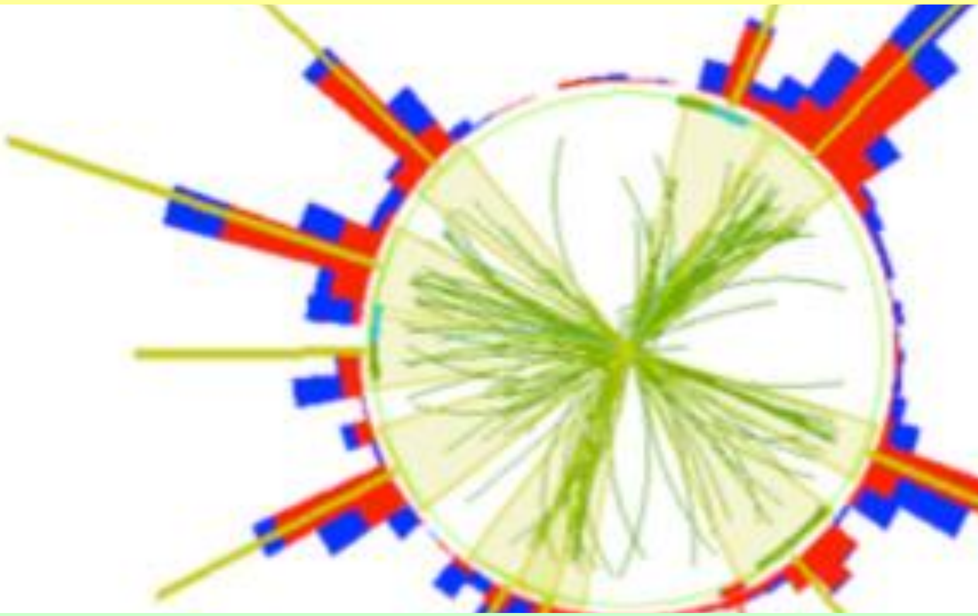
Beyond Feynman Diagrams
Lecture 2



Lance Dixon

Academic Training Lectures

CERN

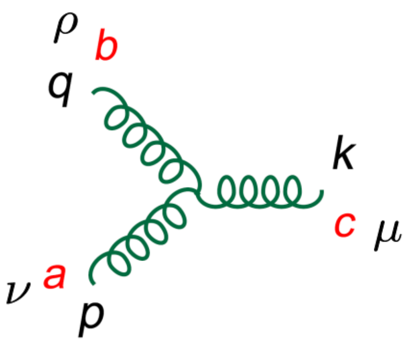
April 24-26, 2013

Modern methods for trees

1. Color organization (briefly)
2. Spinor variables
3. Simple examples
4. Factorization properties
5. BCFW (on-shell) recursion relations

How to organize gauge theory amplitudes

- Avoid tangled algebra of color and Lorentz indices generated by Feynman rules



$$= i g_s f^{abc}[\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho]$$

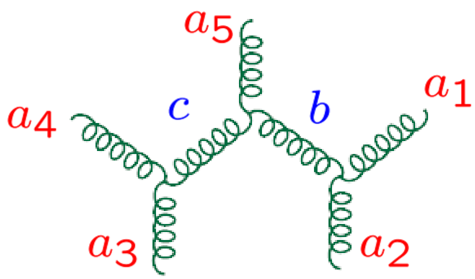
structure constants

- Take advantage of physical properties of amplitudes

- Basic tools:
  - dual (trace-based) color decompositions
  - spinor helicity formalism

Color

Standard color factor for a QCD graph has lots of structure constants contracted in various orders; for example:



$$\propto \; f^{a_1 a_2 b} \, f^{a_3 a_4 c} \, f^{b c a_5}$$

Write every n-gluon tree graph color factor as a sum of traces of matrices T^a in the fundamental (defining) representation of $SU(N_c)$:

$\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$ + all non-cyclic permutations

Use definition: $[T^a, T^b] = i f^{abc} T^c$

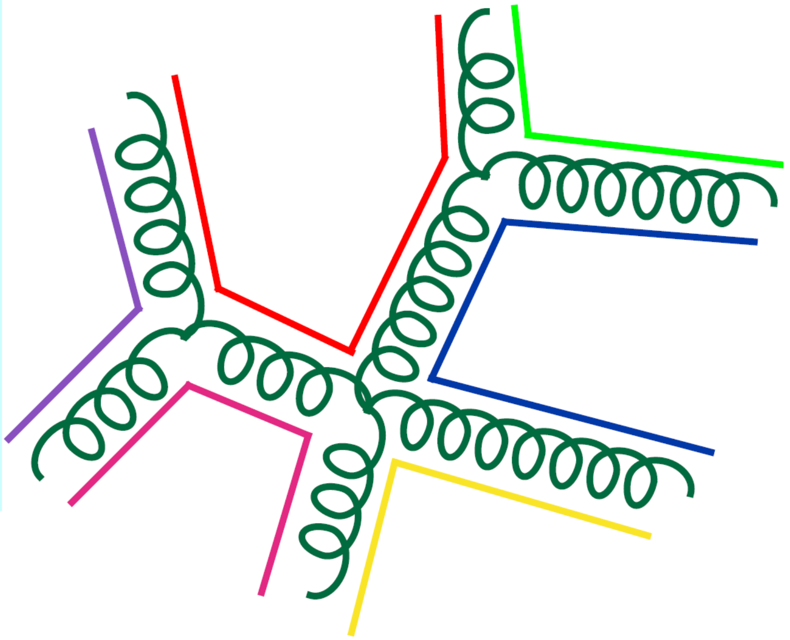
+ normalization: $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ → $f^{abc} = -i \, \mathrm{Tr}([T^a, T^b] T^c)$

Double-line picture ('t Hooft)

- In limit of large number of colors N_c, a gluon is always a combination of a color and a different anti-color.
- Gluon tree amplitudes dressed by lines carrying color indices, $1,2,3,\ldots,N_c$.
- Leads to color ordering of the external gluons.
- Cross section, summed over colors of all external gluons

= Σ |color-ordered amplitudes|^2

- Can still use this picture at $N_c=3$.
- Color-ordered amplitudes are still the building blocks.
- Corrections to the color-summed cross section, can be handled exactly, but are suppressed by $1/N_c^2$



Trace-based (dual) color decomposition

For n-gluon tree amplitudes, the color decomposition is

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g_s^{n-2} \text{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) \, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) + \text{non-cyclic permutations}$$

momenta

color

helicities
$h_i = \pm 1$

color-ordered subamplitude only depends on momenta.
Compute separately for each cyclicly inequivalent helicity configuration $(h_1, h_2, \ldots, h_n)$

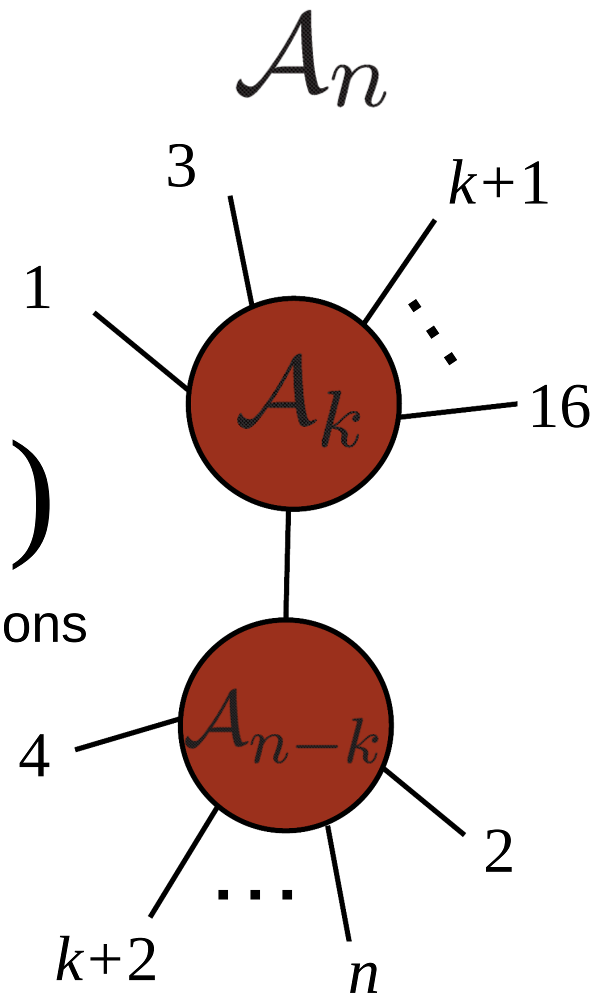
- Because $A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$ comes from planar diagrams with cyclic ordering of external legs fixed to $1,2,\ldots,n$, it only has singularities in cyclicly-adjacent channels $s_{i,i+1}$, …

Similar decompositions for amplitudes with external quarks.

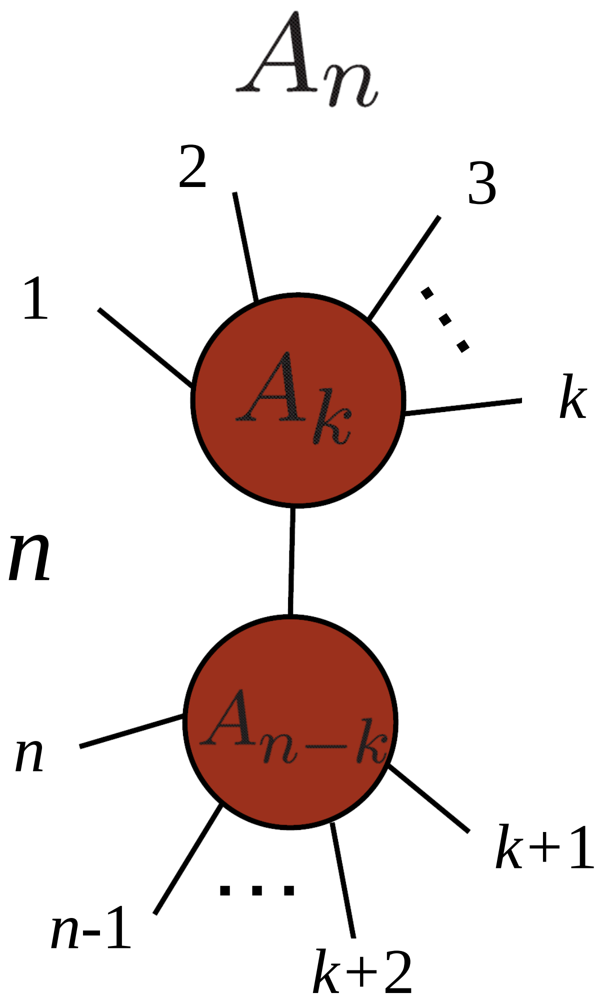
Far fewer factorization channels with color ordering

$\binom{n}{k}$

partitions



only n



Color sums

Parton model says to sum/average over final/initial colors (as well as helicities):

$$d\sigma^{\text{tree}} \propto \sum_{a_i} \sum_{h_i} |\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\})|^2$$

Insert:

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) \;=\; g_s^{n-2} \text{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})\, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$$
$$+ \text{ non-cyclic permutations}$$

and do color sums to get:

$$d\sigma^{\text{tree}} \propto N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} |A_n^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}), \ldots, \sigma(n^{h_n}))|^2 + \mathcal{O}(N_c^{-2})$$

→ Up to $1/N_c^2$ suppressed effects, squared subamplitudes have definite color flow – important for development of parton shower

Spinor helicity formalism

Scattering amplitudes for massless plane waves of definite momentum:
Lorentz 4-vectors k_i^μ $k_i^2=0$

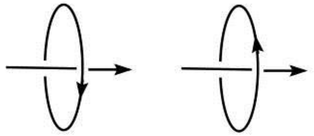
Natural to use Lorentz-invariant products (invariant masses): $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

But for elementary particles with **spin** (*e.g.* all except Higgs!)
there is a better way:

Take "square root" of 4-vectors k_i^μ (spin 1)
use Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin ½)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$ left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

q,g,γ, all have 2 helicity states, $h = \pm$



Massless Dirac spinors

- Positive and negative energy solutions to the massless Dirac equation, $\not{k}\, u(k) = 0, \quad \not{k}\, v(k) = 0$
  are identical up to normalization.
- Chirality/helicity eigenstates are

$$u_\pm(k) = \tfrac{1}{2}(1 \pm \gamma_5) u(k), \quad v_\pm(k) = \tfrac{1}{2}(1 \pm \gamma_5) v(k)$$

- Explicitly, in the Dirac representation

$$u_+(k) = v_-(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{k^+} \\ \sqrt{k^-} e^{i\varphi_k} \\ \sqrt{k^+} \\ \sqrt{k^-} e^{i\varphi_k} \end{bmatrix}, \quad u_-(k) = v_+(k) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{k^-} e^{-i\varphi_k} \\ -\sqrt{k^+} \\ -\sqrt{k^-} e^{-i\varphi_k} \\ \sqrt{k^+} \end{bmatrix}$$

$$e^{\pm i\varphi_k} \equiv \frac{k^1 \pm i k^2}{\sqrt{(k^1)^2 + (k^2)^2}} \qquad k^\pm = k^0 \pm k^3$$

Spinor products

Instead of Lorentz products: $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

Use spinor products: $\bar{u}_-(k_i) u_+(k_j) = \varepsilon^{\alpha\beta} (\lambda_i)_\alpha (\lambda_j)_\beta = \langle i\, j \rangle$

$$\bar{u}_+(k_i) u_-(k_j) = \varepsilon^{\dot\alpha\dot\beta} (\tilde\lambda_i)_{\dot\alpha} (\tilde\lambda_j)_{\dot\beta} = [i\, j]$$

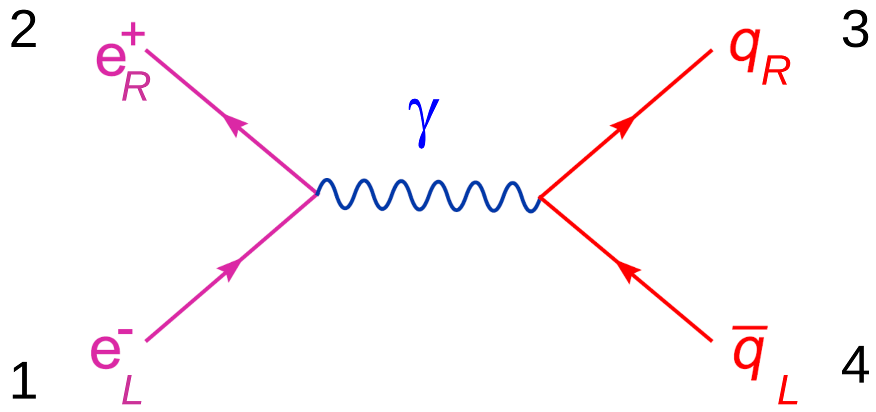
Identity $k_i^\mu (\sigma_\mu)_{\alpha\dot\alpha} = (\not{k}_i)_{\alpha\dot\alpha} = u_+(k_i) \bar{u}_+(k_i) = (\lambda_i)_\alpha (\tilde\lambda_i)_{\dot\alpha}$

$\Rightarrow$ These are **complex square roots** of Lorentz products (for real k_i):

$$\langle i\, j \rangle [j\, i] = \frac{1}{2} \mathrm{Tr} \left[\not{k}_i \, \not{k}_j \right] = 2 k_i \cdot k_j = s_{ij}$$

$$\langle i\, j \rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

~ Simplest Feynman diagram of all



add helicity information, numeric labels

$$\mathcal{A}_4 = 2ie^2 Q_e Q_q \delta^{\bar{i}_4}_{i_3} A_4$$

$$A_4 = \frac{1}{2\,s_{12}} \overline{v_-}(k_2)\gamma^\mu u_-(k_1)\ \overline{u_+}(k_3)\gamma_\mu v_+(k_4)$$

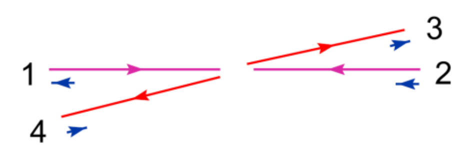
$$= \frac{1}{2\,s_{12}} (\sigma^\mu)_{\alpha\dot\alpha}(\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}\ (\sigma_\mu)^{\dot\beta\beta}(\tilde\lambda_3)_{\dot\beta}(\lambda_4)_\beta$$

$$= \frac{1}{s_{12}} (\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}(\lambda_4)_\alpha(\tilde\lambda_3)_{\dot\alpha}$$

Fierz identity

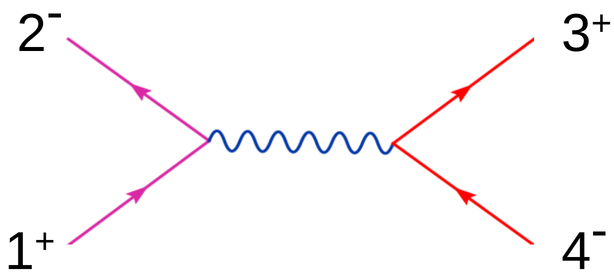
$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma_\mu)^{\dot\beta\beta} = 2\ \delta^\beta_\alpha\,\delta^{\dot\beta}_{\dot\alpha}$$

$$A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}} = e^{i\phi}\frac{s_{13}}{s_{12}} = \frac{-e^{i\phi}}{2}(1-\cos\theta)$$



helicity suppressed as 1 || 3 or 2 || 4

Useful to rewrite answer



Crossing symmetry more manifest
if we switch to all-outgoing helicity labels
(flip signs of incoming helicities)

$$A_4 = \frac{\langle 2\,4\rangle [1\,3]}{s_{12}} = \frac{\langle 2\,4\rangle [1\,3]}{\langle 1\,2\rangle [2\,1]} \frac{\langle 1\,3\rangle}{\langle 1\,3\rangle} = -\frac{\langle 2\,4\rangle [2\,4] \langle 2\,4\rangle}{\langle 1\,2\rangle [2\,4] \langle 4\,3\rangle}$$

$$A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle \langle 3\,4\rangle}$$ "holomorphic"

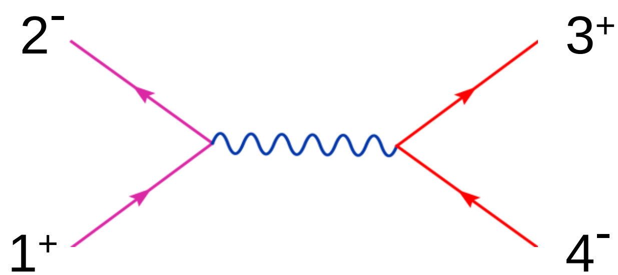
or $$A_4 = \frac{[1\,3]^2}{[1\,2][3\,4]}$$ "antiholomorphic"

useful identities:

$$\langle i\,j\rangle = -\langle j\,i\rangle$$
$$[i\,j] = -[j\,i]$$
$$\langle i\,i\rangle = [i\,i] = 0$$
$$\langle i\,j\rangle [j\,i] = s_{ij}$$
$$\sum_{j=1}^{n} \langle i\,j\rangle [j\,k] = 0$$
$$s_{12} = s_{34}$$
$$s_{13} = s_{24}$$
$$\langle i\,j\rangle\langle k\,l\rangle - \langle i\,k\rangle\langle j\,l\rangle = \langle i\,l\rangle\langle k\,j\rangle$$

Schouten

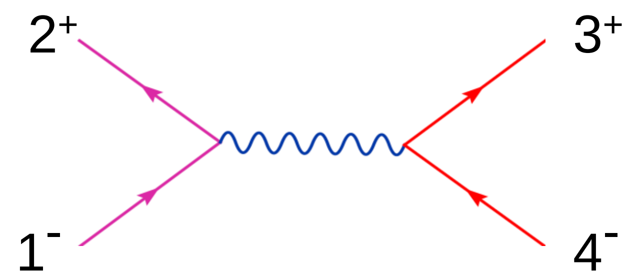
Symmetries for all other helicity config's



C

$1 \leftrightarrow 2$

$\Rightarrow$

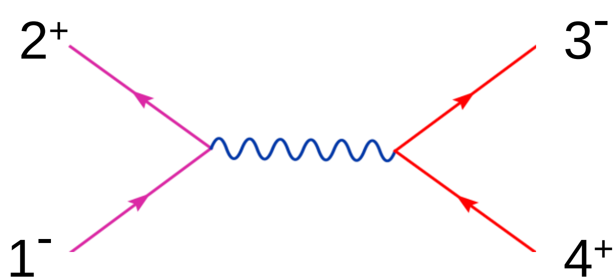


$$A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

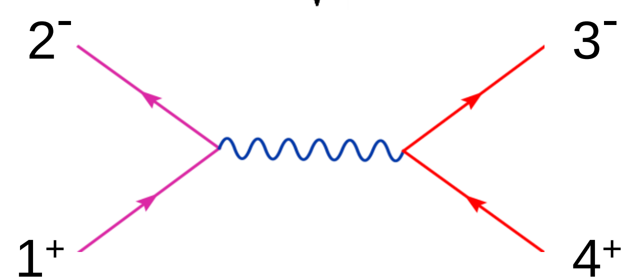
$$A_4 = -\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

P $\Downarrow$ $\langle\,\rangle \leftrightarrow [\,]$

$\Downarrow$



$\Rightarrow$



$$A_4 = \frac{[2\,4]^2}{[1\,2][3\,4]}$$

$$A_4 = -\frac{[1\,4]^2}{[1\,2][3\,4]}$$

Unpolarized, helicity-summed cross sections

(the norm in QCD)

$$\frac{d\sigma(e^+e^- \to q\bar{q})}{d\cos\theta} \propto \sum_{\text{hel.}} |A_4|^2 = 2\left\{\left|\frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2 + \left|\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\right|^2\right\}$$

$$= 2\,\frac{s_{24}^2 + s_{14}^2}{s_{12}^2}$$

$$= \frac{1}{2}\left[(1-\cos\theta)^2 + (1+\cos\theta)^2\right]$$

$$= 1 + \cos^2\theta$$

Helicity formalism for massless vectors

Berends, Kleiss, De Causmaecker, Gastmans, Wu (1981); De Causmaecker, Gastmans, Troost, Wu (1982); Xu, Zhang, Chang (1984); Kleiss, Stirling (1985); Gunion, Kunszt (1985)

$$(\varepsilon_i^+)_\mu = \varepsilon_\mu^+(k_i, q) = \frac{\langle i^+|\gamma_\mu|q^+\rangle}{\sqrt{2}\,\langle i\, q\rangle}$$

$$(\not\varepsilon_i^+)_{\alpha\dot\alpha} = \not\varepsilon_{\alpha\dot\alpha}^+(k_i, q) = \frac{\sqrt{2}\,\tilde\lambda_i^{\dot\alpha}\lambda_q^\alpha}{\langle i\, q\rangle}$$

reference vector q^μ

is null, $q^2 = 0$

$\not q|q^\pm\rangle = 0$

obeys $\varepsilon_i^+ \cdot k_i = 0$ (required transversality)

$\varepsilon_i^+ \cdot q = 0$ (bonus)

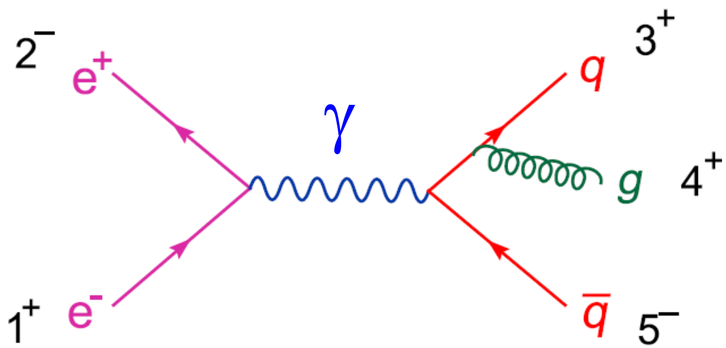
under azimuthal rotation about k_i axis, helicity +1/2 $\tilde\lambda_i^{\dot\alpha} \to e^{i\phi/2}\tilde\lambda_i^{\dot\alpha}$

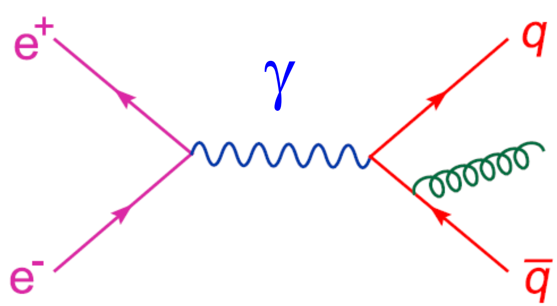
helicity -1/2 $\lambda_i^\alpha \to e^{-i\phi/2}\lambda_i^\alpha$

so $\not\varepsilon_i^+ \propto \frac{\tilde\lambda_i^{\dot\alpha}}{\lambda_i^\alpha} \to e^{i\phi}\ \not\varepsilon_i^+$ as required for helicity +1

Next most famous pair of Feynman diagrams

(to a higher-order QCD person)





$$\mathcal{A}_5 = 2ie^2 g Q_e Q_q (T^{a_4})_{i_3}^{\bar{\imath}_5} A_5$$

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3+\not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\,s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4+\not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\,s_{45}}$$

$e^+e^- \to qg\bar{q}$ (cont.)

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|q^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)|4^-\rangle\langle q\,5\rangle}{s_{45}\langle 4\,5\rangle}$$

Choose $q = k_5$ to remove 2nd graph

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|5^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle}$$

$$= -\frac{\langle 2\,5\rangle[1\,2]\langle 2\,5\rangle[4\,3]}{\langle 1\,2\rangle[2\,1]\langle 3\,4\rangle[4\,3]\langle 4\,5\rangle}$$

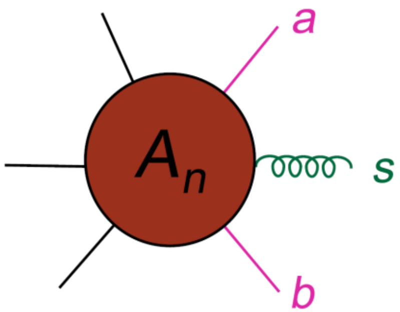
$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle}$$

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$

1. Soft gluon behavior

$k_4 \to 0$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} = \frac{\langle 3\,5\rangle}{\langle 3\,4\rangle\langle 4\,5\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,5\rangle}$$
$$\to \mathcal{S}(3,4^+,5) \times A_4(1^+,2^-,3^+,5^-)$$



Universal "eikonal" factors
for emission of soft gluon *s*
between two hard partons *a* and *b*

$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$
$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

Soft emission is from the classical chromoelectric current:
independent of parton type (*q* *vs.* *g*) and helicity
– only depends on momenta of *a*,*b*, and color charge:

$$\frac{\varepsilon_s^+(q)\cdot k_a}{k_a\cdot k_s} - \frac{\varepsilon_s^+(q)\cdot k_b}{k_b\cdot k_s} \propto \frac{\langle a\,q\rangle}{\langle s\,q\rangle\langle a\,s\rangle} - \frac{\langle b\,q\rangle}{\langle s\,q\rangle\langle b\,s\rangle} = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$ (cont.)

2. Collinear behavior

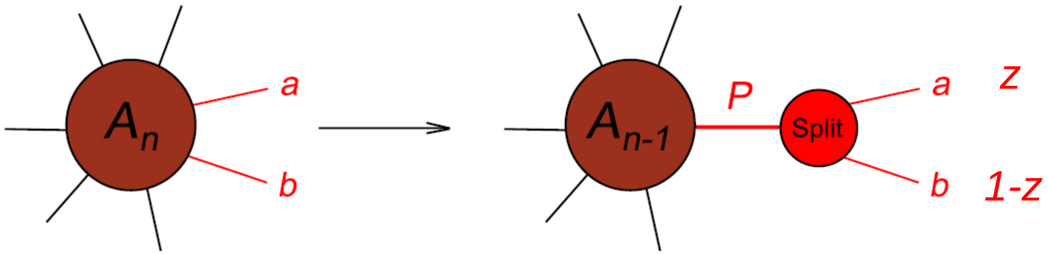
$k_3 \,||\, k_4$: $\quad k_3 = z\, k_P, \quad k_4 = (1-z)\, k_P$

$k_P \equiv k_3 + k_4, \quad k_P^2 \to 0$

$\lambda_3 \approx \sqrt{z}\lambda_P, \quad \lambda_4 \approx \sqrt{1-z}\lambda_P,$ etc.

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{1}{\sqrt{1-z}\,\langle 3\,4\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle P\,5\rangle}$$

$$\to \text{Split}_-(3_q^+, 4_g^+) \times A_4(1^+, 2^-, P^+, 5^-)$$



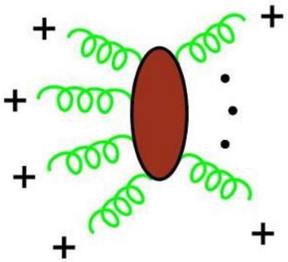
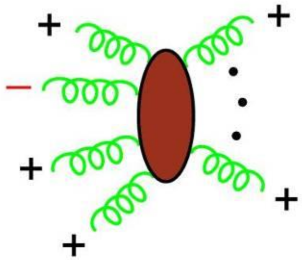
Square root of Altarelli-Parisi splitting probablility

Universal collinear factors, or splitting amplitudes $\text{Split}_{-h_P}(a^{h_a}, b^{h_b})$ depend on parton type and helicity h

Simplest pure-gluonic amplitudes

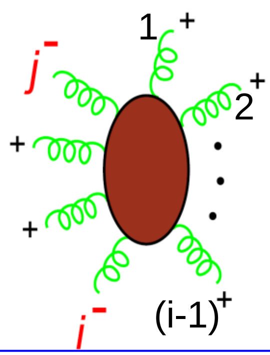
Note: helicity label assumes particle is outgoing; reverse if it's incoming

Strikingly, many vanish:

$$A_n^{\text{tree}}(1^{\pm}, 2^{+}, \ldots, n^{+}) =$$  $=$  $= 0$

Maximally helicity-violating (MHV) amplitudes:

$$A_n^{ij,\ \text{MHV}} = A_n^{\text{tree}}(1^{+}, 2^{+}, \ldots, i^{-}, \ldots, j^{-}, \ldots, n^{+})$$

$=$ 

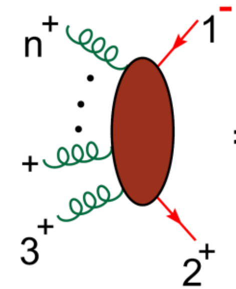
$$= \frac{\langle i\, j \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n\, 1 \rangle}$$

Parke-Taylor formula (1986)

MHV amplitudes with massless quarks

Helicity conservation on fermion line → $A_n^{\text{tree}}(1_{\bar{q}}^{\pm}, 2_q^{\pm}, 3^{h_3}, \ldots, n^{h_n}) \equiv 0$

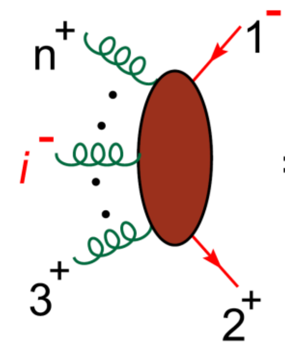
more vanishing ones: $A_n^{\text{tree}}(1_{\bar{q}}^{-}, 2_q^{+}, 3^{+}, \ldots, n^{+}) =$



$= 0$

the MHV amplitudes:

$A_n^{\text{tree}}(1_{\bar{q}}^{-}, 2_q^{+}, \ldots, i^{-}, \ldots, n^{+}) =$



$= \dfrac{\langle 1\, i\rangle^3 \langle 2\, i\rangle}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$

Related to pure-gluon MHV amplitudes by a secret supersymmetry: after stripping off color factors, massless quarks ~ gluinos

Grisaru, Pendleton, van Nieuwenhuizen (1977); Parke, Taylor (1985); Kunszt (1986); Nair (1988)

Properties of MHV amplitudes

1. Soft limit

$k_s \to 0$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a\, s\rangle\langle s\, b\rangle \cdots \langle n\, 1\rangle} = \frac{\langle a\, b\rangle}{\langle a\, s\rangle\langle s\, b\rangle} \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a\, b\rangle \cdots \langle n\, 1\rangle}$$
$$\to \mathrm{Soft}(a, s^+, b) \times A_{n-1}^{ij,\ \mathrm{MHV}}$$

2. Gluonic collinear limits:

$k_a \,||\, k_b \quad (b = a + 1)$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a-1, a\rangle\langle a\, b\rangle\langle b, b+1\rangle \cdots \langle n\, 1\rangle} = \frac{1}{\sqrt{z(1-z)}\langle a\, b\rangle} \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a-1, P\rangle\langle P, b+1\rangle \cdots \langle n\, 1\rangle}$$
$$\to \mathrm{Split}_-(a^+, b^+) \times A_{n-1}^{ij,\ \mathrm{MHV}}$$

So

$$\mathrm{Split}_-(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

and

$$\mathrm{Split}_+(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

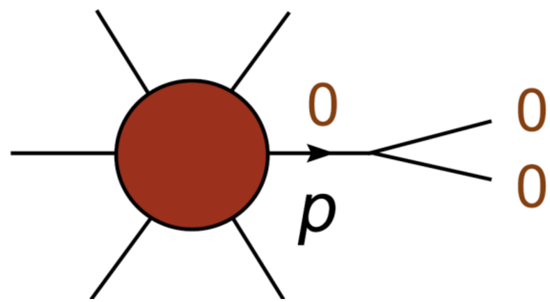
$$\mathrm{Split}_+(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

plus parity conjugates

Spinor Magic

Spinor products precisely capture
square-root + phase behavior in **collinear limit**.
Excellent variables for **helicity amplitudes**

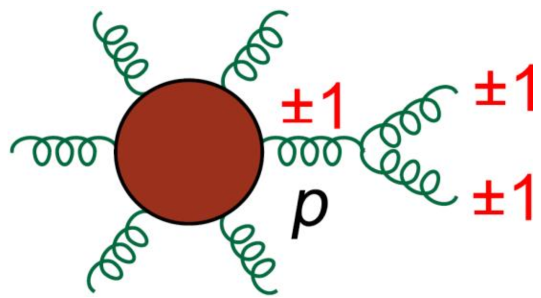
scalars



$$\sim \frac{1}{p^2} \sim \frac{1}{s_{ij}}$$

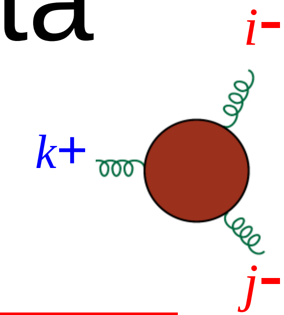
gauge theory

angular momentum mismatch



$$\sim \frac{e^{\pm i\phi}}{\sqrt{p^2}} \sim \frac{1}{\langle i\, j\rangle} \text{ or } \frac{1}{[i\, j]}$$

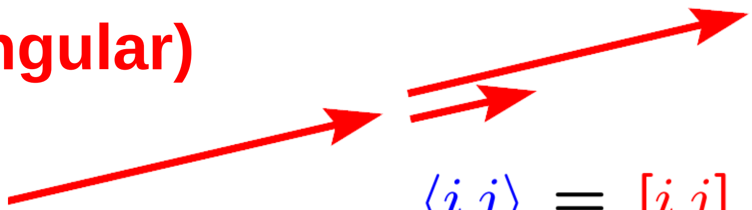
Utility of Complex Momenta



- Makes sense of most basic process: all 3 particles massless

$$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2 = 0 \quad \forall\ i,j \qquad \langle i\, j\rangle [j\, i] = s_{ij}$$

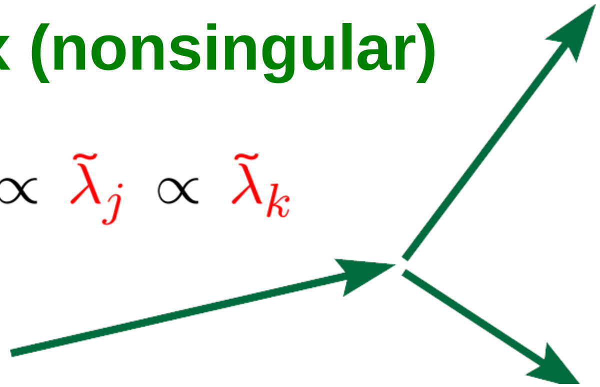
real (singular)



$$\langle i\, j\rangle = [i\, j] = s_{ij} = 0 \qquad \forall\ i,j$$

complex (nonsingular)

$$\tilde\lambda_i \propto \tilde\lambda_j \propto \tilde\lambda_k$$



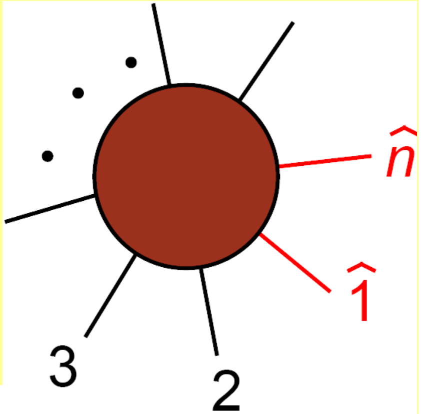
$$[i\, j] = 0 \ \text{ but } \ \langle i\, j\rangle \neq 0$$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\langle 3\, 1\rangle}$$
makes sense

use conjugate kinematics for (++-): $\lambda_i \propto \lambda_j \propto \lambda_k \quad \langle i\, j\rangle = 0\,,\ [i\, j] \neq 0$

Tree-level "plasticity" → BCFW recursion relations

- BCFW consider a family of on-shell amplitudes $A_n(z)$ depending on a complex parameter z which shifts the momenta to complex values



- For example, the $[n,1\rangle$ shift:

$$\lambda_1 \to \hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \tilde{\lambda}_1 \to \tilde{\lambda}_1$$
$$\lambda_n \to \lambda_n \qquad \tilde{\lambda}_n \to \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

- On-shell condition: $$\begin{aligned}(\hat{k}_1)^\mu(\hat{k}_1)_\mu &= (\hat{k}_1)^{\alpha\dot{\alpha}}(\hat{k}_1)_{\dot{\alpha}\alpha} \\ &= \langle(\lambda_1 + z\lambda_n)(\lambda_1 + z\lambda_n)\rangle[1\,1] = 0\end{aligned}$$

similarly, $\hat{k}_n^2 = 0$

- Momentum conservation:

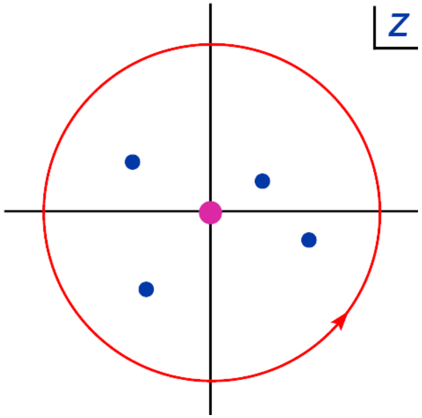
$$\hat{k}_1 + \hat{k}_n = (\lambda_1 + z\lambda_n)\tilde{\lambda}_1 + \lambda_n(\tilde{\lambda}_n - z\tilde{\lambda}_1) = k_1 + k_n$$

Analyticity → recursion relations

$$\hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1$$
$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

$$\Rightarrow \quad A(0) \;\to\; A(z)$$

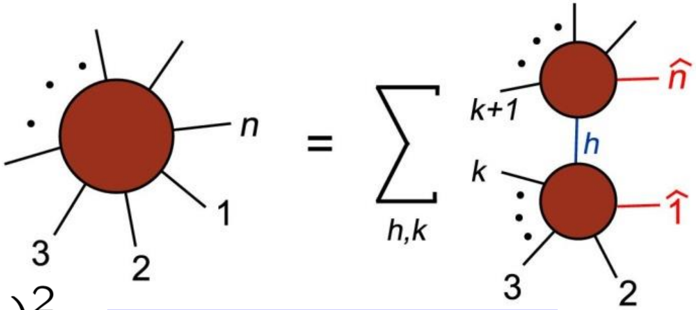
meromorphic function, each pole corresponds to one factorization



Cauchy: If $A(\infty) = 0$ then

$$0 = \frac{1}{2\pi i}\oint dz\,\frac{A(z)}{z} = A(0) + \sum_k \text{Res}\Big[\frac{A(z)}{z}\Big]\Big|_{z=z_k}$$

Where are the poles? Require on-shell intermediate state,



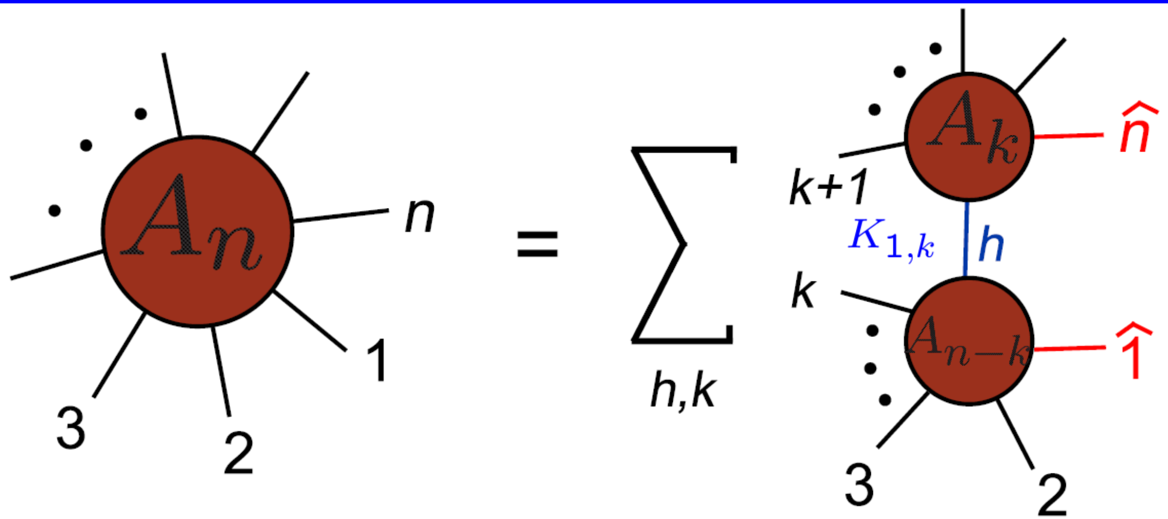
$$\begin{aligned} 0 &= (\hat{k}_1(z) + k_2 + \cdots + k_k)^2 = (z\lambda_n\tilde{\lambda}_1 + K_{1,k})^2 \\ &= z\langle n^- |\, \not{K}_{1,k} | 1^-\rangle + K_{1,k}^2 \end{aligned}$$

$$z_k = -\frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^-\rangle}$$

Final formula

Britto, Cachazo, Feng, hep-th/0412308

$$A_n(1,2,\ldots,n) = \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h}) \times \frac{i}{K_{1,k}^2} A_{n-k+1}(\hat{K}_{1,k}^{h},k+1,\ldots,n-1,\hat{n})$$



A_{k+1} and $A_{n\text{-}k+1}$ are on-shell **color-ordered** tree amplitudes with fewer legs, evaluated with 2 momenta shifted by a **complex** amount

To finish proof, show $A(\infty) = 0$

Britto, Cachazo, Feng, Witten, hep-th/0501052

Propagators:

$$\frac{1}{\widehat{K}^2_{1,k}(z)} = \frac{1}{K^2_{1,k} + z\lambda_n^a (K_{1,k})_{a\dot{a}} \tilde{\lambda}_1^{\dot{a}}} \sim \frac{1}{z}$$

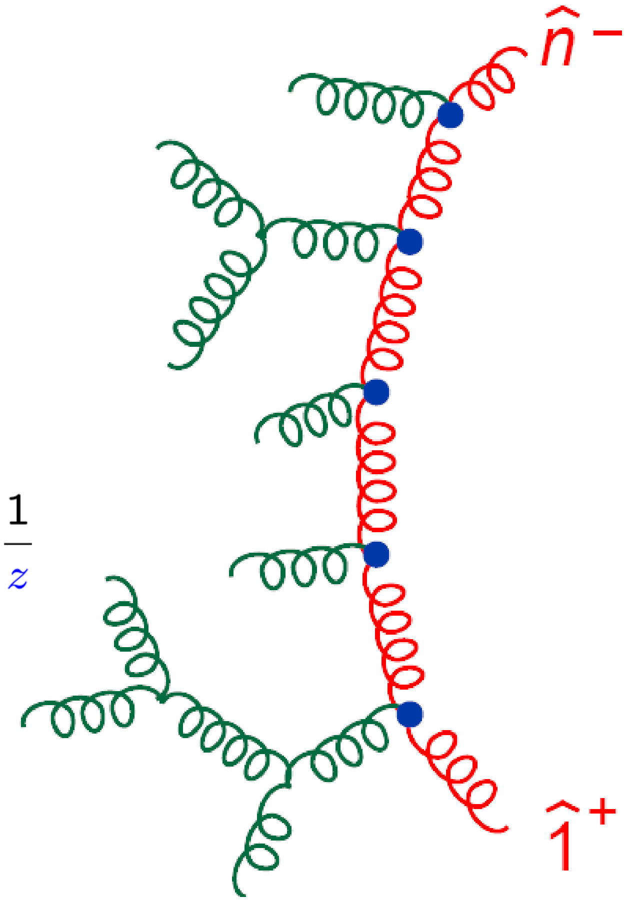
3-point vertices: $\propto \ \widehat{k}^\mu(z) \propto z$

Polarization vectors:

$$\not{\epsilon}_1^+ \propto \frac{\tilde{\lambda}_1 \lambda_q}{\langle \lambda_1 \, \lambda_q \rangle} \propto \frac{1}{z} \qquad \not{\epsilon}_n^- \propto \frac{\lambda_n \tilde{\lambda}_q}{\langle \tilde{\lambda}_n \, \tilde{\lambda}_q \rangle} \propto \frac{1}{z}$$

Total:

$$\frac{1}{z} \times \left(z \frac{1}{z} \right)^r z \times \frac{1}{z} = \frac{1}{z}$$



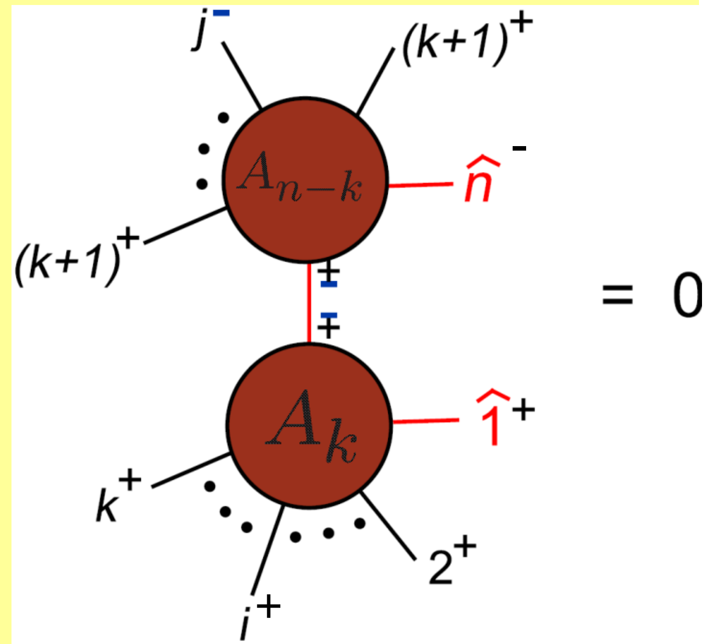
MHV example

- Apply the $[n,1\rangle$ BCFW formula to the MHV amplitude

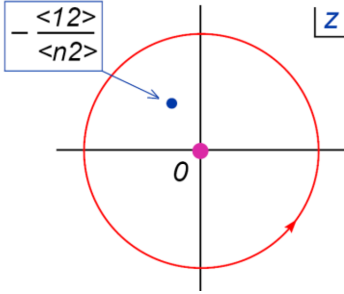
$$A_n^{jn,\,\text{MHV}} = A_n(1^+, 2^+, \ldots, j^-, \ldots, n^-) = \frac{\langle j\, n\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

- The generic diagram vanishes because 2 + 2 = 4 > 3
- So one of the two tree amplitudes is always zero
- The one exception is $k = 2$, which is different because

$$A_3(1^+, 2^+, 3^-) \neq 0$$



MHV example (cont.)



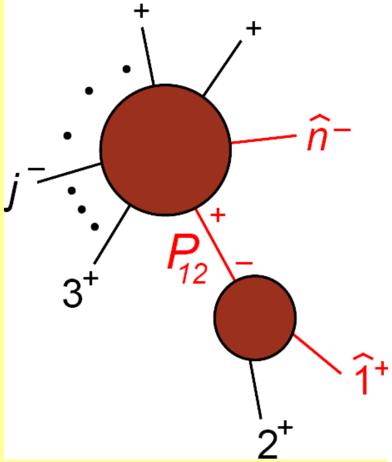
- For $k = 2$, we compute the value of z:

$$z_2 = -\frac{s_{12}}{\langle n^-|(1+2)|1^-\rangle} = -\frac{\langle 1\,2\rangle[2\,1]}{\langle n\,2\rangle[2\,1]} = -\frac{\langle 1\,2\rangle}{\langle n\,2\rangle}$$

- Kinematics are complex collinear

$$\langle \hat{1}\,2\rangle = \langle 1\,2\rangle + z_2\langle n\,2\rangle = 0 \qquad [\hat{1}\,2] = [1\,2] \neq 0$$

$$s_{\hat{1}2} = \langle \hat{1}\,2\rangle[2\,\hat{1}] = 0$$



- The only term in the BCFW formula is:

$$A_{n-1}(\hat{P}_{12}^+, 3^+, \ldots, j^-, \ldots, n^-)\frac{1}{s_{12}}A_3(\hat{1}^+, 2^+, -\hat{P}_{12}^-)$$

$$= \frac{\langle j\,\hat{n}\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1, \hat{n}\rangle\langle \hat{n}\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[\hat{1}\,2]^3}{[2\,\hat{P}][\hat{P}\,\hat{1}]}$$

$$= \frac{\langle j\,n\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1, n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]}$$

note
$A_3(+,+,+) = 0$

MHV example (cont.)

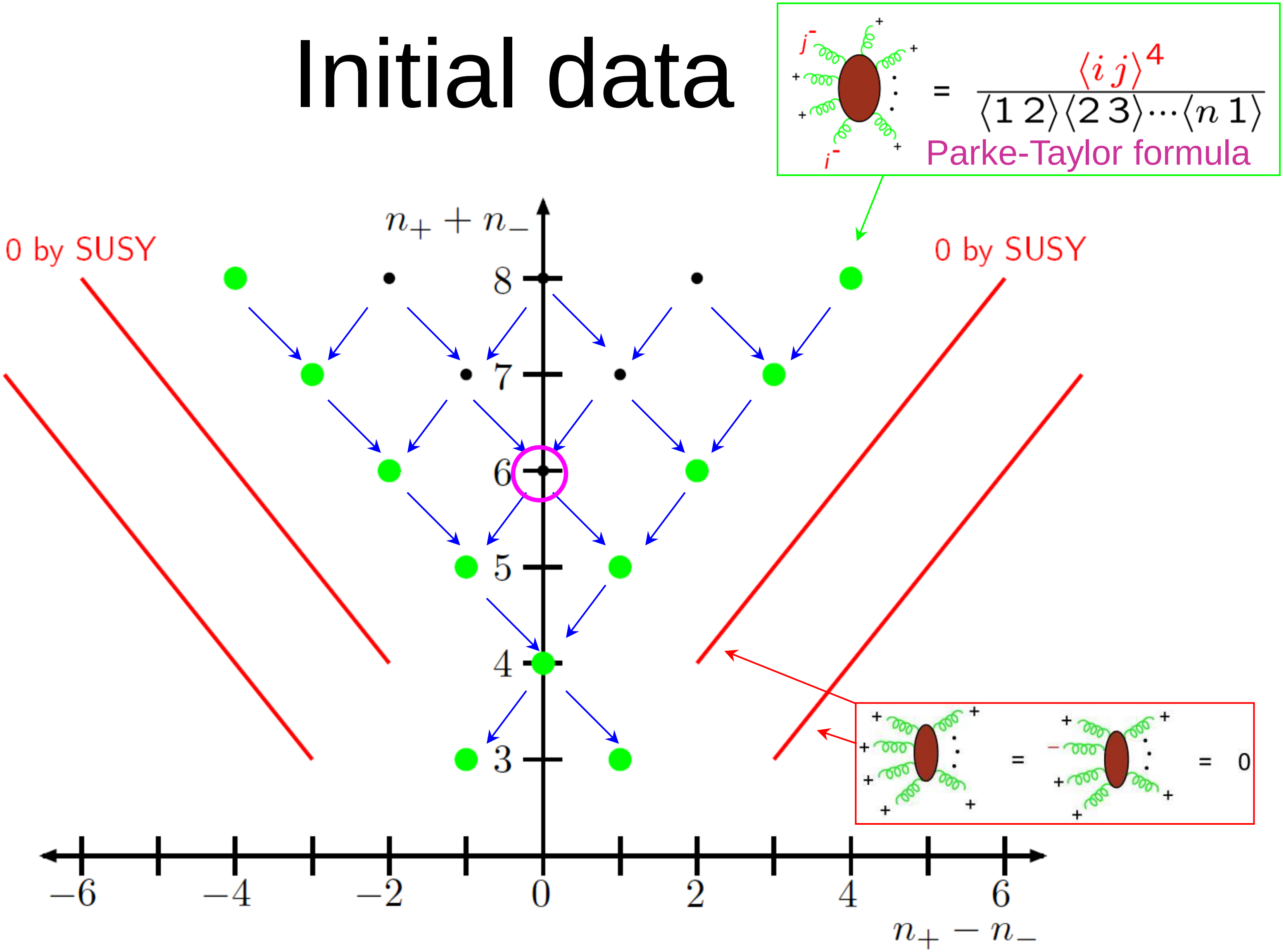
- Using $$\langle n\,\hat{P}\rangle[\hat{P}\,2] \;=\; \langle n^-|(1+2)|2^-\rangle + z\langle n\,n\rangle[1\,2] = \langle n\,1\rangle[1\,2]$$ $$\langle 3\,\hat{P}\rangle[\hat{P}\,1] \;=\; \langle 3^-|(1+2)|1^-\rangle + z\langle 3\,n\rangle[1\,1] = \langle 3\,2\rangle[2\,1]$$

one confirms

$$\frac{\langle j\,n\rangle^4}{\langle \hat{P}\,3\rangle\langle 3\,4\rangle\cdots\langle n-1,n\rangle\langle n\,\hat{P}\rangle}\frac{1}{s_{12}}\frac{[1\,2]^3}{[2\,\hat{P}][\hat{P}\,1]}$$
$$= \frac{\langle j\,n\rangle^4[1\,2]^3}{(\langle 1\,2\rangle[2\,1])([1\,2]\langle 2\,3\rangle)(\langle n\,1\rangle[1\,2])\langle 3\,4\rangle\cdots\langle n-1,n\rangle}$$
$$= \frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1,n\rangle\langle n\,1\rangle}$$
$$= A_n^{jn,\,\mathrm{MHV}}$$

- This proves the Parke-Taylor formula by induction on n.

Initial data



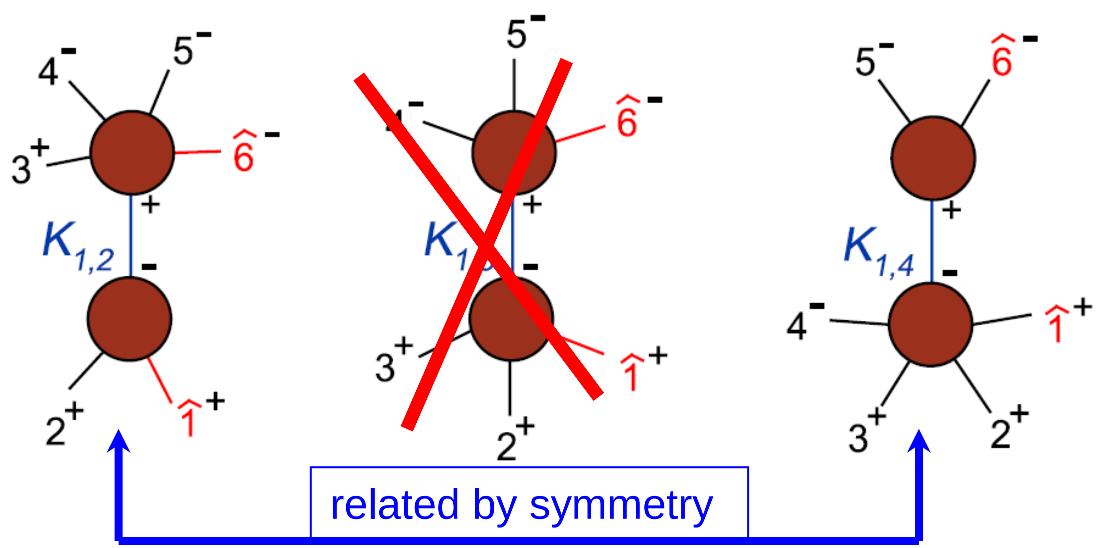
A 6-gluon example

220 Feynman diagrams for *gggggg*

Helicity + color + MHV results + symmetries

$\Rightarrow$ only $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$, $A_6(1^+,2^+,3^-,4^+,5^-,6^-)$

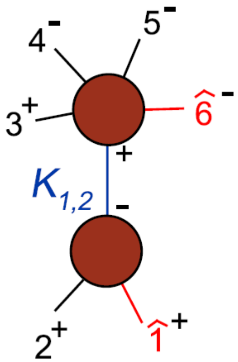
3 BCF diagrams $\rightarrow$ 2 $\rightarrow$ 1



The one $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$ diagram

$$\hat{K} = k_1 + k_2 - \frac{\langle 1\,2\rangle}{\langle 6\,2\rangle}|6\rangle[1|$$

$$|\hat{6}] = |6] + \frac{\langle 1\,2\rangle}{\langle 6\,2\rangle}|1]$$



$$= -\frac{i}{s_{12}} \frac{[\hat{1}\,2]^3}{[2\,\hat{K}][\hat{K}\,\hat{1}]} \frac{[\hat{K}\,3]^3}{[3\,4][4\,5][5\,\hat{6}][\hat{6}\,\hat{K}]}$$

$$= -\frac{i}{s_{12}} \frac{[1\,2]^3}{([2\,\hat{K}]\langle\hat{K}\,6\rangle)(\langle 6\,\hat{K}\rangle[\hat{K}\,1])} \frac{(\langle 6\,\hat{K}\rangle[\hat{K}\,3])^3}{[3\,4][4\,5][5\,\hat{6}]([\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle)}$$

$$= i\,\frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{612}\langle 2^-|(6+1)|5^-\rangle}$$

$$\langle 6\,\hat{K}\rangle[\hat{K}\,a] = \langle 6\,1\rangle\,[1\,a] + \langle 6\,2\rangle\,[2\,a]$$
$$= \langle 6^-|(1+2)|a^-\rangle$$

$$[5\,\hat{6}] = [5\,6] + \frac{\langle 1\,2\rangle[5\,1]}{\langle 6\,2\rangle} = \frac{\langle 5^+|(6+1)|2^+\rangle}{\langle 6\,2\rangle}$$

$$[\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle = \langle 6^+|(1+2)|6^+\rangle + s_{12} = s_{612}$$

Simple final form

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) \;=\; \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\,\langle 1\,2\rangle\,[3\,4]\,[4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,[5\,6]\,[6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

Simpler than form found in 1980s — Mangano, Parke, Xu (1988)
despite (because of?) spurious singularities $\langle 2^-|(6+1)|5^-\rangle$

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) \;=\; \frac{([1\,2]\,\langle 4\,5\rangle\,\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\,\langle 5\,6\rangle\,\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

Relative simplicity much more striking for *n>6*