The Tsallis Distribution at the LHC

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Thanks to:

Dawit Worku,

G.I. Lykasov,

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A.S. Pravan.

Transverse Momentum Distribution

STAR collaboration, B.I. Abelev at al.
arXiv: nucl-ex/0607033; Phys. Rev. **C75**, 064901 (2007)
PHENIX collaboration, A. Adare et al.
arXiv: 1102.0753 [nucl-ex]; Phys. Rev. **C83**, 064903 (2011)
ALICE collaboration, K. Aamodt et al.
arXiv: 1101.4110 [hep-ex]; Eur. Phys. J. **C71**, 1655 (2011)
CMS collaboration, V. Khachatryan et al.
arXiv: 1102.4282 [hep-ex]; JHEP **05**, 064 (2011)
ATLAS collaboration, G. Aad et al.
arXiv: 1012.5104 [hep-ex]; New J. Phys. **13** (2011) 053033.

Transverse Momentum Distribution

STAR, ALICE, CMS, ATLAS use:

$$\frac{\mathrm{d}^2 N}{\mathrm{d}p_\mathrm{T}\mathrm{d}y} = p_\mathrm{T} \times \frac{\mathrm{d}N}{\mathrm{d}y} \frac{(n-1)(n-2)}{nT(nT+m_0(n-2))} \left(1 + \frac{m_\mathrm{T} - m_0}{nT}\right)^{-n}$$

What is the connection with the Tsallis distribution?
Also, the physical significance of the parameters n and T has never been discussed by STAR, ALICE, ATLAS, CMS.

Tsallis Distribution

Possible generalization of Boltzmann-Gibbs statistics

Constantino Tsallis
Rio de Janeiro, TBPF
J. Stat. Phys. 52 (1988) 479-487

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POSSIBLE GENERALIZATION OF BOLTZMANN-GIBBS STATISTICS

by

Constantino TSALLIS

RIO DE JANEIRO

1987

Multifractal concepts and structures are quickly acquiring importance in many active areas (e.g., non-linear dynamical systems, growth models, commensurate/incommensurate structures). This is due to their utility as well as to their elegance. Within this framework, the quantity which is normally scaled is p_i^q, where p_i is the probability associated to an event and q any real number [1]. We shall use this quantity to generalize the standard expression of the entropy S in information theory, namely $S = -k \sum_{i=1}^{W} p_i \ln p_i$, where $W \in \mathbb{N}$ is the total number of possible (microscopic) configurations and $\{p_i\}$ the associated probabilities. We *postulate* for the entropy

$$S_q \equiv k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \qquad (q \in \mathbb{R}) \tag{1}$$

where k is a conventional positive constant and $\sum_{i=1}^{W} p_i = 1$. We immediately verify that

$$S_1 \equiv \lim_{q\to1} S_q = k \lim_{q\to1} \frac{1 - \sum_{i=1}^{W} p_i e^{(q-1)\ln p_i}}{q-1} = -k \sum_{i=1}^{W} p_i \ln p_i \tag{1'}$$

where we have used the replica-trick type of expansion. We illustrate definition (1) in Fig. 1. S_q may be rewritten as follows:

$$S_q = \frac{k}{q-1} \sum_{i=1}^{W} p_i \left(1 - p_i^{q-1}\right) \tag{2}$$

In the grand canonical ensemble the particle number, energy density and pressure are given by

$$
\begin{aligned}
N &= gV \int \frac{d^3p}{(2\pi)^3} \exp\left(-\frac{E-\mu}{T}\right), \\
\epsilon &= g \int \frac{d^3p}{(2\pi)^3} E \exp\left(-\frac{E-\mu}{T}\right), \\
P &= g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} \exp\left(-\frac{E-\mu}{T}\right),
\end{aligned}
$$

where T and μ are the temperature and the chemical potential, V is the volume and g is the degeneracy factor.

In particular, the particle number is:

$$
\begin{aligned}
E\frac{d^3N}{d^3p} &= \frac{gVE}{(2\pi)^3}\, e^{-\frac{E-\mu}{T}}, \\
\frac{d^2N}{m_T dm_T dy} &= \frac{gVm_T \cosh y}{(2\pi)^2}\, e^{-\frac{m_T \cosh y - \mu}{T}},
\end{aligned}
$$

at mid-rapidity, $y = 0$ and zero chemical potential this becomes

$$
\left.\frac{d^2N}{m_T dm_T dy}\right|_{y=0} = \frac{gVm_T}{(2\pi)^2}\, e^{-\frac{m_T}{T}}
$$

m_T scaling! Hopeless!

For high energy physics a consistent form of Tsallis statistics for the particle number, energy density and pressure is given by

$$
\begin{aligned}
N &= gV \int \frac{d^3p}{(2\pi)^3} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} , \\
\epsilon &= g \int \frac{d^3p}{(2\pi)^3} E \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} , \\
P &= g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} .
\end{aligned}
$$

where T and μ are the temperature and the chemical potential, V is the volume and g is the degeneracy factor. The Tsallis distribution introduces a new parameter q which for transverse momentum spectra is always close to 1.

Thermodynamic consistency

$$dE = -pdV + TdS + \mu dN$$

Inserting $E = \epsilon V$, $S = sV$ and $N = nV$ leads to

$$d\epsilon = Tds + \mu dn$$

$$dP = nd\mu + sdT$$

In particular

$$n = \left.\frac{\partial P}{\partial \mu}\right|_T, \quad s = \left.\frac{\partial P}{\partial T}\right|_\mu, \quad T = \left.\frac{\partial \epsilon}{\partial s}\right|_n, \quad \mu = \left.\frac{\partial \epsilon}{\partial n}\right|_s.$$

must be satisfied.

As an example look at the number of particles:

$$\left.\frac{\partial P}{\partial \mu}\right|_T = g\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3E}\frac{\partial}{\partial \mu}\left[1+(q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}}.$$

$$= g\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3E}\frac{q}{T}\left[1+(q-1)\frac{E-\mu}{T}\right]^{-\frac{2q-1}{q-1}}$$

To continue, use

$$\frac{d}{dE}\left[1+(q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} =$$

$$-\frac{q}{q-1}\left[1+(q-1)\frac{E-\mu}{T}\right]^{\frac{-2q+1}{q-1}}\frac{q-1}{T}$$

The expressions for the energy density and the pressure are indeed thermodynamically consistent:

$$
\begin{aligned}
\left.\frac{\partial P}{\partial \mu}\right|_T &= -g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3} \frac{d}{EdE} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} \\
&= -g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3} \frac{d}{pdp} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} \\
&= g \int \frac{d\cos\theta d\phi dp}{(2\pi)^3} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} \frac{d}{dp} \frac{p^3}{3} \\
&= n
\end{aligned}
$$

after an integration by parts and using $p\,dp = E\,dE$.

Hence the expressions for the particle number, the energy density and the pressure are thermodynamically consistent, relations of the type

$$N = V \left. \frac{\partial P}{\partial \mu} \right|_T$$

are indeed satisfied.

In the Tsallis distribution the total number of particles is given by:

$$N = gV \int \frac{d^3p}{(2\pi)^3} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} .$$

The corresponding momentum distribution is given by

$$E\frac{dN}{d^3p} = gVE\frac{1}{(2\pi)^3} \left[1 + (q-1)\frac{E-\mu}{T}\right]^{-\frac{q}{q-1}} ,$$

which, in terms of the rapidity and transverse mass variables, $E = m_T \cosh y$, becomes (at mid-rapidity for μ = 0)

$$\left.\frac{d^2N}{dp_T\,dy}\right|_{y=0} = gV\frac{p_T m_T}{(2\pi)^2} \left[1 + (q-1)\frac{m_T}{T}\right]^{-fracqq-1} ,$$

J.C. and D. Worku, J. Phys. **G39** (2012) 025006; arXiv:1203.4343[hep-ph].

For the connection with the Boltzmann distribution: rewrite the Tsallis distribution using

$$[1+(q-1)x]^{1/(1-q)} = \exp\left(\frac{1}{1-q}\ln\left[1+(q-1)x\right]\right),$$

and consider the limit $q \to 1$

$$\begin{aligned}\lim_{q\to 1} \quad & [1+(q-1)x]^{1/(1-q)} \\ & \approx \exp\frac{1}{(1-q)}(q-1)x \\ & = \exp\left(-x\right),\end{aligned}$$

The Tsallis distribution reduces to the Boltzmann distribution in the limit where $q \to 1$

$$\lim_{q\to 1}\frac{d^2N}{dp_T\,dy} = \\ gV\frac{p_T m_T\cosh y}{(2\pi)^2}\exp\left(-\frac{m_T\cosh y-\mu}{T}\right).$$

In all cases q is close to one, typically between 1.05 and 1.2.

Comparison of the Tsallis form with the STAR, ALICE, ATLAS, CMS distributions

$$\frac{\mathrm{d}^2 N}{\mathrm{d}p_\mathrm{T}\mathrm{d}y} = gV\frac{p_T m_T}{(2\pi)^2}\left[1+(q-1)\frac{m_T}{T}\right]^{q/(1-q)},$$

$$\frac{\mathrm{d}^2 N}{\mathrm{d}p_\mathrm{T}\mathrm{d}y} = p_\mathrm{T}\times\frac{\mathrm{d}N}{\mathrm{d}y}\frac{(n-1)(n-2)}{nT(nT+m_0(n-2))}\left[1+\frac{m_\mathrm{T}-m_0}{nT}\right]^{-n}$$

For the comparison use the following substitution:

$$n\rightarrow\frac{q}{q-1}$$

$$nT\rightarrow\frac{T+m_0(q-1)}{q-1}$$

After this substitution one obtains

$$\frac{d^2N}{dp_T\, dy} = \quad p_T \frac{\mathrm{d}N}{\mathrm{d}y} \frac{(n-1)(n-2)}{nT(nT+m_0(n-2))}$$
$$\left[\frac{T}{T+m_0(q-1)}\right]^{-q/(q-1)}$$
$$\left[1+(q-1)\frac{m_T}{T}\right]^{-q/(q-1)} .$$

To be compared with

$$\frac{\mathrm{d}^2N}{\mathrm{d}p_\mathrm{T}\mathrm{d}y} = gV\frac{p_T m_T}{(2\pi)^2}\left[1+(q-1)\frac{m_T}{T}\right]^{-q/(q-1)} .$$

Apart from several constant factors, which can be absorbed in the volume V, only a factor of m_T differs! However, m_0 shouldn't appear as it destroys m_T scaling. The inclusion of the factor m_T leads to a more consistent interpretation of the variables q and T.

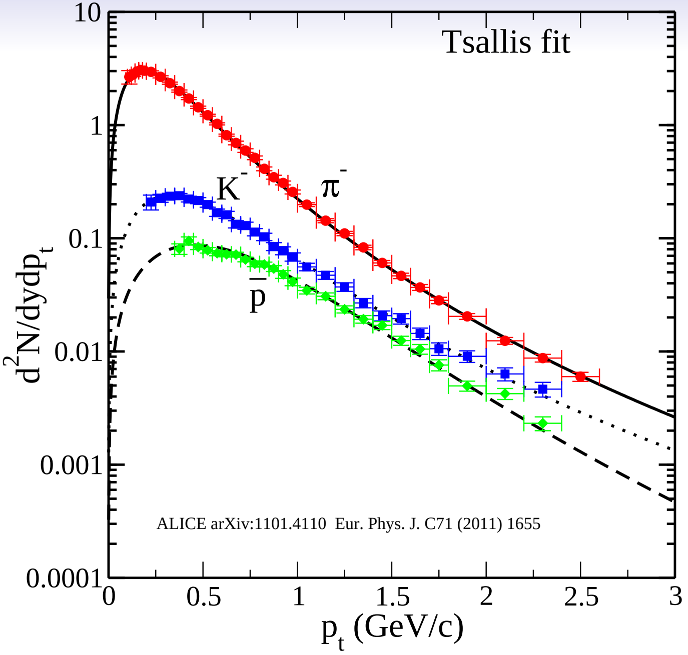
Interpretation of Tsallis Parameter q

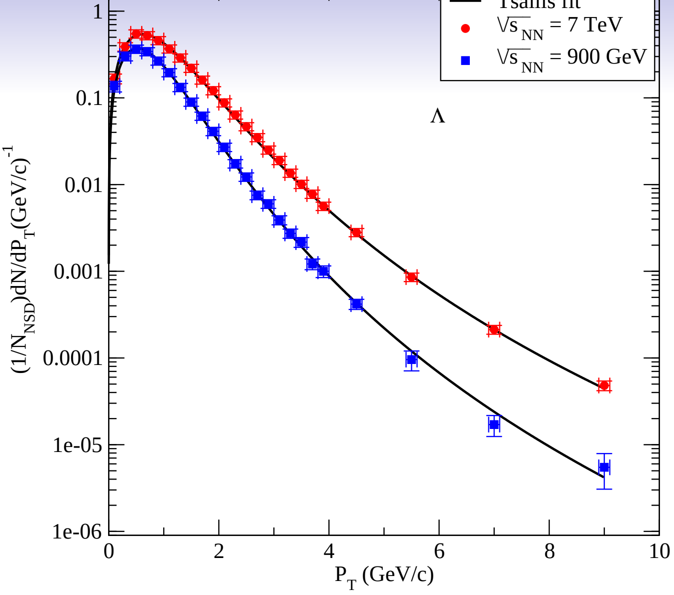
G. Wilk and Z. Wlodarczyk, Phys. Rev. Lett. 84 (2000) 2770

$$\left\langle \frac{1}{T_B} \right\rangle = \frac{1}{T}$$

and also

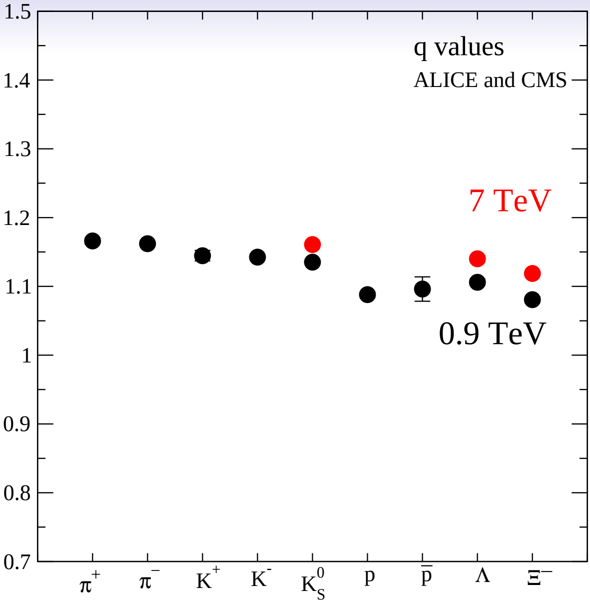
$$\frac{\left\langle \left(\frac{1}{T_B} \right)^2 \right\rangle - \left\langle \frac{1}{T_B} \right\rangle^2}{\left\langle \frac{1}{T_B} \right\rangle^2} = q - 1$$

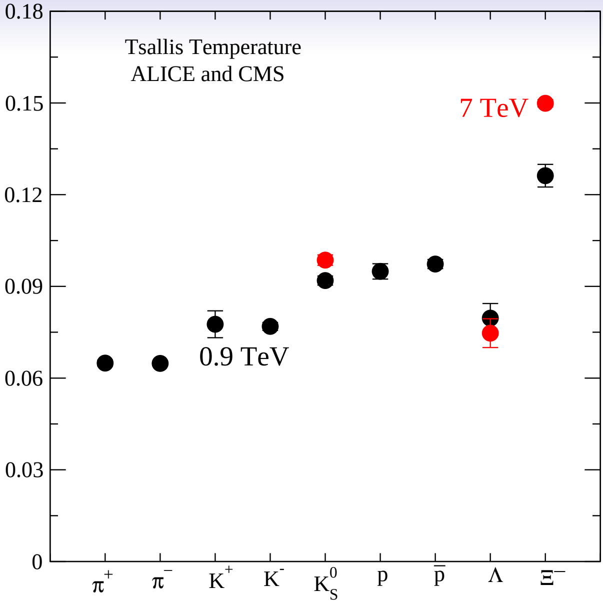

Tsallis fit
K-
π-
p̄
d²N/dydp_t
p_t (GeV/c)
ALICE arXiv:1101.4110 Eur. Phys. J. C71 (2011) 1655
10
1
0.1
0.01
0.001
0.0001
0
0.5
1
1.5
2
2.5
3



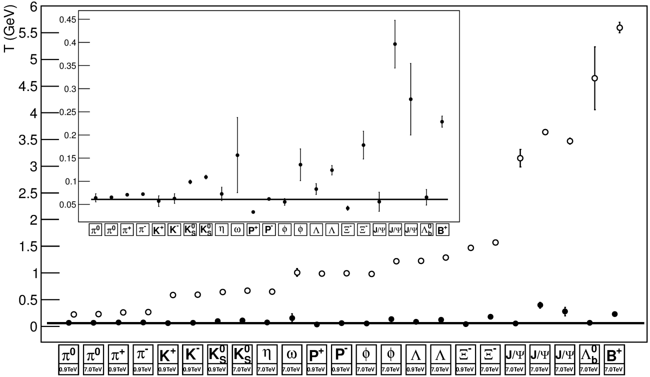
CMS collaboration

$p-p$ ALICE and CMS	
Particle	q
π^+	1.1661 $\pm$0.0041
π^-	1.1620 $\pm$0.0031
K^+	1.1446 $\pm$0.0076
K^-	1.1425 $\pm$0.0029
K^0_S	1.1352 $\pm$0.0024
K^0_S (7 TeV)	1.1608 $\pm$0.0025
p	1.0880 $\pm$0.0042
$\bar{p}$	1.0961 $\pm$0.0177
Λ	1.1060 $\pm$0.0045
Λ (7 TeV)	1.1401 $\pm$0.0042
Ξ^-	1.0807 $\pm$0.0043
Ξ^- (7 TeV)	1.1188 $\pm$0.0029

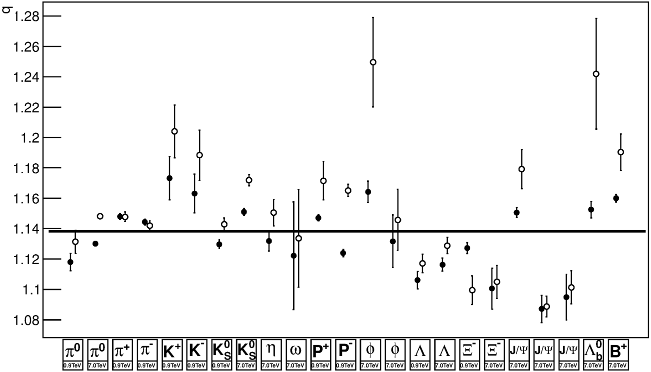

q values
ALICE and CMS
7 TeV
0.9 TeV
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
π^+
π^-
K^+
K^-
K^0_S
p
$\bar{p}$
Λ
Ξ^-



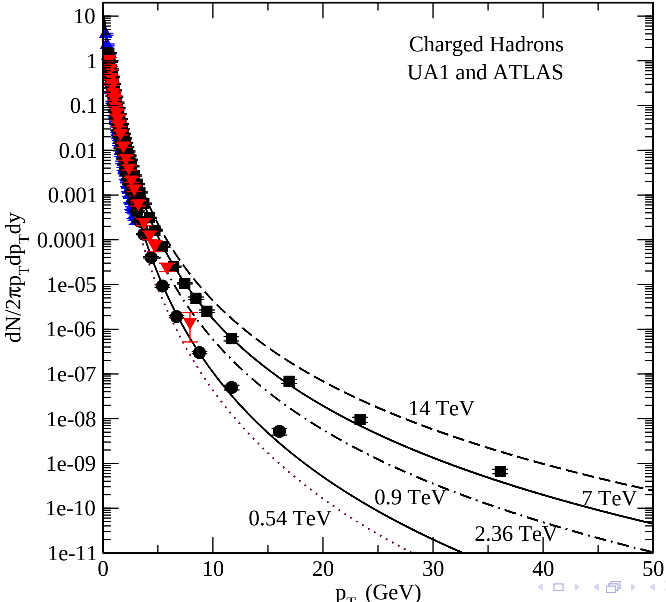
Tsallis Temperature
ALICE and CMS
7 TeV
0.9 TeV
0.18
0.15
0.12
0.09
0.06
0.03
0
π^+
π^-
K^+
K^-
K^0_S
p
$\bar{p}$
Λ
Ξ^-

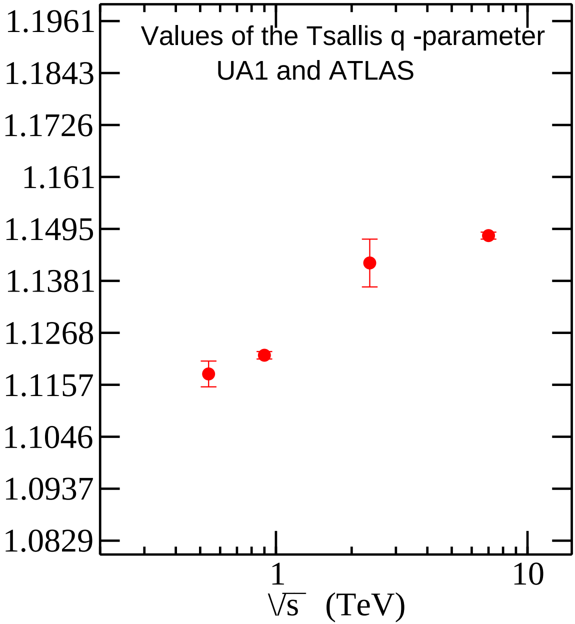


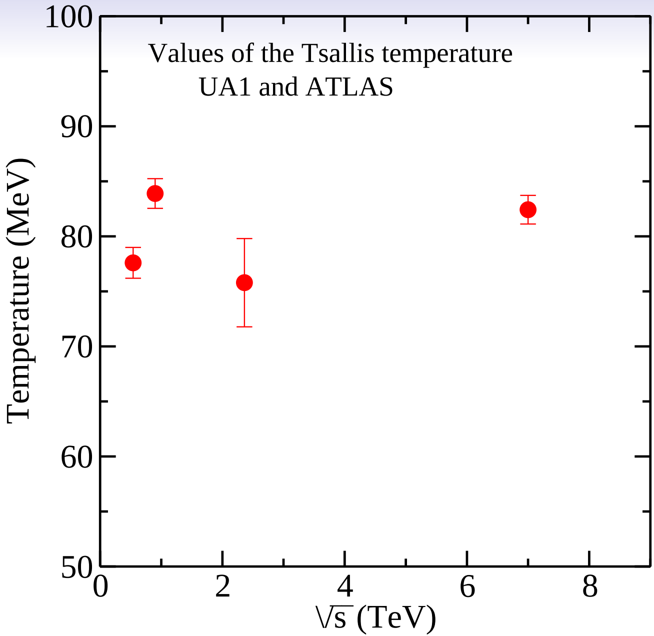
Open symbols effective temperature *T* obtained using ALICE distribution. Closed symbols use Tsallis distribution.
L. Marques, E. Andrade-II and A. Deppman arXiv:1210.1725 [hep-ph

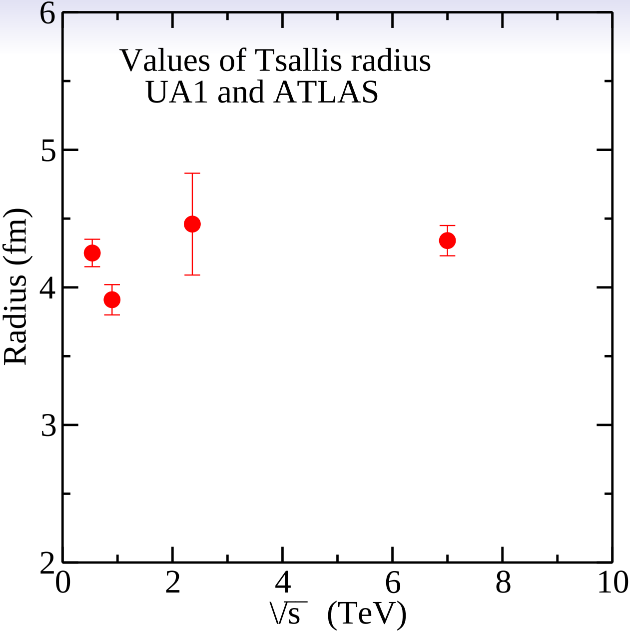


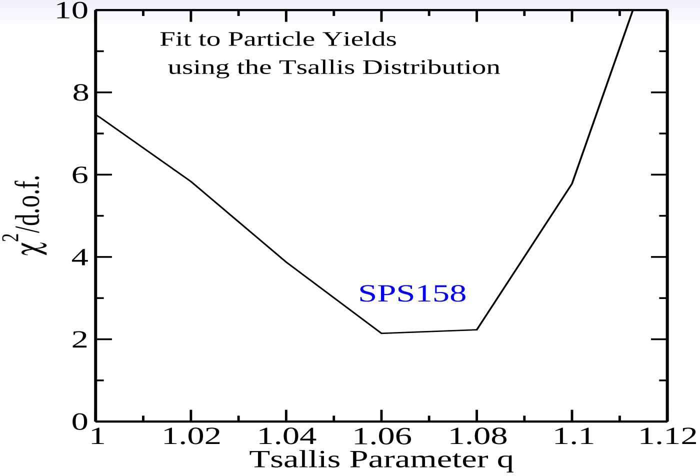
Open symbols: parameter *q* obtained from the ALIICE distribution. Closed symbols use Tsallis distribution.
L. Marques, E. Andrade-II and A. Deppman arXiv:1210.1725 [hep-ph]


Charged Hadrons
UA1 and ATLAS
14 TeV
7 TeV
2.36 TeV
0.9 TeV
0.54 TeV
$dN/2\pi p_T dp_T dy$
p_T (GeV)
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
1e-11
0
10
20
30
40
50


Values of the Tsallis q -parameter
UA1 and ATLAS
1.1961
1.1843
1.1726
1.161
1.1495
1.1381
1.1268
1.1157
1.1046
1.0937
1.0829
1
10
√s (TeV)


Values of the Tsallis temperature
UA1 and ATLAS
Temperature (MeV)
100
90
80
70
60
50
0
2
4
6
8
√s (TeV)


Values of Tsallis radius
UA1 and ATLAS
Radius (fm)
6
5
4
3
2
0
2
4
6
8
10
√s (TeV)



J. C., G. Hamar, P. Levai, S. Wheaton
Journal of Physics **G 36** (2009) 064018.

Conclusion:
Use

$$\frac{d^2N}{dp_T dy} = gV\frac{p_T m_T}{(2\pi)^2}\left[1+(q-1)\frac{m_T}{T}\right]^{-\frac{q}{q-1}}, \tag{1}$$

instead of

$$\frac{d^2N}{dp_T dy} = p_T \times \frac{\mathrm{d}N}{\mathrm{d}y}\frac{(n-1)(n-2)}{nT(nT+m_0(n-2))}\left[1+\frac{m_{\mathrm{T}}-m_0}{nT}\right]^{-n} \tag{2}$$