High frequency homogenization: Connecting the Microstructure to the Macroscale

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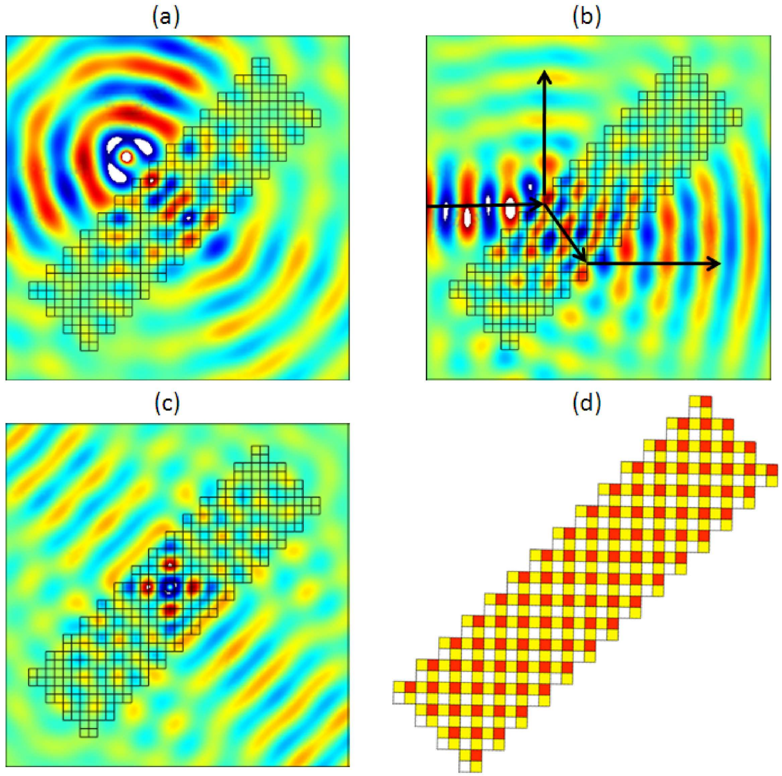
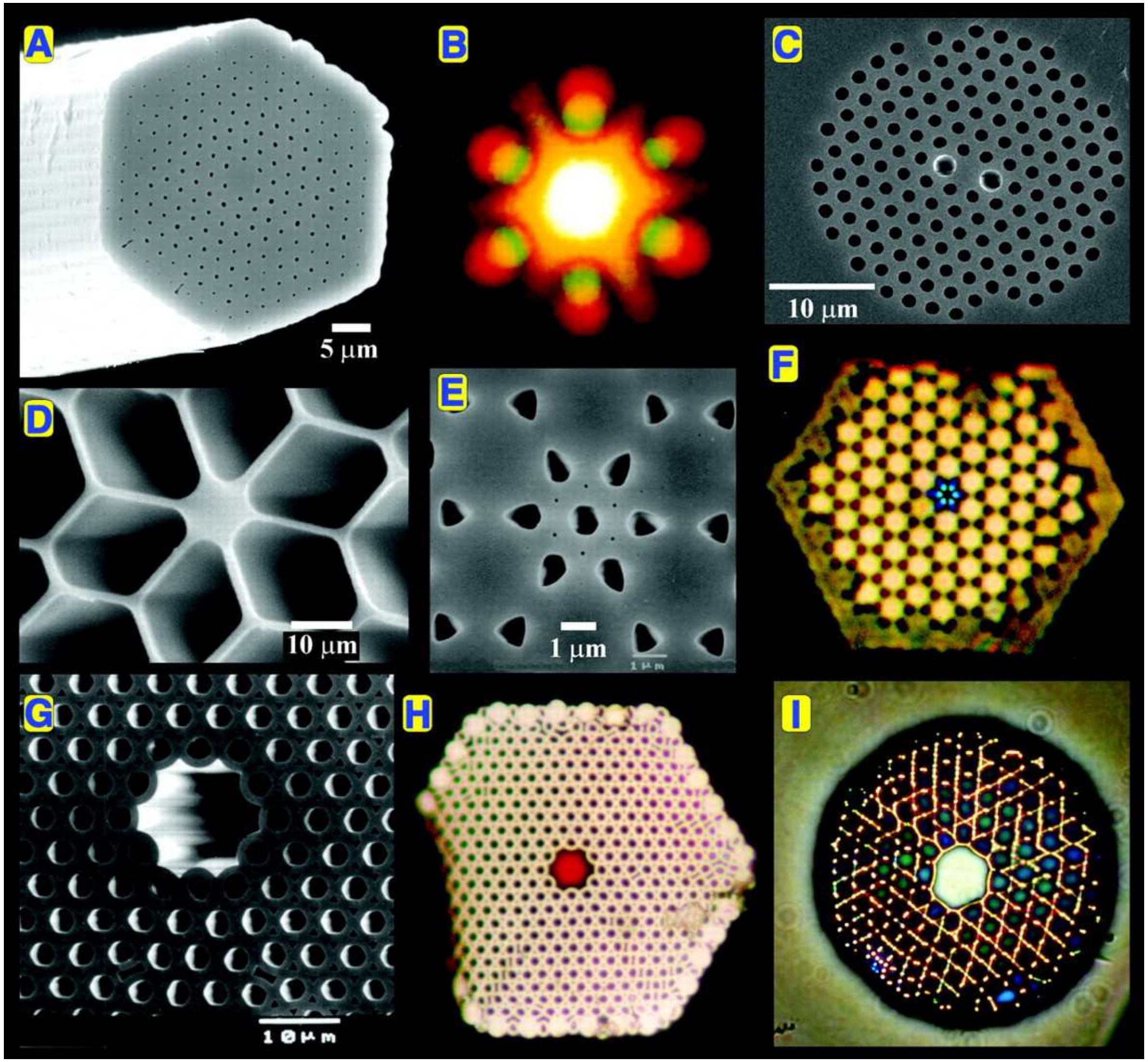
Key aim to replace a complex medium with a smeared out “effective” replacement.



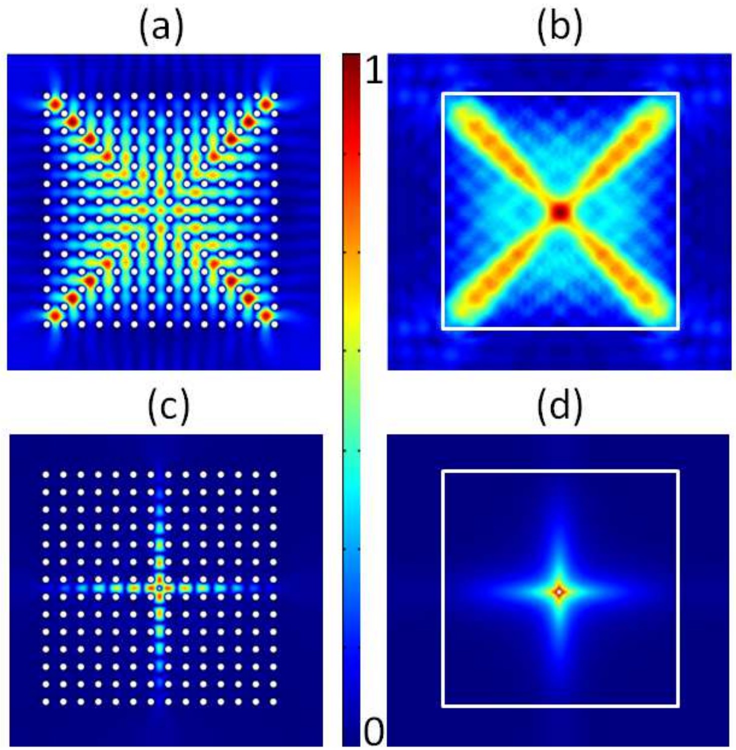
- Examples showing the range of effects that can be designed: Anisotropy, endoscope/ lensing, Dirac cones and cloaking, shielding.
- Homogenization theory: conventional theory valid for low frequencies and long waves. High frequency means multiple scattering and wavelength close to micro-scale. Need a new idea...
- Connect to discrete media.
- Concluding remarks.

A real structure: Photonic crystals (Optics)



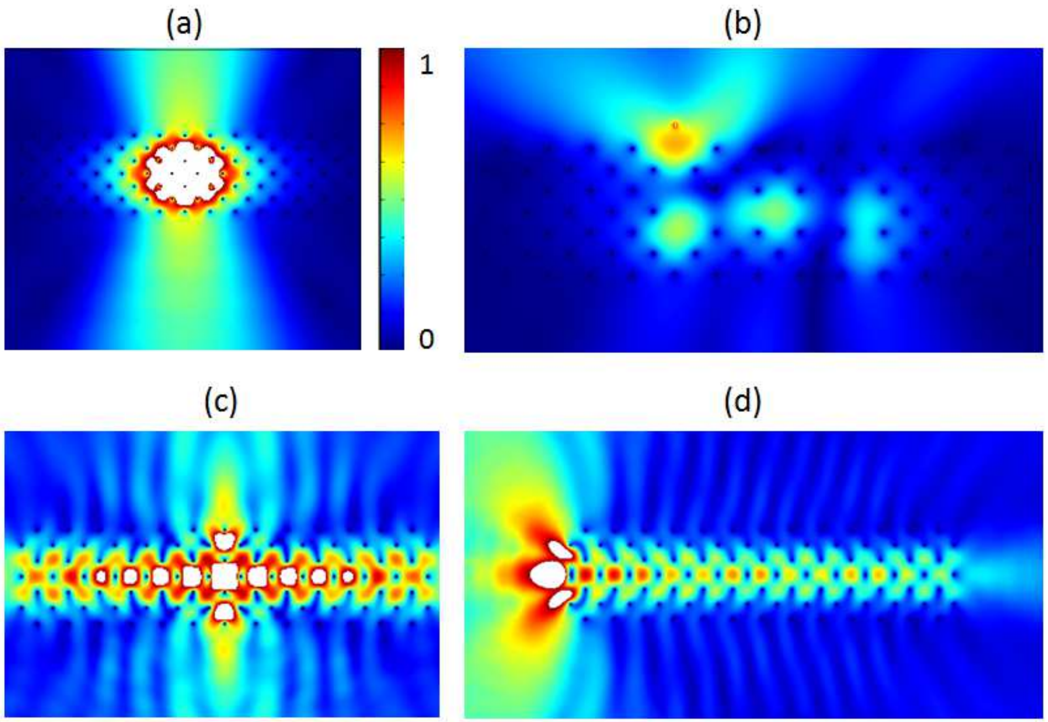
Micrographs of various Photonic Crystal Fibre structures taken from the review of Russell (Science 2003). The regular array of holes allow for excellent (low-loss) waveguides in optics and have a host of applications: sensors, high bandwidth guides, optical filters etc.

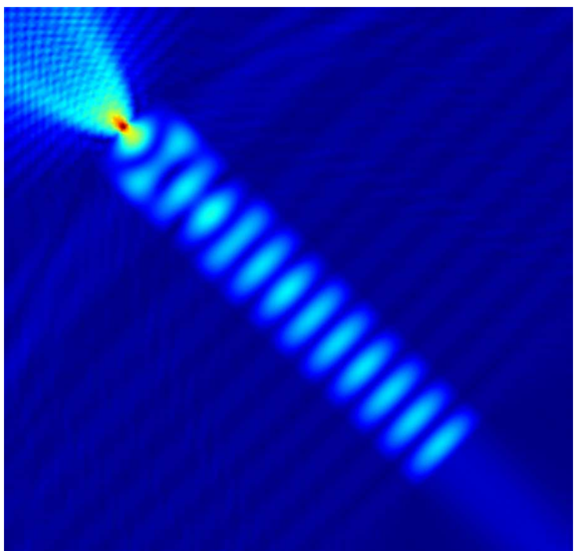
Zoology of features: Guiding waves along specific directions: anisotropy



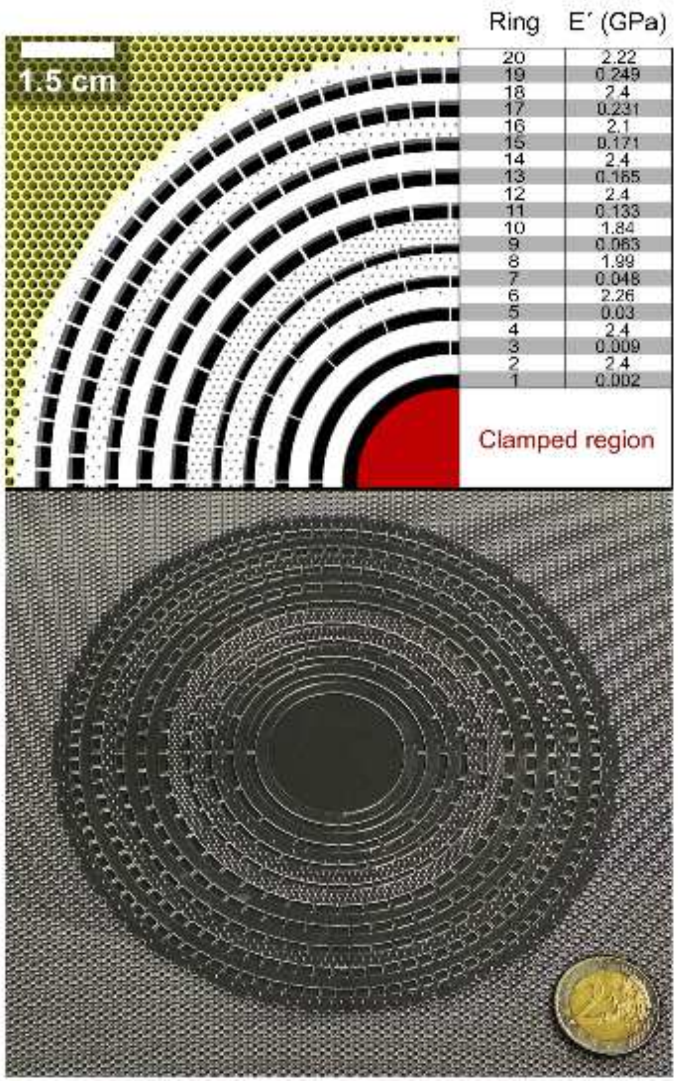
Left side full FE computations with point excitation at centre (two different frequencies) and right hand side the equivalent continuous material.
A large decrease in computing effort and increase in understanding.

Endoscope/ lensing





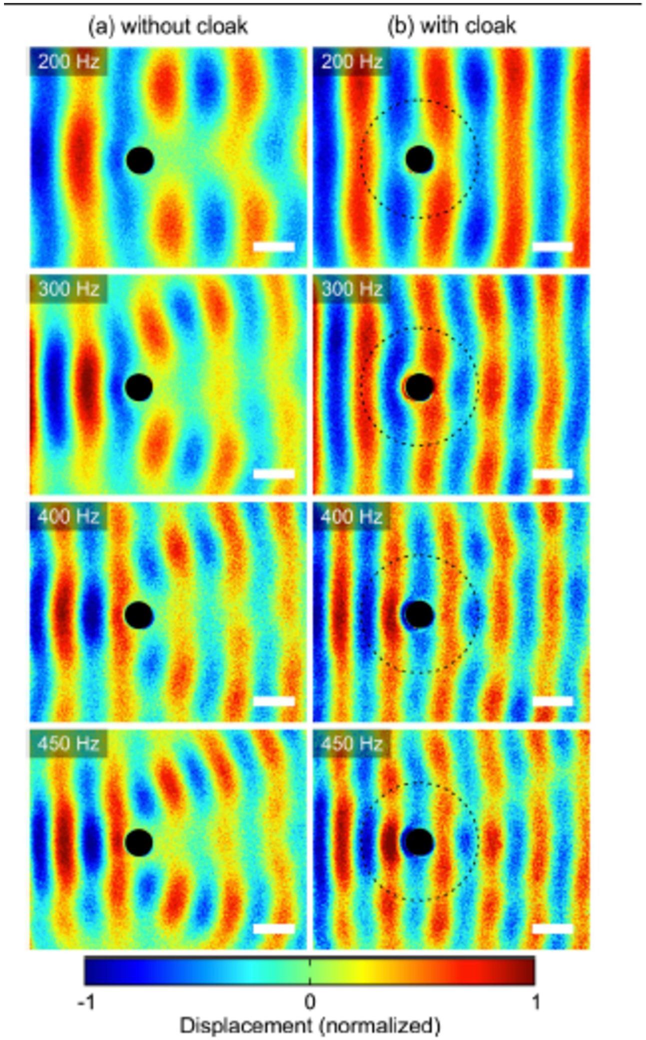
Shielding/ cloaking Experiments



Ring	E′ (GPa)
20	2.22
19	0.249
18	2.4
17	0.231
16	2.1
15	0.171
14	2.4
13	0.165
12	2.4
11	0.133
10	1.84
9	0.063
8	1.99
7	0.048
6	2.26
5	0.03
4	2.4
3	0.009
2	2.4
1	0.002

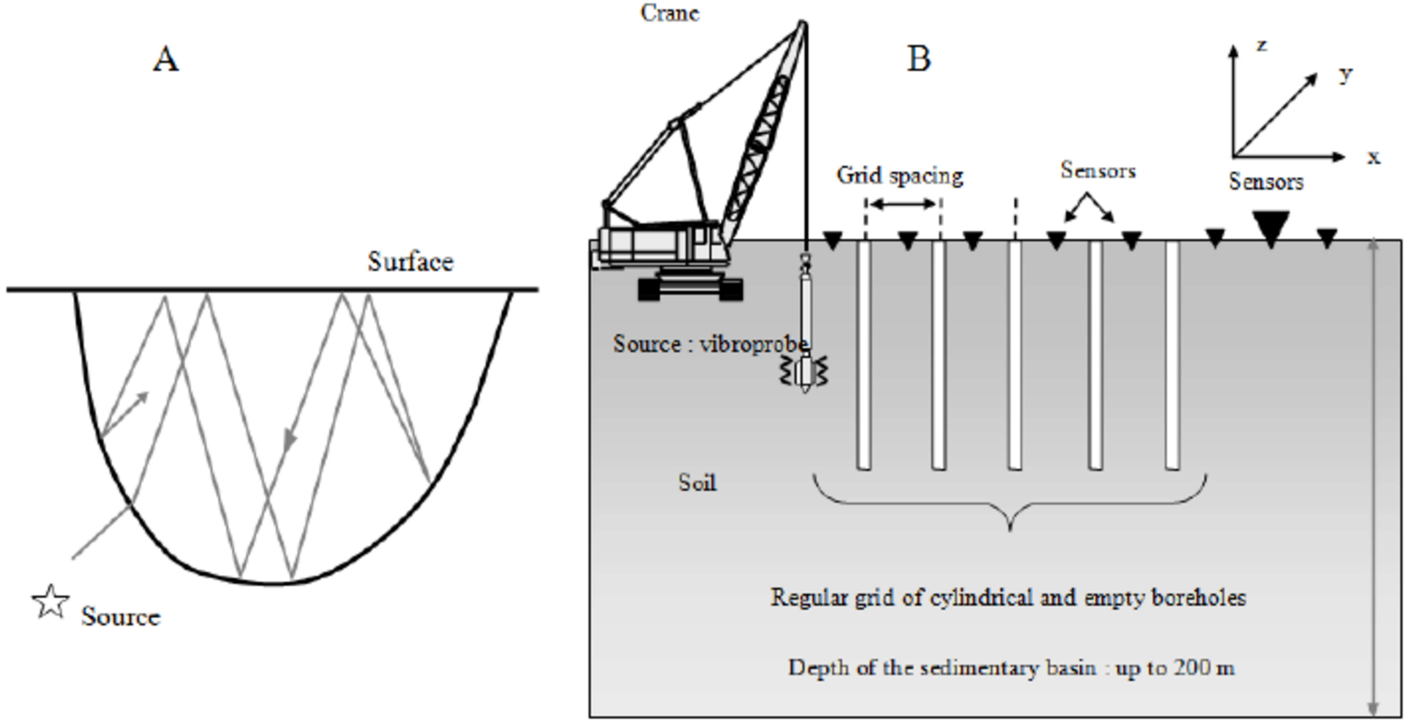
Based on a suggestion by Guenneau & coworkers (Farhat et al PRL 2009), Stenger et al (PRL 2012) have made a real elastic cloak! It works over a relatively broad range of frequencies.

Shielding/ cloaking



Taken from Stenger et al (PRL 2012).

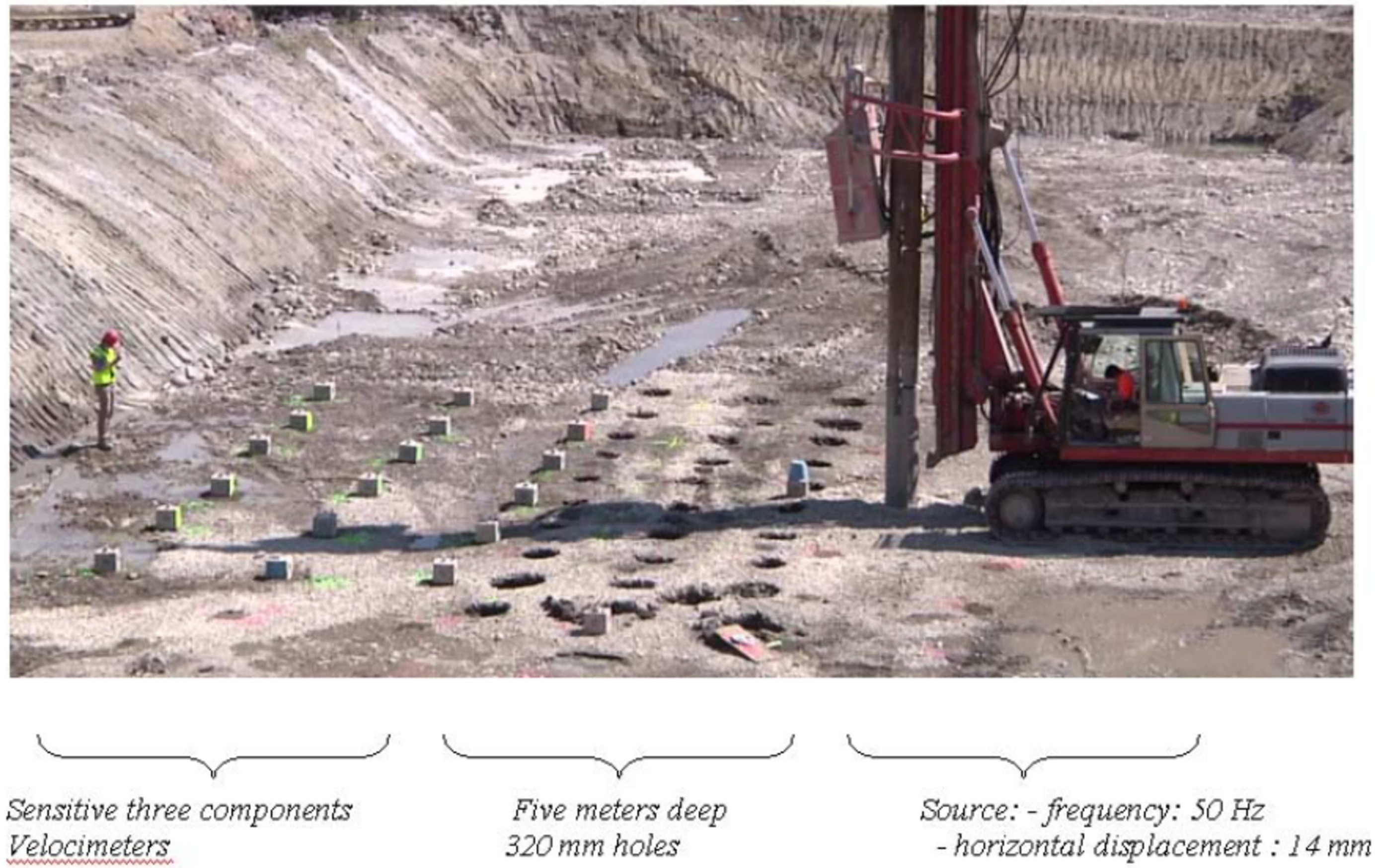
Seismic shield



Taken from Brule, Javelaud, Enoch & Guenneau.

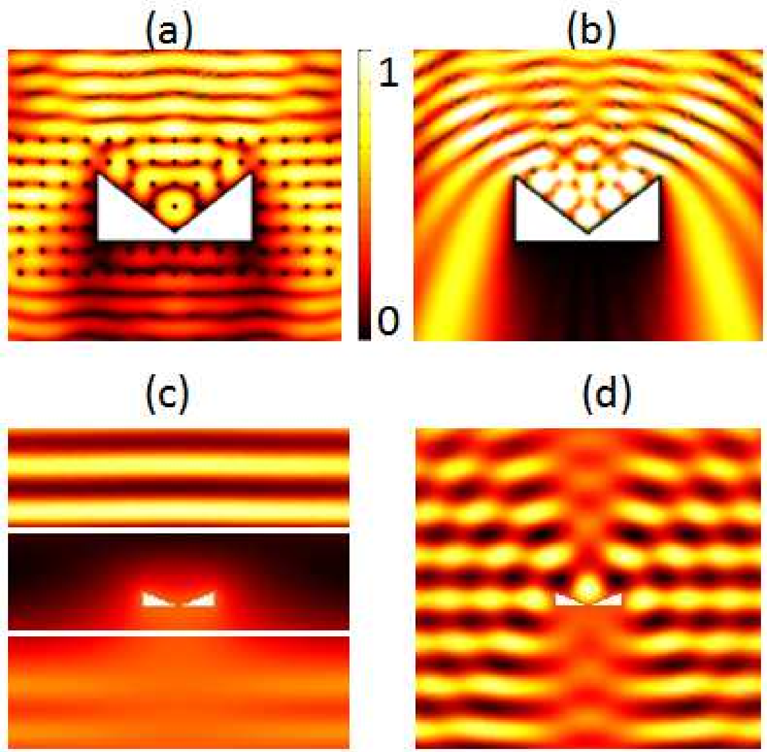
Large scale experiments

Fig. S1-2. Photograph of the seismic metamaterial experiment from the Ménard company.



Taken from Brule, Javelaud, Enoch & Guenneau. Engineering by Menard company.

Cloaking



Here an obstacle is placed within a structured medium and the waves pass through (almost) unhindered.

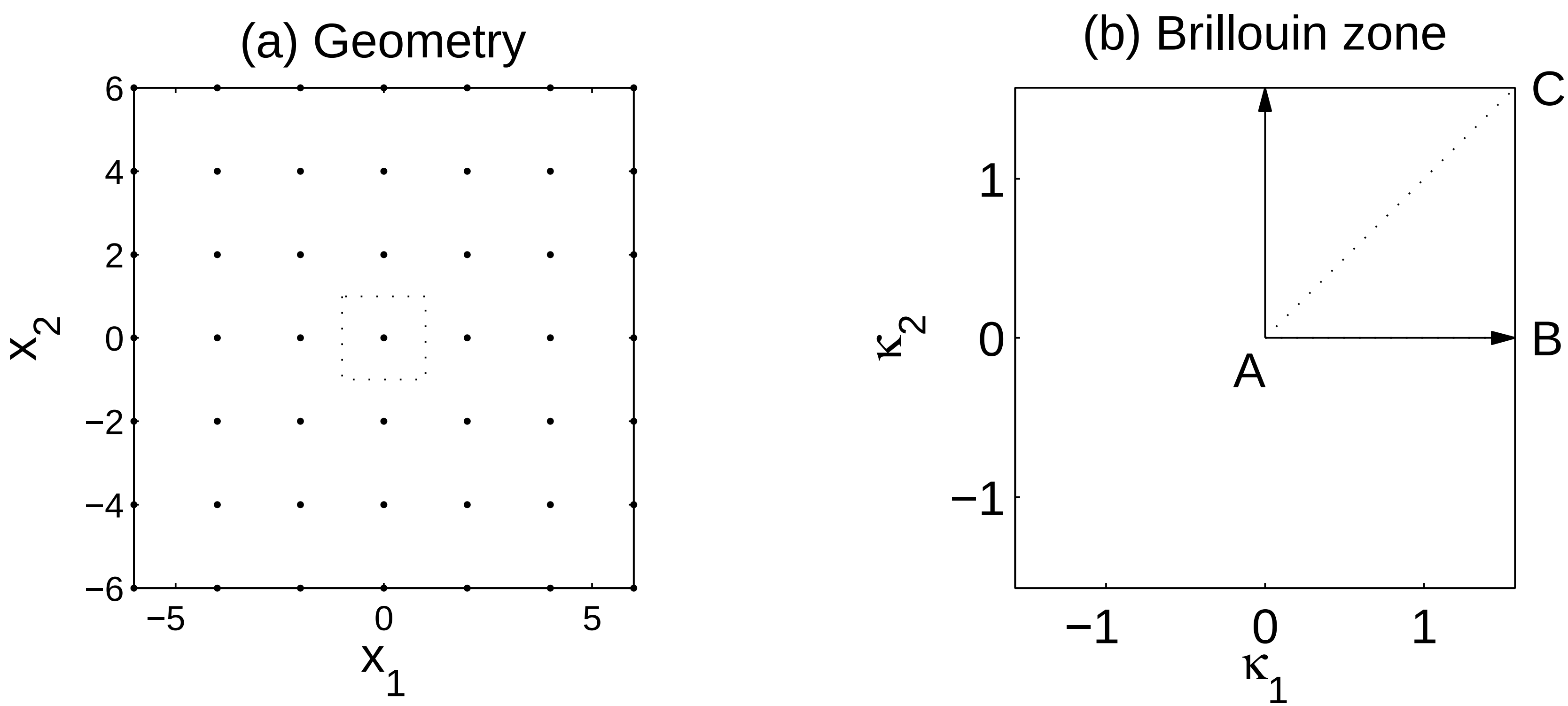
How does it work? How do we replace a structured medium with a continuous one that has the same/similar features? Let us digress to dispersion curves...

Elastic Plates - description of Brillouin zone

Now consider

$$\nabla^2\left(\beta\nabla^2 u\right) - \mu\omega^2\hat{u} = 0;$$

which is a fourth order equation for waves in elastic plates.



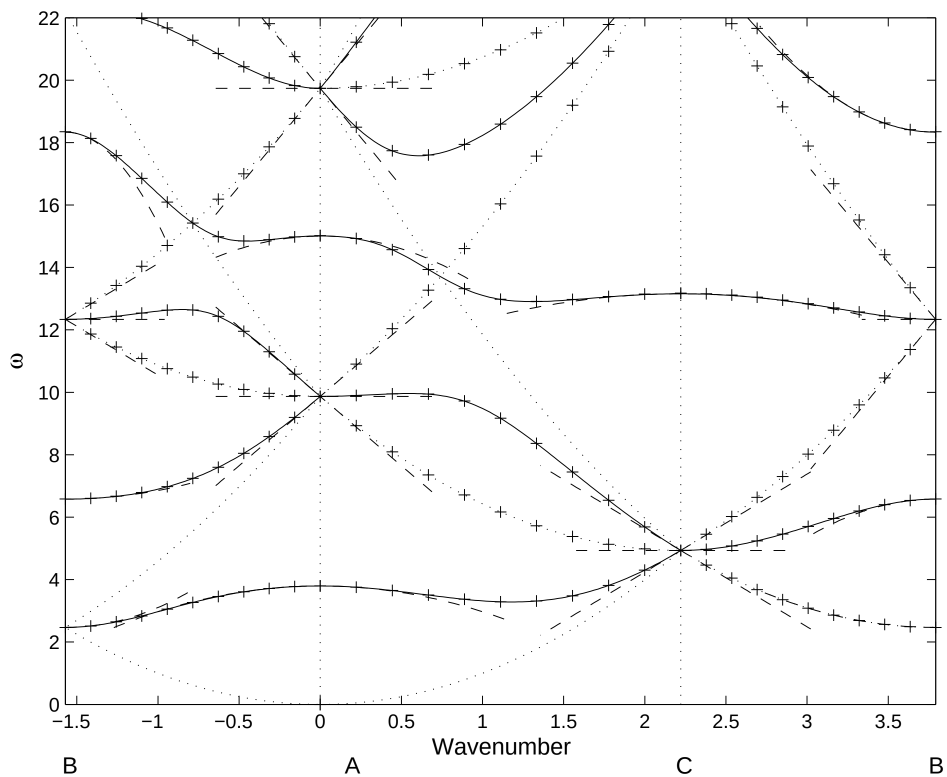
A doubly periodic simply supported plate (the dots represent the simple supports) in panel (a) with the elementary cell shown by the dotted lines and in (b) the irreducible Brillouin zone with the lettering for wavenumber positions shown.

Dispersion curves and their interpretation for the Physics

There is an exact dispersion relation

$$D(\kappa_1, \kappa_2, \omega) = \sum_{n_1, n_2} \frac{1}{[(\pi n_1 - \kappa_1)^2 + (\pi n_2 - \kappa_2)^2]^2 - \omega^2} = 0,$$

some complications due to singularities.



Everything can be understood from the dispersion curves... a model that captures the behaviour is HFH.

Homogenization theory

A huge research area with many thousands of articles, numerous books etc.
Almost all of this is either static or quasi-static: long wave and low frequency, so the wavelengths are much longer than the microscale.
For instance, taking a piecewise constant elastic string on $-\infty < x < \infty$

$$l^2 \frac{d^2 u}{dx^2} + \Omega^2 \frac{u}{c^2(\xi)} = 0, \qquad \text{with} \quad \Omega = \frac{\omega l}{\hat{c}_0}. \tag{0.1}$$

with $\xi = x/l$ and speed

$$c(\xi) = \begin{cases} 1/r & \text{for} \quad n \leq \xi < n+1 \\ 1 & \text{for} \quad n-1 \leq \xi < n \end{cases}$$

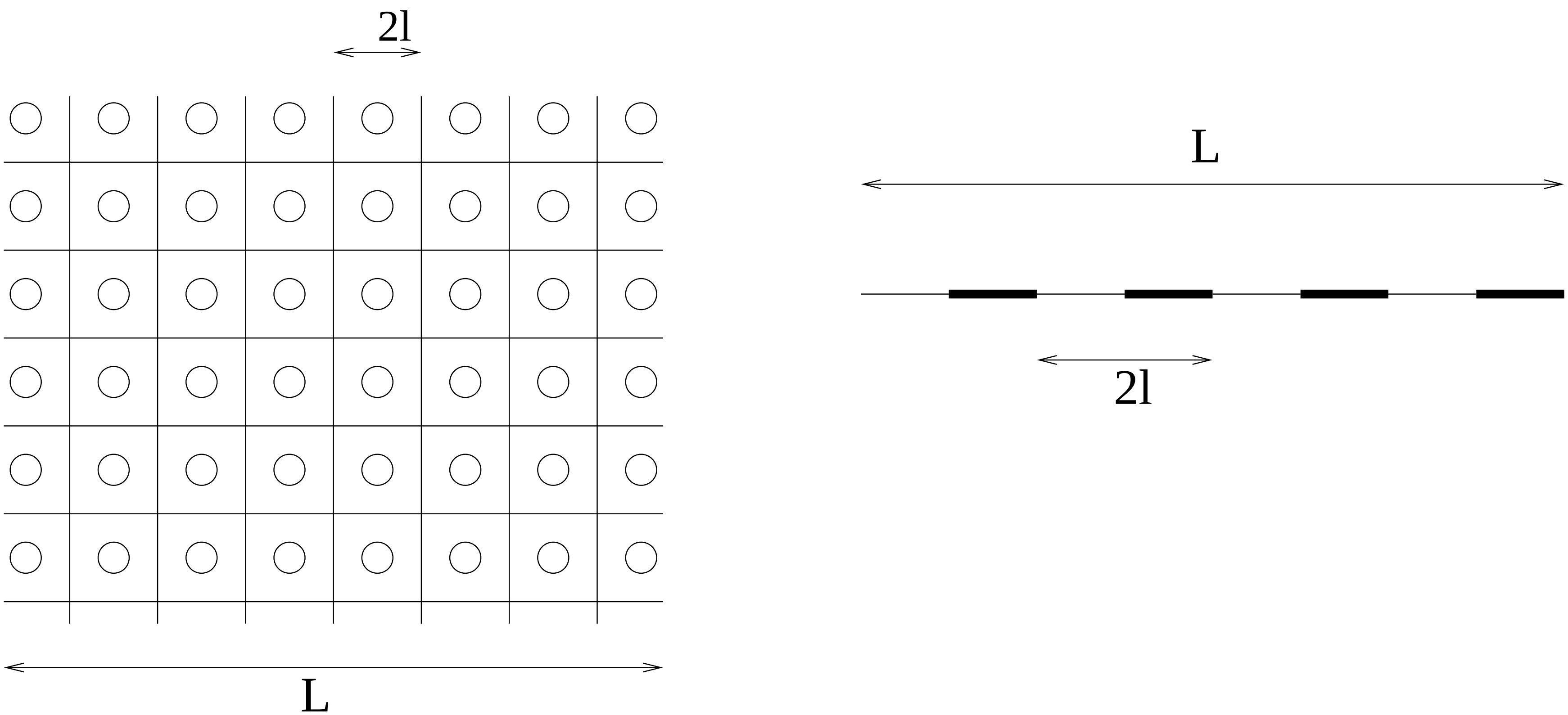
(n even). Then setting $u = u(\xi, X) = u_0(X) + ...$ ($X = x/L$ with L a long scale) and $\Omega^2 \sim \epsilon^2 \Omega_2^2$ where $\epsilon = l/L \ll 1$ gives (after some algebra..)

$$u_{0XX}(X) + u_0(X) \frac{\Omega_2^2}{2} \int_{-1}^{1} \frac{1}{c^2(\xi)} d\xi = 0.$$

One just replaces the inverse speed squared effectively by its “effective” speed which is just the average. Naturally attractive approach giving equations only on the long scale and details built into an average.
We need to get further than this!

General theory

Turn to a continuum with a double periodic microstructure - aim to generate a continuum description only on the macroscale. Then return to Bloch cases to verify model, then to non-periodic cases....



Consider a medium with two lengthscales L and l where $L \gg l$, set $\epsilon = l/L \ll 1$ for future use.
The microstructure is characterized by stiffness $\hat{a}(x_1/l, x_2/l)$ and density $\hat{\rho}(x_1/l, x_2/l)$ that are periodic on the microscale $\boldsymbol{\xi} = (x_1/l, x_2/l)$.

Consider a wave equation, for, say, SH waves in anti-plane elasticity with periodic density with time harmonic dependence $\exp(-i\omega t)$ assumed understood, as

$$l^2\nabla_{\mathbf{x}} \cdot [a(\boldsymbol{\xi})\nabla_{\mathbf{x}} u(\mathbf{x})] + \Omega^2 \rho(\boldsymbol{\xi}) u(\mathbf{x}) = 0 \quad \text{with} \quad \Omega = \frac{\omega l}{\hat{c}_0}$$

with $\hat{c}_0 = \sqrt{\hat{a}_0/\hat{\rho}_0}$ a characteristic wave speed.
Adopt a multiple scales approach treating the disparate lengthscales $\mathbf{X} = \mathbf{x}/L$, and $\boldsymbol{\xi} = \mathbf{x}/l$ as new independent variables to get

$$\nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}} u(\mathbf{X},\boldsymbol{\xi})] + \Omega^2 \rho(\boldsymbol{\xi}) u(\mathbf{X},\boldsymbol{\xi})$$

$$+\epsilon[2a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}} + \nabla_{\boldsymbol{\xi}} a(\boldsymbol{\xi})] \cdot \nabla_{\mathbf{X}} u(\mathbf{X},\boldsymbol{\xi}) + \epsilon^2 a(\boldsymbol{\xi})\nabla_{\mathbf{X}}^2 u(\mathbf{X},\boldsymbol{\xi}) = 0$$

As noted when looking at Bloch waves there are standing waves that are locally periodic-periodic on the microscale - then $u(\mathbf{X},\boldsymbol{\xi})$ periodic in $\boldsymbol{\xi}$, but not necessarily in $\mathbf{X}$.

$$u|_{\xi_1=1} = u|_{\xi_1=-1}, \qquad u|_{\xi_2=1} = u|_{\xi_2=-1},$$

$$u_{\xi_1}|_{\xi_1=1} = u_{\xi_1}|_{\xi_1=-1}, \qquad u_{\xi_2}|_{\xi_2=1} = u_{\xi_2}|_{\xi_2=-1}.$$

Asymptotic theory

Adopt the ansatz:

$$u(\mathbf{X}, \boldsymbol{\xi}) = u_0(\mathbf{X}, \boldsymbol{\xi}) + \epsilon u_1(\mathbf{X}, \boldsymbol{\xi}) + \epsilon^2 u_2(\mathbf{X}, \boldsymbol{\xi}) + \ldots, \quad \Omega^2 = \Omega_0^2 + \epsilon\Omega_1^2 + \epsilon^2\Omega_2^2 + \ldots$$

Each $u_i(\mathbf{X}, \boldsymbol{\xi})$ for $i = 1, 2\ldots$, is periodic in $\boldsymbol{\xi}$.
Importantly, this is not limited to $\Omega^2 \ll 1$ as in classical homogenization for which

$$u(\mathbf{X}, \boldsymbol{\xi}) = u_0(\mathbf{X}) + \ldots, \quad \Omega^2 = \epsilon^2\Omega_2^2 + \ldots$$

Now solve order-by-order in ϵ.
At leading order

$$\nabla_{\boldsymbol{\xi}} \cdot [a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}} u_0] + \Omega_0^2\rho(\boldsymbol{\xi})u_0 = 0$$

A discrete spectrum of eigenvalues Ω_0^2 for which there is no phase shift across the structure and standing wave is formed. Solution is (simple eigenvalue)

$$u_0(\mathbf{X}, \boldsymbol{\xi}) = f_0(\mathbf{X})U_0(\boldsymbol{\xi}, \Omega_0) \tag{0.2}$$

where $U_0(\boldsymbol{\xi}, \Omega_0)$ is a periodic function of $\boldsymbol{\xi}$, is known as is Ω_0. $f_0(\mathbf{X})$ is unknown and varies only on the macroscale.

Continuum equation

Some algebra.... to second order ... then

$$\nabla_{\boldsymbol{\xi}}\cdot[a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}u_2]+\Omega_0^2\rho(\boldsymbol{\xi})u_2 = -a(\boldsymbol{\xi})U_0\nabla_{\mathbf{X}}^2 f_0-[2a(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}}+\nabla_{\boldsymbol{\xi}}a(\boldsymbol{\xi})]\cdot\nabla_{\mathbf{X}}u_1-\Omega_2^2\rho(\boldsymbol{\xi})f_0U_0$$

contains both $f_0(\mathbf{X})$ and the eigenvalue correction, Ω_2^2. Invoking an orthogonality condition, integrating over the cell yields an eigenvalue problem for f_0 and Ω_2^2 as the homogenized partial differential equation

$$T_{ij}\frac{\partial^2 f_0}{\partial X_i \partial X_j}+\Omega_2^2 f_0=0, \quad \text{with} \quad T_{ij}=\frac{t_{ij}}{\iint_S \rho(\boldsymbol{\xi})U_0^2 dS} \quad \text{for} \quad i,j=1,2.$$

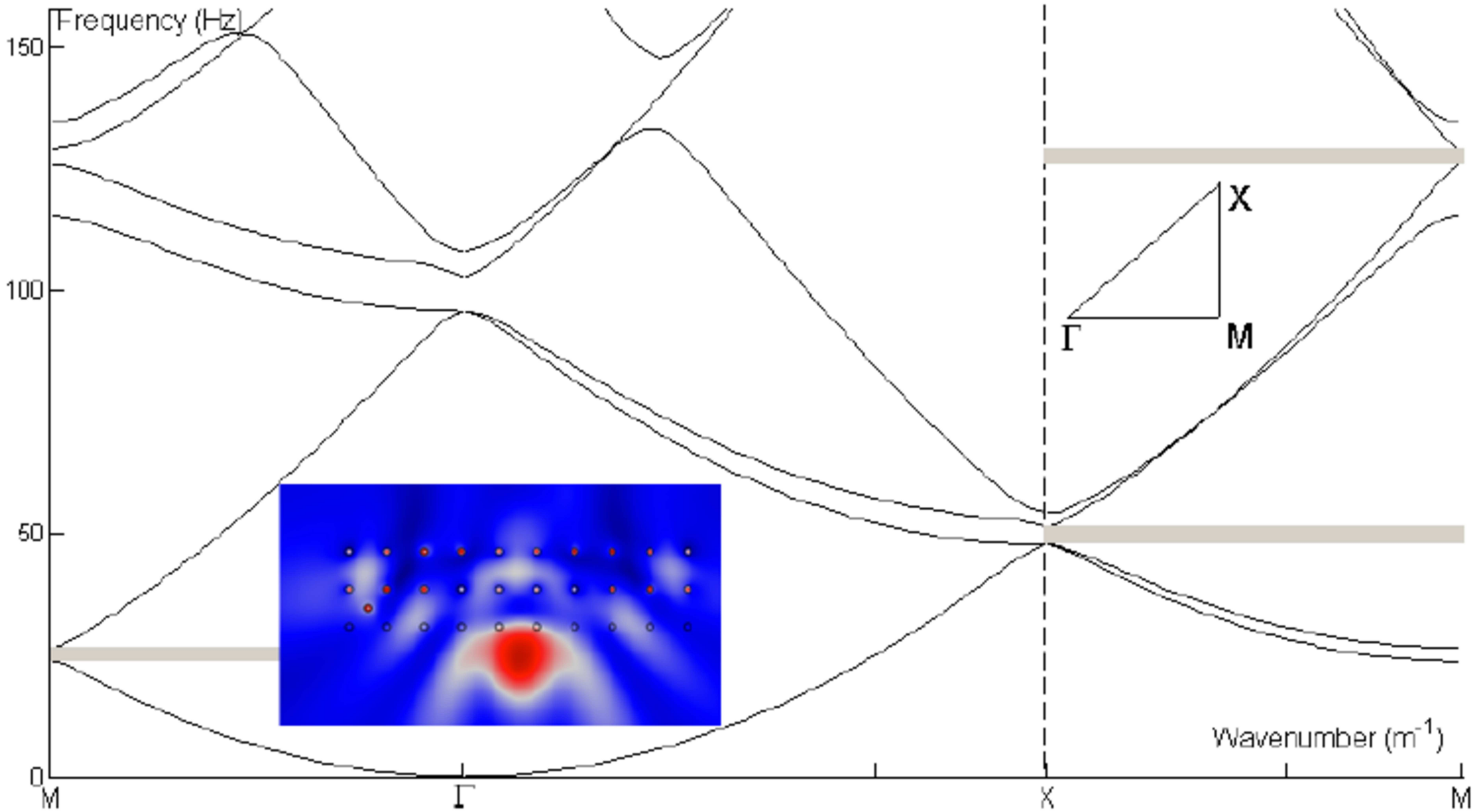
$$t_{11}=-2\int_{-1}^{1}[a(\boldsymbol{\xi})U_0^2]_{\xi_1=1}d\xi_2+\iint_S(2a(\boldsymbol{\xi})V_{1\xi_1}^{(1)}+a_{\xi_1}(\boldsymbol{\xi})V_1^{(1)})U_0 dS,$$

$$t_{12}=t_{21}=\frac{1}{2}\iint_S\left(2a(\boldsymbol{\xi})(V_{1\xi_2}^{(1)}+V_{1\xi_1}^{(2)})+a_{\xi_2}(\boldsymbol{\xi})V_1^{(1)}+a_{\xi_1}(\boldsymbol{\xi})V_1^{(2)}\right)U_0 dS,$$

$$t_{22}=-2\int_{-1}^{1}[a(\boldsymbol{\xi})U_0^2]_{\xi_2=1}d\xi_1+\iint_S(2a(\boldsymbol{\xi})V_{1\xi_2}^{(2)}+a_{\xi_2}(\boldsymbol{\xi})V_1^{(2)})U_0 dS.$$

This is entirely on the macroscale with the microstructure built in through integrated quantities. Thus the medium is “homogenized”, but at high frequencies.

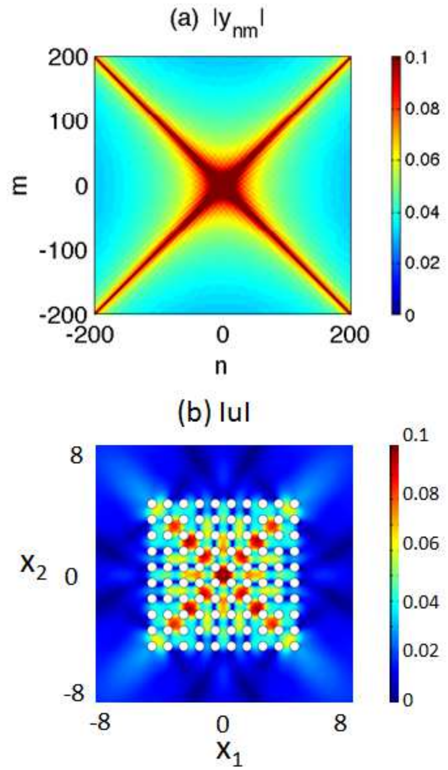
Seismic shielding again

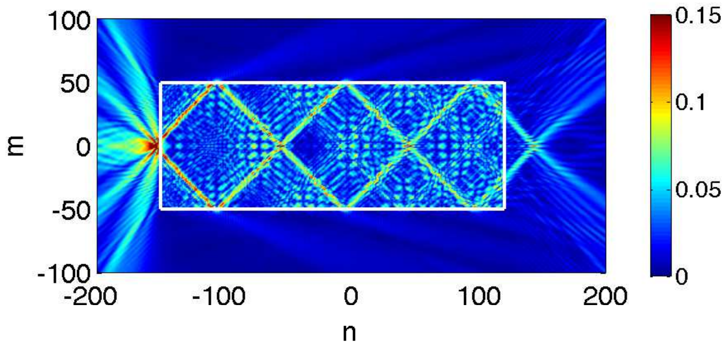


Note that the dispersion relations tell us where/ how to design the structure.

Discrete systems

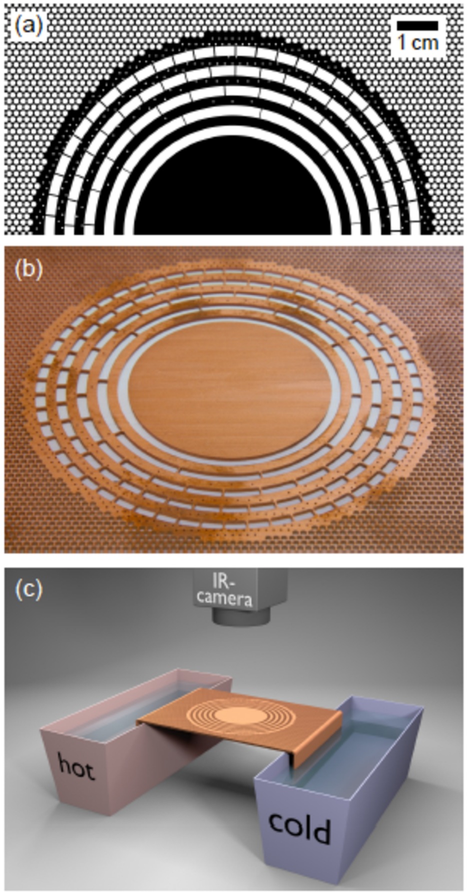
Worth noting that similar ideas work for discrete systems



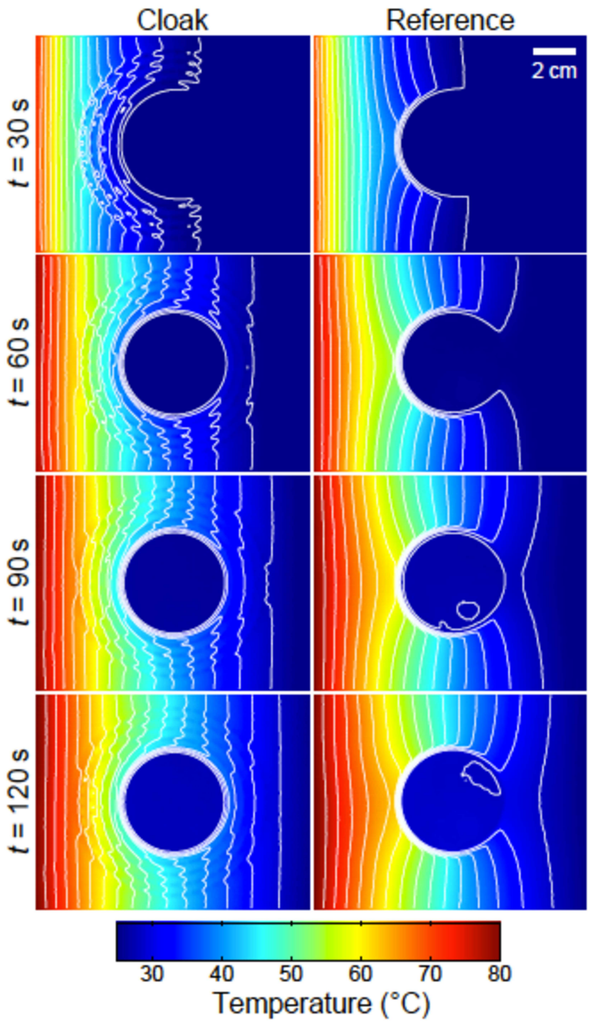


Thermal cloaking

Can one protect delicate instrumentation through a thermal shield or a thermal cloak using similar ideas!?



Experiments



Taken from Schittney, Kadic, Guenneau and Wegener 2013

Concluding remarks

- A versatile theory for high frequency homogenization now exists for: continuous media, lattice frames, discrete mass/ spring models. In fact for anything that is periodic or nearly so.
- Gives physical insight - new classes of (relevant) spectral problems. An area of applied analysis involves spectral theory and embedded modes.
- Enables the design of smart materials or smart surfaces to shield, cloak or redirect waves.
- Extensions to other microstructures: hexagonal/ triangular etc all possible. Nonlinear, stochastic etc etc Importantly to diffusion/ thermal effects and to elasticity. Extend the plates etc.

The homogenization method is in Craster, Kaplunov and Pichugin (Proc R Soc Lond A, 2010) & for discrete media in Craster, Kaplunov & Postnova (QJMAM, 2010). For nets in Nolde, Craster, Kaplunov (JMPS 2011) and for optics in Craster, Kaplunov, Nolde and Guenneau (JOSA 2011 & Wave Motion 2012).
Elastic plates / Maxwell in Antonakakis & Craster (Proc R Soc Lond 2012, 2013) and for dispersion X anisotropy in Craster, Antonakakis, Guenneau (Phys Rev B 2012).
The thermal shielding is in Schittny, Kadic, Guenneau & Wegener (2013).
Elastic plate experiments for cloaking in Stenger, Wilhelm and Wegener (PRL, 2012) and theory in Farhat, Guenneau, Enoch (PRL 2009) .