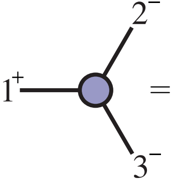

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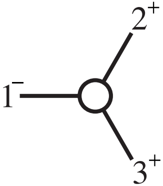

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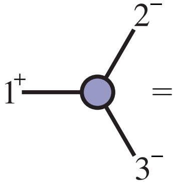

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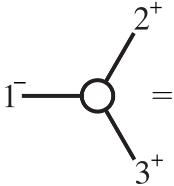

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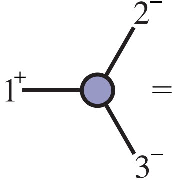

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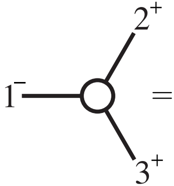

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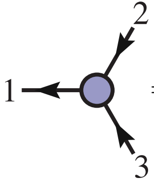

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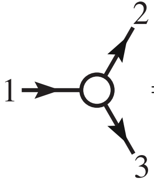

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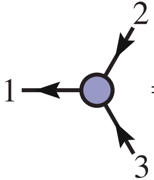

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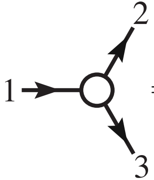

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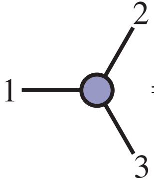

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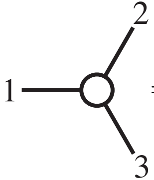

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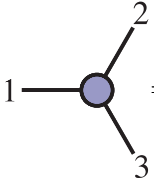

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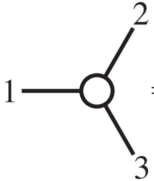

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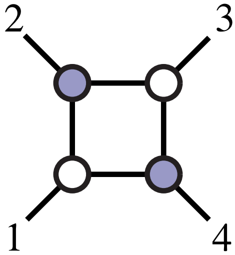

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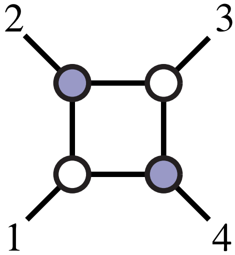

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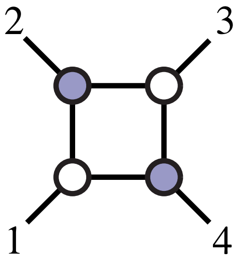

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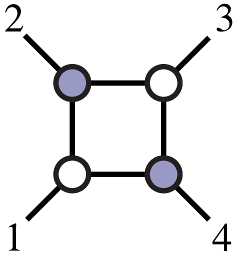

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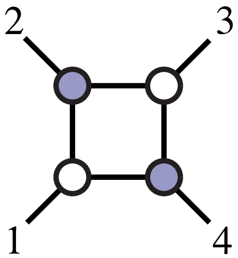

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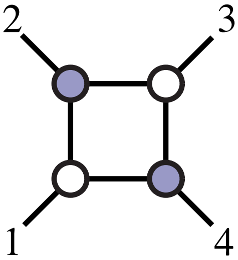

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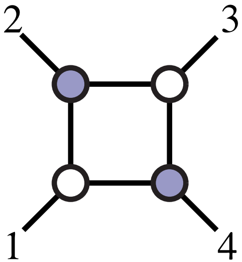

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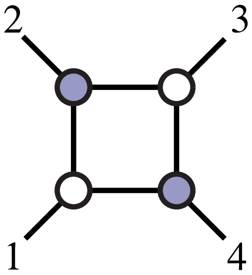

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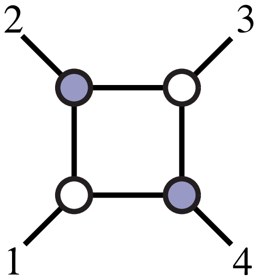

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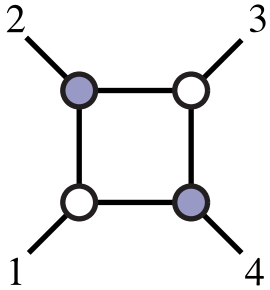

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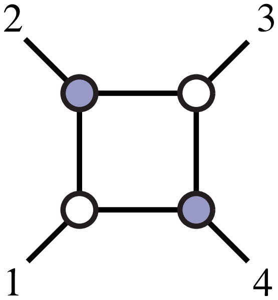

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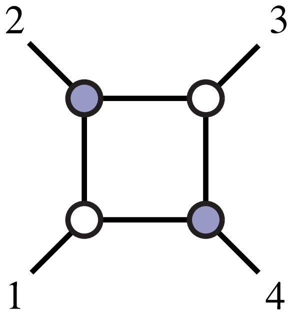

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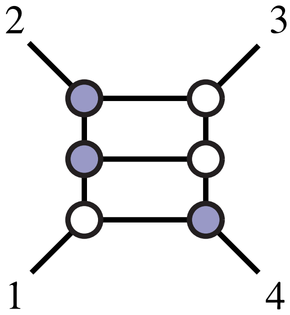

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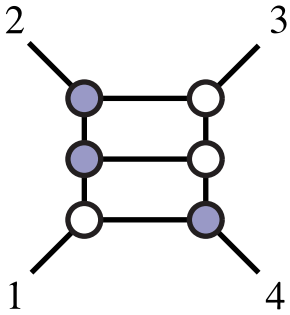

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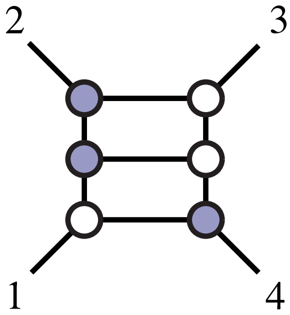

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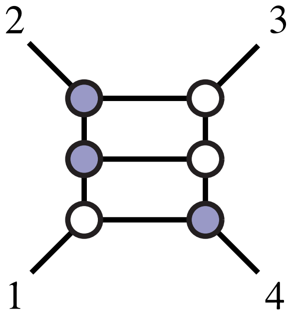

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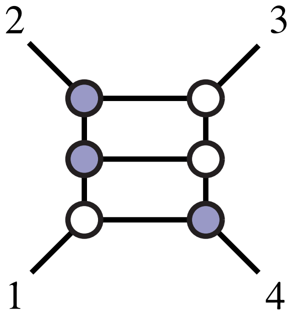

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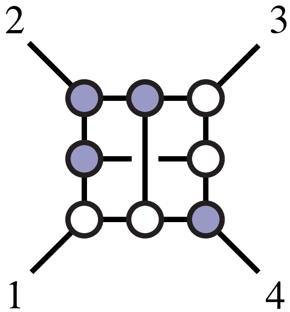

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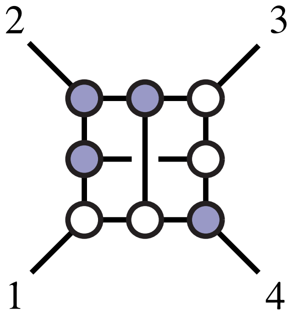

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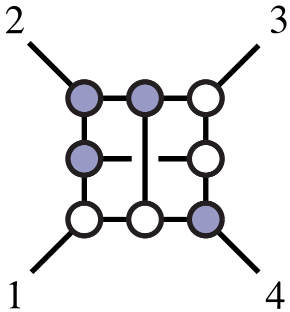

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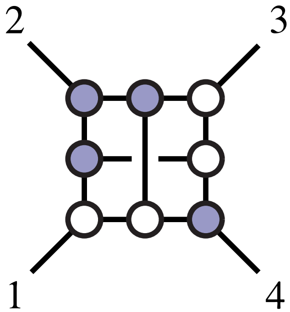

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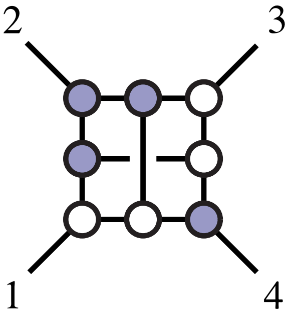

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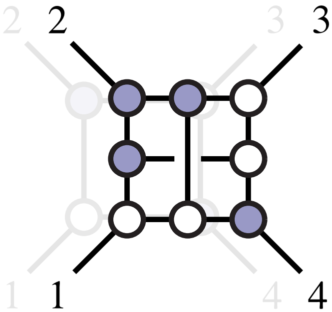

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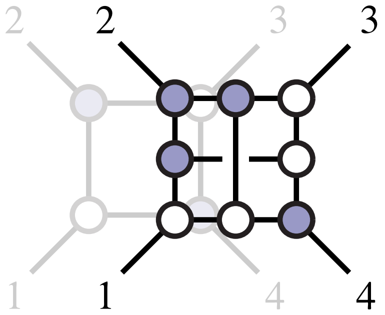

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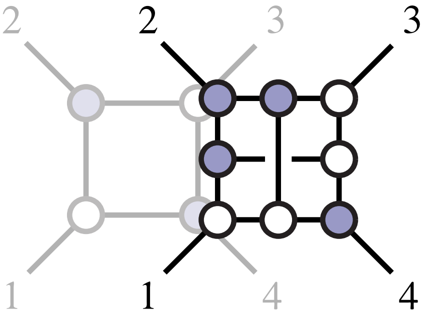

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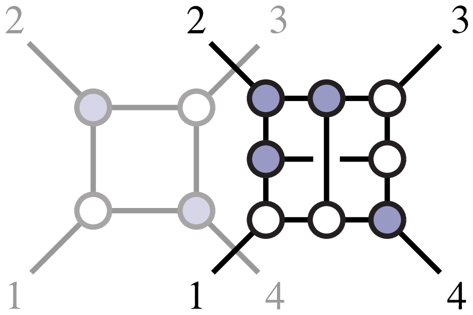

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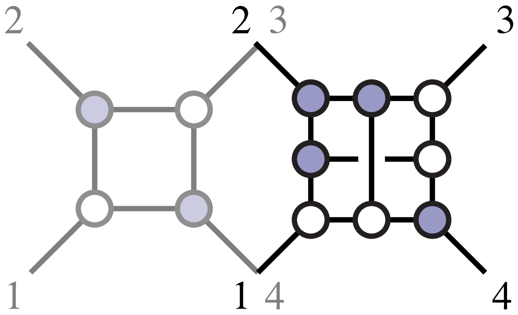

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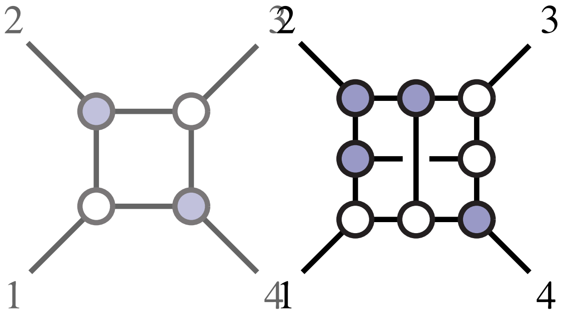

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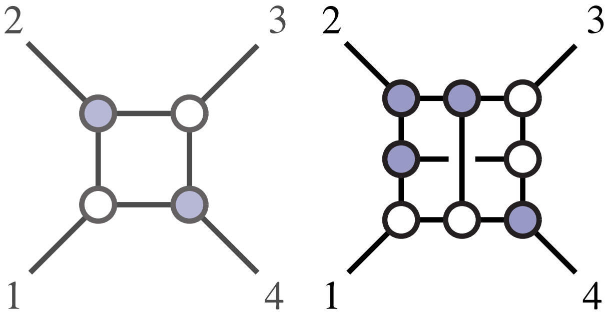

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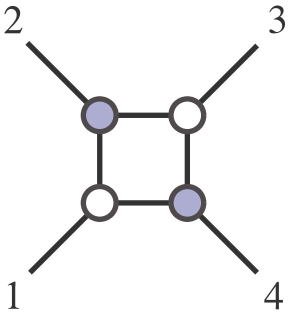

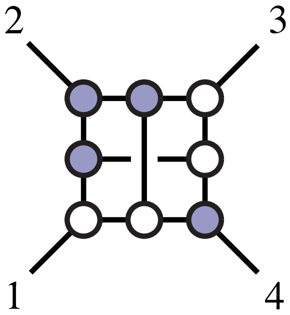

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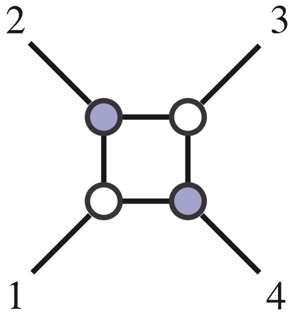

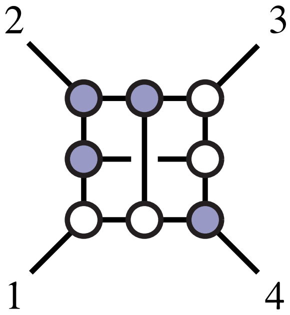

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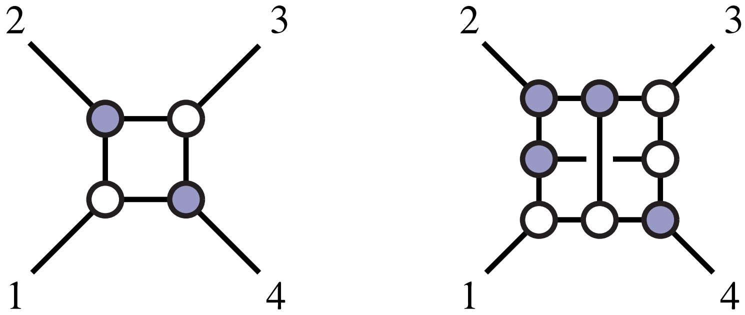

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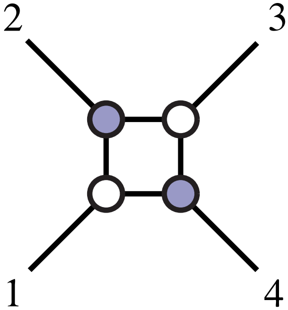

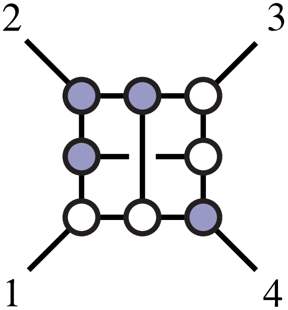

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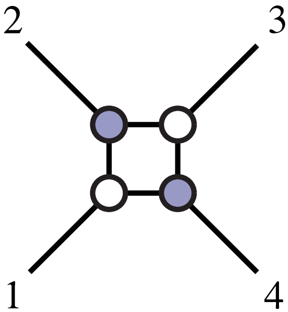

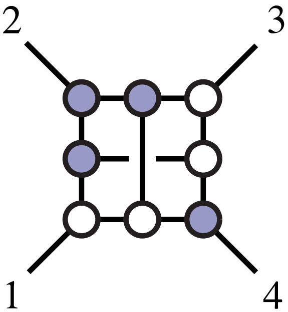

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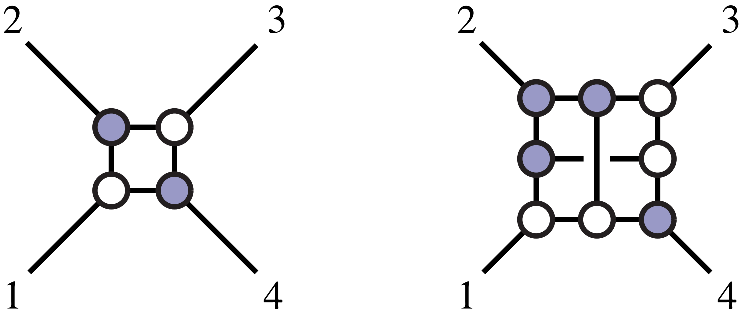

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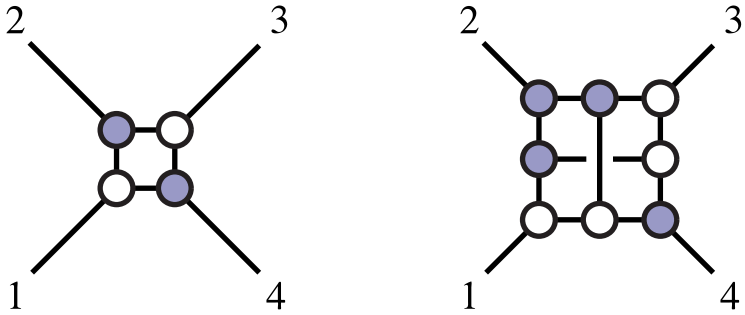

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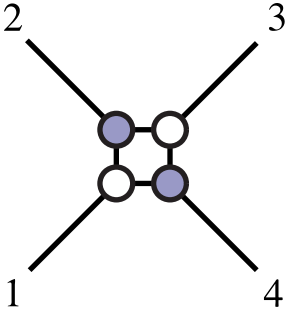

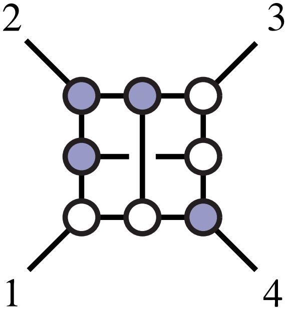

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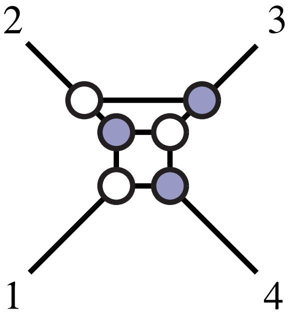

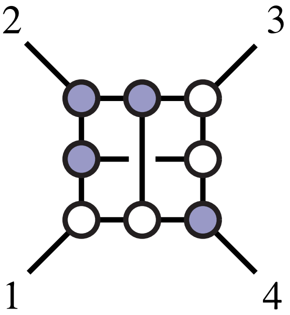

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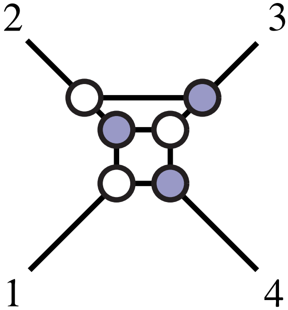

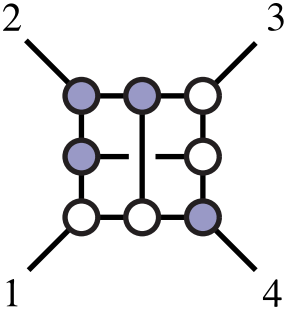

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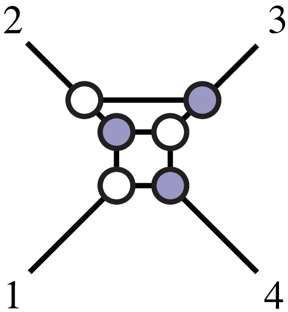

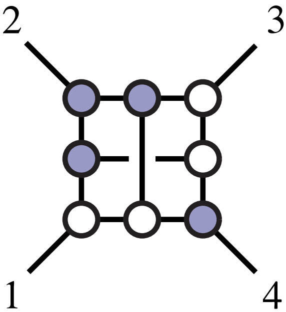

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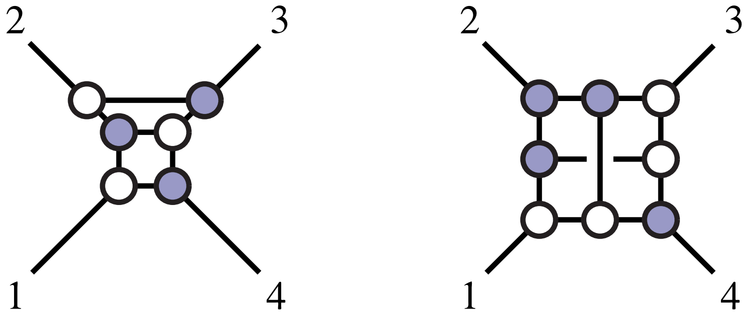

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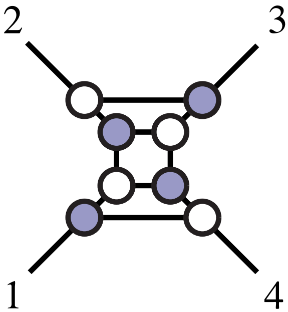

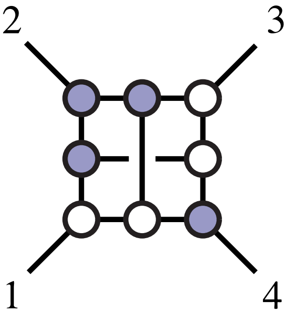

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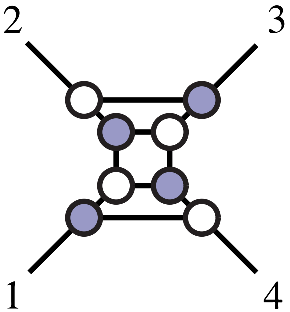

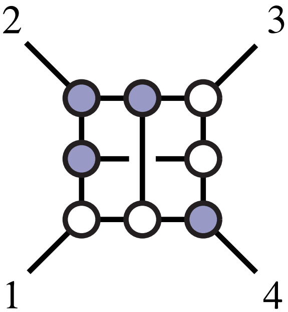

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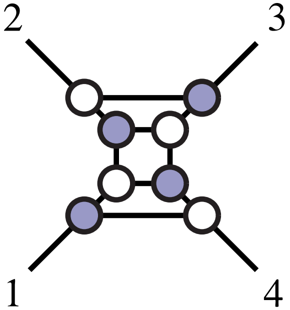

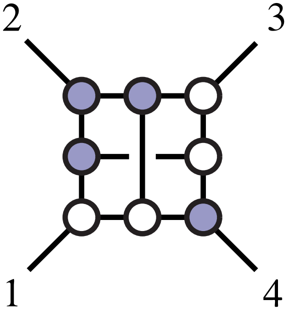

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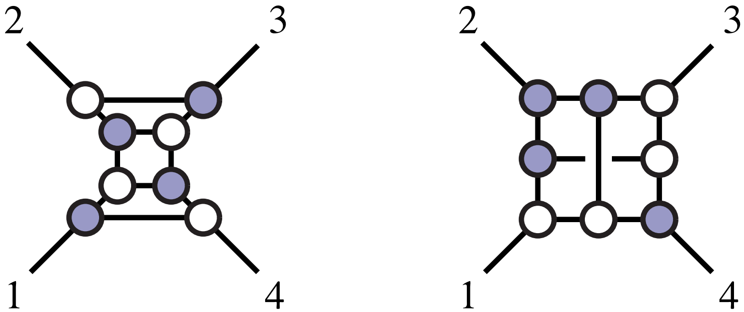

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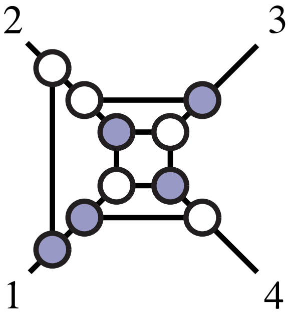

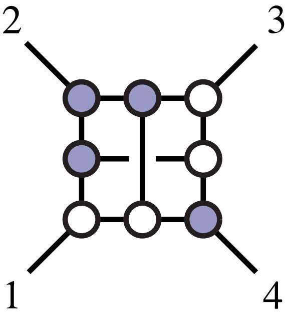

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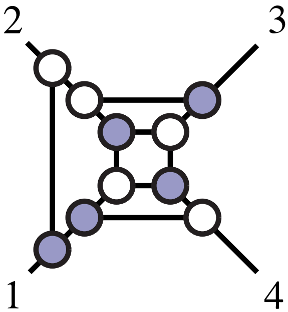

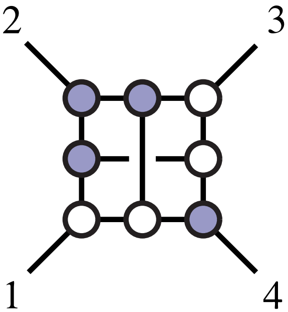

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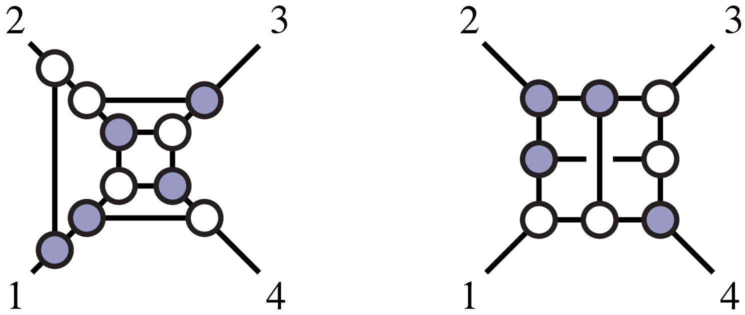

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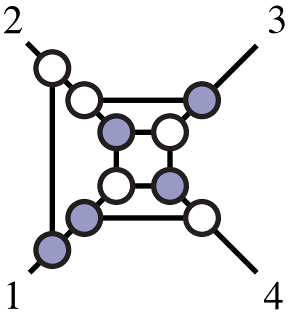

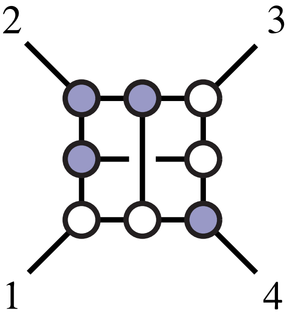

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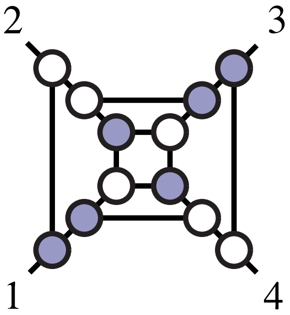

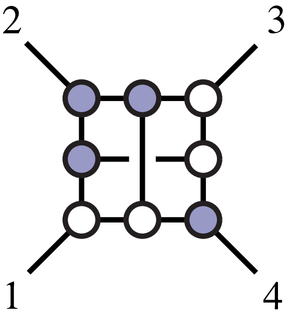

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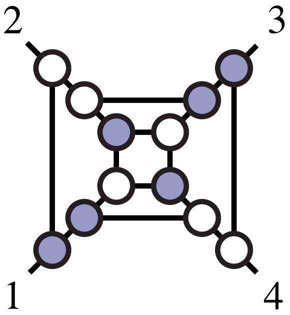

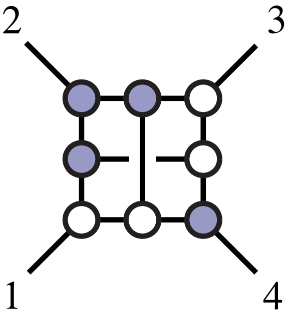

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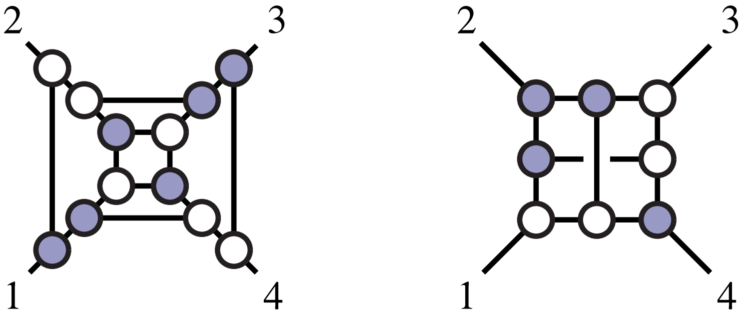

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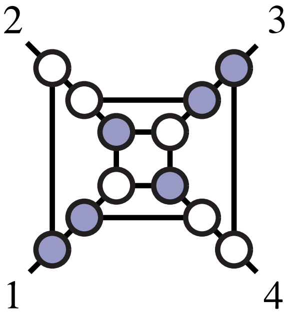

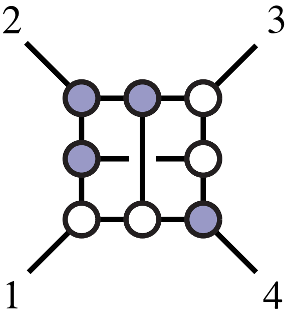

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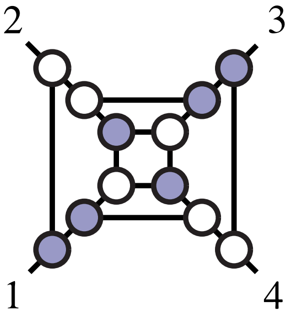

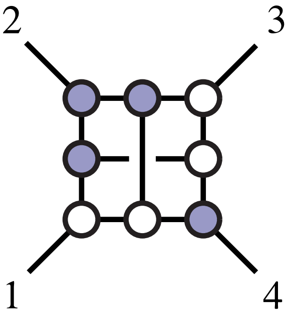

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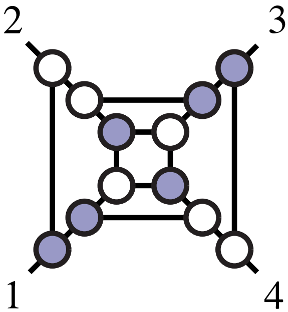

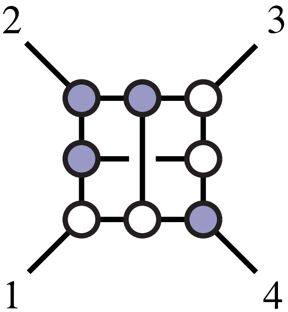

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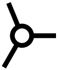

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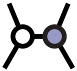

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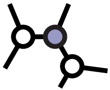

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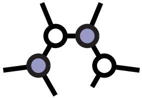

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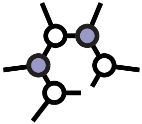

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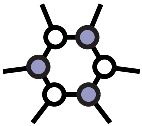

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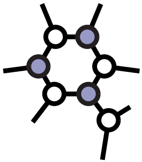

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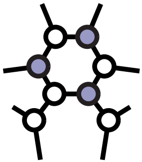

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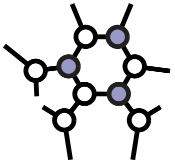

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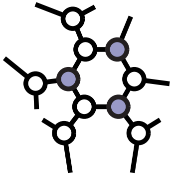

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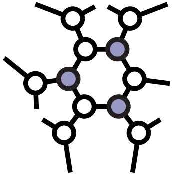

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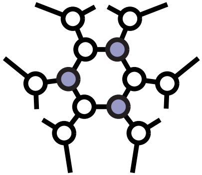

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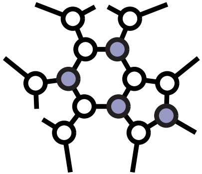

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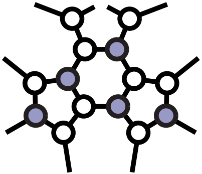

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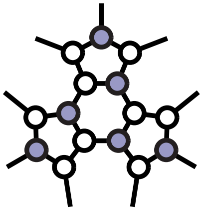

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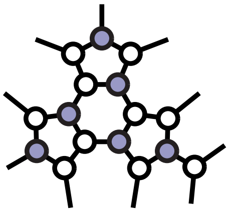

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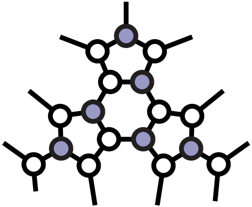

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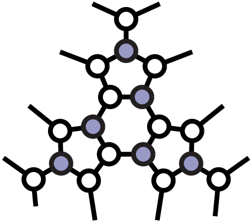

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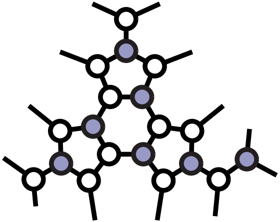

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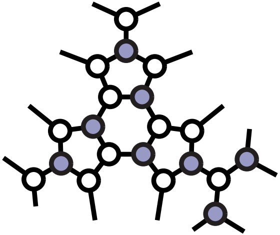

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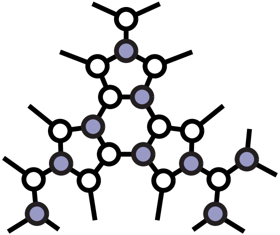

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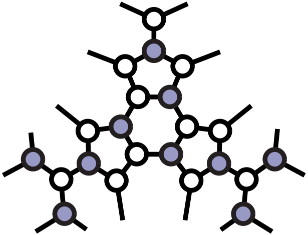

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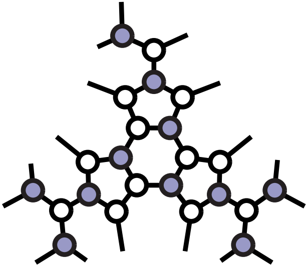

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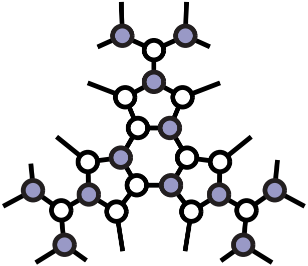

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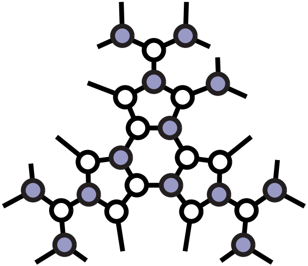

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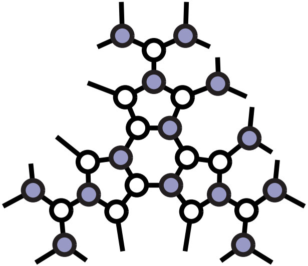

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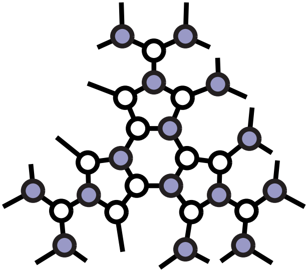

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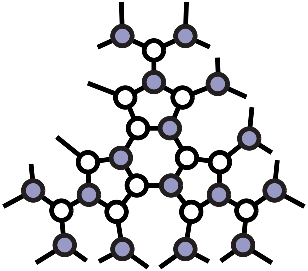

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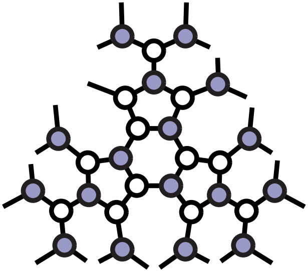

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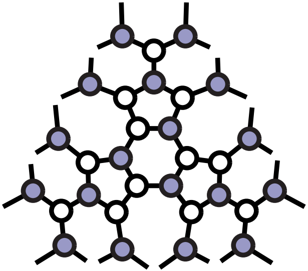

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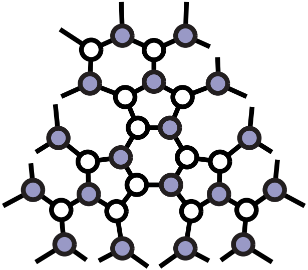

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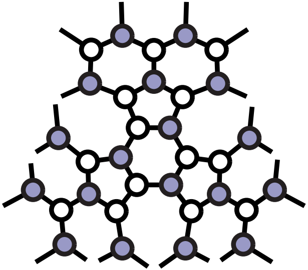

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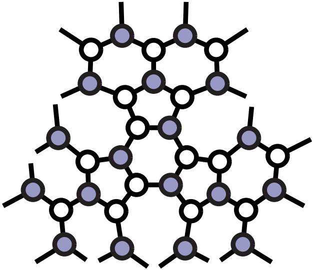

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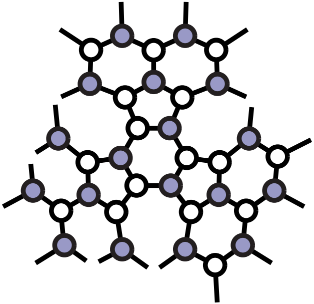

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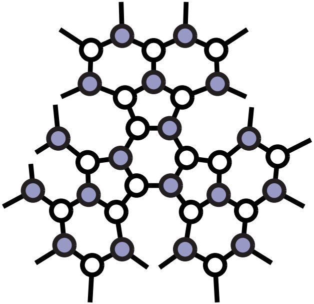

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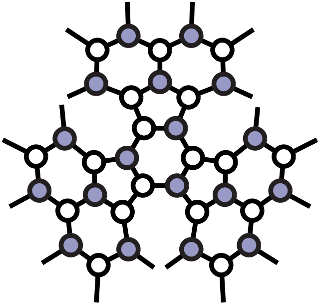

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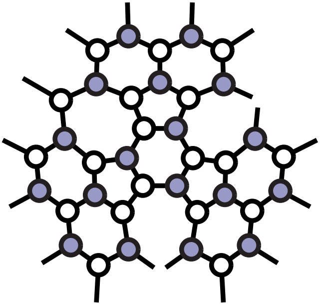

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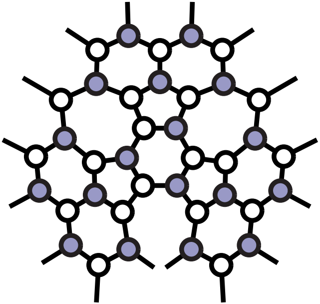

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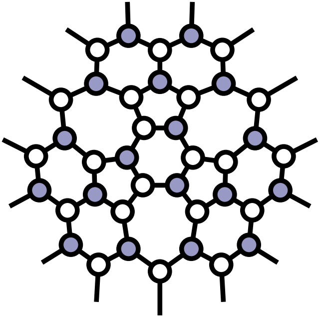

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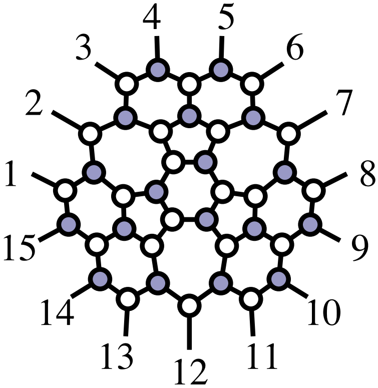

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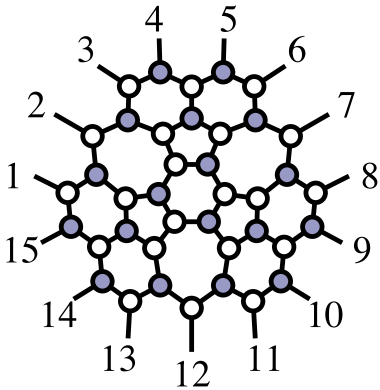

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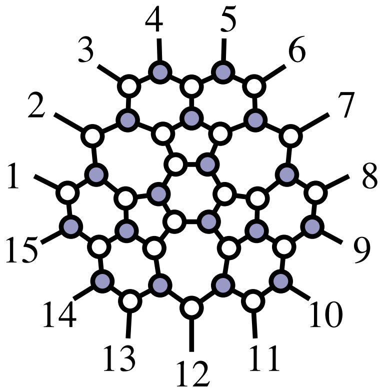

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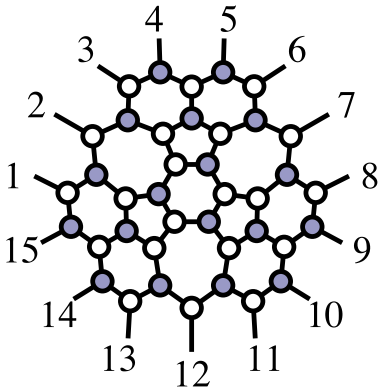

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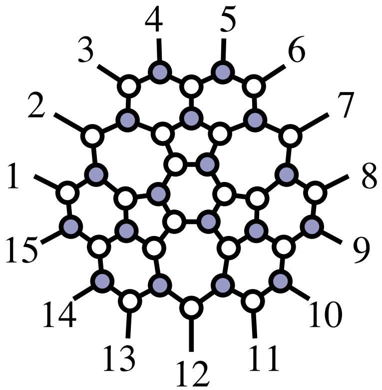

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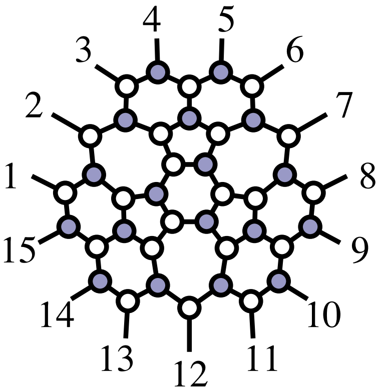

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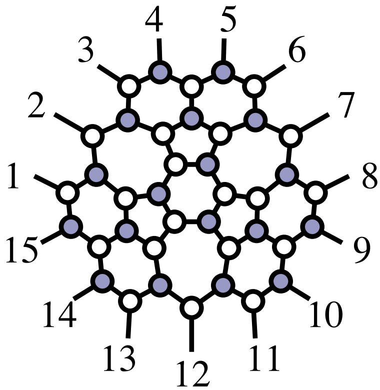

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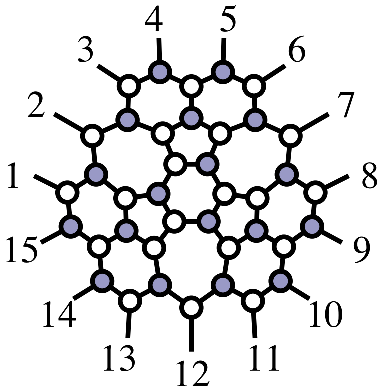

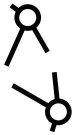

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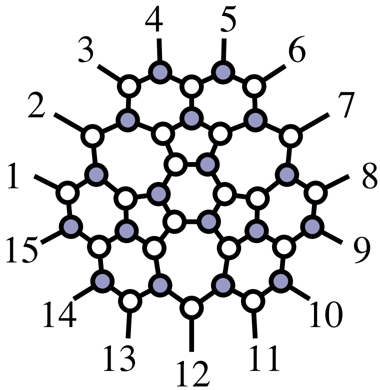

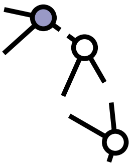

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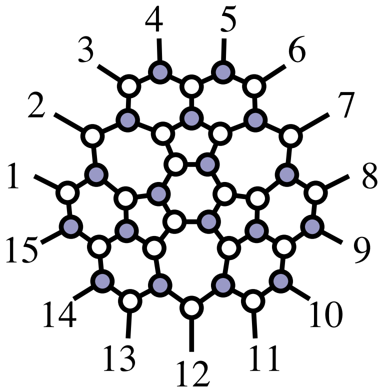

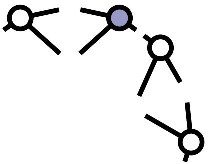

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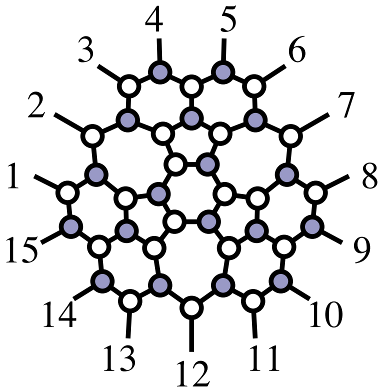

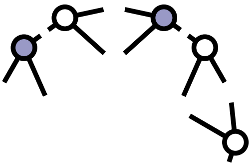

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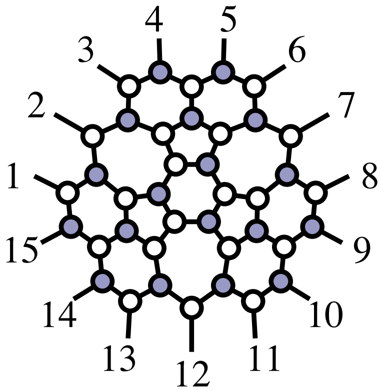

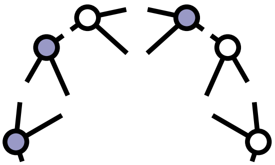

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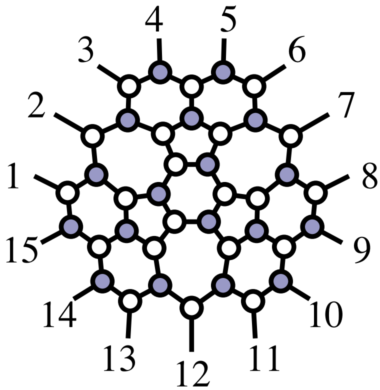

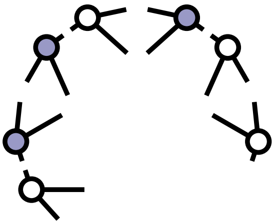

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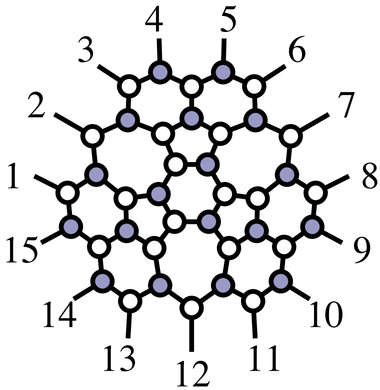

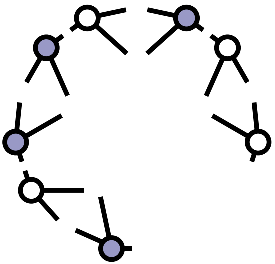

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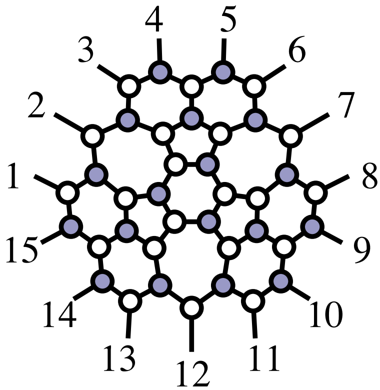

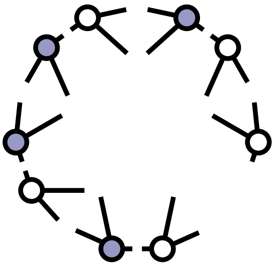

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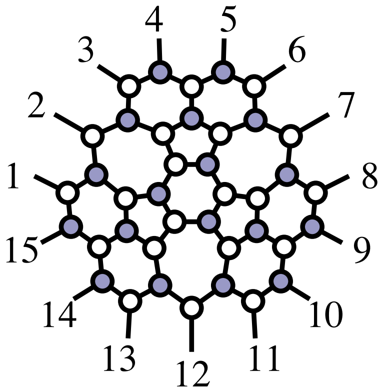

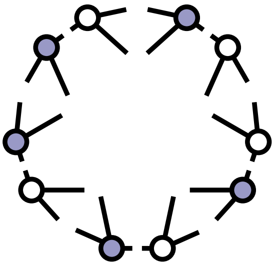

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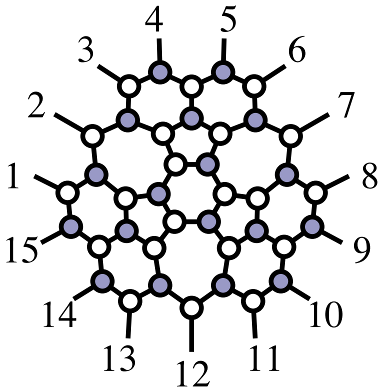

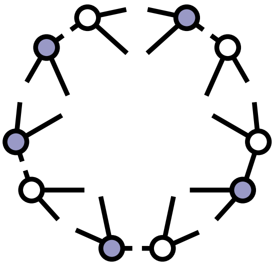

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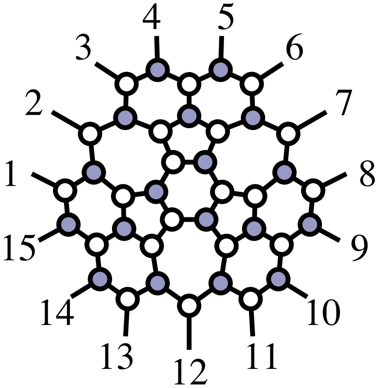

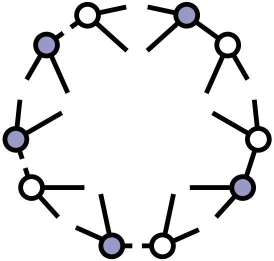

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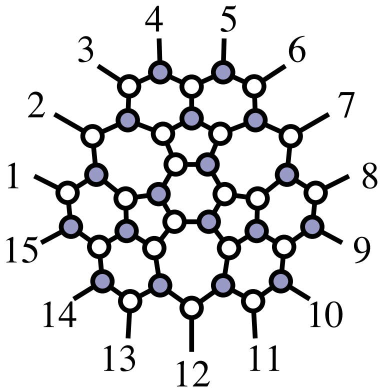

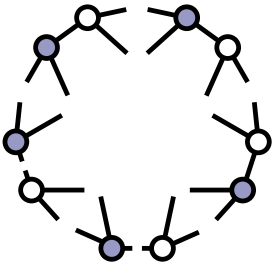

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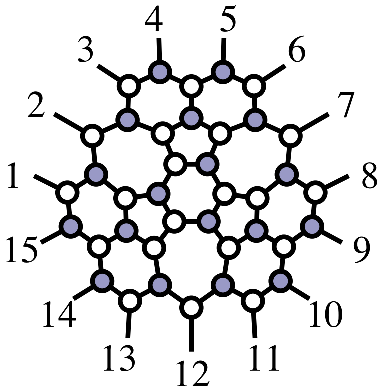

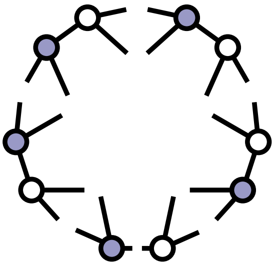

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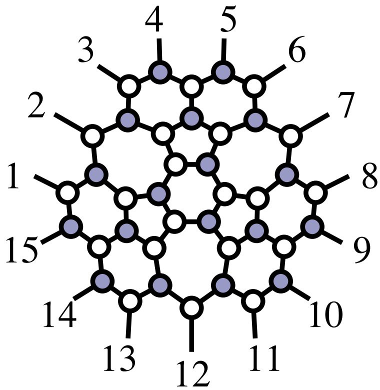

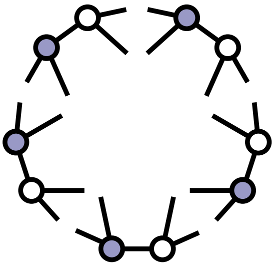

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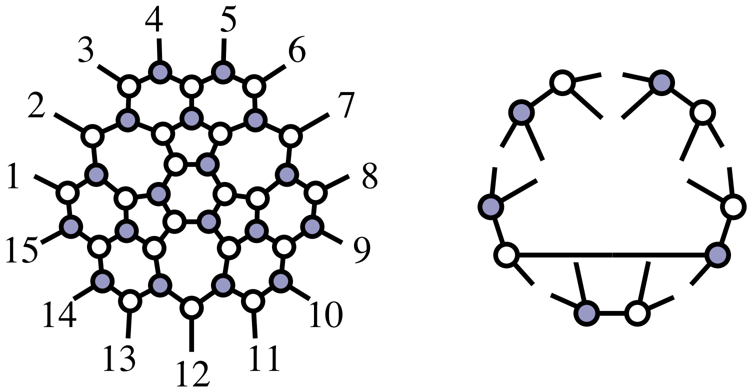

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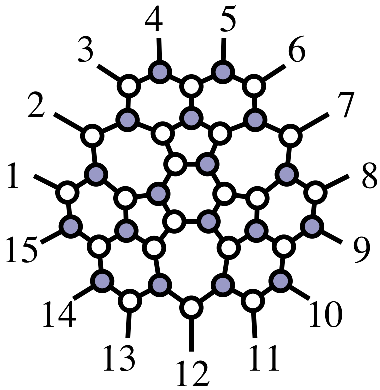

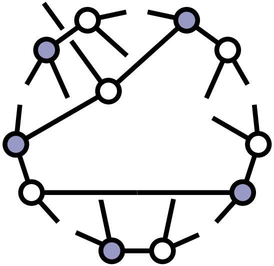

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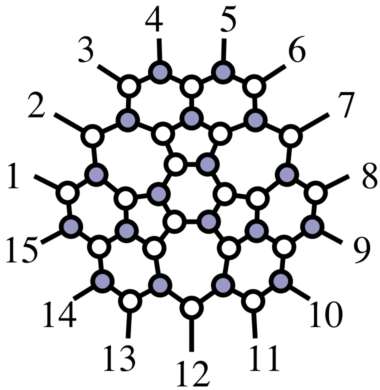

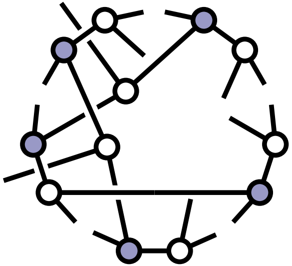

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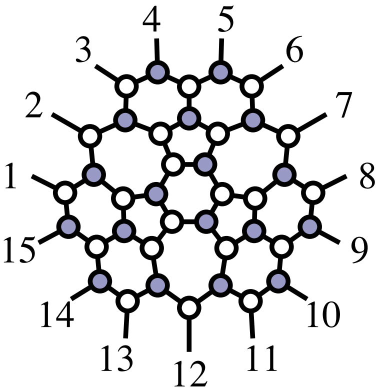

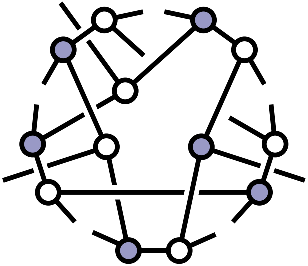

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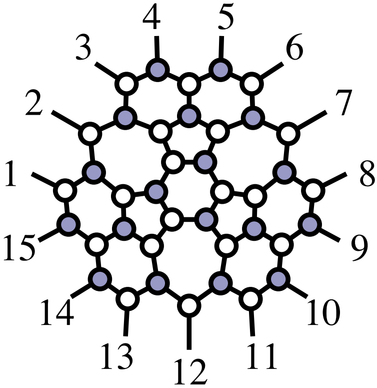

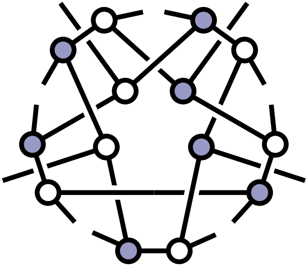

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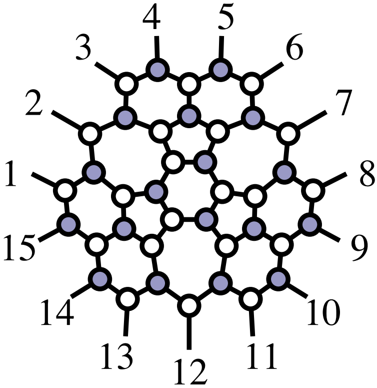

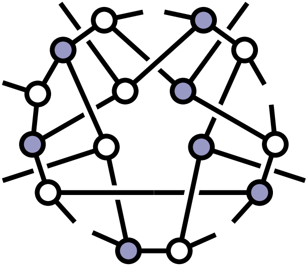

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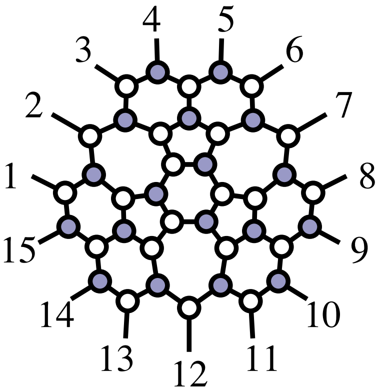

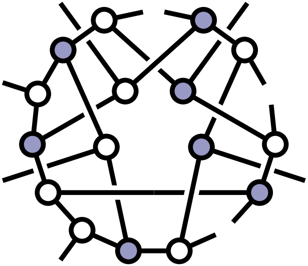

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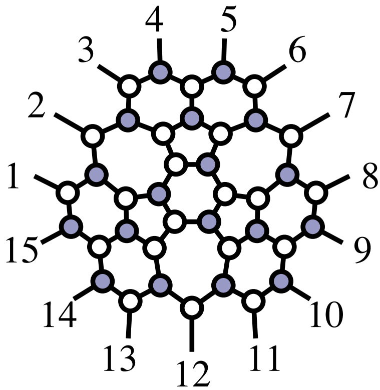

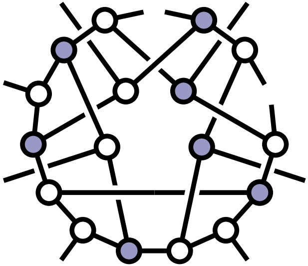

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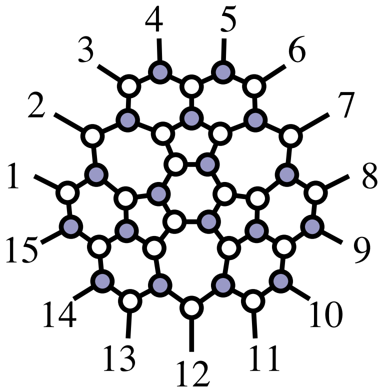

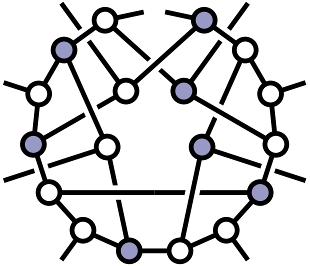

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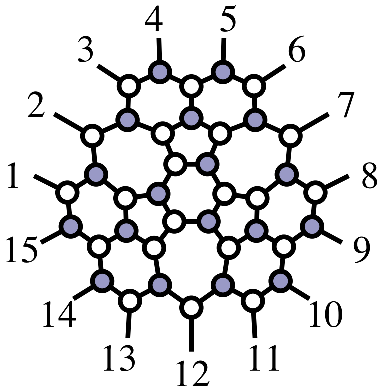

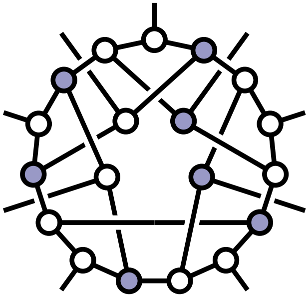

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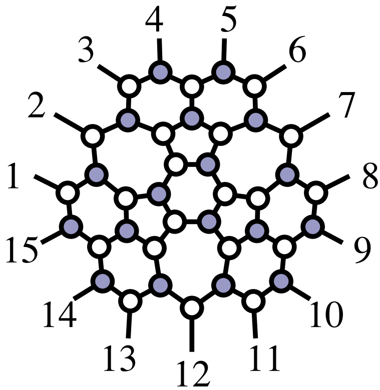

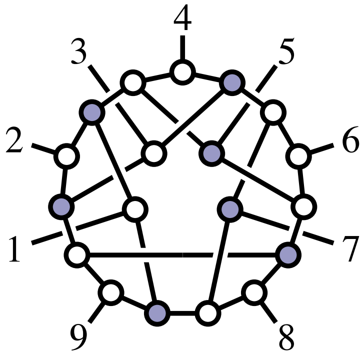

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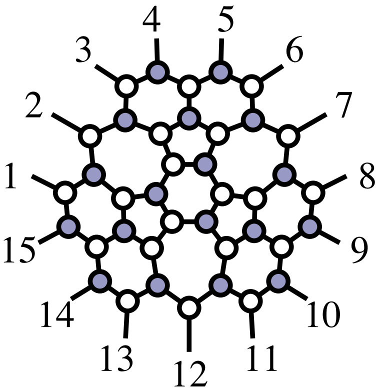

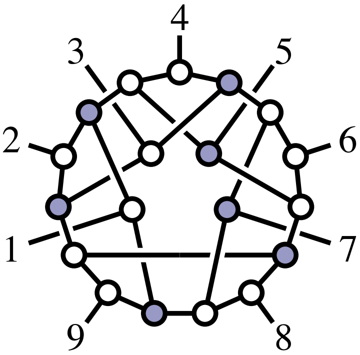

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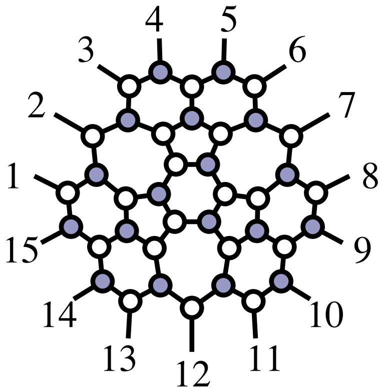

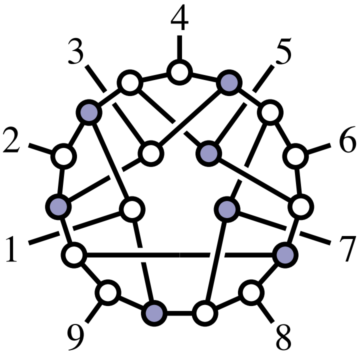

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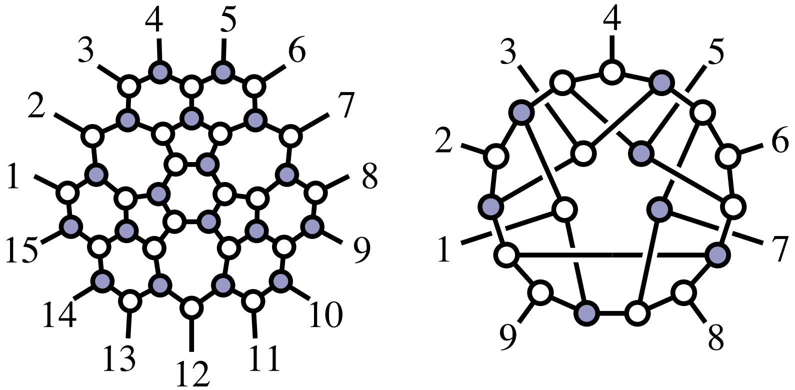

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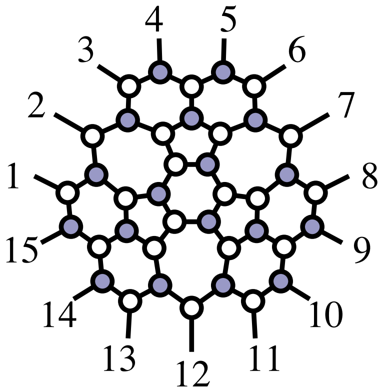

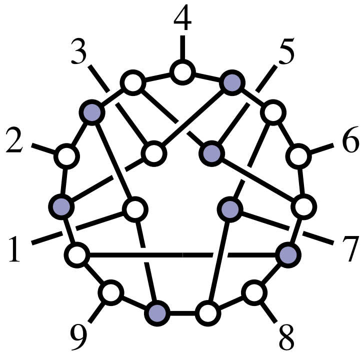

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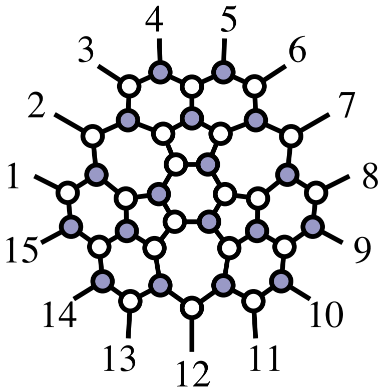

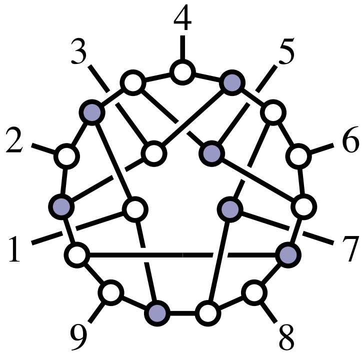

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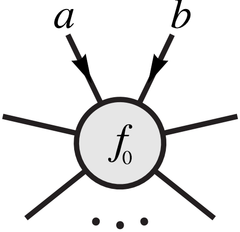

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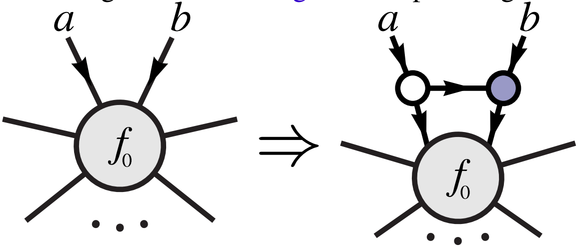

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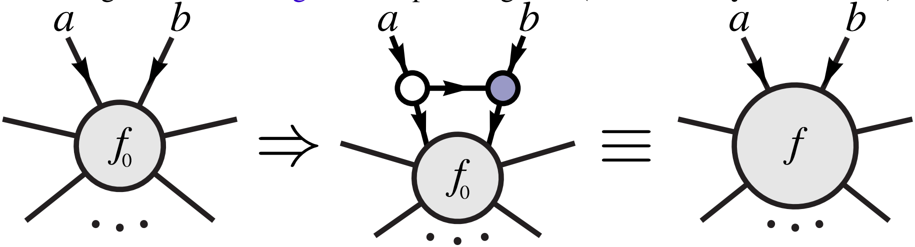

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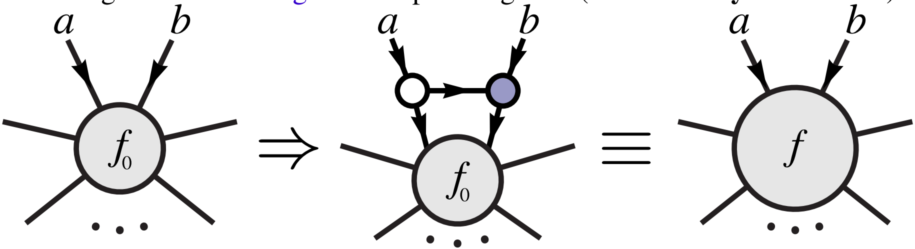

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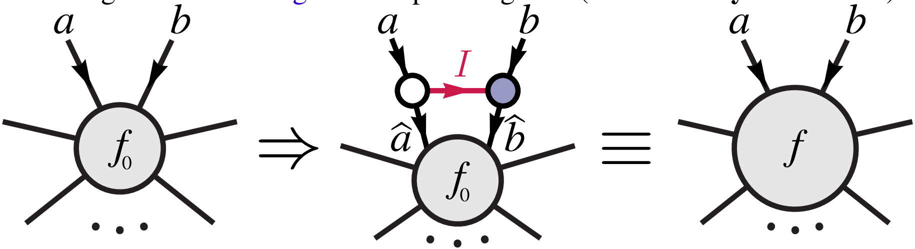

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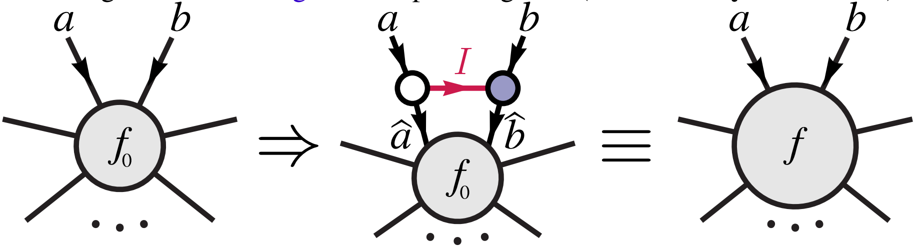

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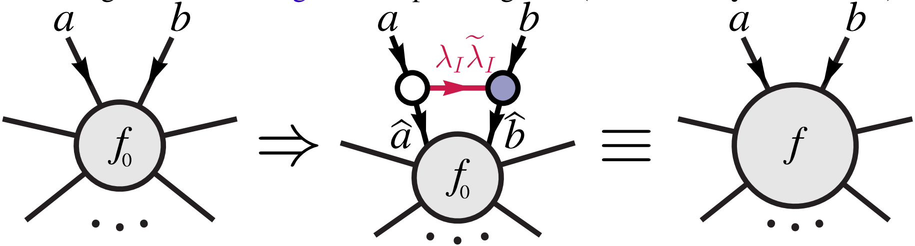

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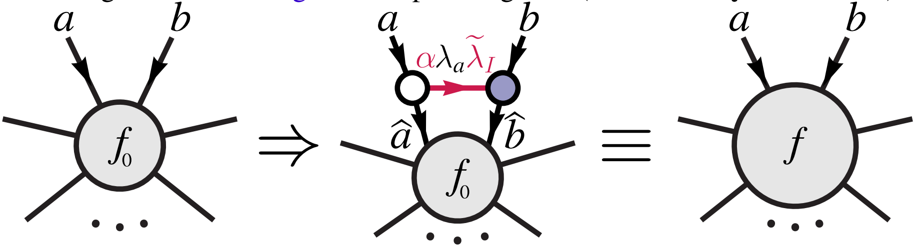

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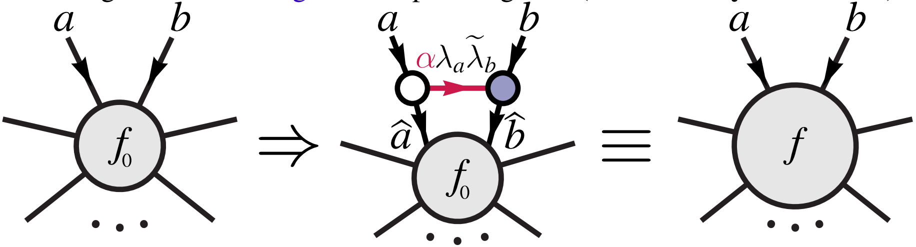

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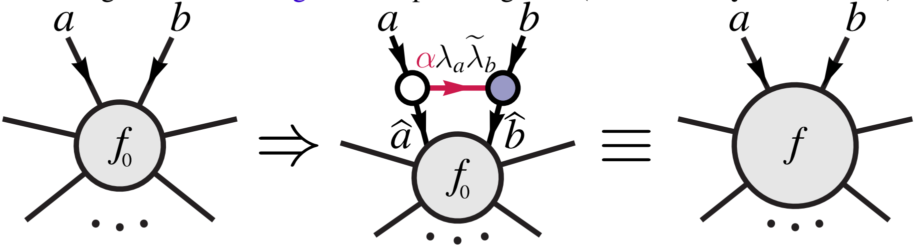

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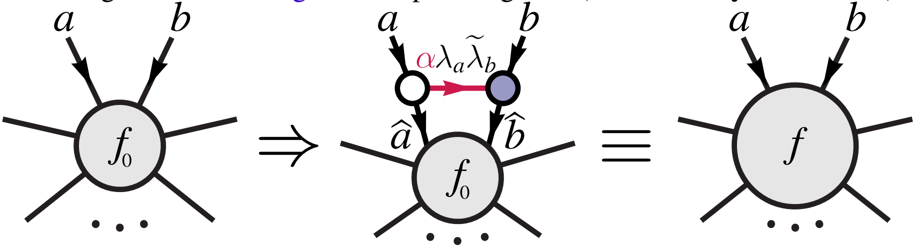

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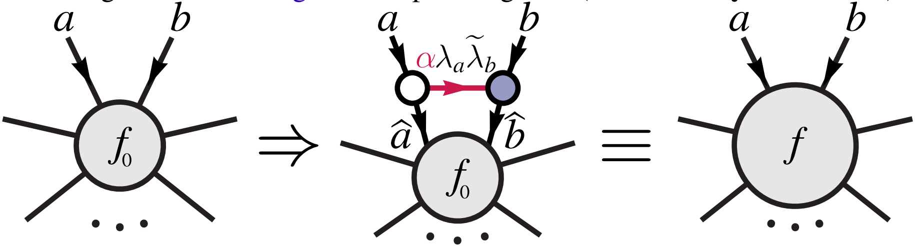

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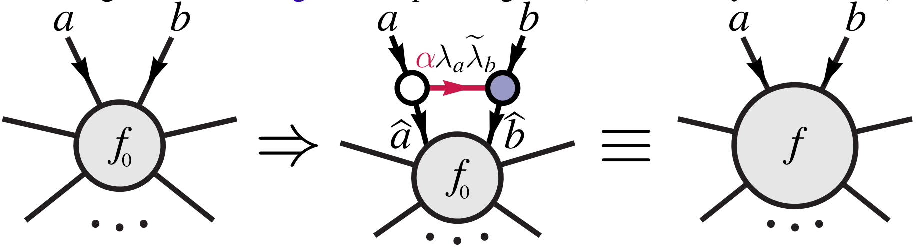

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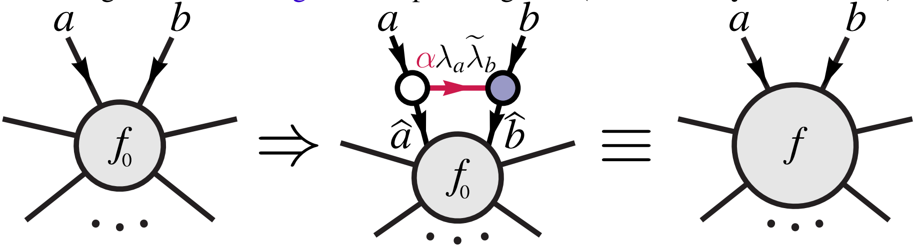

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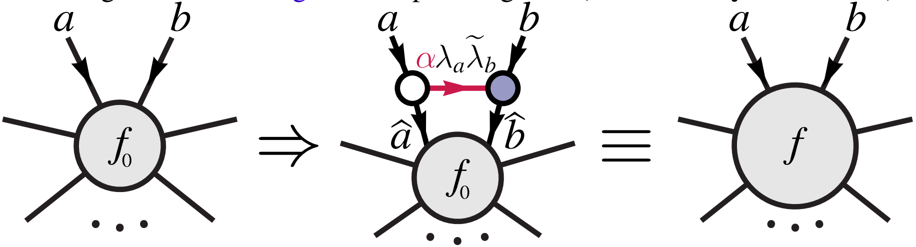

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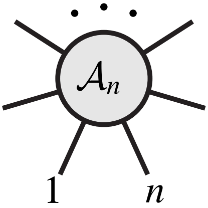

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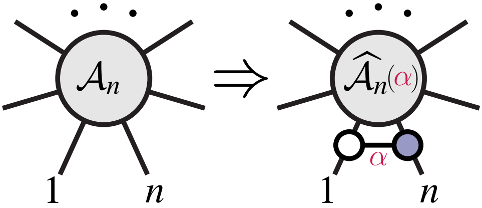

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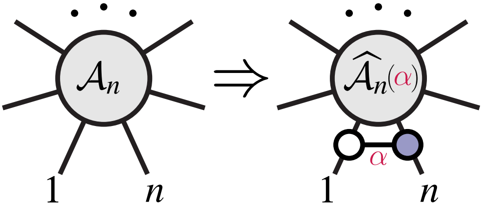

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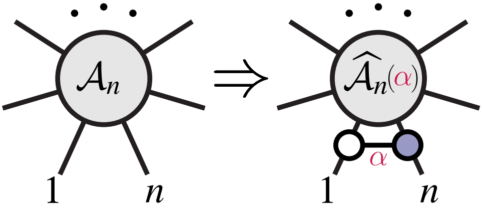

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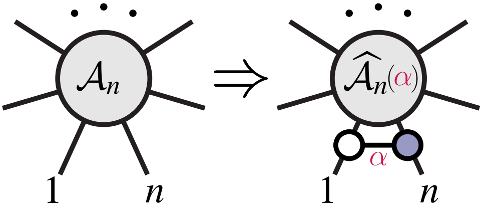

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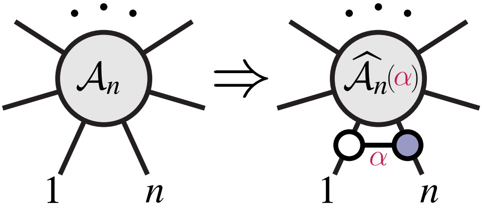

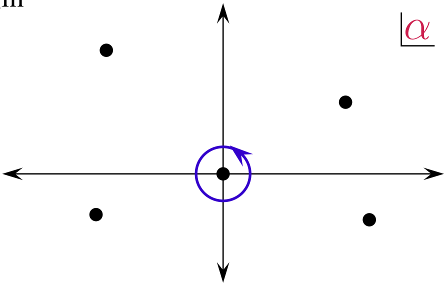

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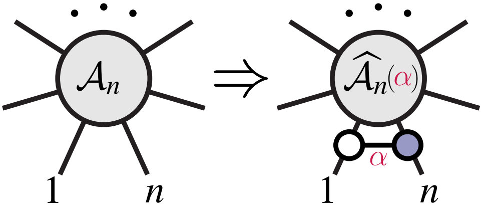

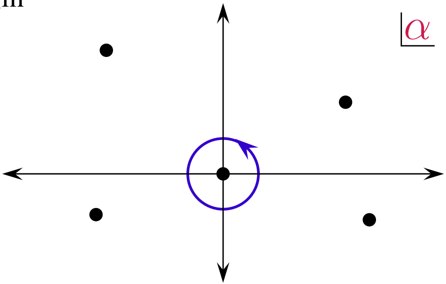

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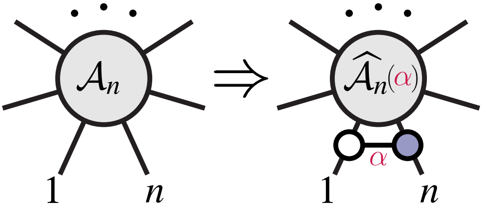

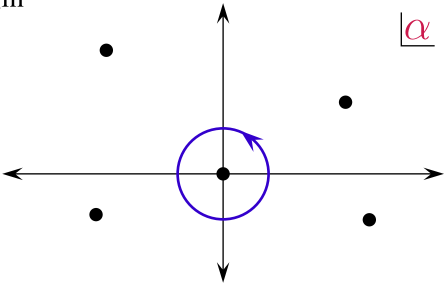

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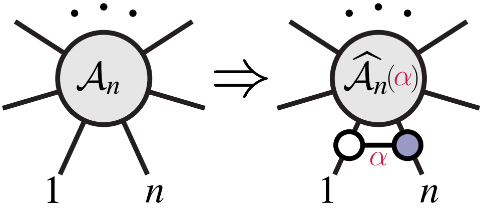

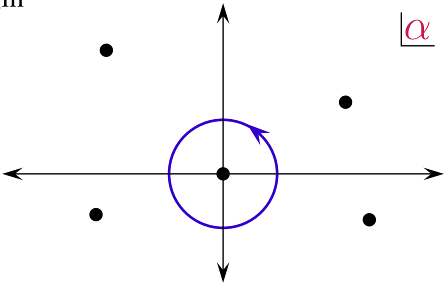

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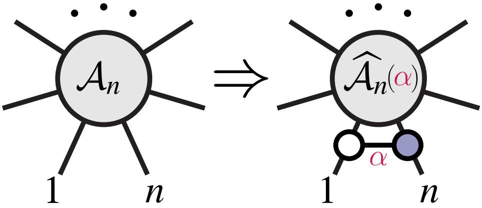

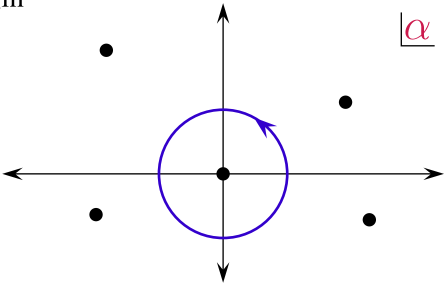

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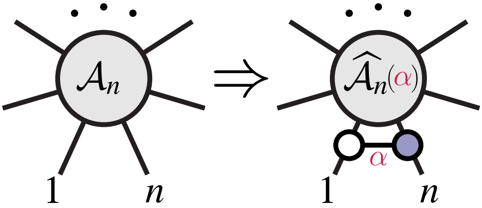

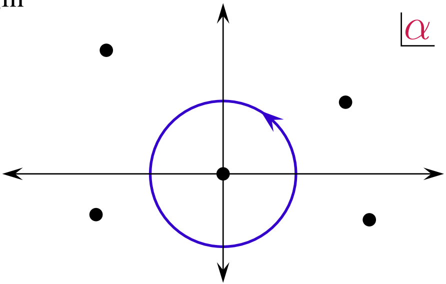

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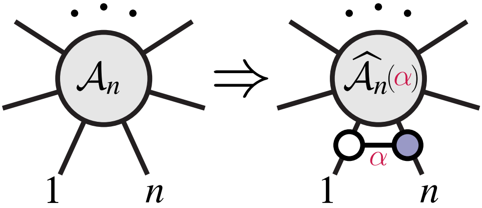

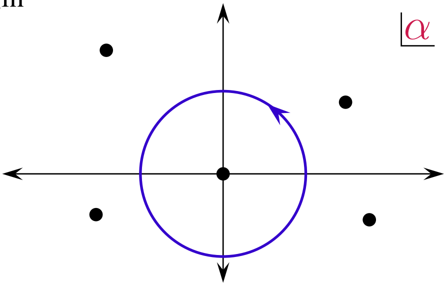

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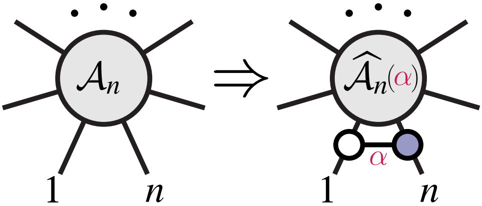

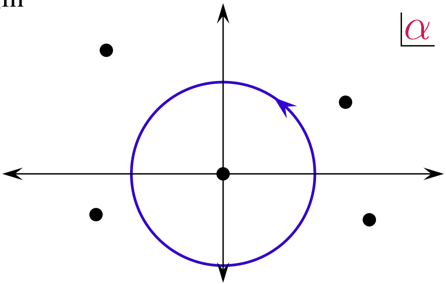

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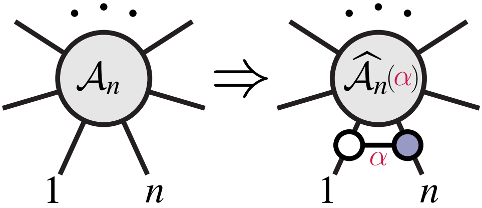

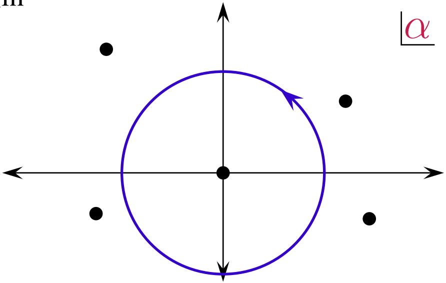

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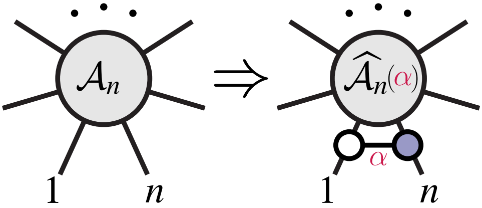

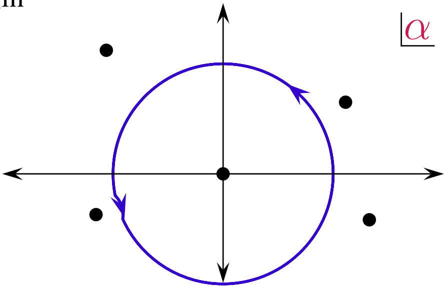

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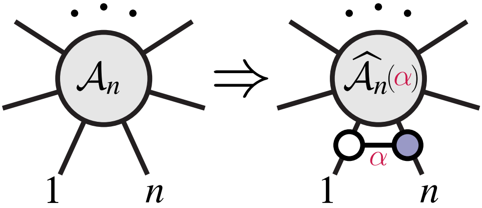

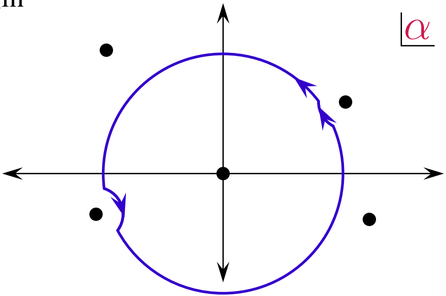

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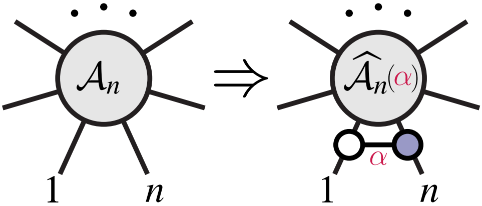

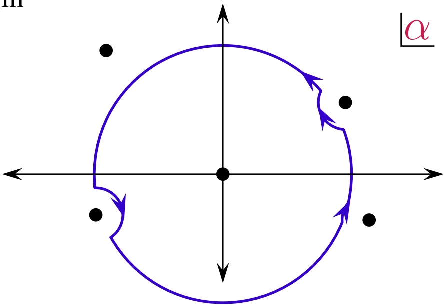

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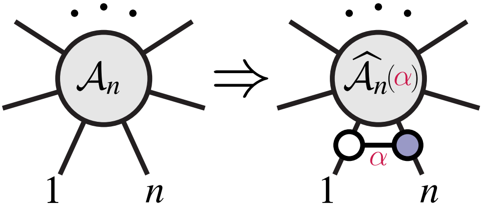

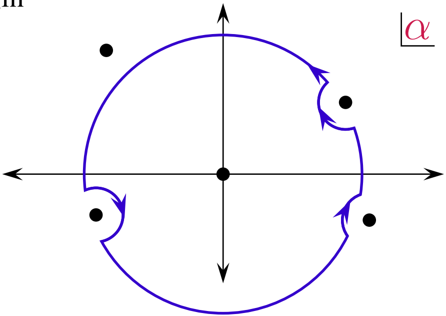

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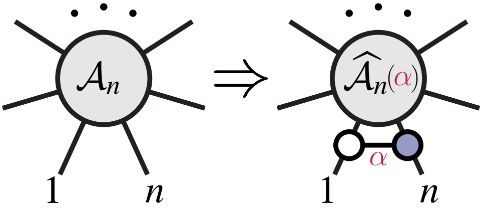

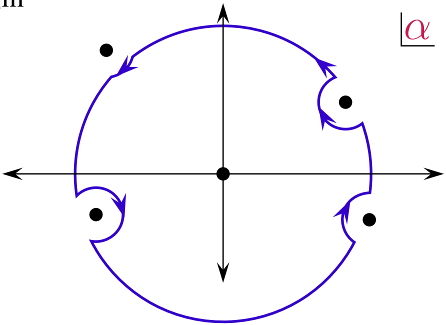

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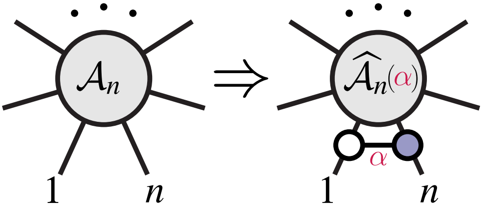

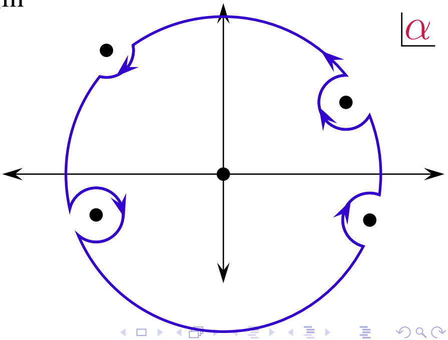

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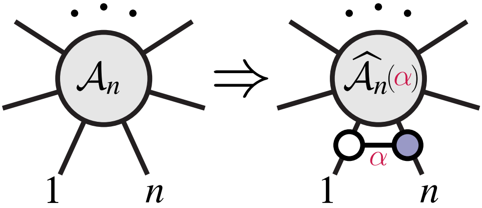

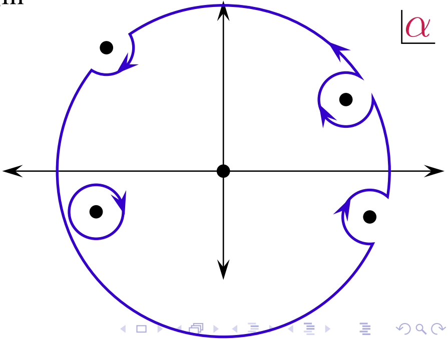

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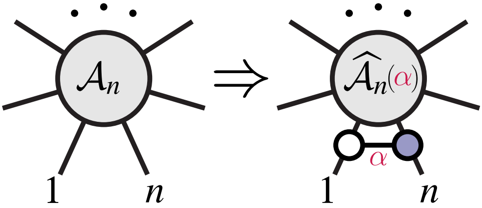

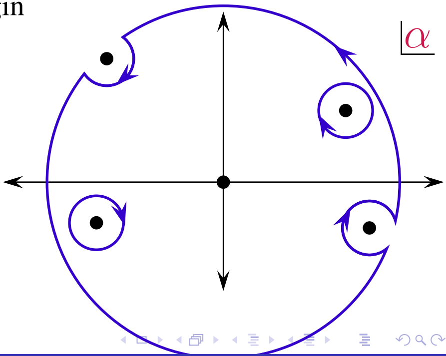

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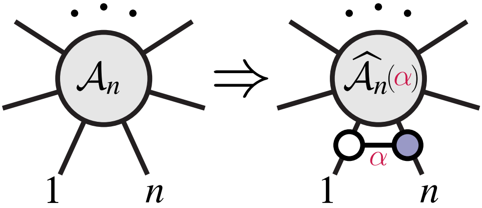

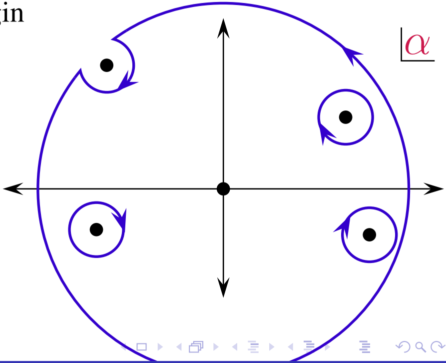

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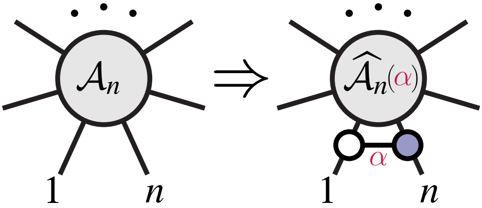

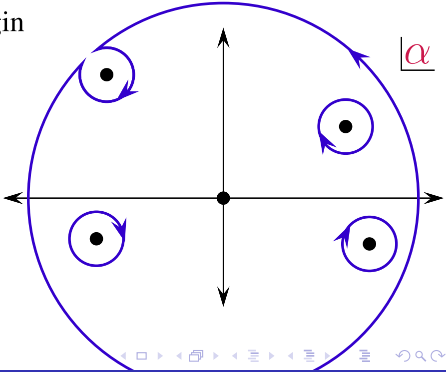

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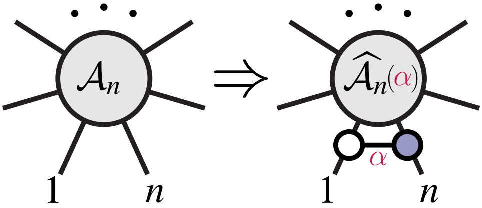

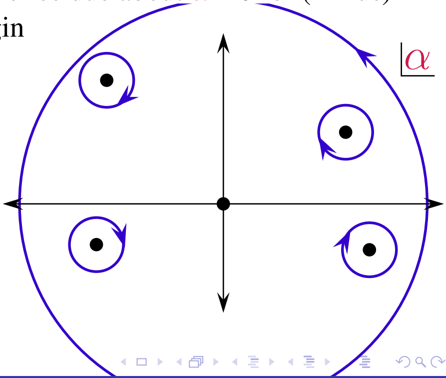

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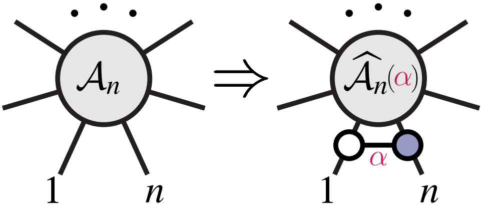

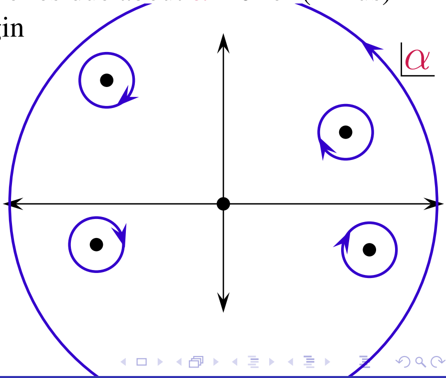

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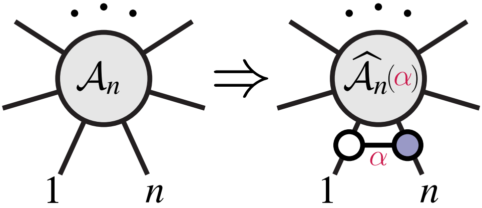

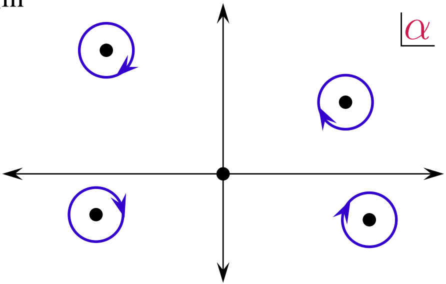

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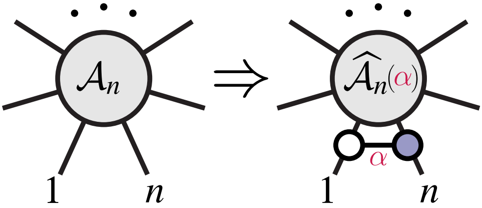

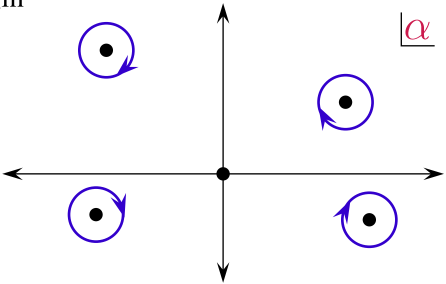

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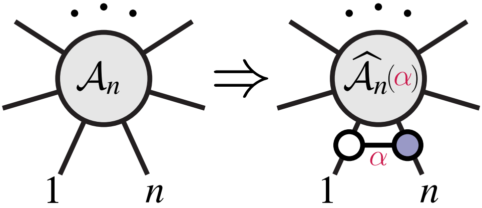

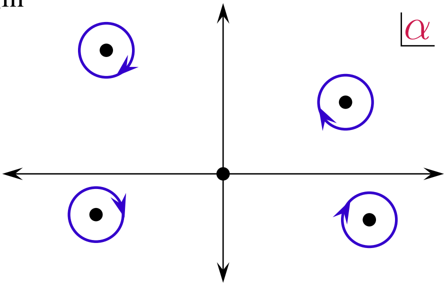

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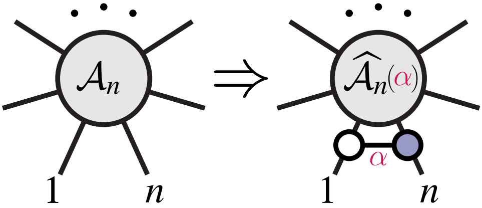

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Consider adding a BCFW bridge to an (unknown) scattering amplitude; the undeformed amplitude $\mathcal{A}_n$ is recovered as the residue about $\alpha = 0$:

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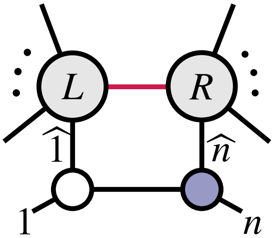

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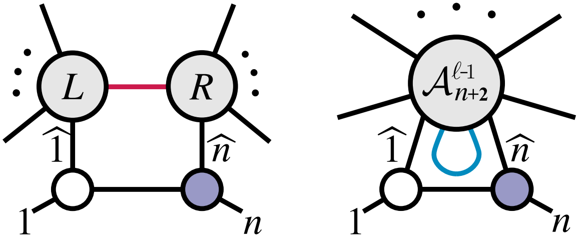

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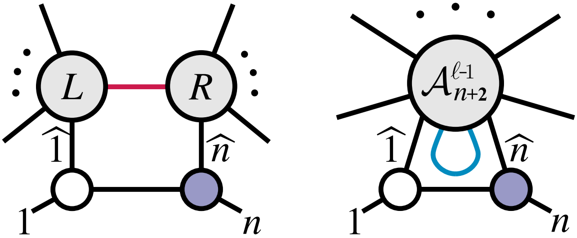

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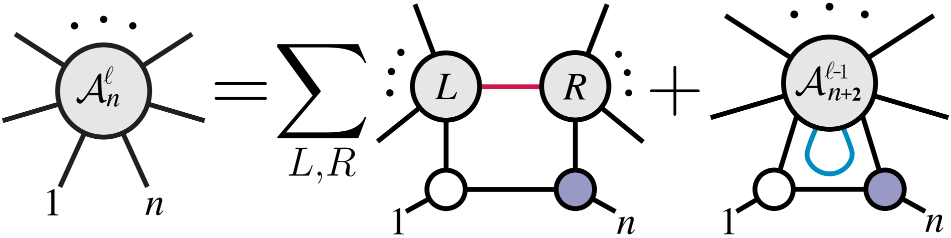

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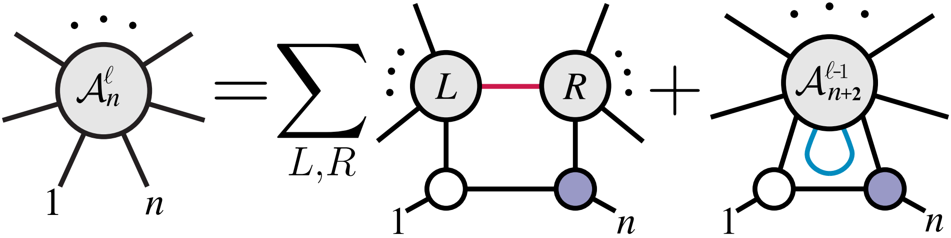

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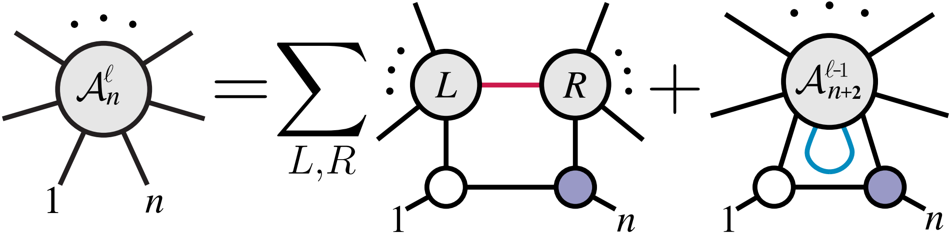

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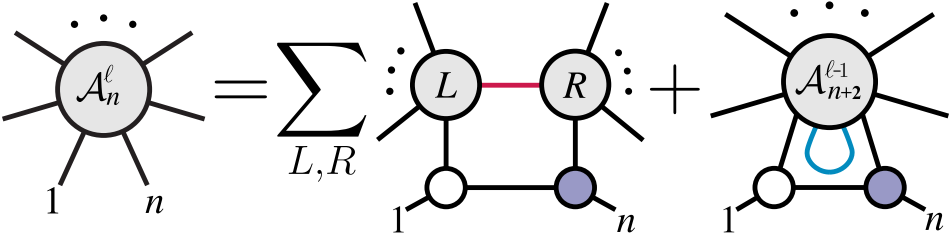

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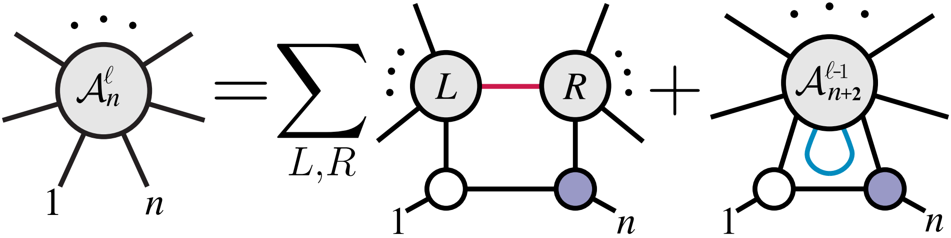

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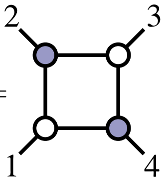
$\mathcal{A}_4^{(2)} =$ 

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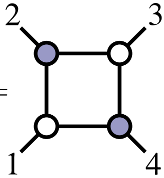
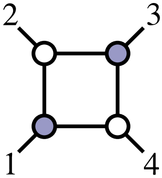
$\mathcal{A}_4^{(2)} =$ 

$=$ 

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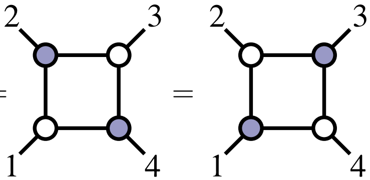
$\mathcal{A}_4^{(2)} =$ 

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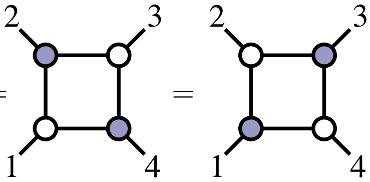

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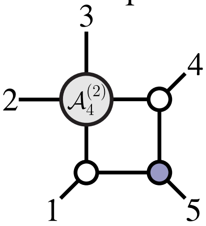

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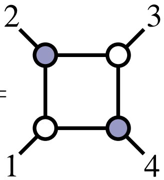

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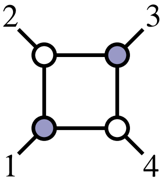

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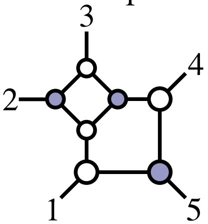

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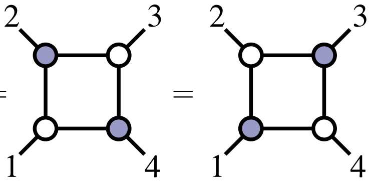

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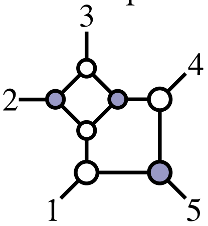

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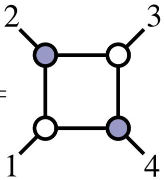
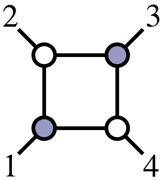
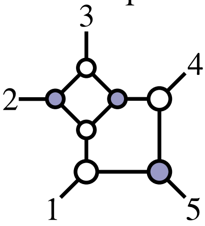
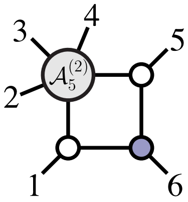
$\mathcal{A}_4^{(2)} =$ 

$=$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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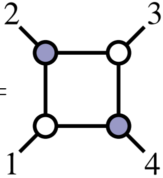
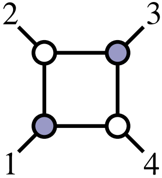
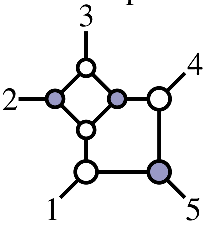
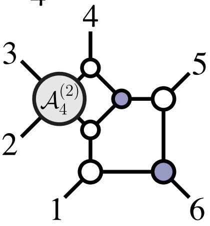
$\mathcal{A}_4^{(2)} =$ 

$=$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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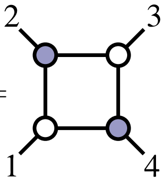
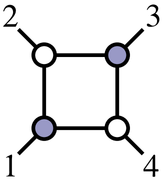
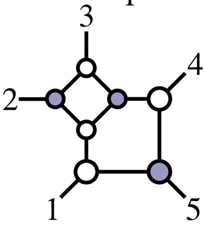
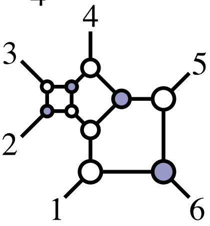
$\mathcal{A}_4^{(2)} =$ 

$=$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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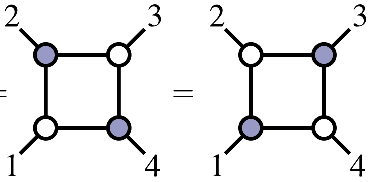
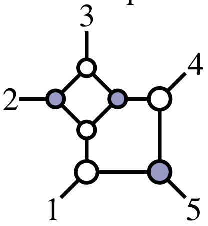
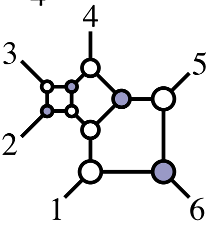
$\mathcal{A}_4^{(2)} =$ 

$\mathcal{A}_5^{(2)} =$ 

$\mathcal{A}_6^{(2)} =$ 

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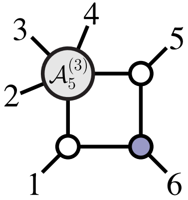

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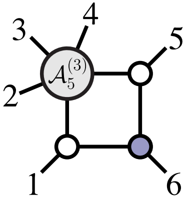
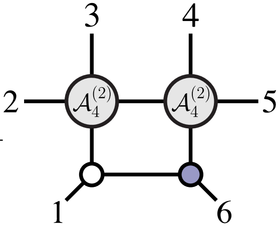
$\mathcal{A}_6^{(3)} =$  $+$ 

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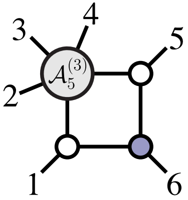
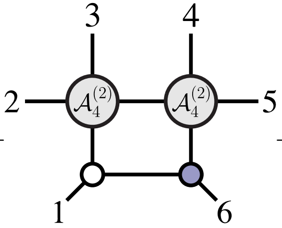
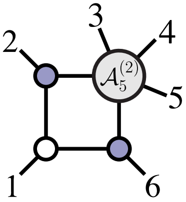
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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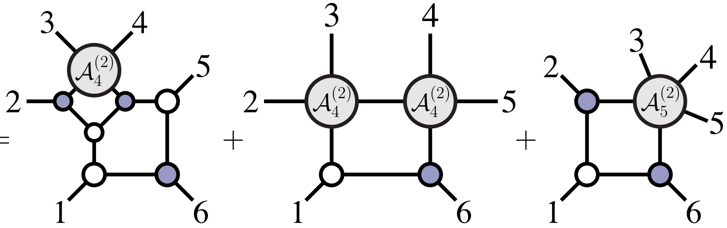
$\mathcal{A}_6^{(3)} =$ 

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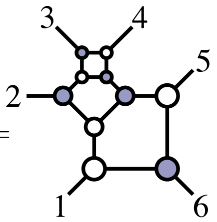
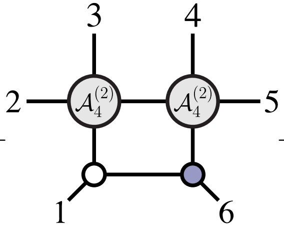
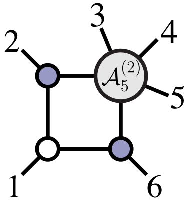
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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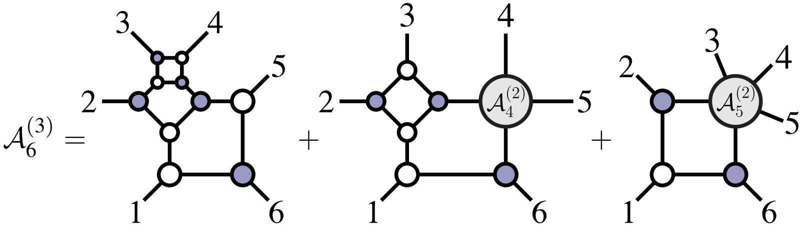

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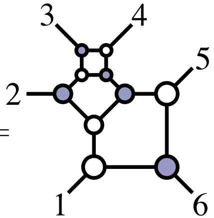
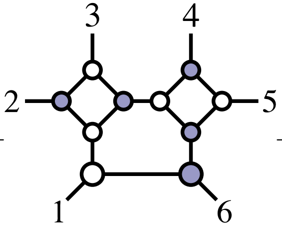
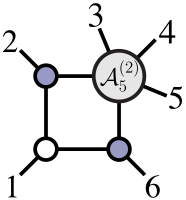
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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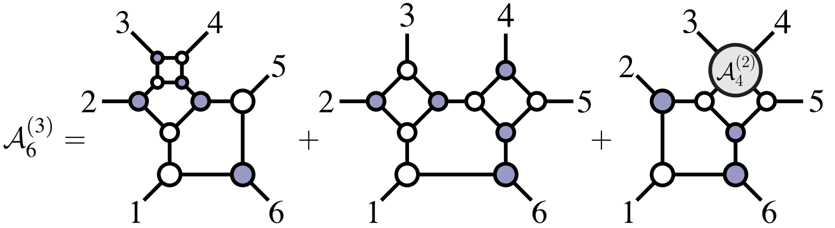

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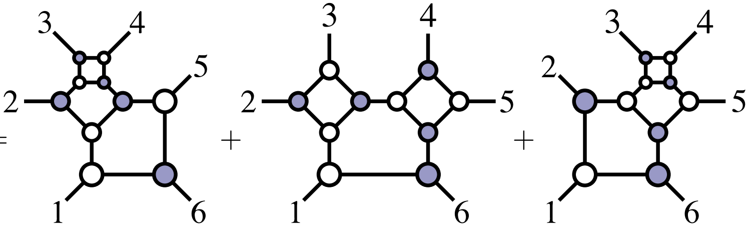

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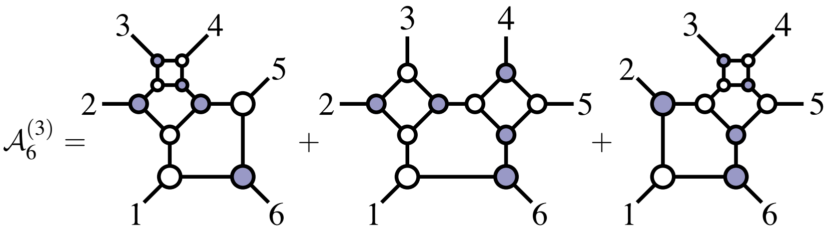

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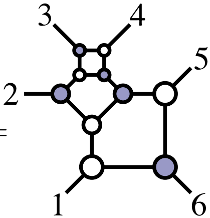
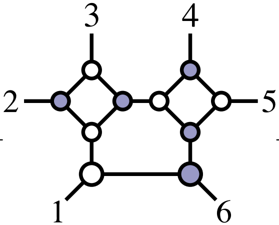
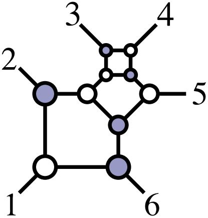
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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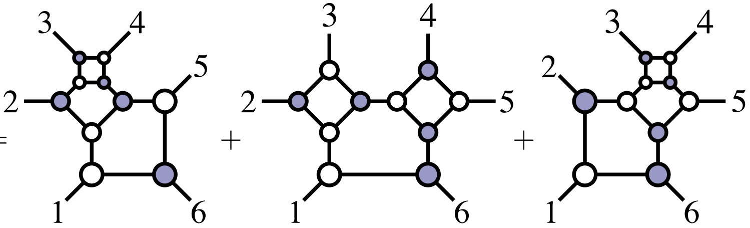

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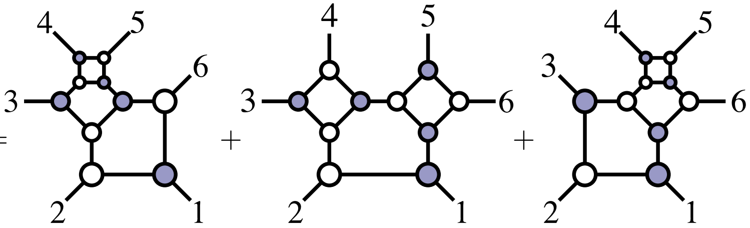
$\mathcal{A}_6^{(3)} =$ 

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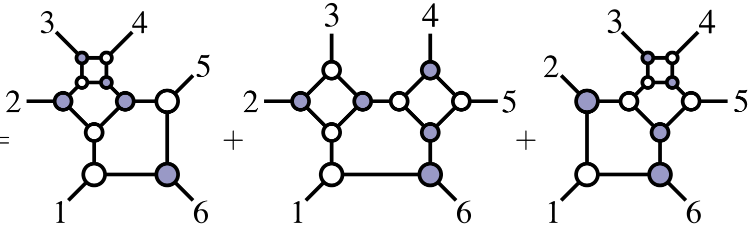
$\mathcal{A}_6^{(3)} =$ 

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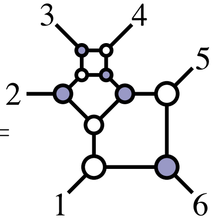
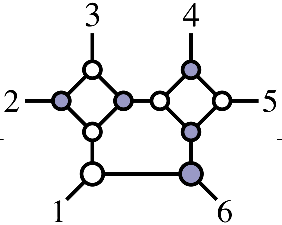
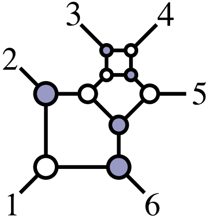
$\mathcal{A}_6^{(3)} =$  $+$  $+$ 

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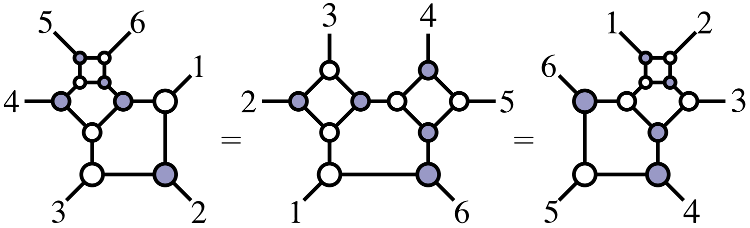

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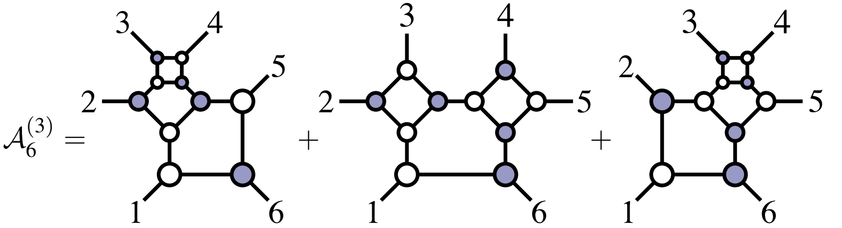

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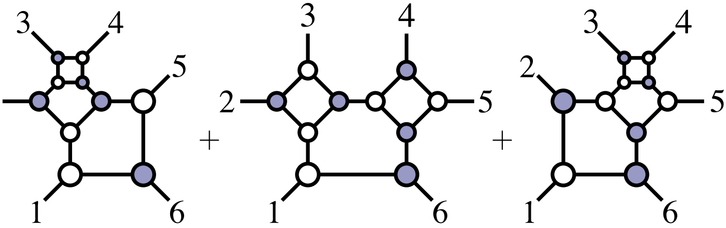

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How can we characterize and compute on-shell diagrams?

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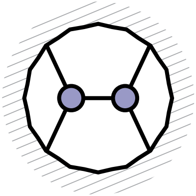

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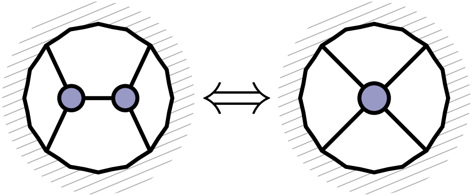

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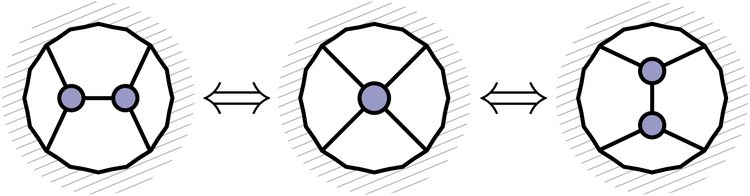

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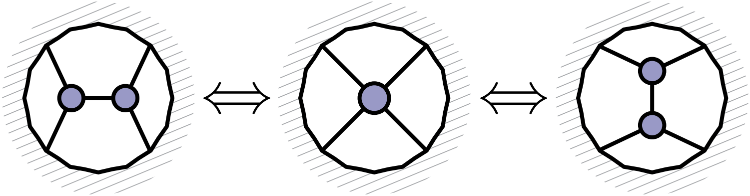

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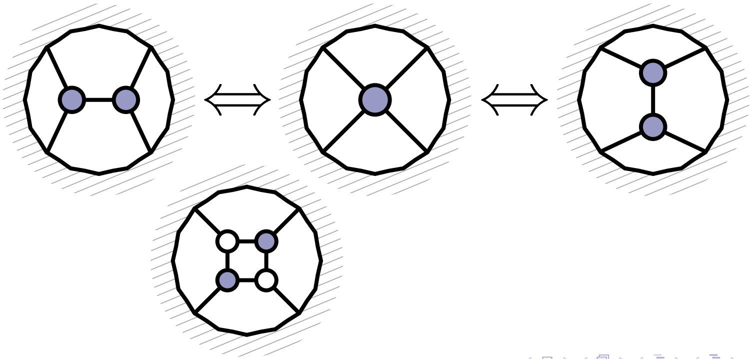

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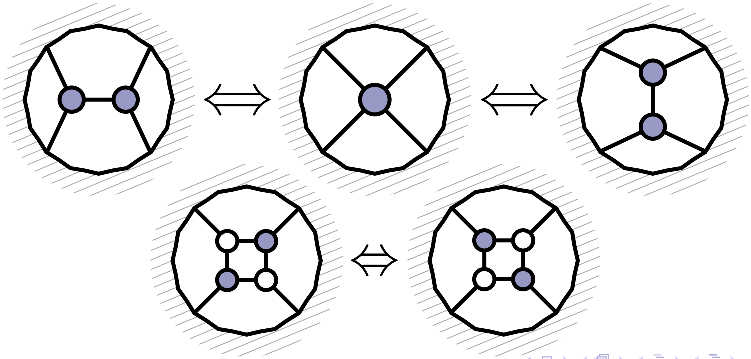

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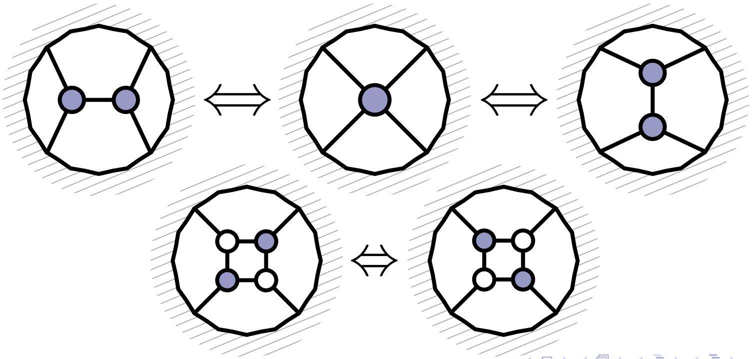

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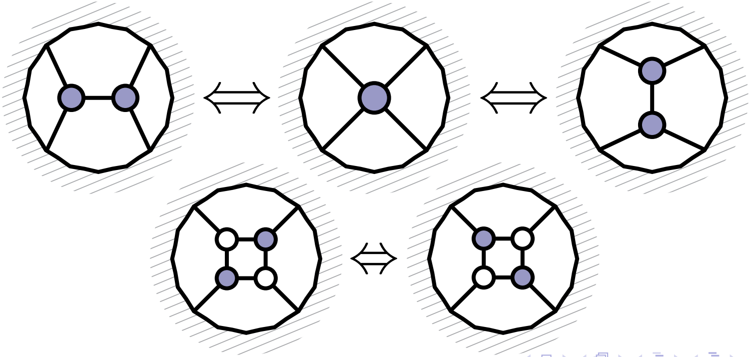

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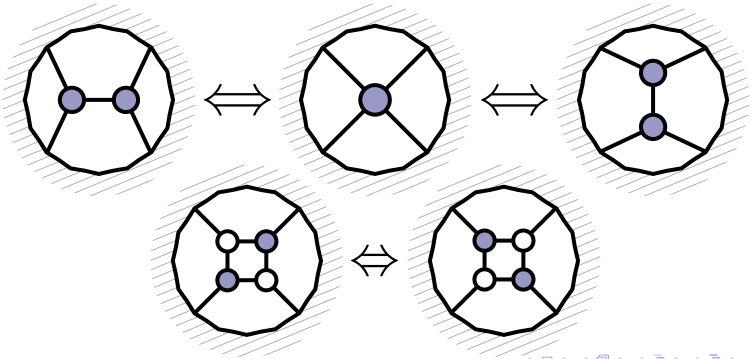

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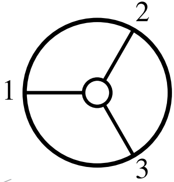

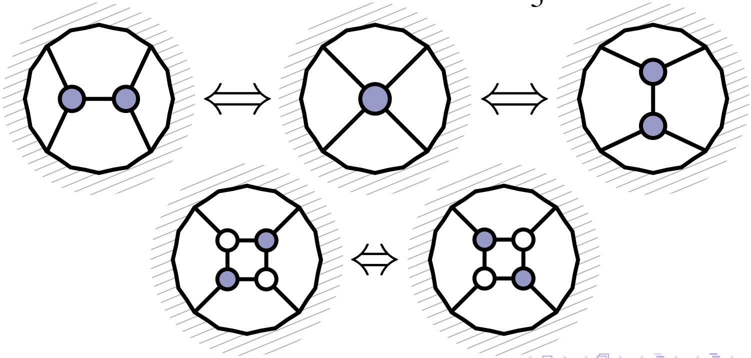

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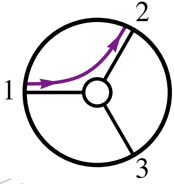

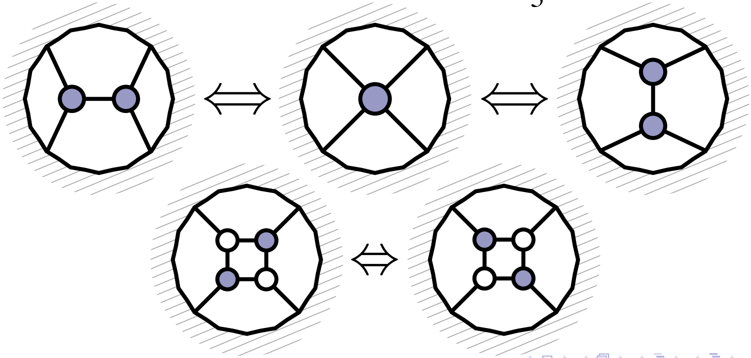

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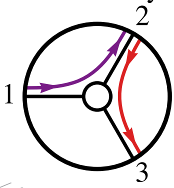

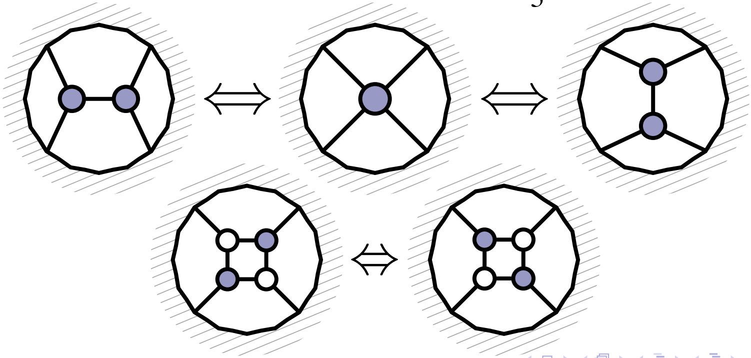

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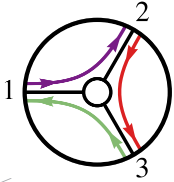

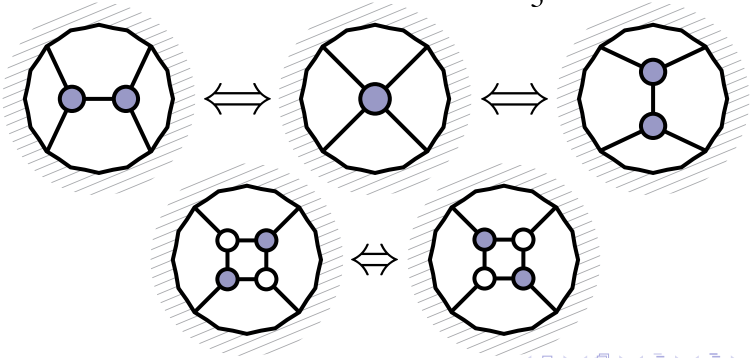

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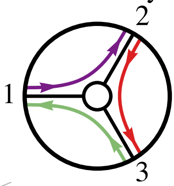

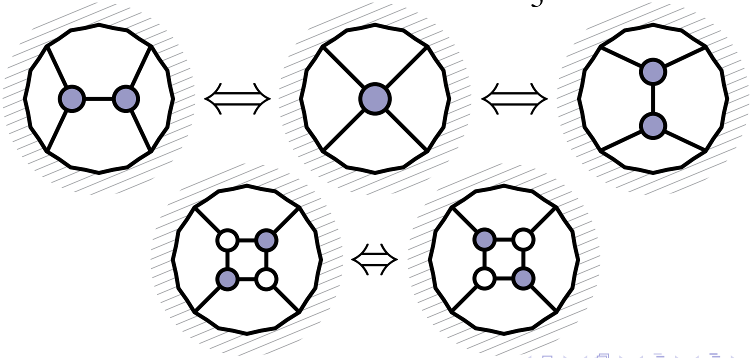

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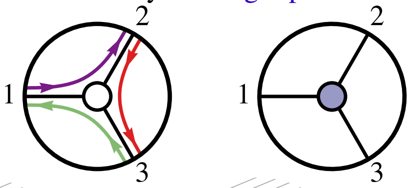

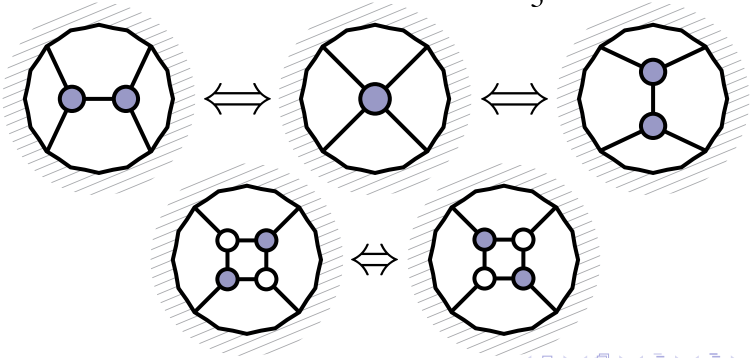

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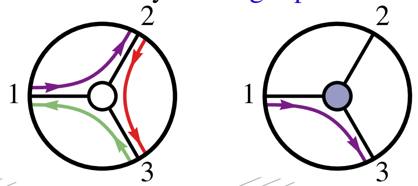

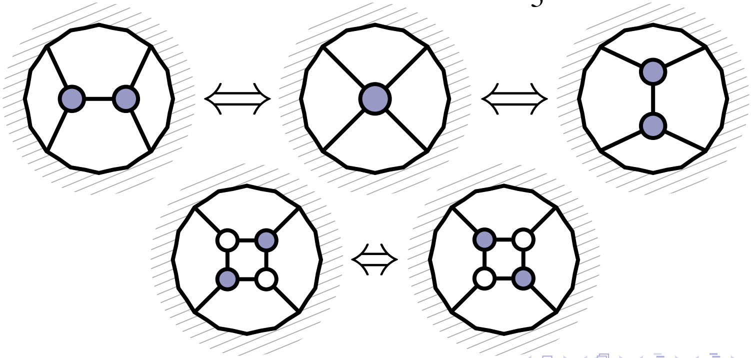

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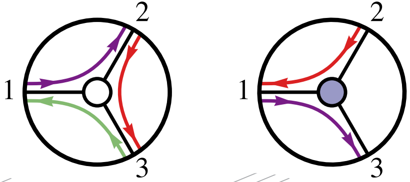

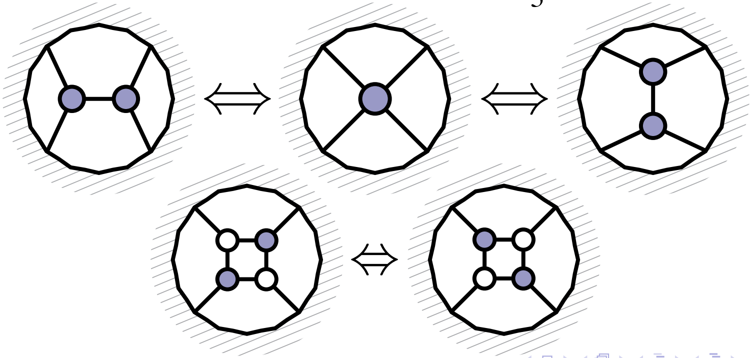

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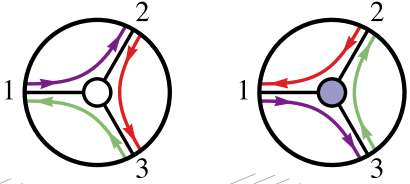

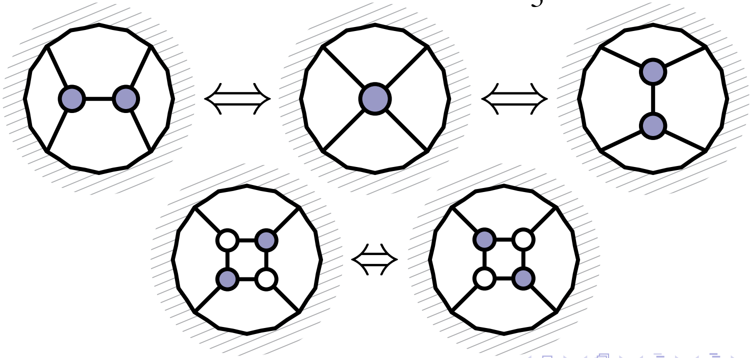

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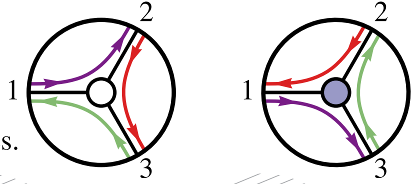

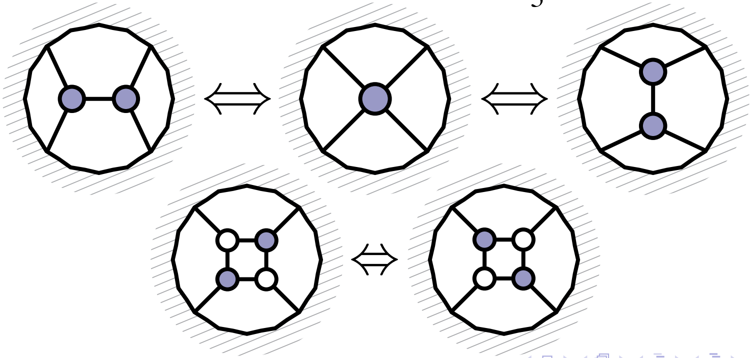

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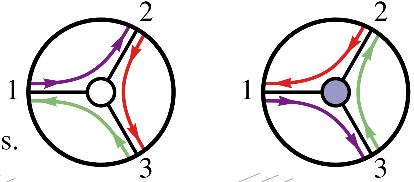

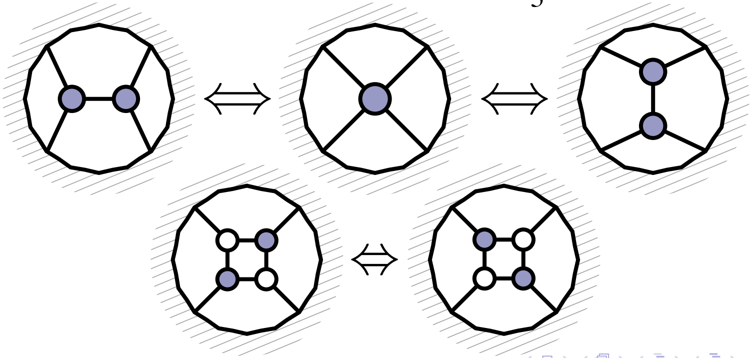

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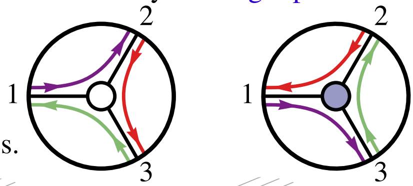

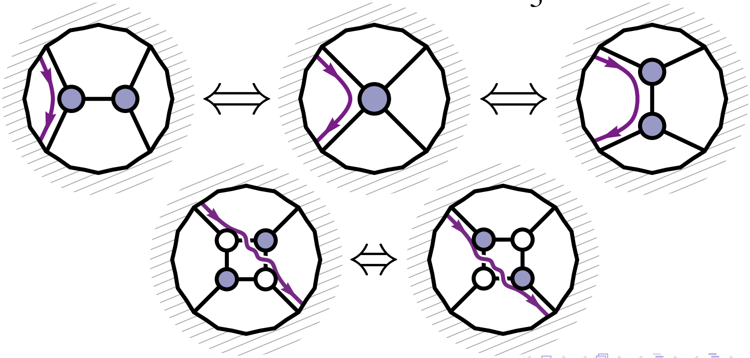

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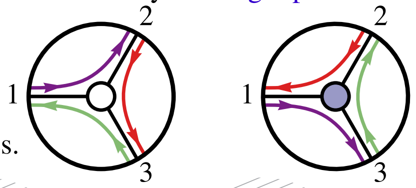

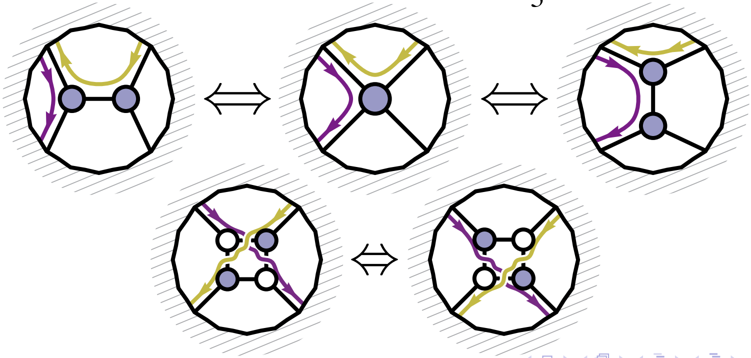

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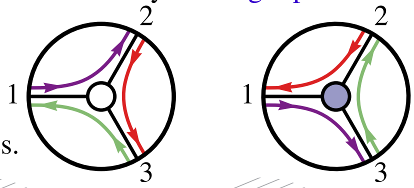

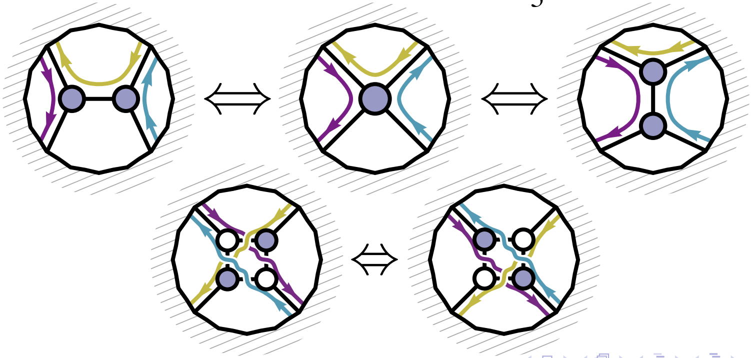

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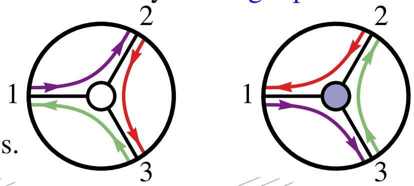

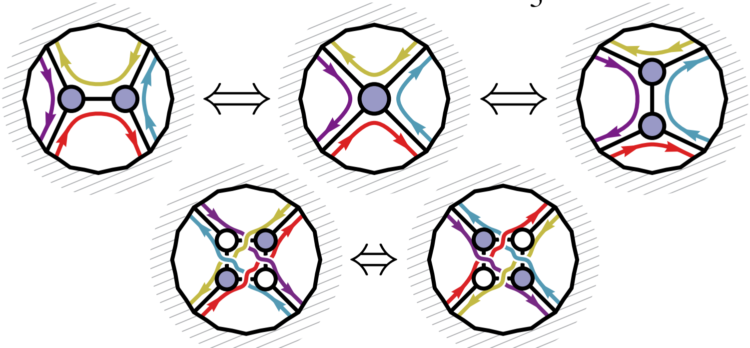

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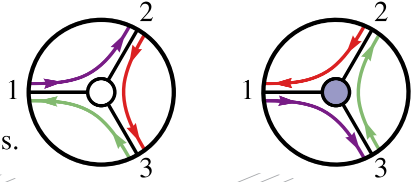

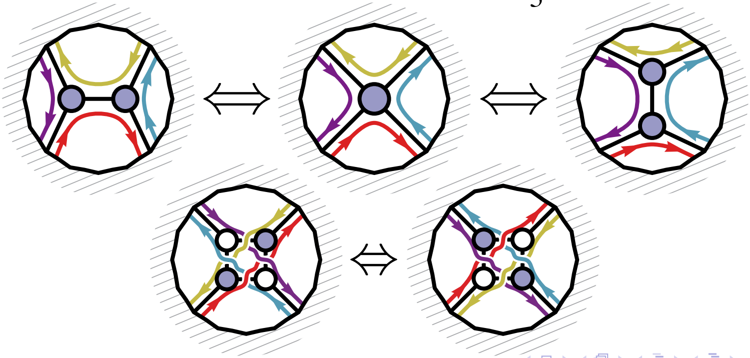

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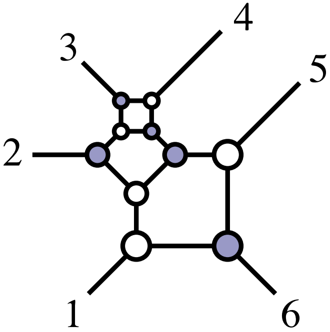

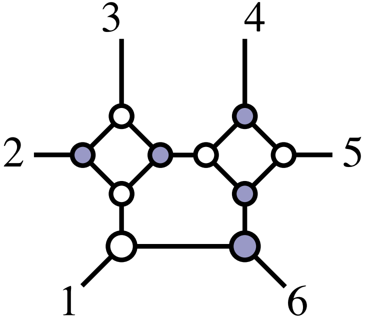

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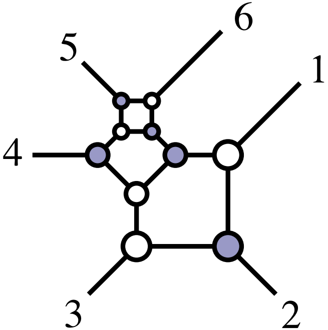

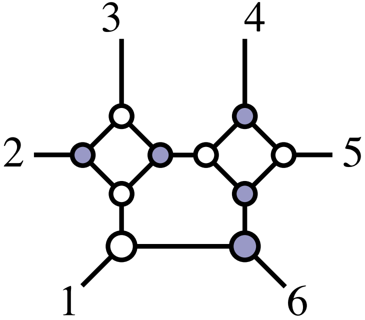

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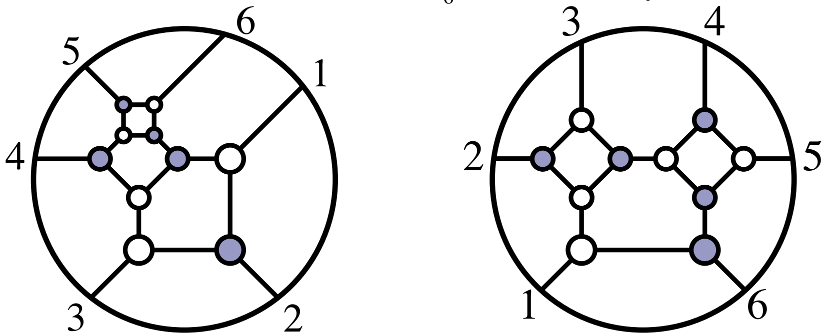

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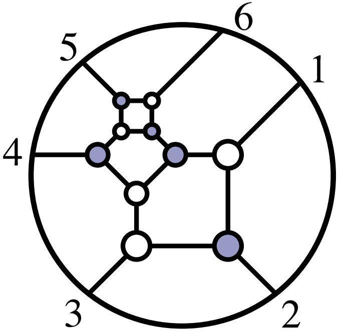

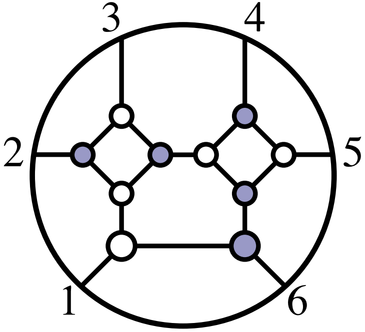

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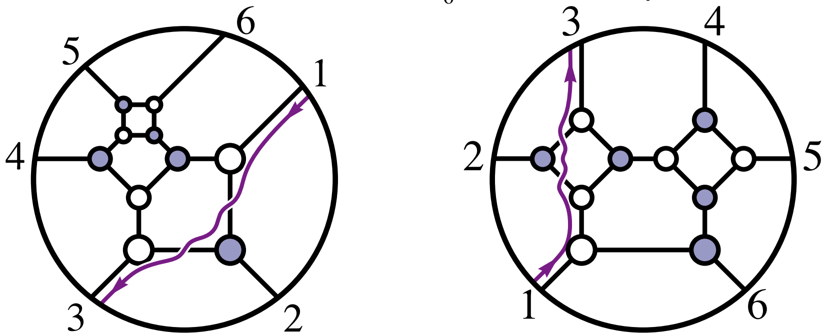

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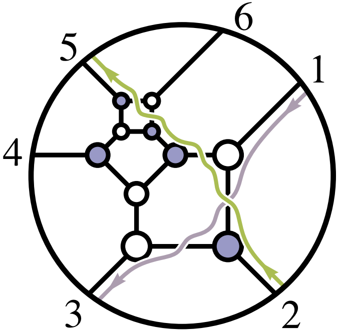

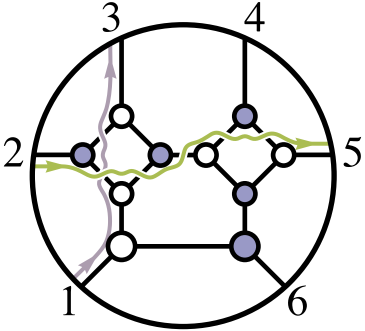

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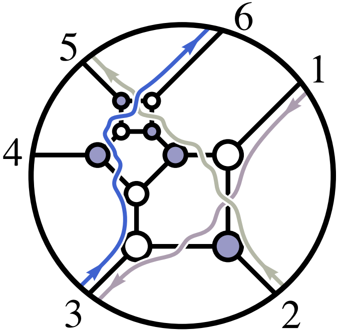

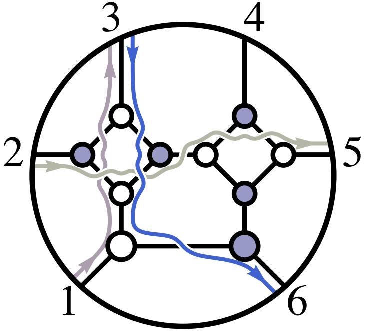

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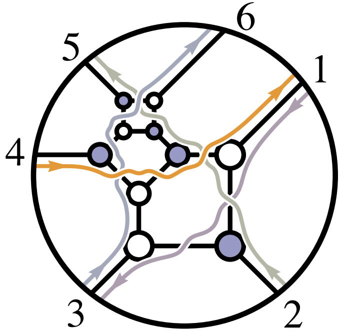

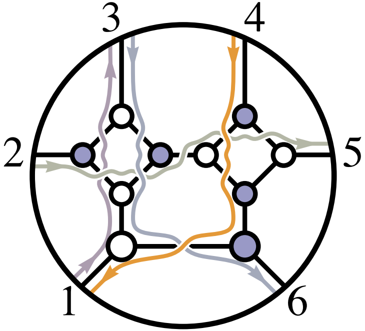

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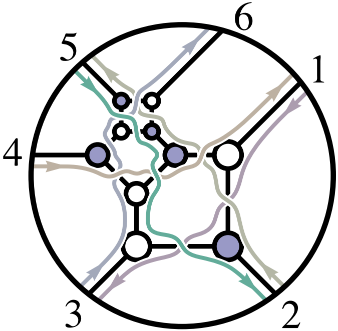

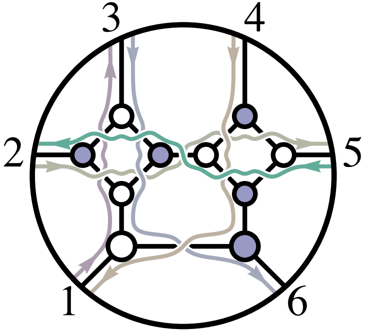

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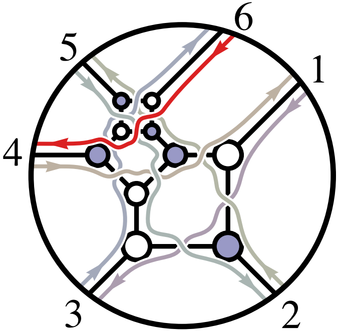

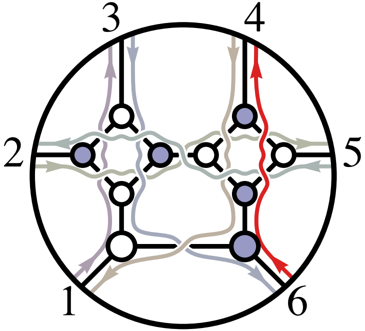

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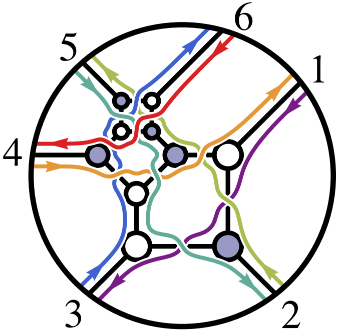

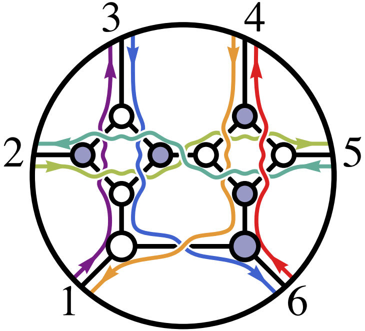

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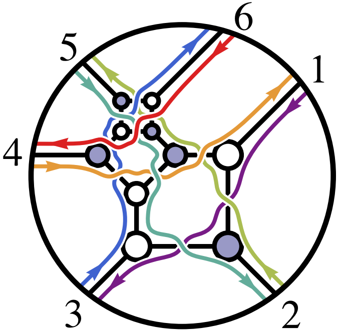

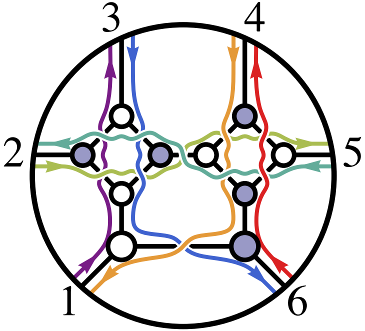

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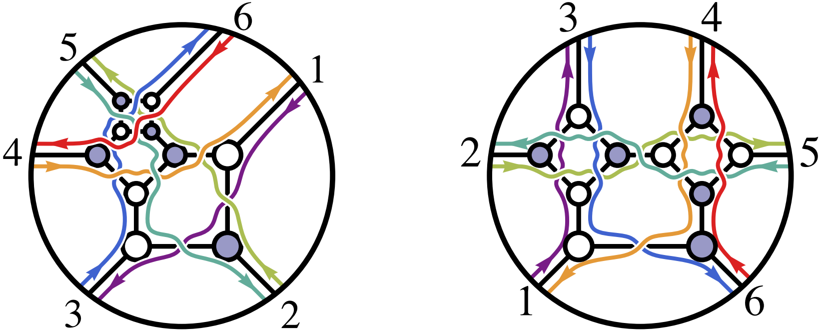

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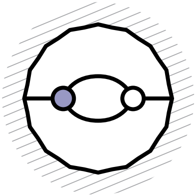

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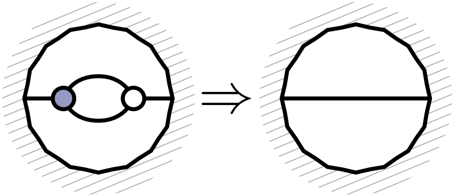

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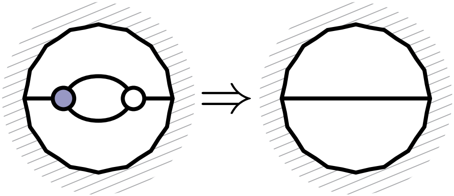

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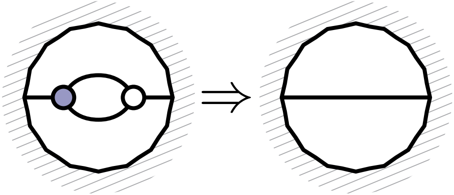

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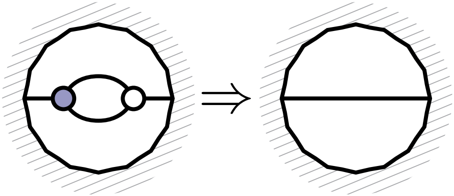

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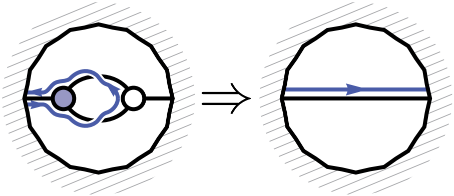

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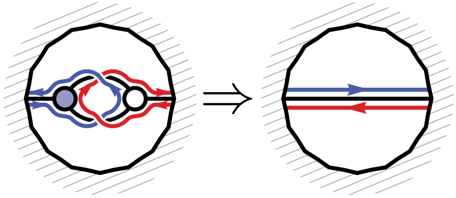

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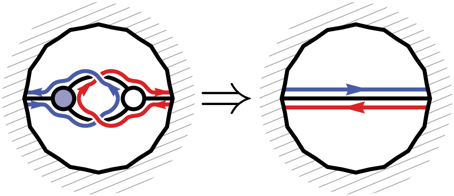
Such factors of $d\alpha/\alpha$ arising from bubble deletion encode loop integrands!



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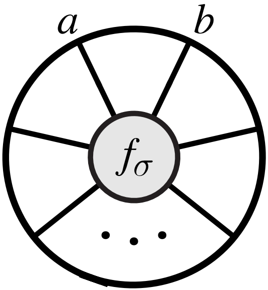

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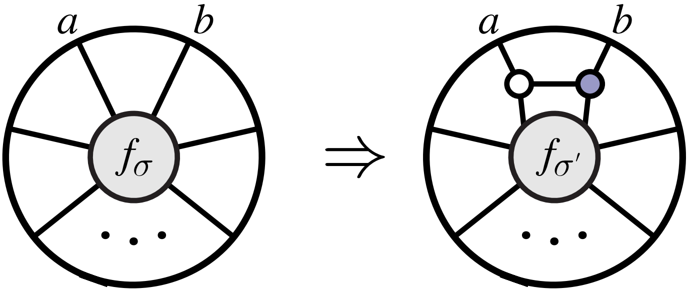

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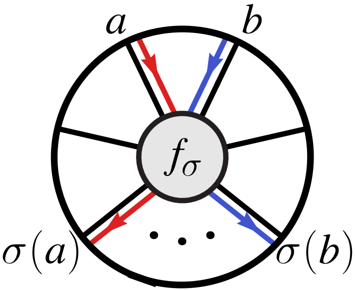

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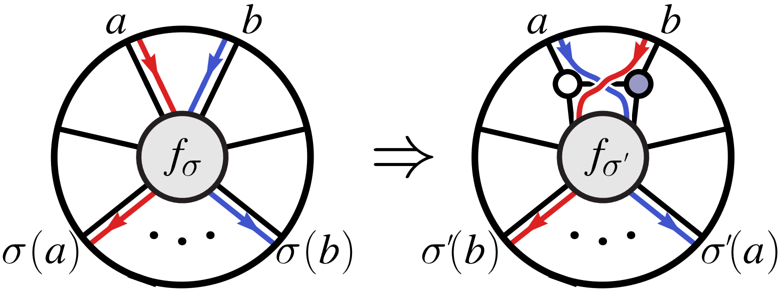

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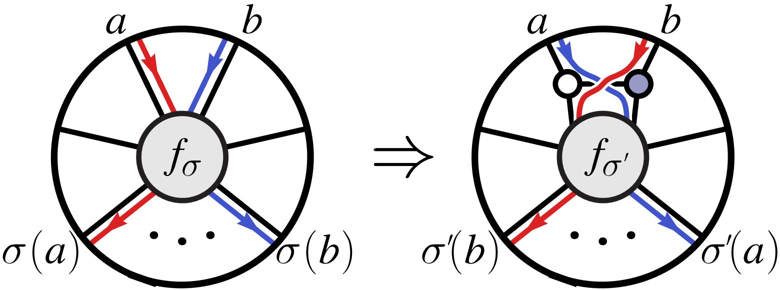

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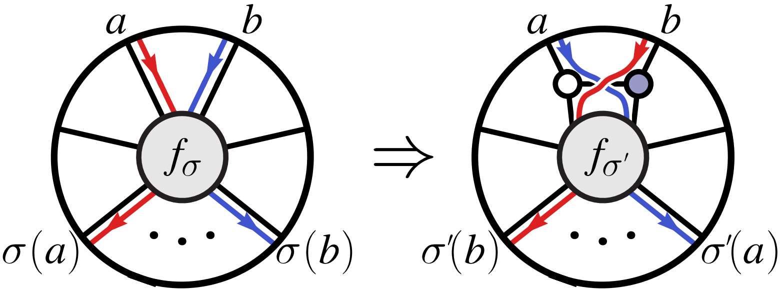

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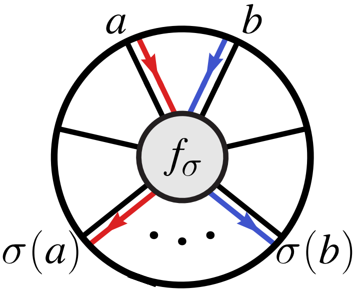

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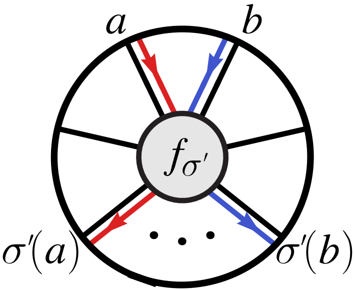
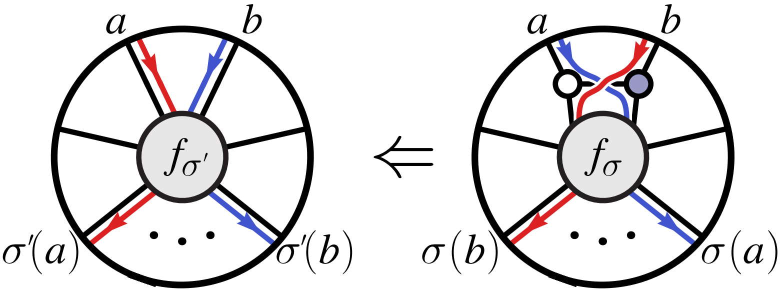


Canonical Coordinates for Computing On-Shell Functions

Recall that attaching BCFW 'bridges' can lead to very rich on-shell diagrams.

Read the other way, we can 'peel-off' bridges and thereby decompose a permutation into transpositions according to $\sigma = (a\,b) \circ \sigma'$



Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions

'Bridge' Decomposition

$$\sigma:\begin{pmatrix}1&2&3&4&5&6\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\3&5&6&7&8&10\end{pmatrix}$$

Canonical Coordinates for Computing On-Shell Functions

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'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ f_\sigma & \{3 & 5 & 6 & 7 & 8 & 10\} & \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ f_\sigma & \{3 & 5 & 6 & 7 & 8 & 10\} & \end{array}$$

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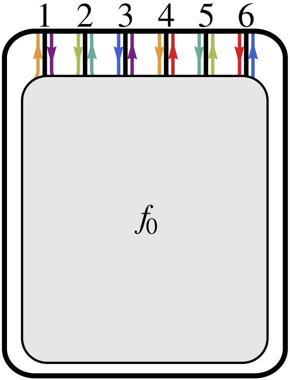

'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \tau \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & \end{array}$$

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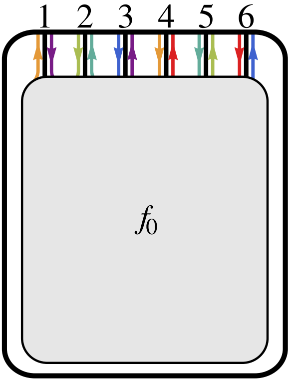


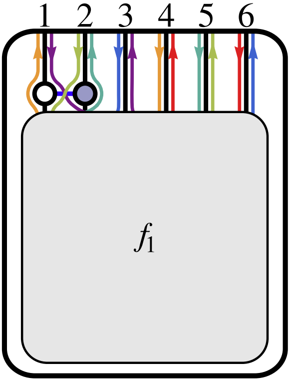
'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \tau \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & (1\,2) \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1} f_1$$



'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \tau \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & \\ & & & & & & & (1\,2) \\ f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & \end{array}$$

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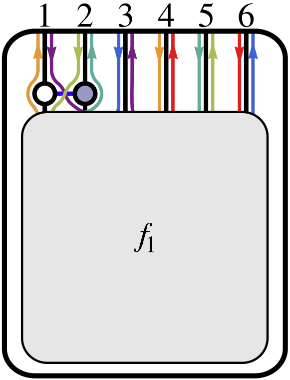


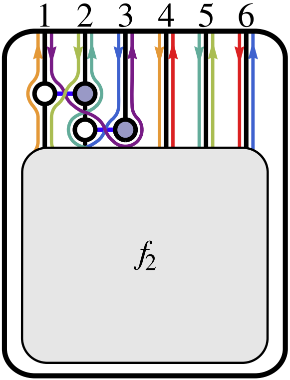
'Bridge' Decomposition

$$\begin{array}{lccccccl} & 1 & 2 & 3 & 4 & 5 & 6 & \tau \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ f_0\ \{ & 3 & 5 & 6 & 7 & 8 & 10\} & (1\,2) \\ f_1\ \{ & 5 & 3 & 6 & 7 & 8 & 10\} & (2\,3) \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2} f_2$$



'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	

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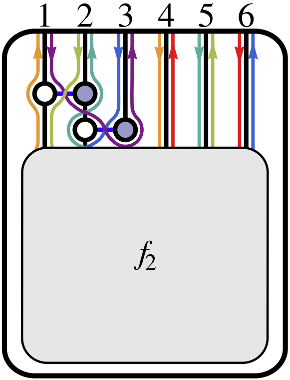

'Bridge' Decomposition

$$\begin{array}{ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & & \tau \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & & (1\,2) \\
f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & & (2\,3) \\
f_2 & \{5 & 6 & 3 & 7 & 8 & 10\} & & (1\,2)
\end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\, b)$ such that $\sigma(a) < \sigma(b)$:

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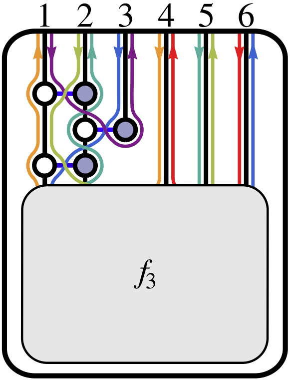

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	

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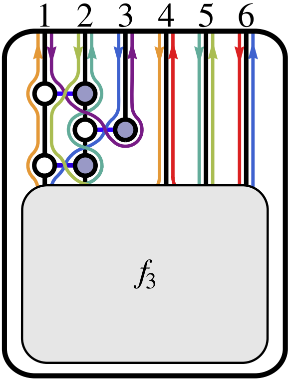

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	
							$(1\,2)$
f_1	{5	3	6	7	8	10}	
							$(2\,3)$
f_2	{5	6	3	7	8	10}	
							$(1\,2)$
f_3	{6	5	3	7	8	10}	
							$(2\,4)$

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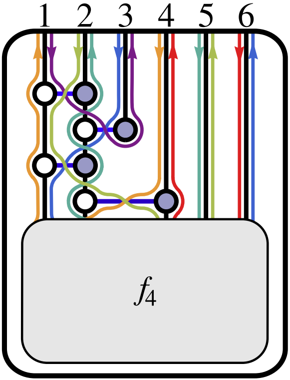

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	

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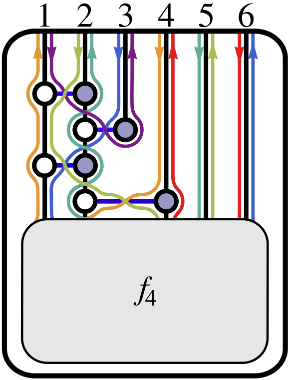

'Bridge' Decomposition

$$\begin{array}{c|cccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & (1\,2) \\
f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & (2\,3) \\
f_2 & \{5 & 6 & 3 & 7 & 8 & 10\} & (1\,2) \\
f_3 & \{6 & 5 & 3 & 7 & 8 & 10\} & (2\,4) \\
f_4 & \{6 & 7 & 3 & 5 & 8 & 10\} & (1\,2)
\end{array}$$

Canonical Coordinates for Computing On-Shell Functions

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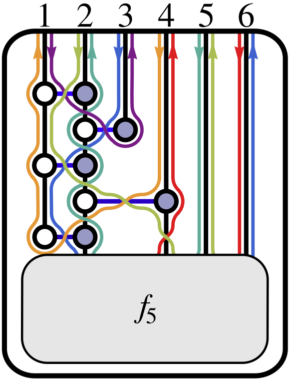

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	

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There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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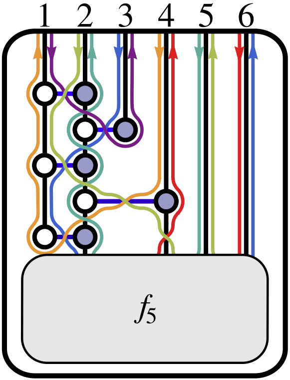

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)

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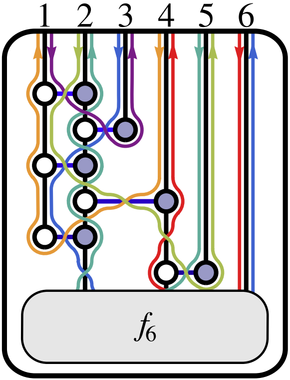

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
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f_6	{7	6	3	8	5	10}	

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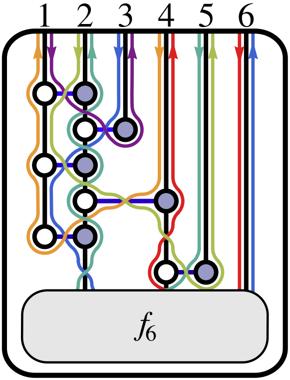

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)

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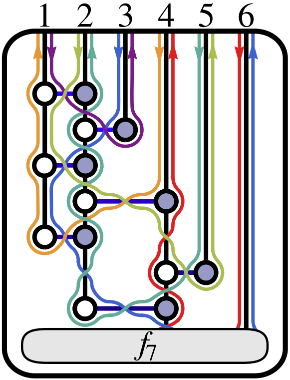

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	

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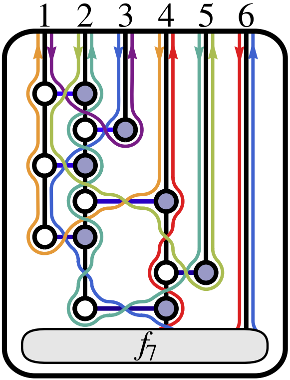

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$

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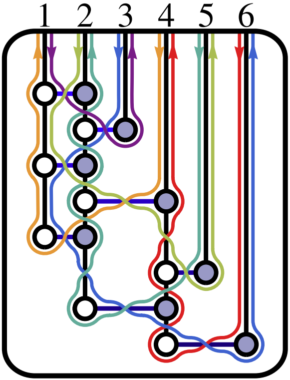

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

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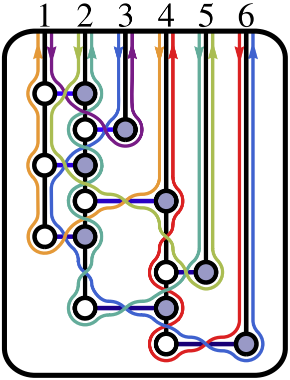

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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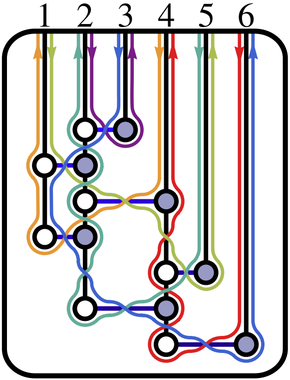
There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$



‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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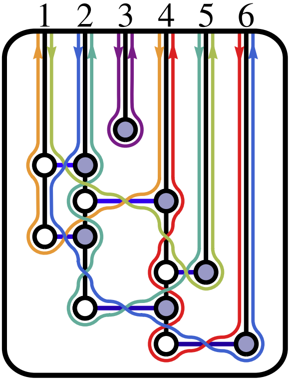

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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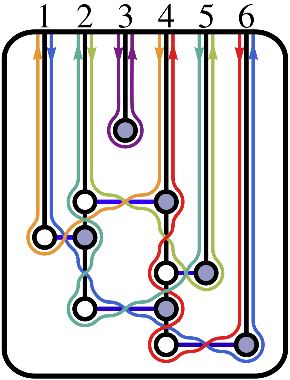

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

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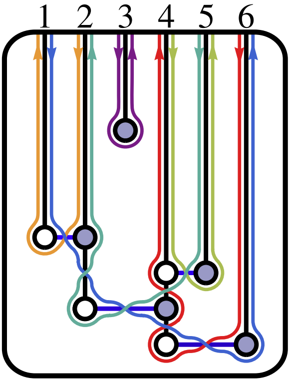

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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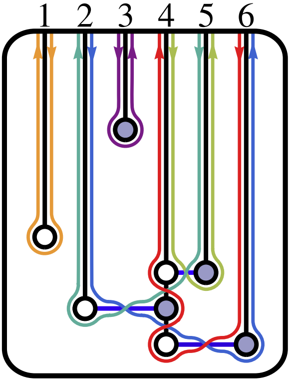

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_5	{7	6	3	5	8	10}	
							$(4\,5)$
f_6	{7	6	3	8	5	10}	
							$(2\,4)$
f_7	{7	8	3	6	5	10}	
							$(4\,6)$
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

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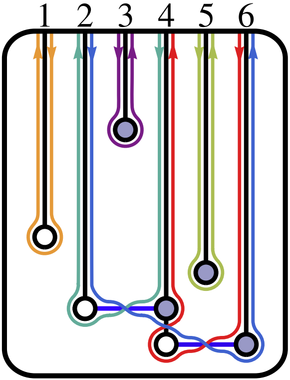

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6}	

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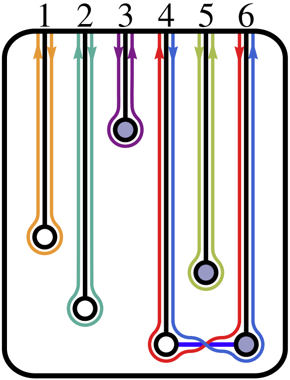

‘Bridge’ Decomposition

$$\begin{array}{lcccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & & \tau \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & & (4\,6) \\
f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} & &
\end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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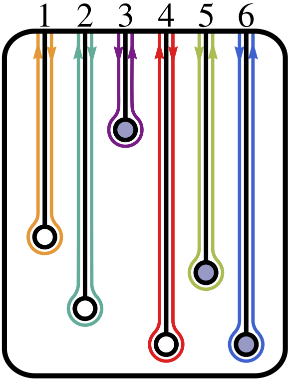

'Bridge' Decomposition

1	2	3	4	5	6	τ
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	

$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\}$

Canonical Coordinates for Computing On-Shell Functions

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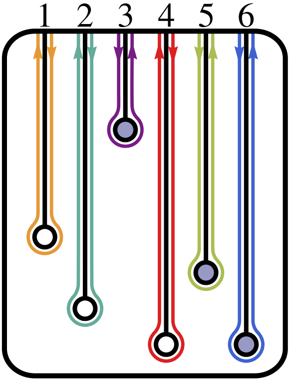

‘Bridge’ Decomposition

1	2	3	4	5	6	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	τ

$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$

Canonical Coordinates for Computing On-Shell Functions

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$$f_8 = \prod_{a=\sigma(a)+n}\Big(\delta^4(\widetilde{\eta}_a)\,\delta^2(\widetilde{\lambda}_a)\Big)\prod_{b=\sigma(b)}\Big(\delta^2(\lambda_b)\Big)$$

'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\,\} & \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

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$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

'Bridge' Decomposition

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \end{array}$$

$$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau\equiv(a\,b)$ such that $\sigma(a)<\sigma(b)$:

$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_8=\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \left(\begin{array}{cccccc}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\end{array}\right)\end{array}$$

'Bridge' Decomposition

$$\begin{array}{ccccccc}1&2&3&4&5&6&\\ \downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\tau\end{array}$$

$$f_8\ \{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$$

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$$f_8 = \delta^{3\times 4}\big(C\cdot\widetilde{\eta}\big)\,\delta^{3\times 2}\big(C\cdot\widetilde{\lambda}\big)\,\delta^{2\times 3}\big(\lambda\cdot C^{\perp}\big)$$

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\!\!\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$
$$C \equiv \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

'Bridge' Decomposition

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \quad \tau$$

$$f_8\;\{7\;\;8\;\;3\;\;10\;\;5\;\;6\;\}$$

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$$f_7 = \frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

$$(4\,6):\;\; C_6 \mapsto C_6 + \alpha_8 C_4$$

'Bridge' Decomposition

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \\ & & & & & & & \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\,\} & \end{array}$$

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$$f_6 = \frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,4):\;\; C_4 \mapsto C_4 + \alpha_7 C_2$$

'Bridge' Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_6 & \{7 & 6 & 3 & 8 & 5 & 10\} & (2\,4) \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} & \end{array}$$

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$$f_5 = \frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$(4\,5)$: $C_5 \mapsto C_5 + \alpha_6 C_4$

'Bridge' Decomposition

$$\begin{array}{lccccccl} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_5 & \{7 & 6 & 3 & 5 & 8 & 10\} & (4\,5) \\ f_6 & \{7 & 6 & 3 & 8 & 5 & 10\} & (2\,4) \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} & \end{array}$$

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$$f_0=\frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}f_8$$

$$f_4=\frac{d\alpha_5}{\alpha_5}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\ \left(\begin{array}{cccccc}1&\alpha_5&0&0&0&0\\0&1&0&\alpha_7&\alpha_6\alpha_7&0\\0&0&0&1&\alpha_6&\alpha_8\end{array}\right)\end{array}$$

$$(1\,2):\;\; C_2\mapsto C_2+\alpha_5 C_1$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_4	$\{6$	7	3	5	8	$10\}$	$(1\,2)$
f_5	$\{7$	6	3	5	8	$10\}$	$(4\,5)$
f_6	$\{7$	6	3	8	5	$10\}$	$(2\,4)$
f_7	$\{7$	8	3	6	5	$10\}$	$(4\,6)$
f_8	$\{7$	8	3	10	5	$6\}$	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_3 = \frac{d\alpha_4}{\alpha_4} \cdots \frac{d\alpha_8}{\alpha_8} \delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & \alpha_5 & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,4): \;\; C_4 \mapsto C_4 + \alpha_4 C_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$

$$f_2 = \frac{d\alpha_3}{\alpha_3}\cdots\frac{d\alpha_8}{\alpha_8}\,\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\,\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\,\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(1\,2):\quad C_2 \mapsto C_2 + \alpha_3 C_1$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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Canonical Coordinates for Computing On-Shell Functions

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$$C \equiv \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right) \end{array}$$

$$(2\,3):\;\; C_3 \mapsto C_3 + \alpha_2 C_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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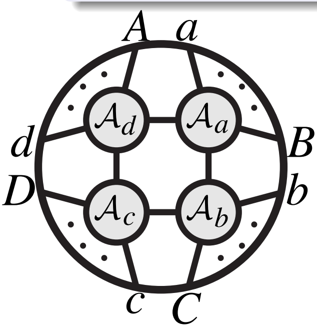
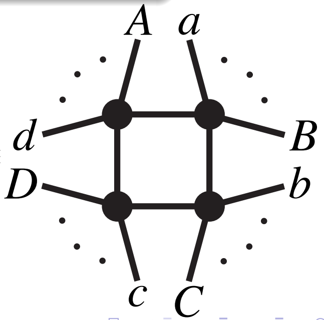
$f_{a,b,c,d} \equiv$ 

$I_{a,b,c,d} \equiv$ 

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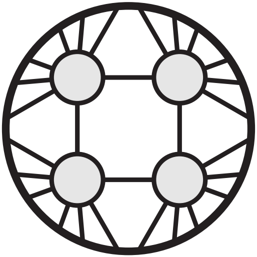
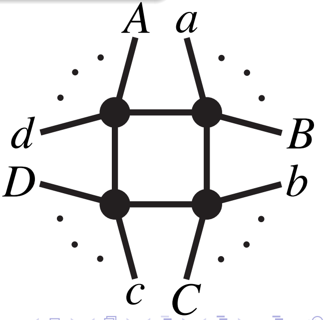
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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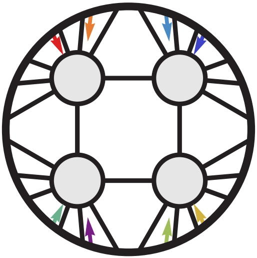
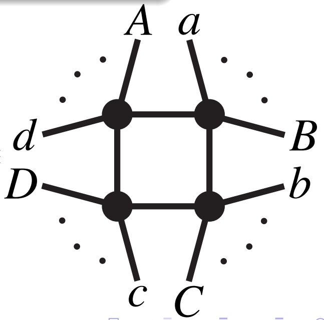
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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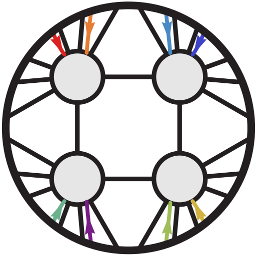
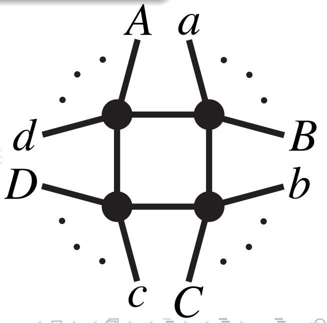
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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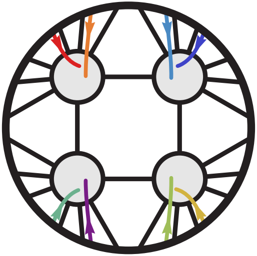
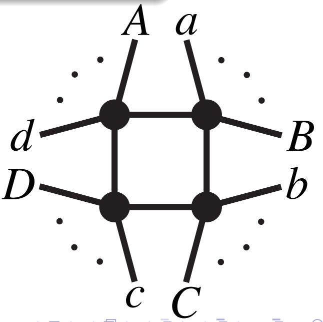
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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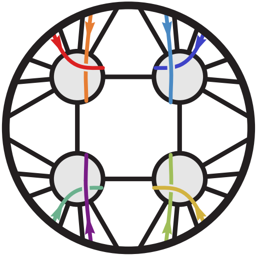
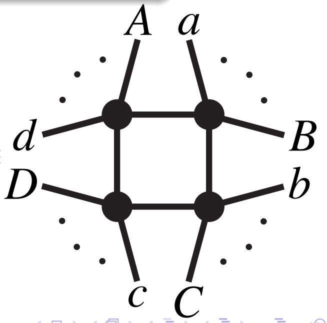
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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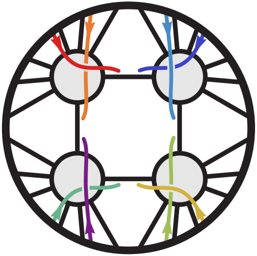
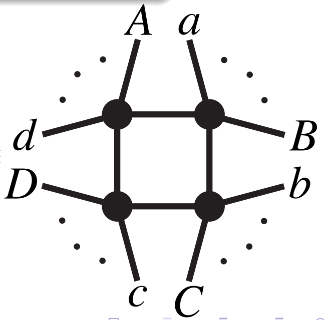
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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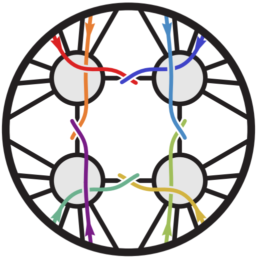
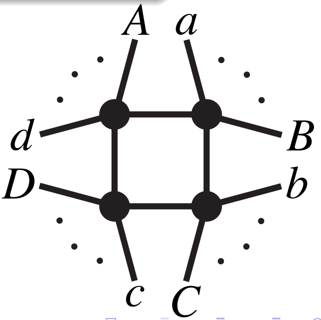
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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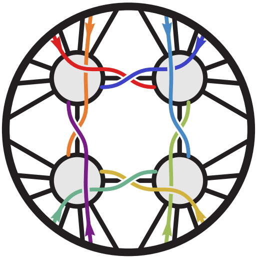
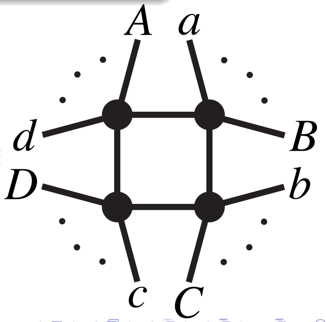
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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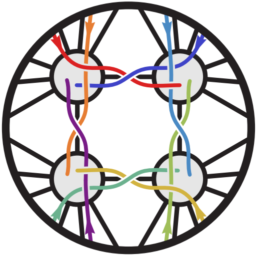
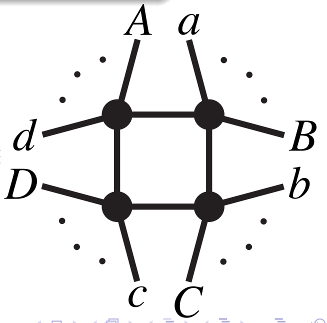
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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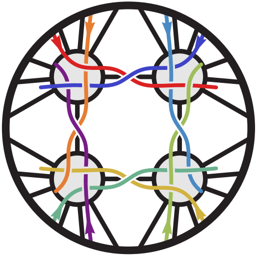
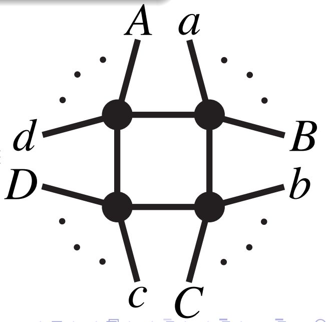
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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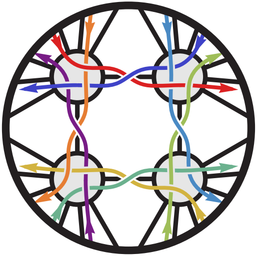

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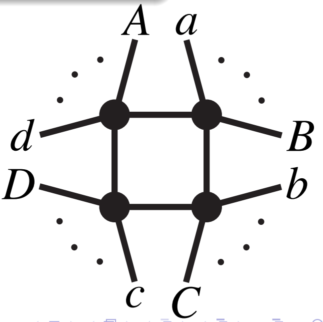

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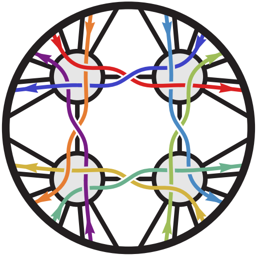
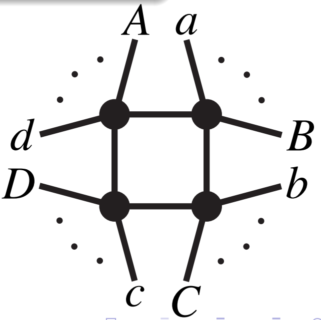
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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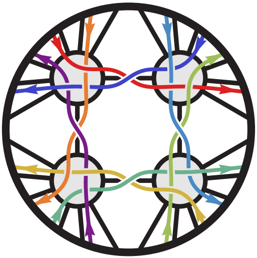

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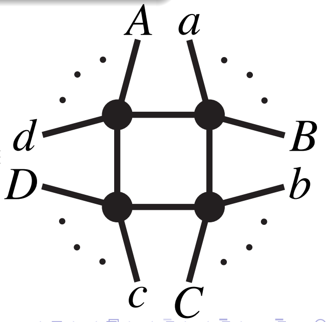

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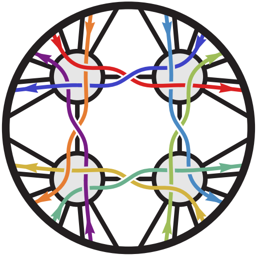

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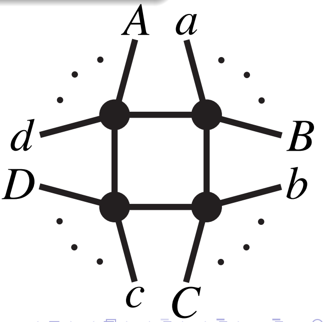

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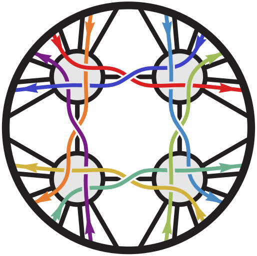
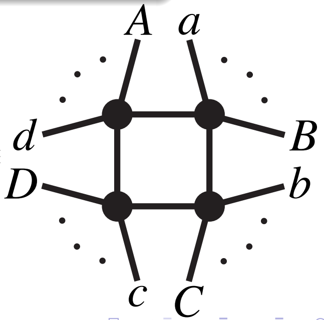
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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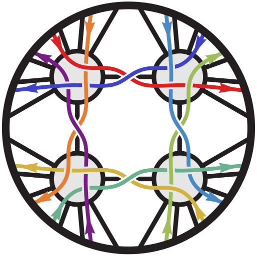
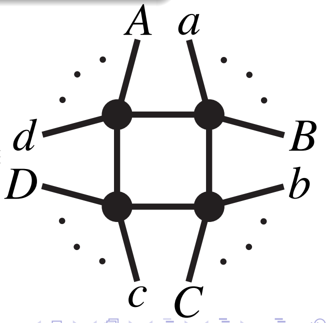
$f_{a,b,c,d} \Leftrightarrow$ 

$I_{a,b,c,d} \equiv$ 

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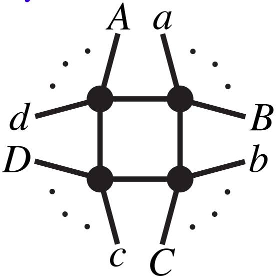

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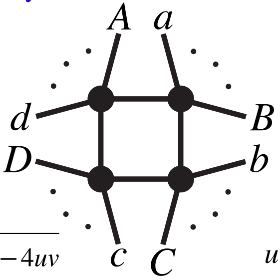

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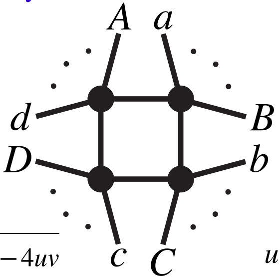

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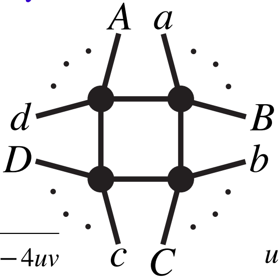

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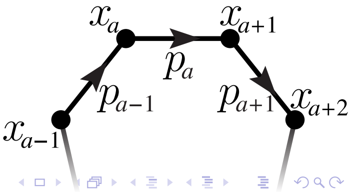

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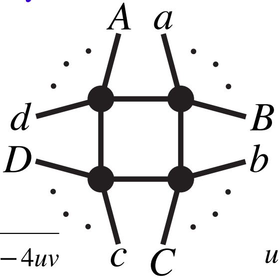

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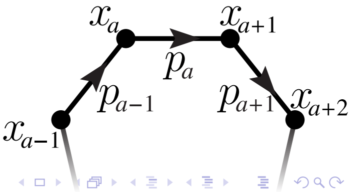

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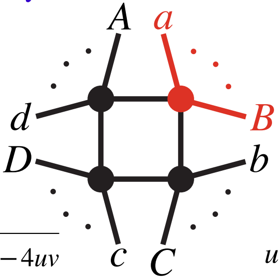

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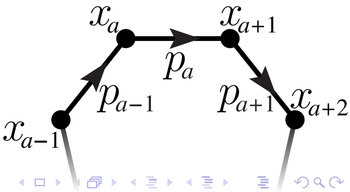

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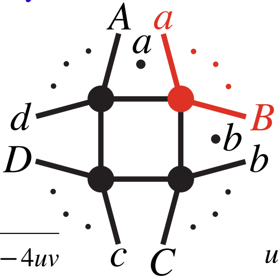

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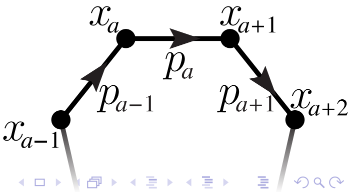

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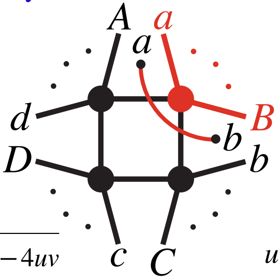

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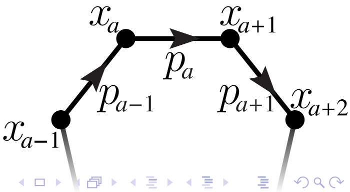

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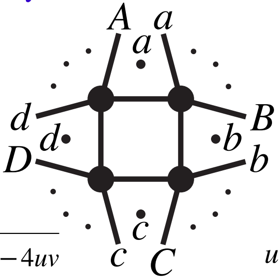

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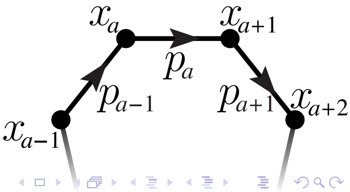

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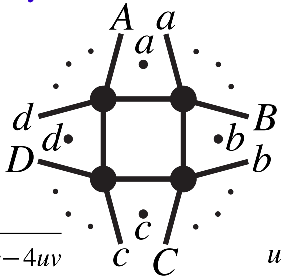

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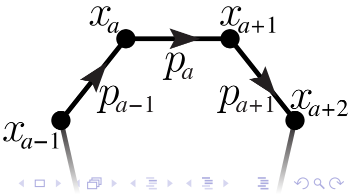

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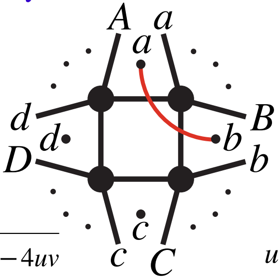

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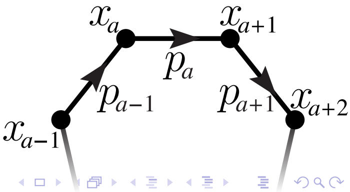

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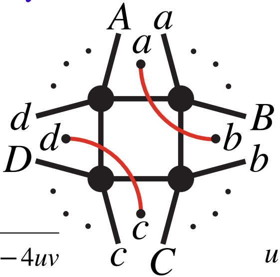

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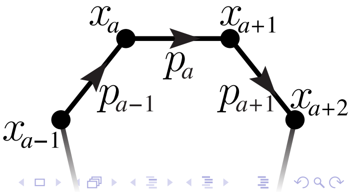

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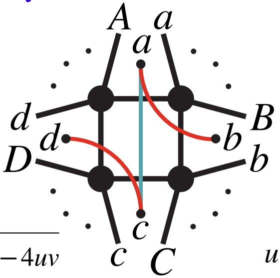

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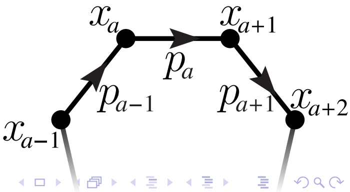

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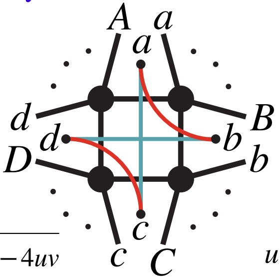

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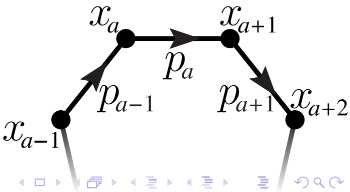

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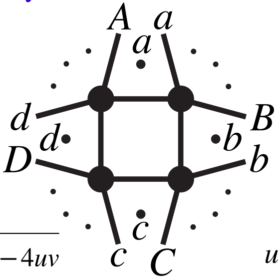

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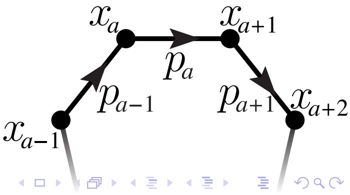

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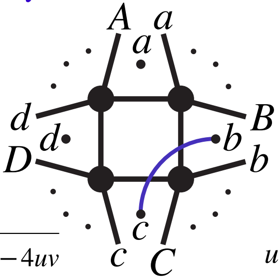

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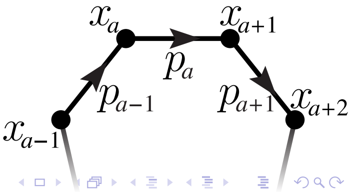

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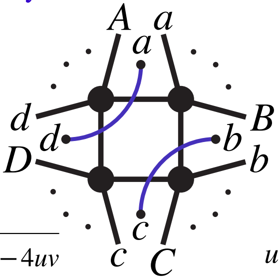

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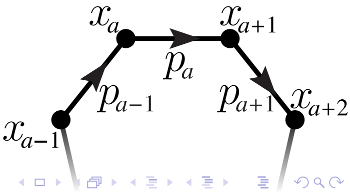

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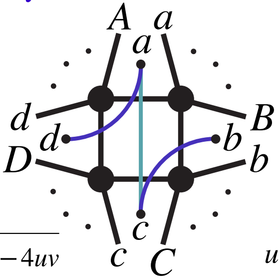

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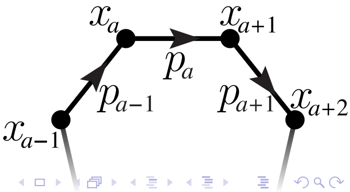

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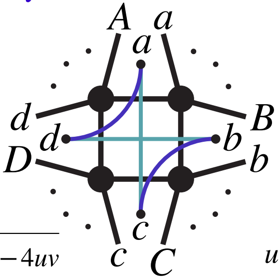

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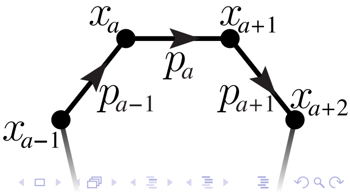

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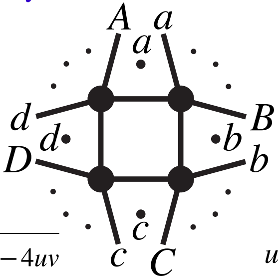

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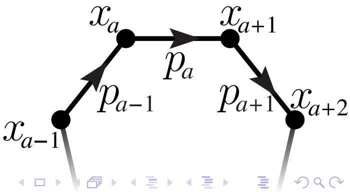

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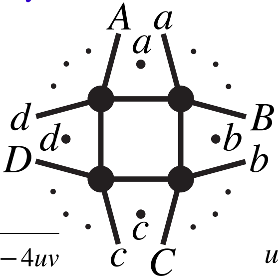

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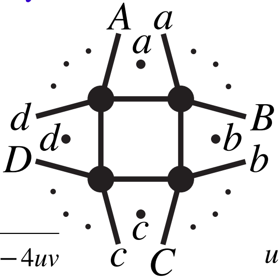

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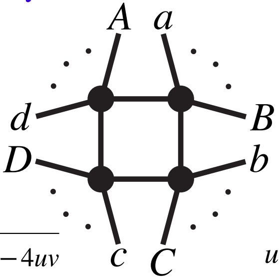

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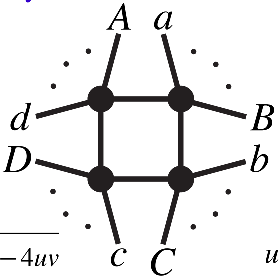

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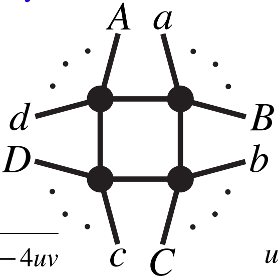

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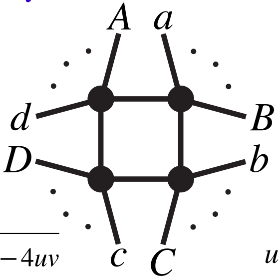

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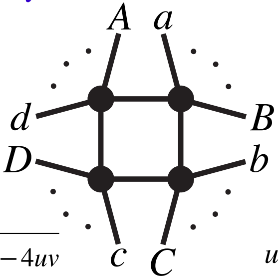

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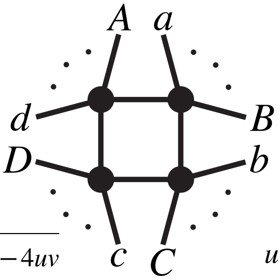

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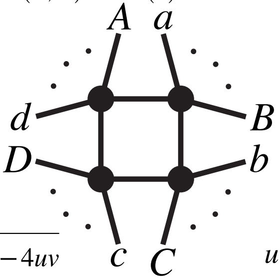

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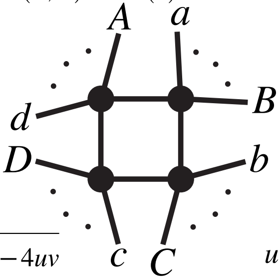

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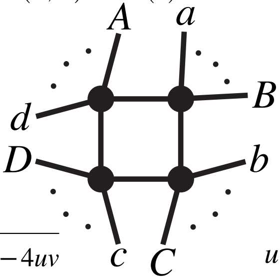

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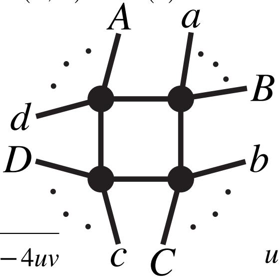

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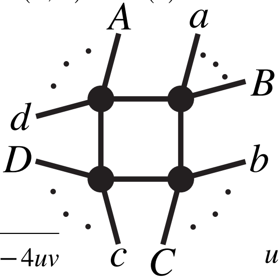

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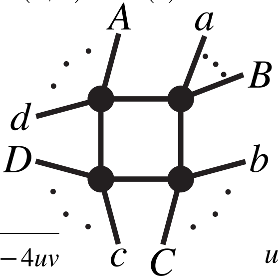

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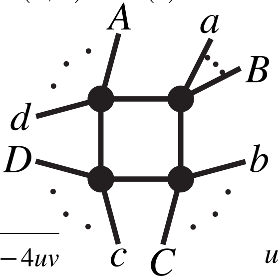

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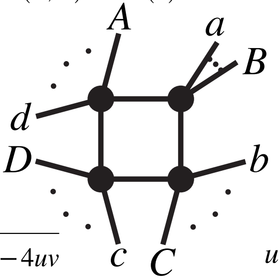
$I_{a,b,c,d}\equiv$  $\equiv -\int\limits_{\ell\in\mathbb{R}^{3,1}} d^4\ell\,\frac{(a,c)(b,d)\Delta}{(\ell,a)(\ell,b)(\ell,c)(\ell,d)}$,

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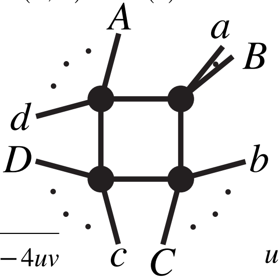

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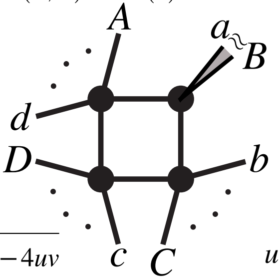

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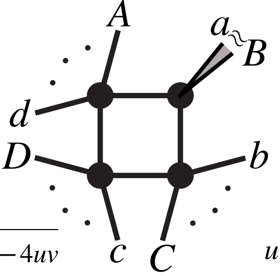

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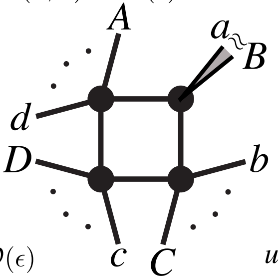

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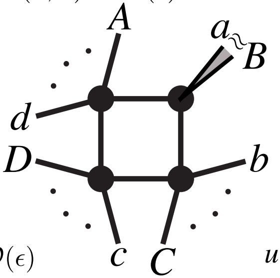

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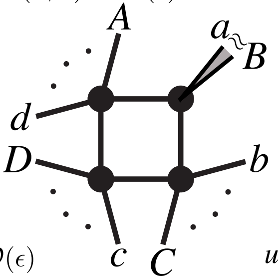

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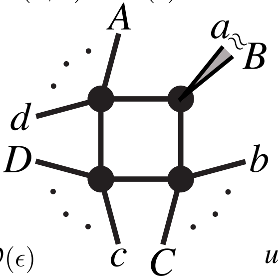

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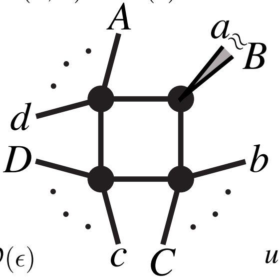

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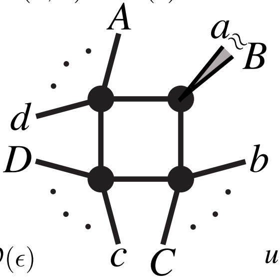

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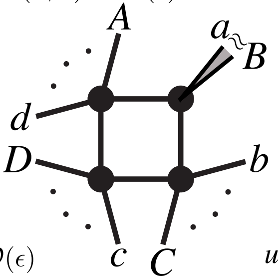

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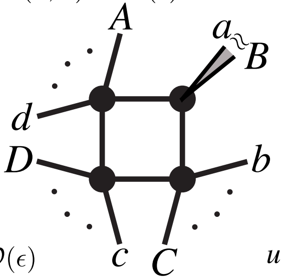

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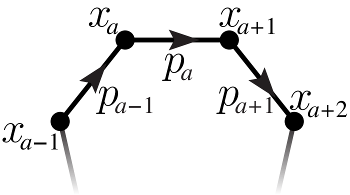

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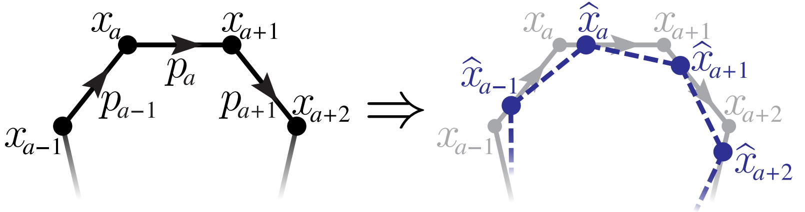

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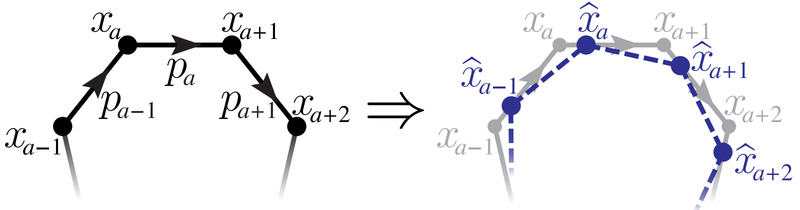

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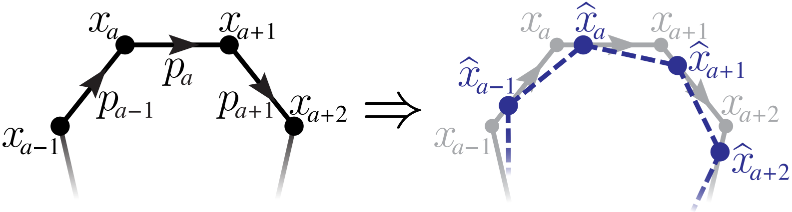

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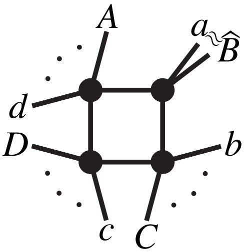

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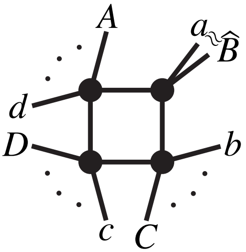

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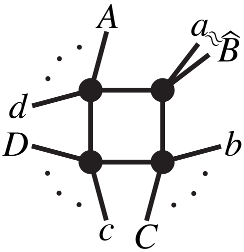

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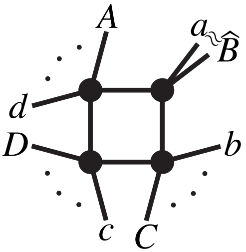

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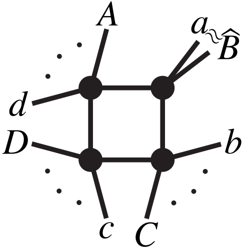

$$= -\mathrm{Li}_2(1-v) - \frac{1}{2}\log(u')\log(v) - \frac{1}{2}\log(\epsilon)\log(v) + \mathcal{O}(\epsilon).$$

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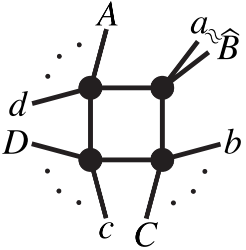

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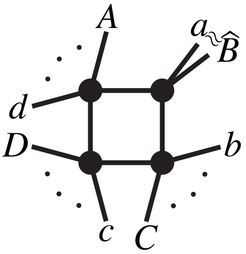

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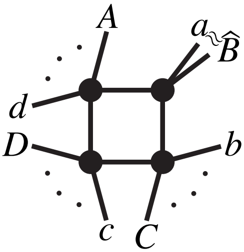

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A 'Box'-Expansion for One-Loop *Integrands*

Th *Chiral* Box Expansion for *Integrands*

$$\mathcal{A}_n^{(k),1} = \sum_{1\leq a<b<c<d}\left(f^1_{a,b,c,d}\mathcal{I}^1_{a,b,c,d} + f^2_{a,b,c,d}\mathcal{I}^2_{a,b,c,d}\right) + \mathcal{A}_n^{(k),0}\sum_a \mathcal{I}_a^{\text{div.}}$$

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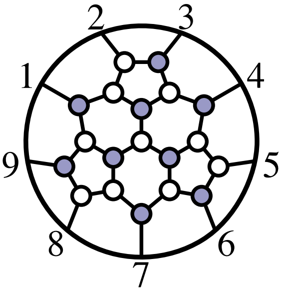
The Combinatorics and Geometry of On-Shell Physics



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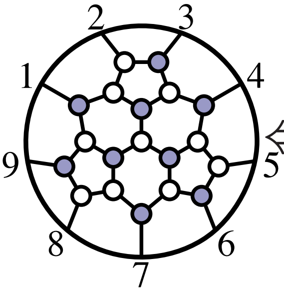
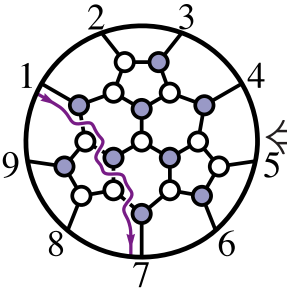


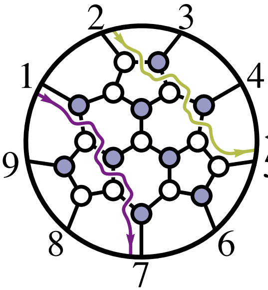
$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & & & & & & & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



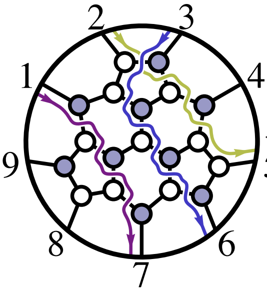
$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & & & & & & & & \\ 7 & & & & & & & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & & & & & & & \\ 7 & 5 & & & & & & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & & & & & & \\ 7 & 5 & 6 & & & & & & \end{pmatrix}$$

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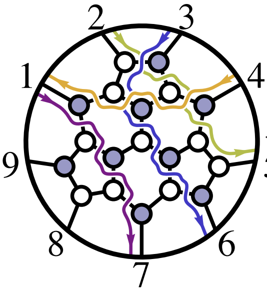

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & & & & & \\ 7 & 5 & 6 & 10 & & & & & \end{pmatrix}$$

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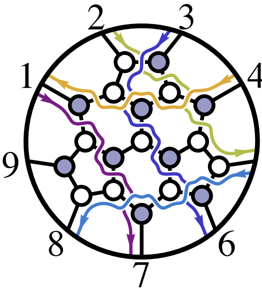

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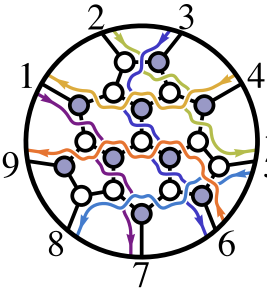

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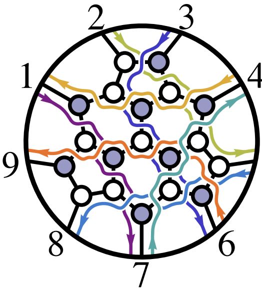

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & & \end{pmatrix}$$

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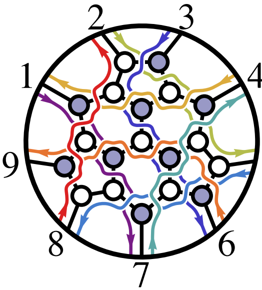

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & \end{pmatrix}$$

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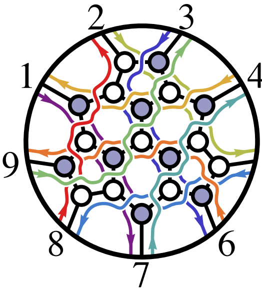
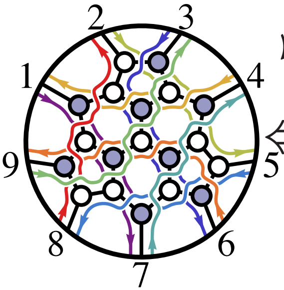


$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_+(4,9)$$



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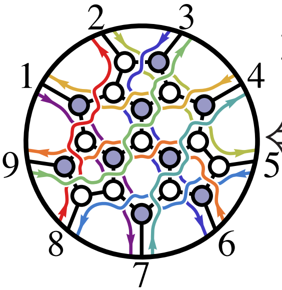
$$C(\alpha)\equiv\left(\begin{array}{ccccccccc}1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&\alpha_1&0&0&0&0&0&1\end{array}\right)\in G_+(4,9)$$



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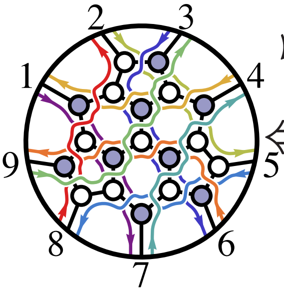

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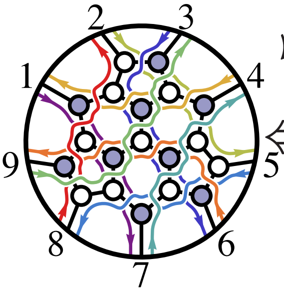

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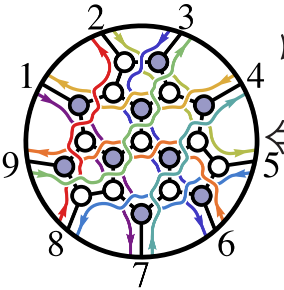
$$C(\alpha)\equiv\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & 0 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

$\Updownarrow$



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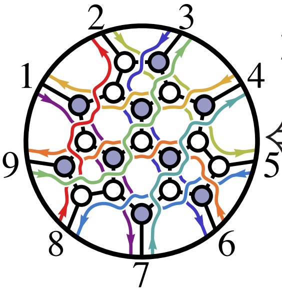
$$C(\alpha)\equiv\begin{pmatrix} 1 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & 0 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$



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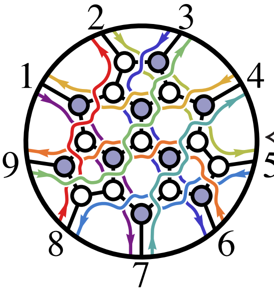

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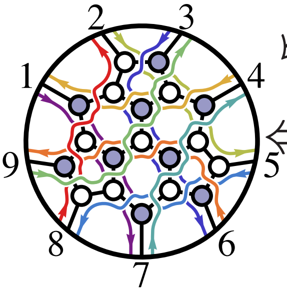

$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

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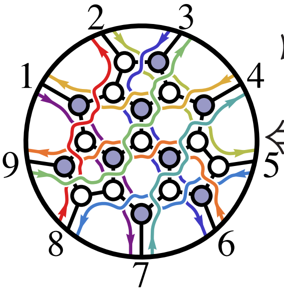
$$\Updownarrow$$

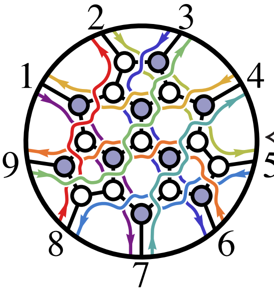


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha) \equiv \begin{pmatrix} 1 & \alpha_8 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_4 & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 \\ -\alpha_9 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix} \in G_+(4,9)$$

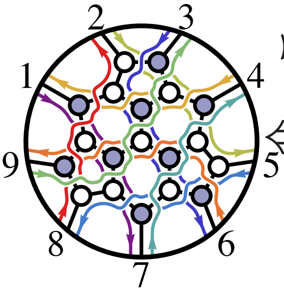
$$\Updownarrow$$



$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha) \equiv \begin{pmatrix} 1 & \alpha_8 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4 & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 \\ -\alpha_9 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix} \in G_+(4,9)$$



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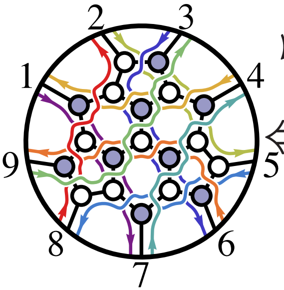

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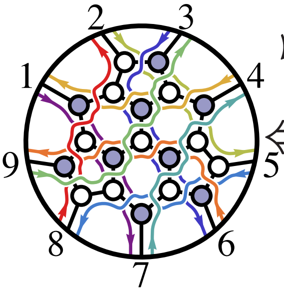

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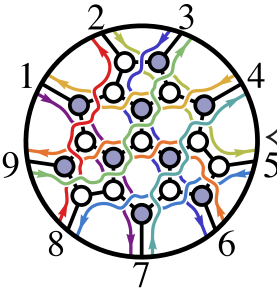

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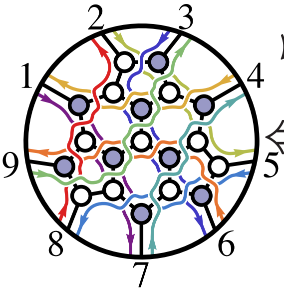
The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & (\alpha_5+\alpha_{14}\,\alpha_8) & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & (\alpha_4+\alpha_{13}\,\alpha_{10}) & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\,\alpha_6) \\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$



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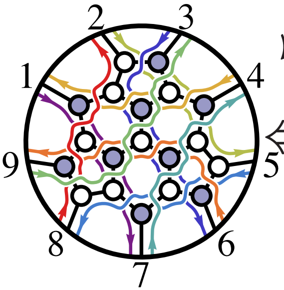

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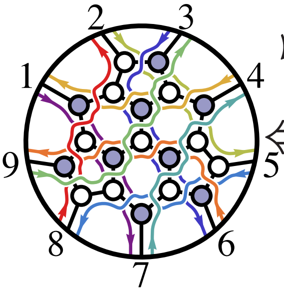

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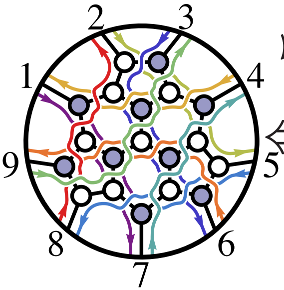

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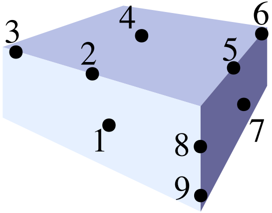

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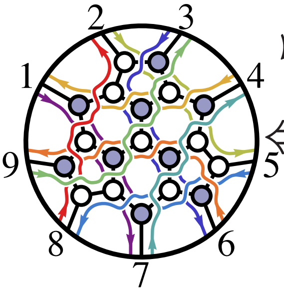

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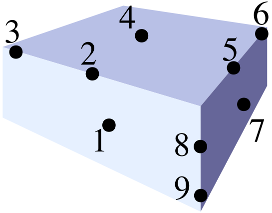

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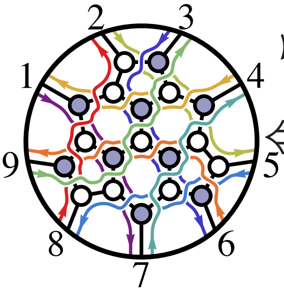

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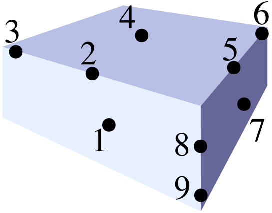

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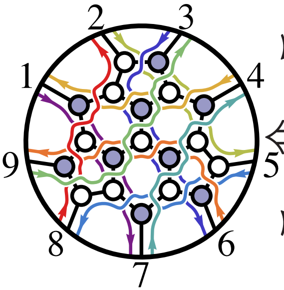

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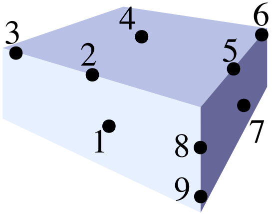

$$f_\sigma\equiv\int\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\,\delta^{k\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\,\delta^{k\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\,\delta^{2\times(n-k)}\big(\lambda\cdot C(\alpha)^{\perp}\big)$$

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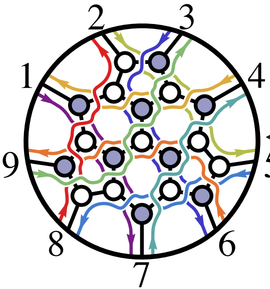

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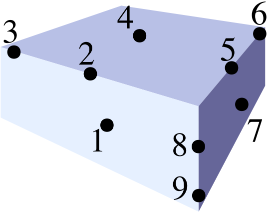

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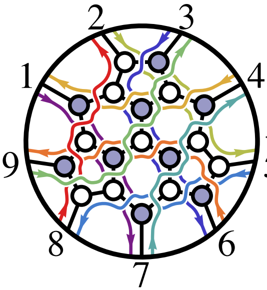

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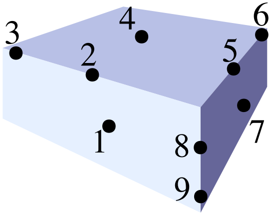

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Closed Books and Open Problems

Further applications and future directions:

- the classification of identities among on-shell diagrams
- on-shell diagrams in theories with less (or no) supersymmetry ($\mathcal{N}<4$)
- applications to 2- and 3-dimensional theories: ABJM and Yang-Baxter
- embedding non-planar on-shell diagrams in the Grassmannian
- a generalization of the 'chiral-box' expansion to higher-loops
- ...

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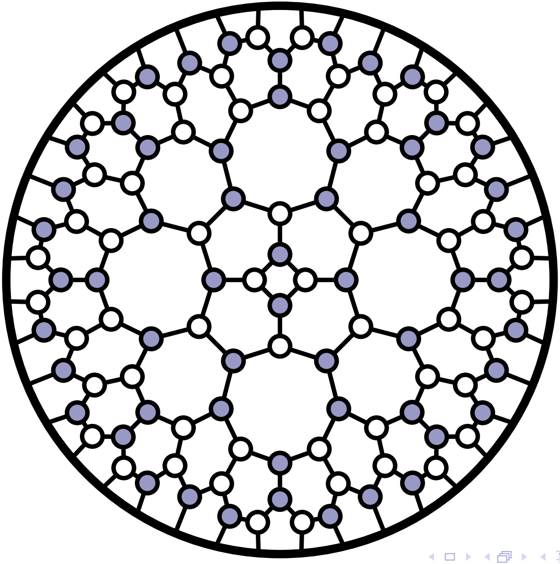
A Contribution to the 40-Point N^{18}MHV Amplitude



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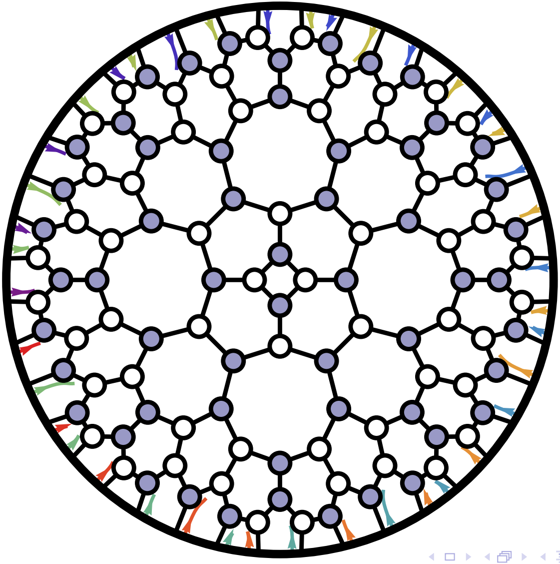

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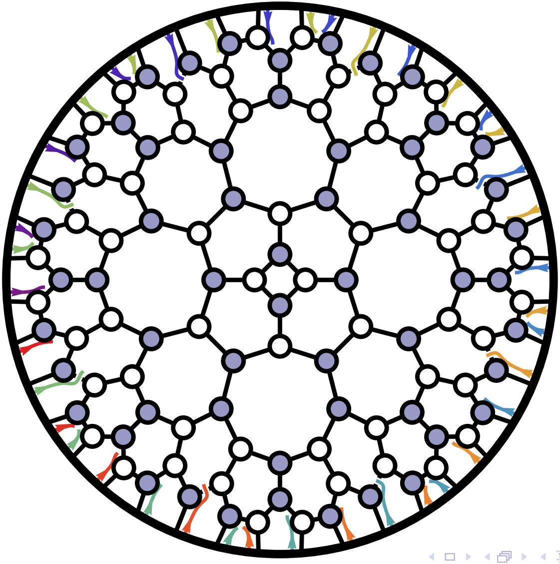

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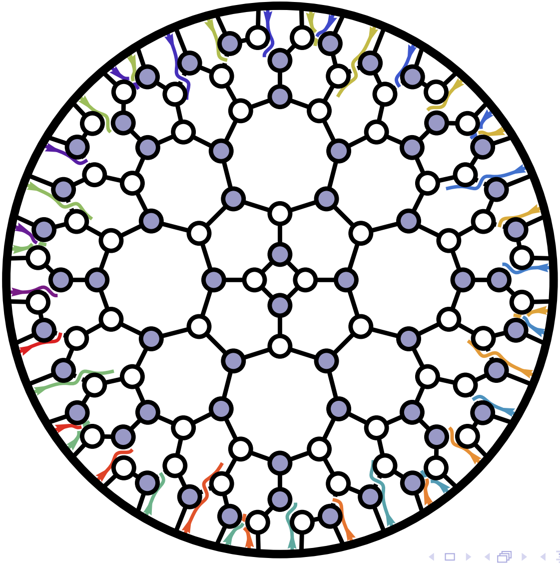

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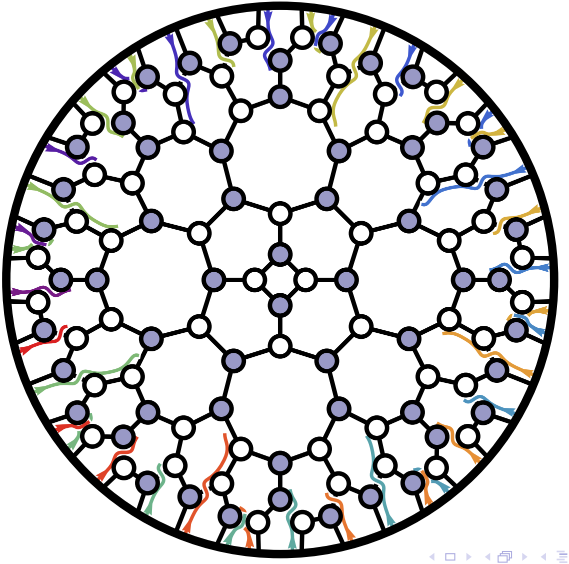

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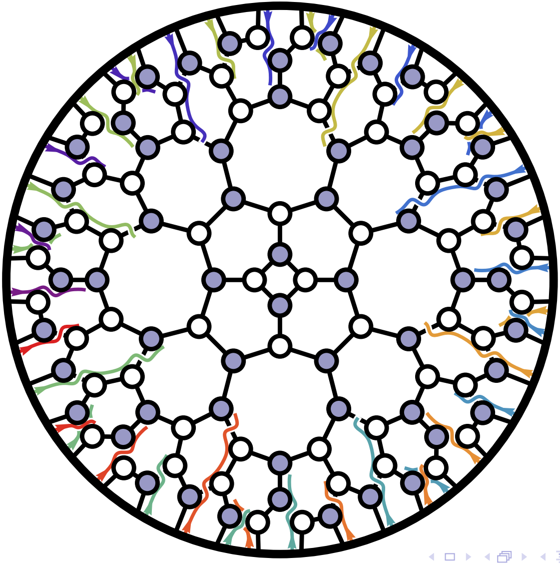

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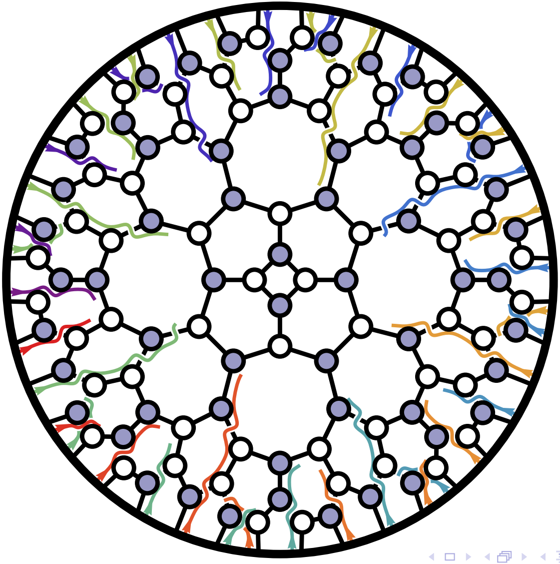

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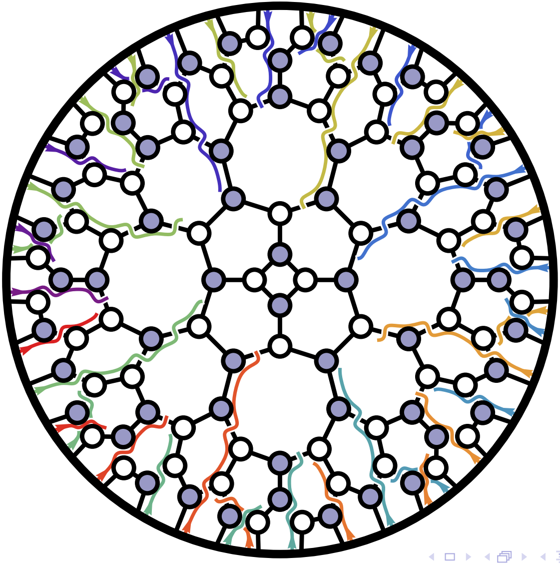

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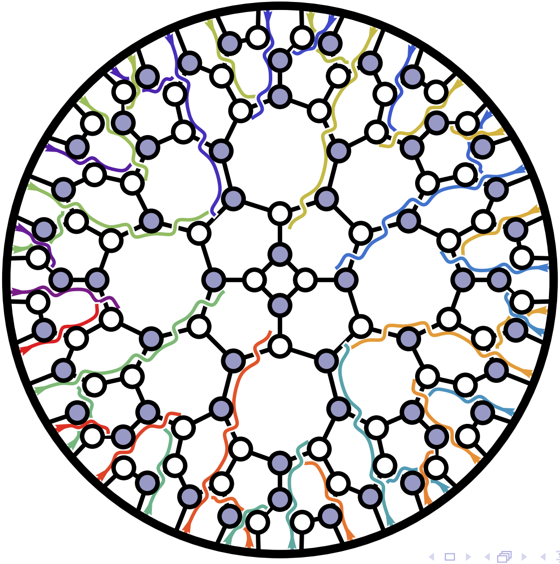

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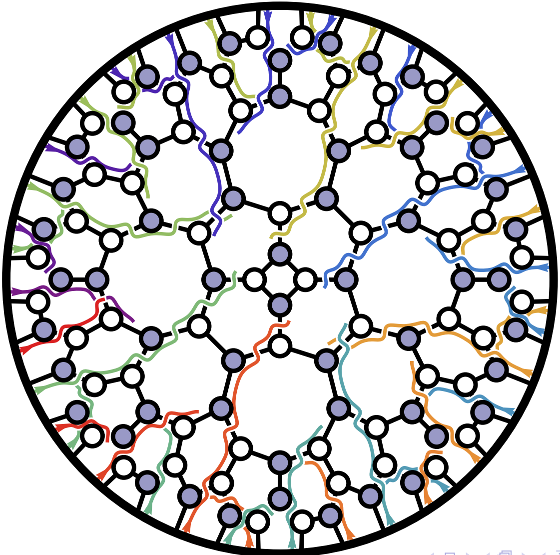

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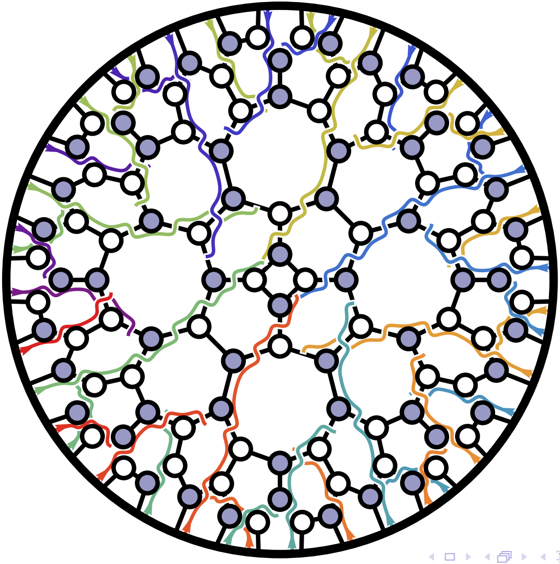
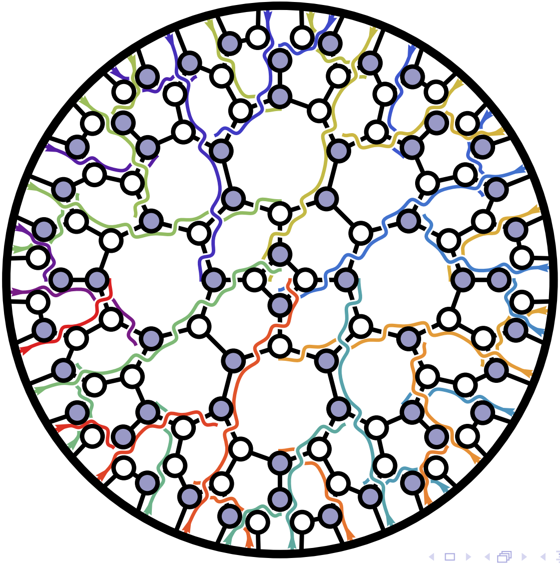


A Contribution to the 40-Point N^{18}MHV Amplitude



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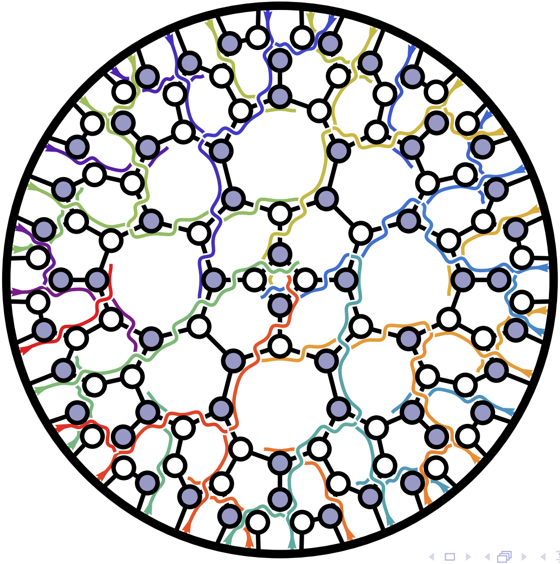

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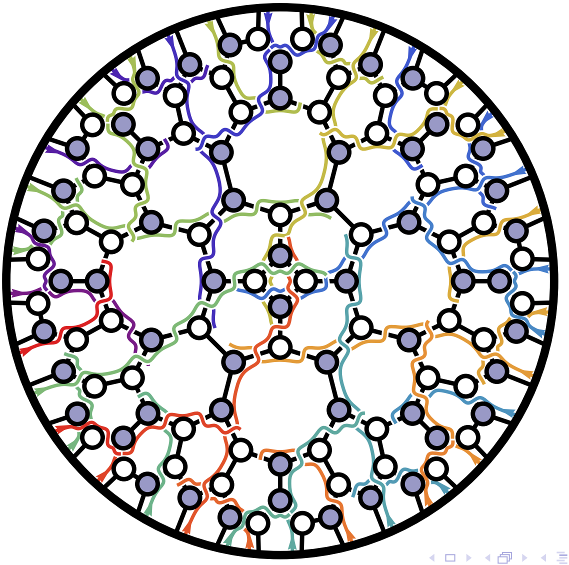

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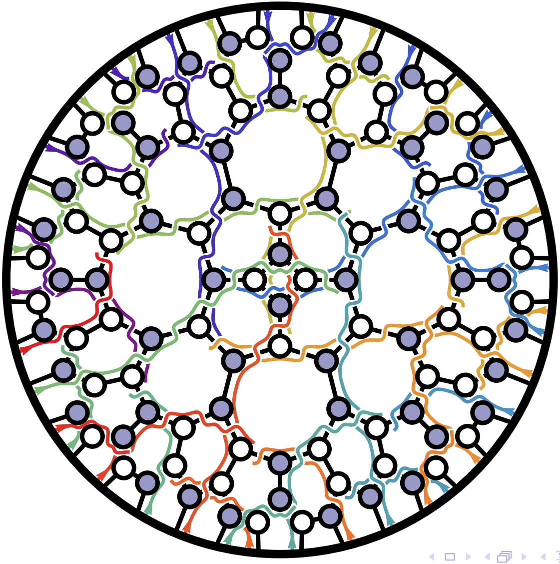

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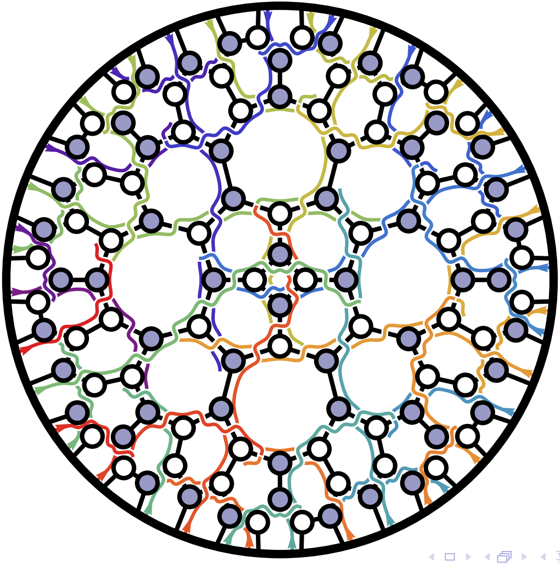

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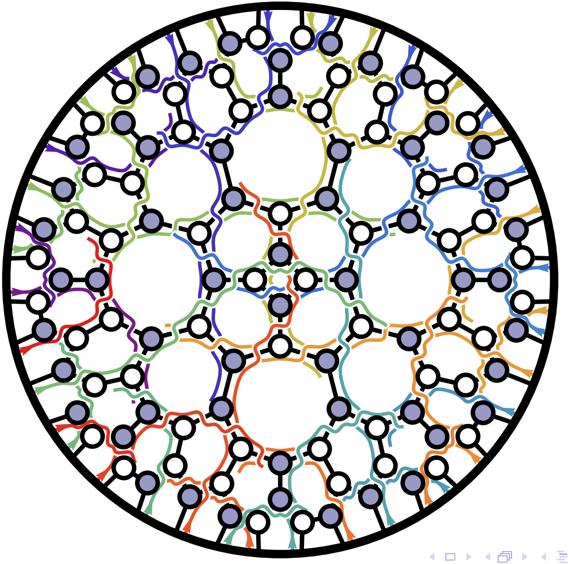
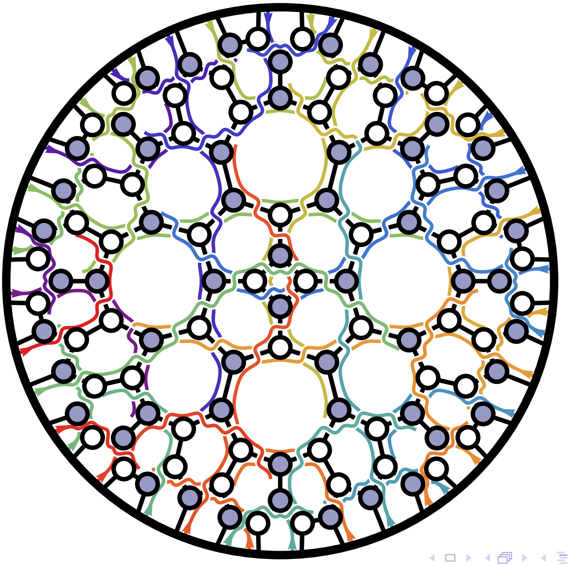


A Contribution to the 40-Point $\mathrm{N}^{18}\mathrm{MHV}$ Amplitude



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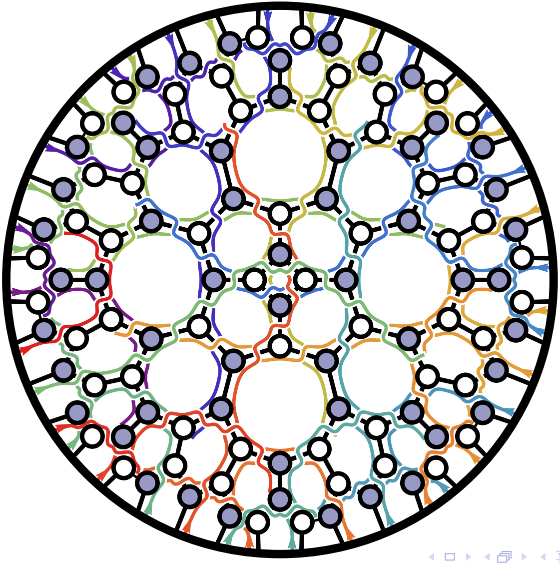

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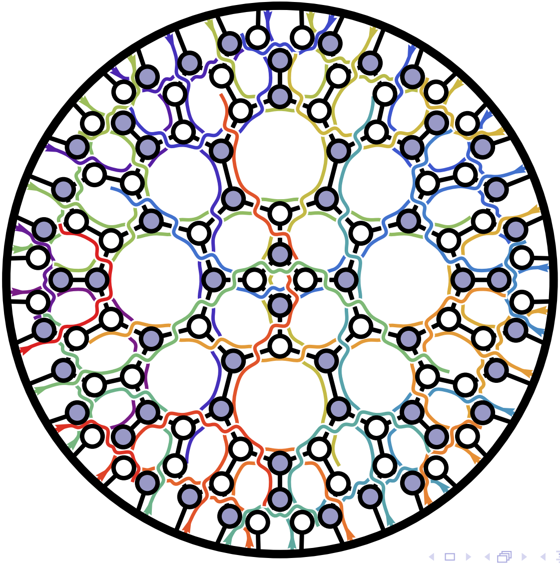

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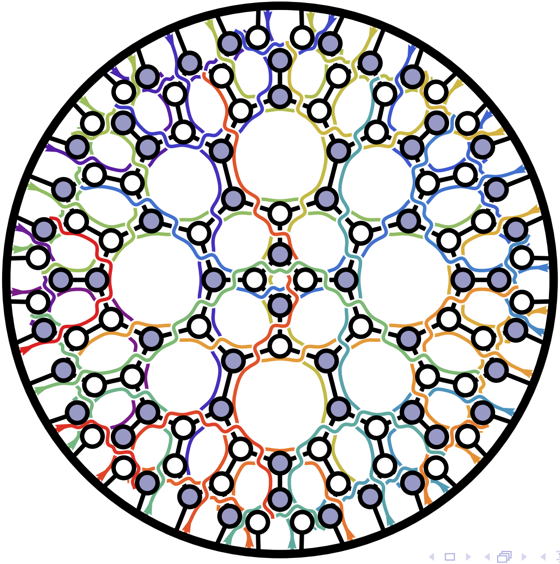

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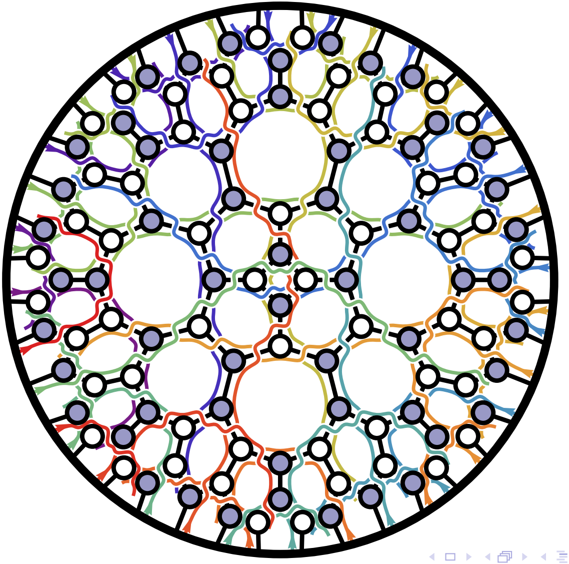

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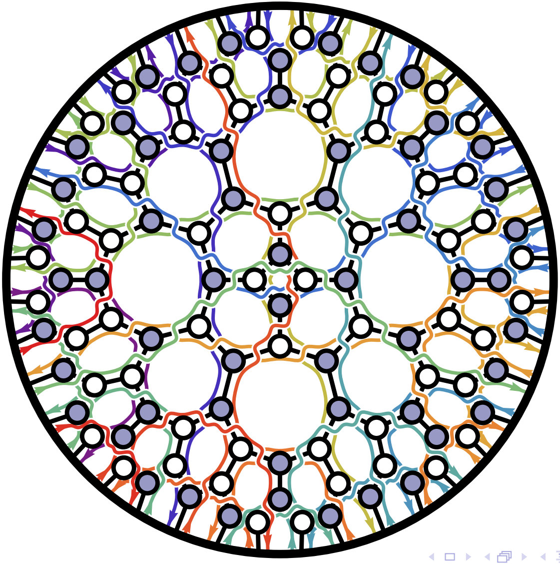

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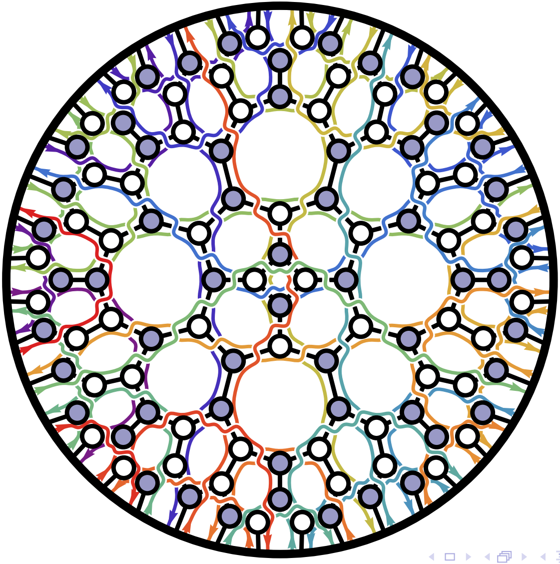

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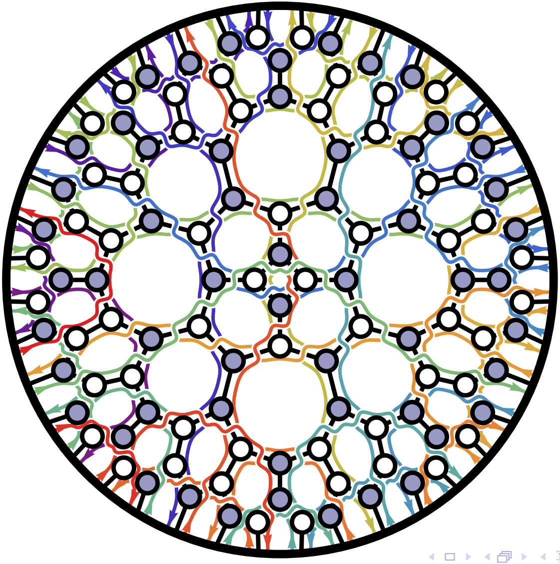

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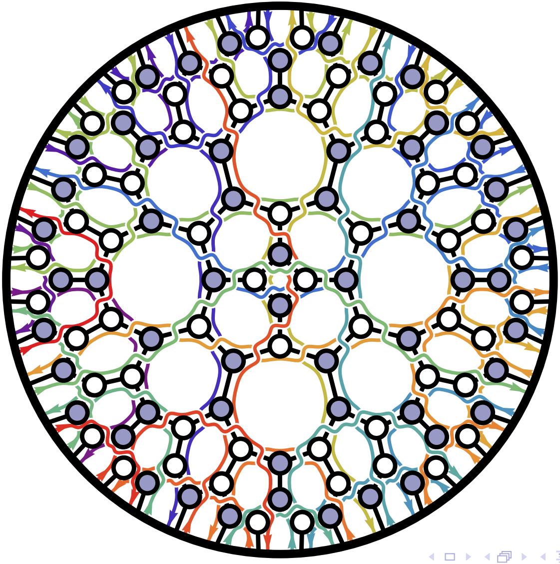

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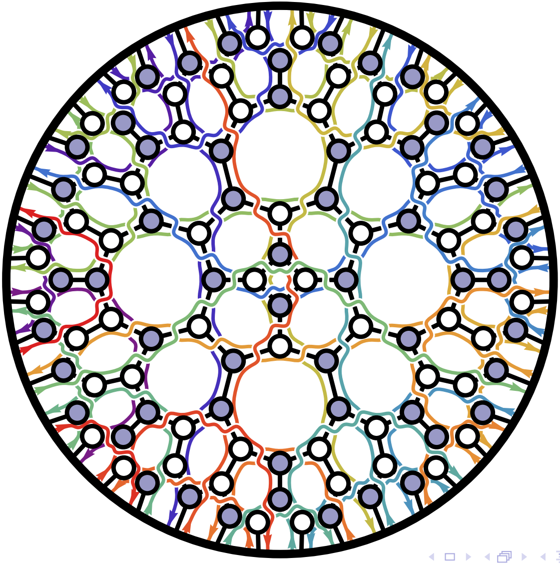

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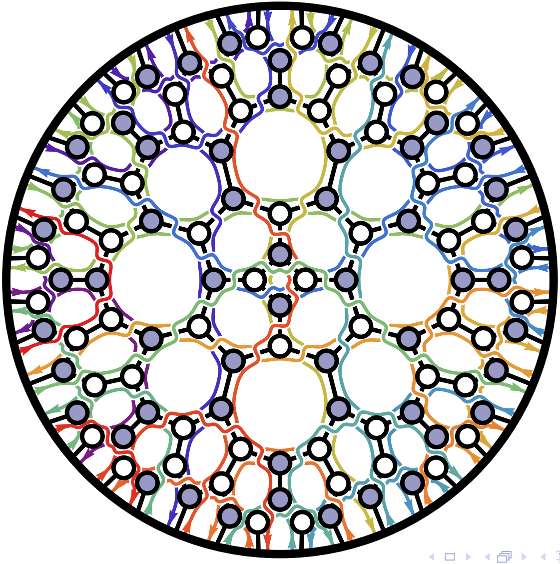

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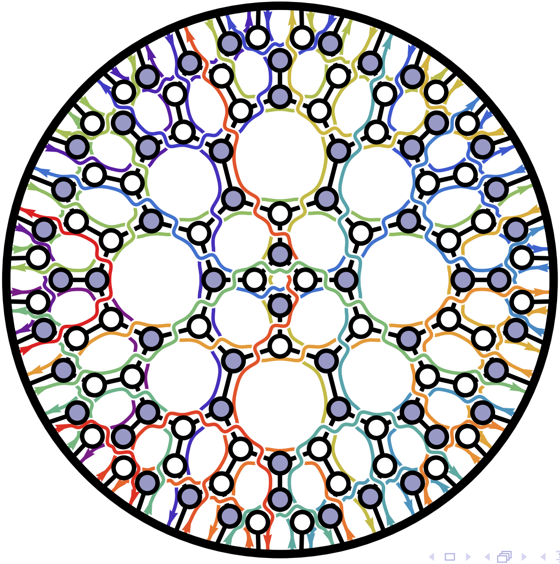
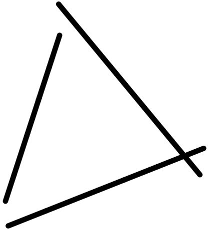


One-Loop Quad-Cuts in Momentum-Twistor Space



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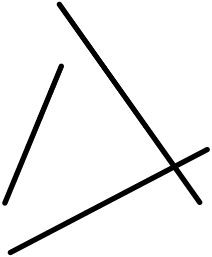

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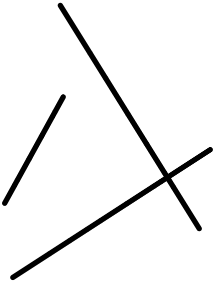

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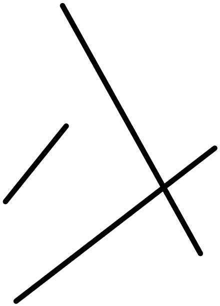

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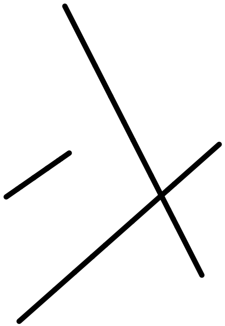

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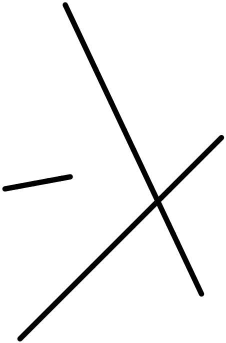

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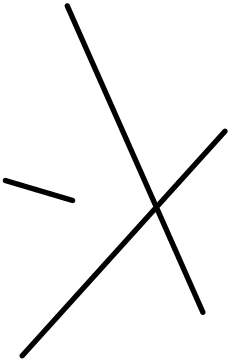

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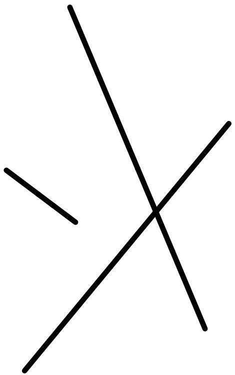

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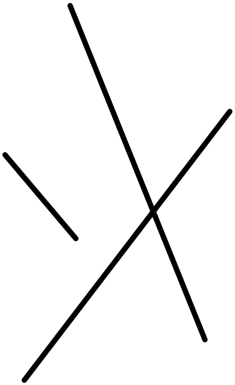

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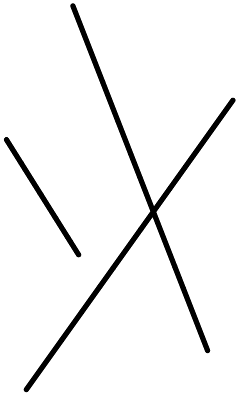

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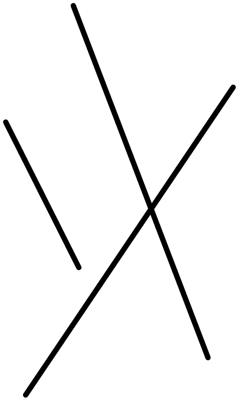

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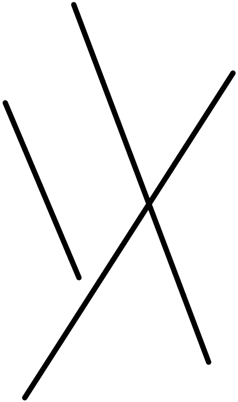

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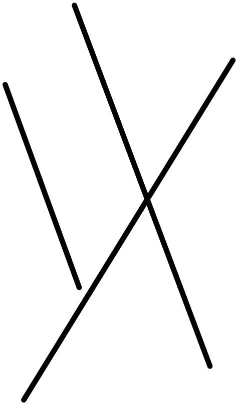

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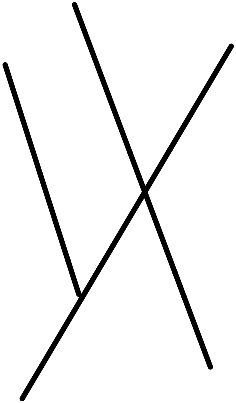

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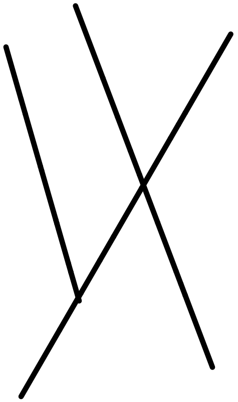

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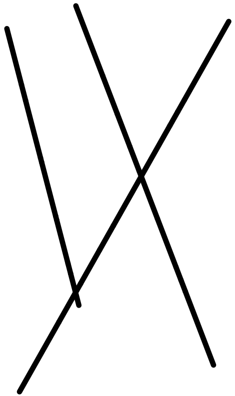

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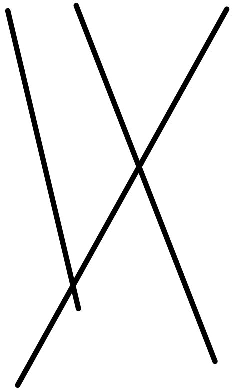

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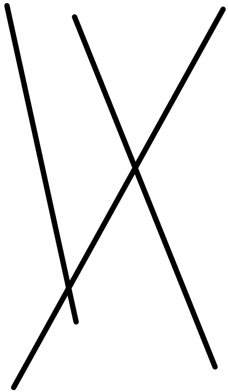

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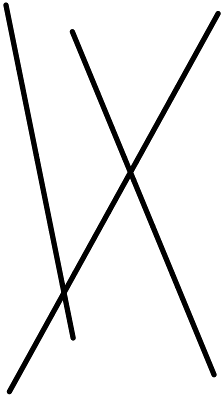

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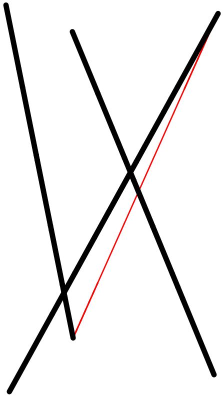

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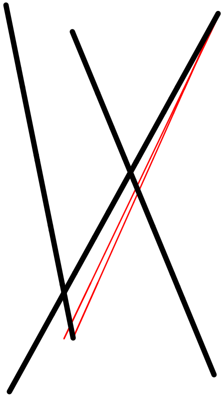

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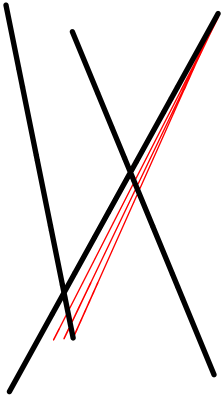

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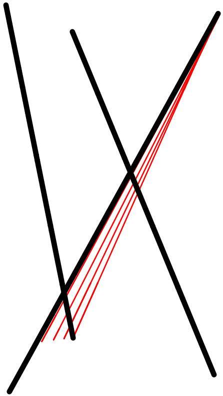

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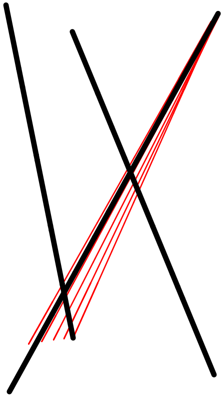

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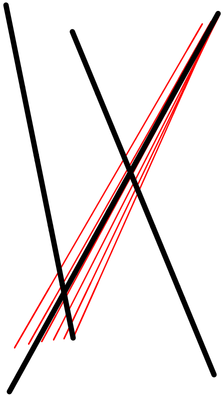

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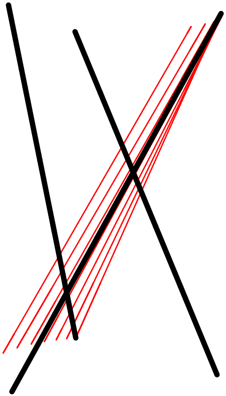

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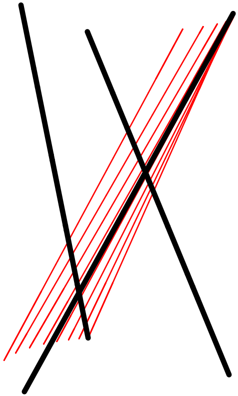

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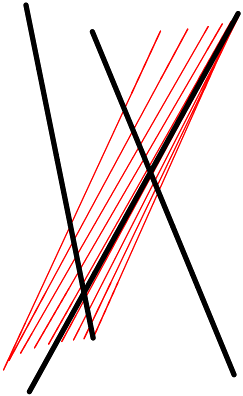

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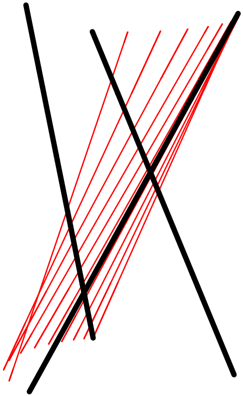

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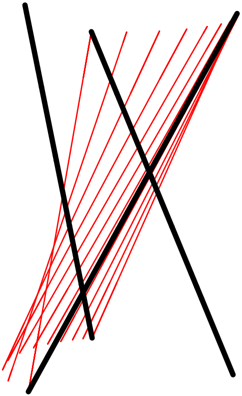

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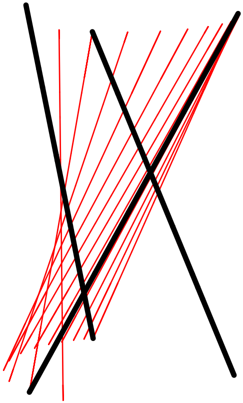

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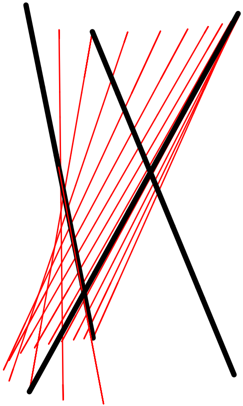

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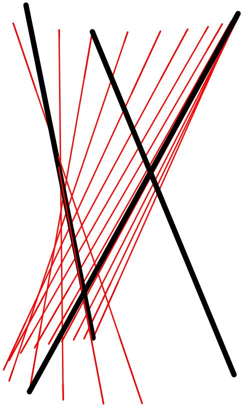

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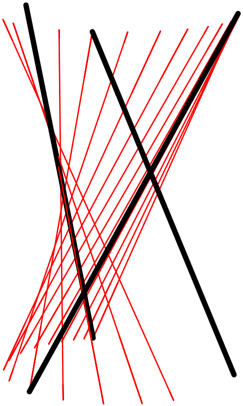

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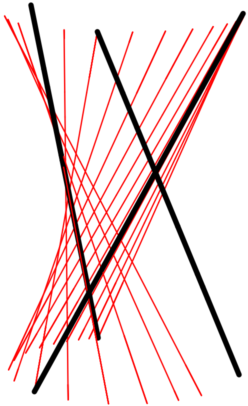

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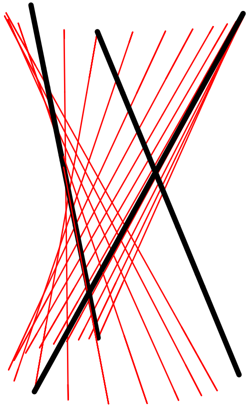

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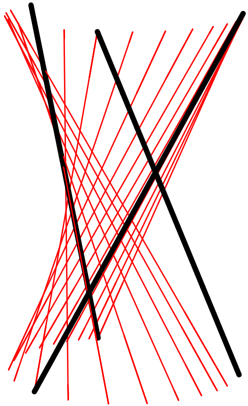

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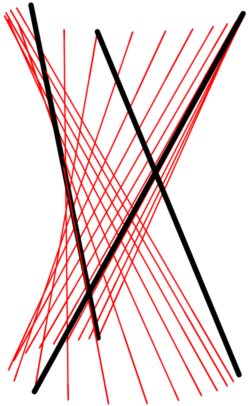

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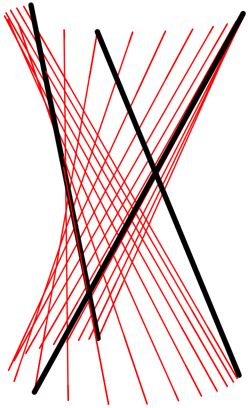

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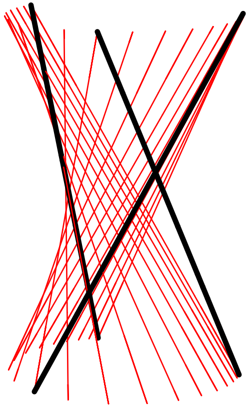

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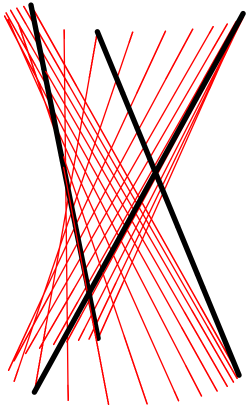

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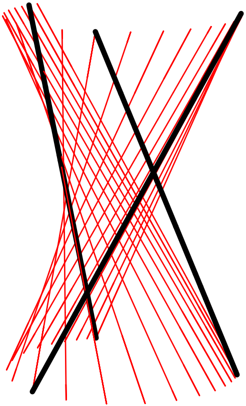

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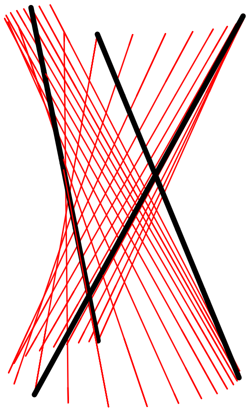

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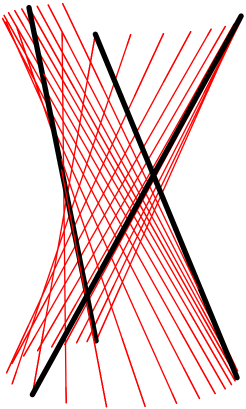

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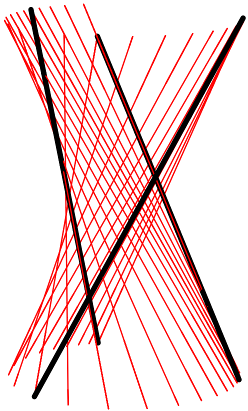

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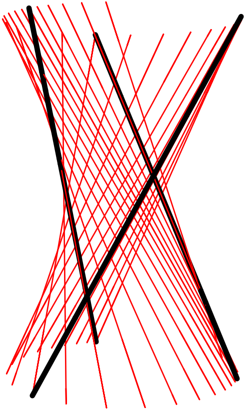

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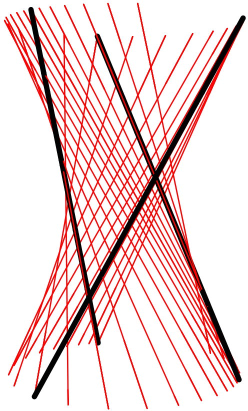

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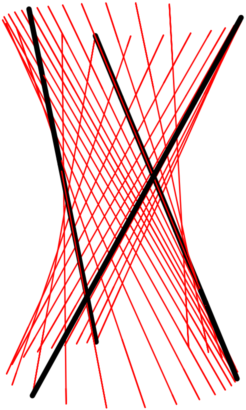
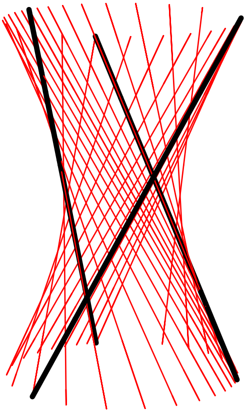


One-Loop Quad-Cuts in Momentum-Twistor Space



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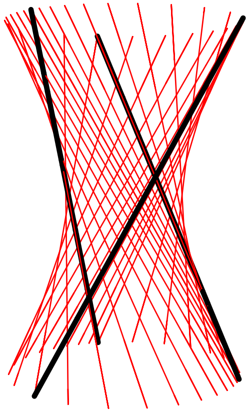

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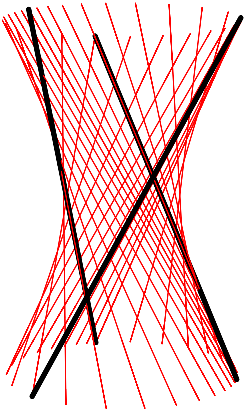

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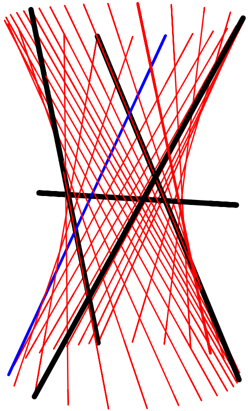

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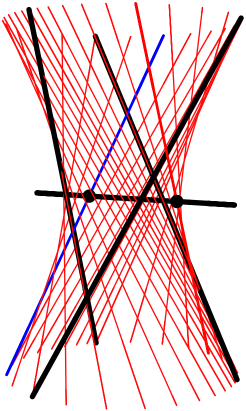

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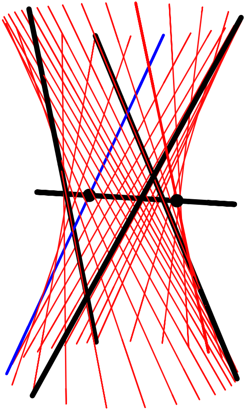

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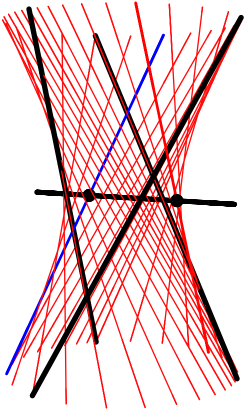

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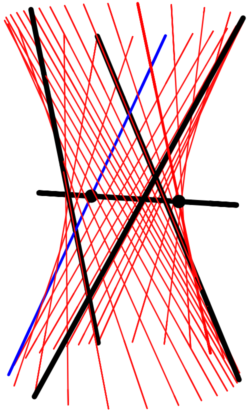

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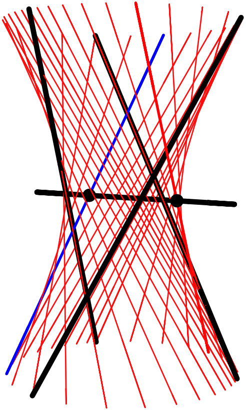

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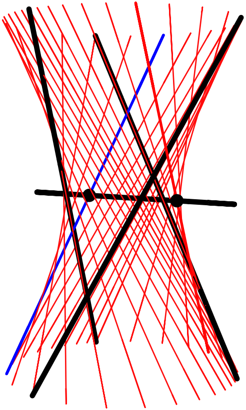

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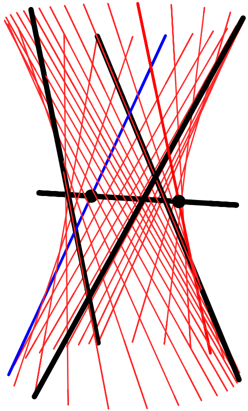

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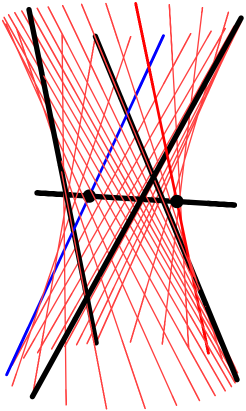

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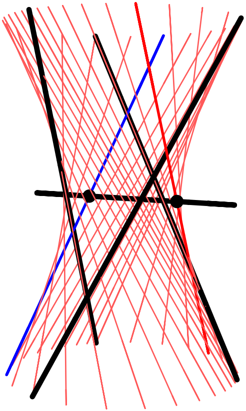

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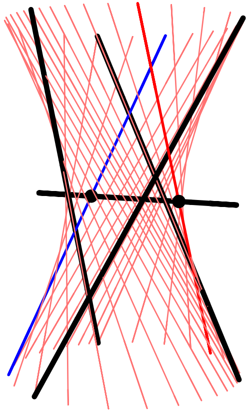

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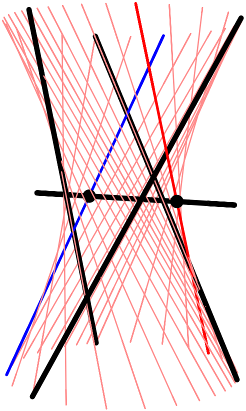

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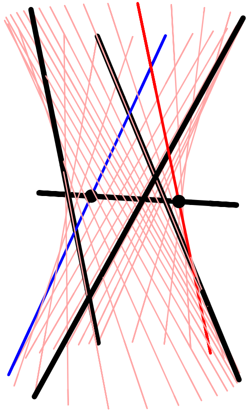

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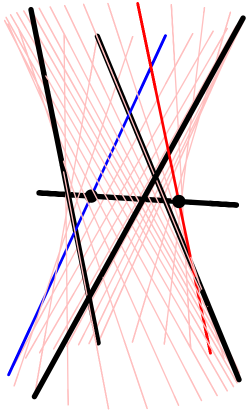

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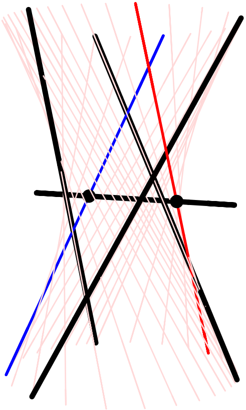

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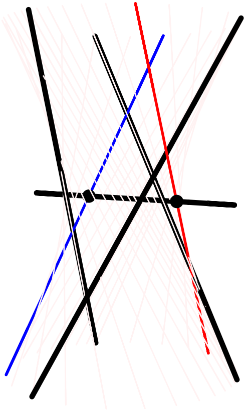

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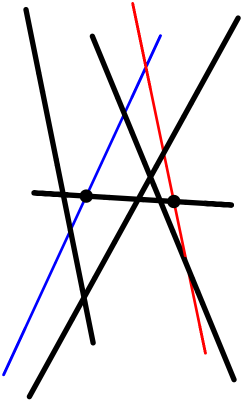

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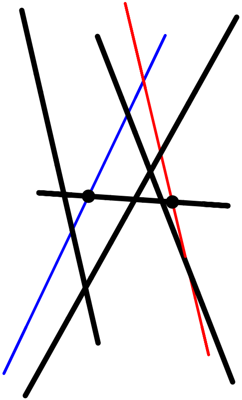

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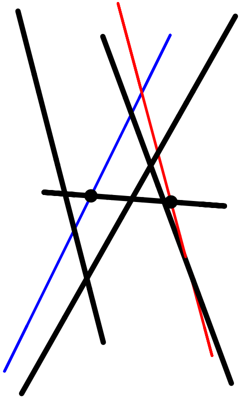

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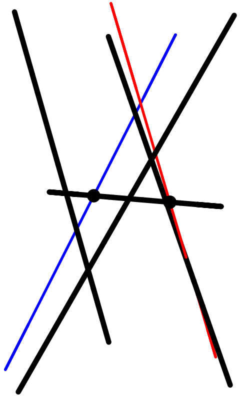

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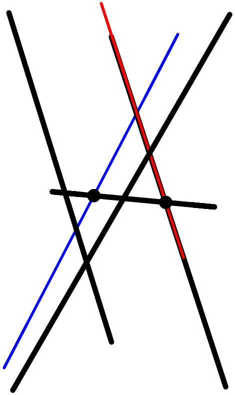

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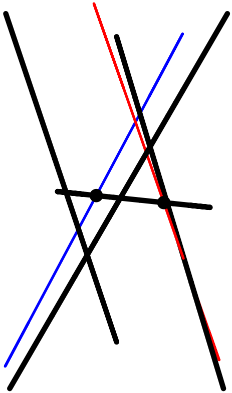

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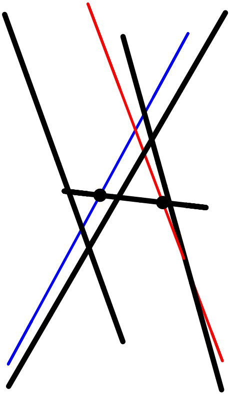

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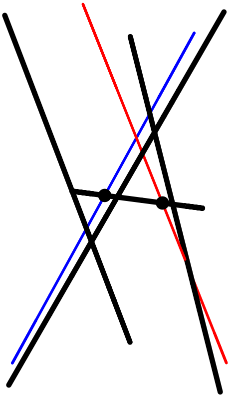

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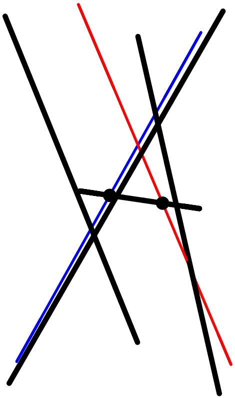

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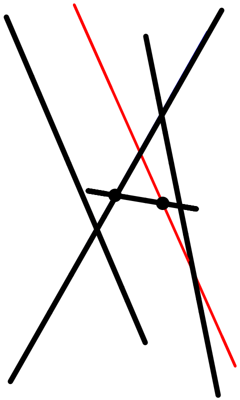

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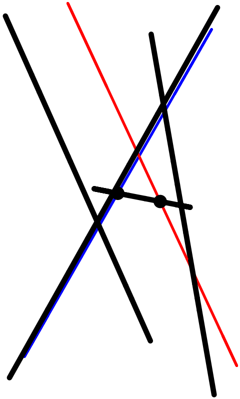

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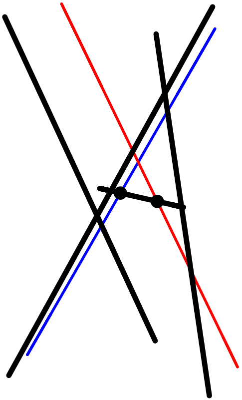

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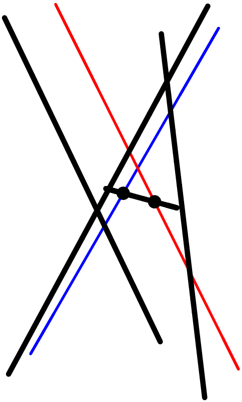

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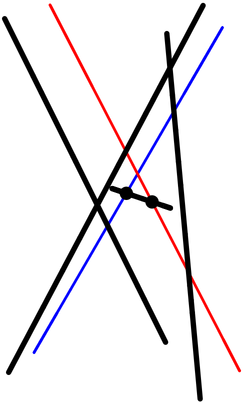

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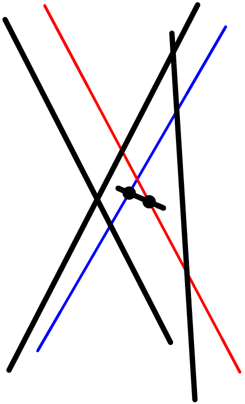

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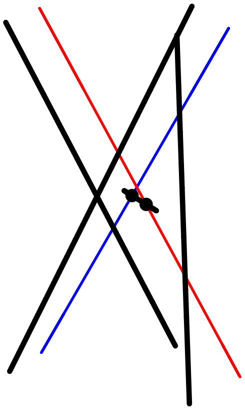

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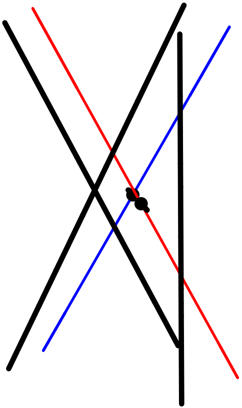

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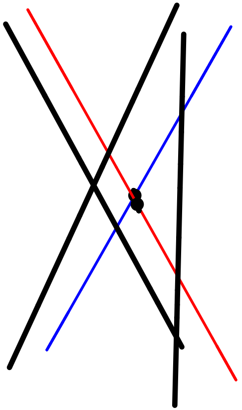

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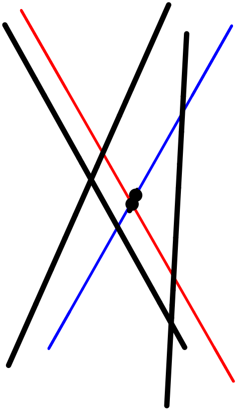

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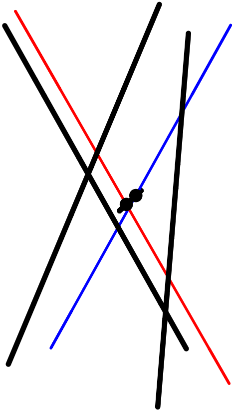

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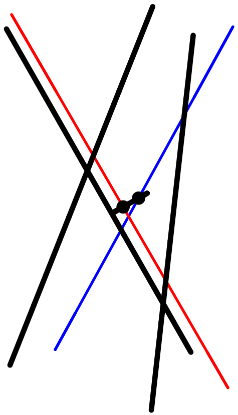

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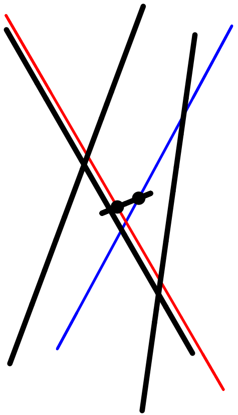

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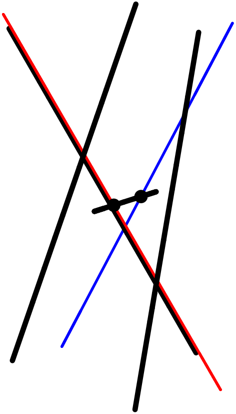

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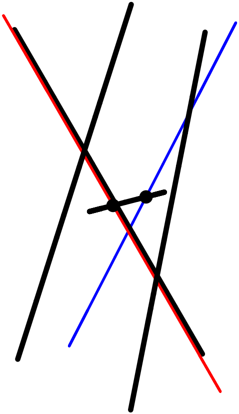

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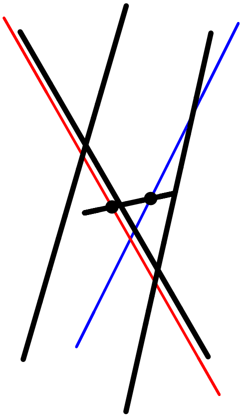

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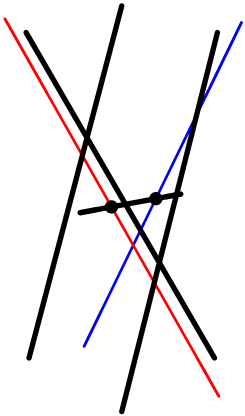

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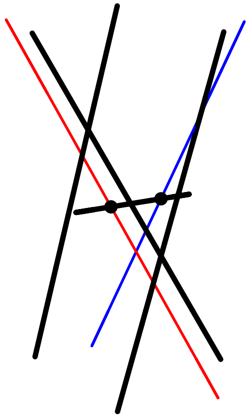

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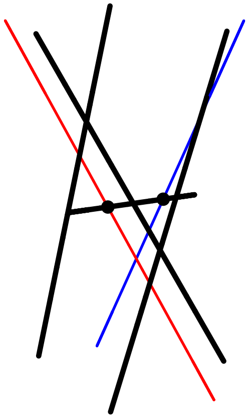

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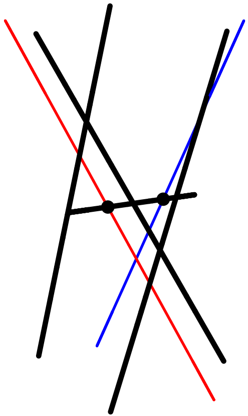

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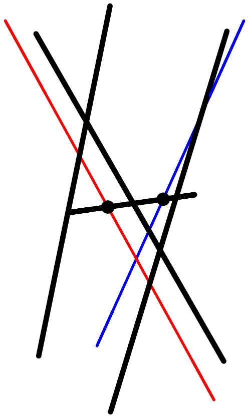

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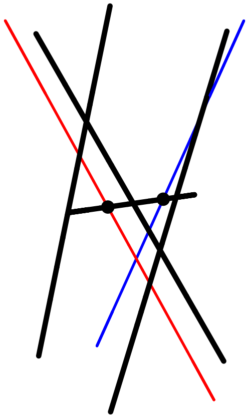

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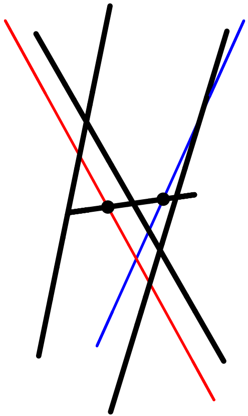

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