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# Integral Forms in String Theory and Supersymmetric Theories


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# 1. Overview on Integral Forms

# Integration on Manifolds

The usual integration on standard manifolds can be viewed as follows

There is a map between

$$\omega \in \Omega^\bullet(\mathcal{M}) \Longrightarrow \mathcal{C}^\infty\left(\widehat{\mathcal{M}}\right) = \Omega^\bullet(\mathcal{M})$$

by identifying the forms with anticommuting variables

$$\boxed{dx^i \to \theta^i}$$

Then, on the new space we can integrate functions $\widehat{\omega} \in \mathcal{C}^\infty\left(\widehat{\mathcal{M}}\right)$

Since there is a natural measure $\widehat{\mu} = dx^{i_1} \wedge \ldots \wedge dx^{i_n} \wedge d\theta^{i_1} \wedge \ldots \wedge d\theta^{i_n}$

which gives

$$\boxed{\int_{\widehat{\mathcal{M}}} \widehat{\omega} = \int_{\mathcal{M}} \omega}$$

# Complexes of Superforms

Given 1-superforms, we have commuting variables $d\theta^i \wedge d\theta^j = d\theta^j \wedge d\theta^i$

So the complex

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \dots \xrightarrow{d} \Omega^n \xrightarrow{d} \dots$$

is infinite. There is no top form.

- How can we define the top form?
- How can we define a sensible integration theory?
- How can we define a cohomological theory for the differential d?

It is convenient to introduce a new basic object

$$\delta(d\theta^i)$$

# The new object has the following properties

If we denote by $dx^I$ the 1-forms associated to the commuting coordinates of the manifold

$$dx^I \wedge dx^J = -dx^J \wedge dx^I, \qquad dx^I \wedge d\theta^j = d\theta^j \wedge dx^I,$$
$$d\theta^i \wedge d\theta^j = d\theta^j \wedge d\theta^i, \qquad \delta(d\theta) \wedge \delta(d\theta') = -\delta(d\theta') \wedge \delta(d\theta),$$
$$d\theta\,\delta(d\theta) = 0, \qquad d\theta\,\delta'(d\theta) = -\delta(d\theta).$$

the last three equations follow from the usual definition of the Dirac delta function and from the changing-variable formula for Dirac delta's

$$\delta(ax+by)\delta(cx+dy) = \frac{1}{\mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\delta(x)\delta(y)$$

# Then, we can introduce the new set of forms

$$dx^{[K_1} \cdots dx^{K_l]} d\theta^{(i_{l+1}} \cdots d\theta^{i_r)} \delta(d\theta^{[i_{r+1}}) \cdots \delta(d\theta^{i_{r+s}]})$$

# and the complexes

$$\cdots \xrightarrow{d} \Omega^{(r|q)} \xrightarrow{d} \Omega^{(r+1|q)} \cdots \xrightarrow{d} \Omega^{(p+1|q)} \xrightarrow{d} \cdots$$

Form Number

Picture Number

$$\Omega^{(p+1|q)}$$

Notice that the picture number corresponds to the number of Delta functions, while the form number corresponds to the total form degree. It can be negative by considering the derivatives of Delta functions.

The integral is given by integrating the Dirac delta functions and then performing the Berezin integral

$$\boxed{\int_{\mathbb{C}^{(p+1|q)}} \omega^{(p+1|q)} = \epsilon^{i_1 \cdots i_q} \partial_{\theta^{i_1}} \cdots \partial_{\theta^{i_q}} \int_{\mathbb{C}^{p+1}} f(x, \theta)}$$

# The complete complex

$$
\begin{tikzcd}
 & 0 \arrow[r, "\mathrm{d}"] & \Omega^{(0|0)} \arrow[r, "\mathrm{d}"] \arrow[d, shift left, "Y"] & \cdots & \Omega^{(r|0)} \arrow[d, shift left, "Y"] & \cdots \arrow[r, "\mathrm{d}"] & \Omega^{(m|0)} \arrow[r, "\mathrm{d}"] \arrow[d, shift left, "Y"] & \Omega^{(m+1|0)} \arrow[d, "Y"] & \cdots \\
 & \vdots \arrow[u, "Z"] & \vdots \arrow[u, shift left, "Z"] & & \vdots \arrow[u, shift left, "Z"] & & \vdots \arrow[u, shift left, "Z"] & \vdots & \\
\cdots & \Omega^{(-1|s)} \arrow[r, "\mathrm{d}"] & \Omega^{(0|s)} \arrow[r, "\mathrm{d}"] & \cdots & \Omega^{(r|s)} & \cdots \arrow[r, "\mathrm{d}"] & \Omega^{(m|s)} \arrow[r, "\mathrm{d}"] & \Omega^{(m+1|s)} & \cdots \\
 & \vdots & \vdots \arrow[d, shift left, "Y"] & & \vdots \arrow[d, shift left, "Y"] & & \vdots \arrow[d, shift left, "Y"] & \vdots \arrow[d, "Y"] & \\
\cdots & \Omega^{(-1|n)} \arrow[r, "\mathrm{d}"] \arrow[u, "Z"] & \Omega^{(0|n)} \arrow[r, "\mathrm{d"}] \arrow[u, shift left, "Z"] & \cdots & \Omega^{(r|n)} \arrow[u, shift left, "Z"] & \cdots \arrow[r, "\mathrm{d}"] & \Omega^{(m|n)} \arrow[r, "\mathrm{d}"] \arrow[u, shift left, "Z"] & 0 &
\end{tikzcd}
$$

Top Form

where there exist the two operators

$$Y_{1\dots n} \stackrel{\text{def}}{=} \prod_{\alpha=1}^{n} Y_\alpha : \mathcal{H}^{(\cdot|0)} \longrightarrow \mathcal{H}^{(\cdot|n)},$$

$$Z_{1\dots n} \stackrel{\text{def}}{=} \prod_{\alpha=1}^{n} Z_\alpha : \mathcal{H}^{(\cdot|n)} \longrightarrow \mathcal{H}^{(\cdot|0)}.$$

moving vertically in the complex of forms $\Longrightarrow$ PCO's

# 2. Integral Forms in String Theory

# RNS string theory

It is based on supersymmetry on the worldsheet parametrized by a set of bosonic and fermionic coordinates $(z, \psi)$ and it is coupled to worldsheet supergravity. This requires a gauge fixing for the worldsheet metric and for the worldsheet gravitinos leading to usual (b,c) ghosts and $(\beta, \gamma)$ ghosts.

The ghosts c and $\gamma$ can be viewed as the differential dz and $d\psi$, and in the computation of RNS amplitudes, one need to insert appropriate operators in order to saturate the ghost anomalies. In the case of commuting ghosts $(\beta,\gamma)$, that requires the introduction of PCO written in terms of Dirac delta functions $\delta(\gamma)$ and $\delta(\beta)$, which can be unerstood geometrically by integral forms of the type $\delta(d\psi)$.

A complete analysis has been recently performed by E. Witten (see **arXiv:1209.2199**, **arXiv:1209.2459** and **arXiv:1209.5461**) and it is based on the correct manipulations of integral forms and Stokes theorem in the case of supermanifolds.

# Pure Spinor String Theory

We start from the fields $x^m, \theta^\alpha, p_\alpha$ ($\theta^\alpha$ are Majorana-Weyl spinors in $d = (9,1)$) and the free field action

$$S = \int d^2z \Big( \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta^\alpha + \hat{p}_\alpha \partial \hat{\theta}^\alpha \Big)$$

*i)* The total conformal charge is $c_T = (10)_x + (-32)_{p,\theta}$

*ii)* Inserting $p_\alpha = p^*_\alpha \equiv \frac{1}{2} \partial x_m \gamma^m_{\alpha\beta} \theta^\beta + \frac{1}{8} (\gamma^m_{\alpha\beta} \theta^\beta)(\theta \gamma_m \partial \theta)$ in $S$,

$$S\big|_{p=p^*} = S_{Green-Schwarz}$$

*iii)* So, $d_\alpha \equiv p_\alpha - p^*_\alpha \approx 0$ must be identified with the fundamental constraint.

$$\boxed{Q = \oint dz \lambda^\alpha d_\alpha}$$

## By requiring the nilpotency of Q

$$\{Q,Q\} = \oint \lambda^\alpha(z) \oint \lambda^\beta(w) d_\alpha(z) d_\beta(w) = \oint \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta \Pi_m = 0$$

$$\boxed{\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0}$$

Pure Spinor Constraints

The space of the zero modes for the ghost fields for the Pure Spinor String theory is

- Conifold Space with Base SO(10)/U(5)
- It is a local Calabi-Yau with conifold singularity
- Integration Measure for this space?

# Viewed from String Amplitude computations:

Vertex operators: $\int dz d\bar{z} \mathcal{V}^{(0,0)}_{z\bar{z}}$ $\oint dz \mathcal{V}^{(1,0)}_{z}$, $\oint d\bar{z} \mathcal{V}^{(0,1)}_{\bar{z}}$ and $\mathcal{U}^{(1,1)}$

$$\boxed{\langle\langle \quad \mathcal{U}^{(1,1)}_1 \mathcal{U}^{(1,1)}_2 \mathcal{U}^{(1,1)}_3 \prod_{j=1}^{n} \int dz_j d\bar{z}_j \mathcal{V}^{(0,0)}_j \quad \rangle\rangle}$$

To compute these amplitudes we need to perform the OPE contractions until we get to the following expression

$$\langle \mathcal{M} \rangle = \int d^{16}\theta_0 d^{10} x_0 \mathcal{D}\lambda_0 \mu(\theta_0, \lambda_0) \mathcal{M}(x_0, \theta_0, \lambda_0)$$

To integrate over these variables we need to define a suitable measure

for that we need a good understanding of the integration on supermanifolds

# Multiloop Amplitudes

The complete N-point g-loop amplitudes can be written in this way

$$\int_{\mathcal{M}_g} dm_i \int [d\mu_\lambda][d\theta][dx] \int \prod_{l=1}^{g} [d\mu_w][dd_l] \prod_{n=1}^{3g-3} \int \mu(z_n) b_B(z_n) \prod_{k=1}^{r} \mathcal{U}^{(1)}(z_k) \times$$
$$\prod_{i=3g-3+1}^{11g} Z(B_{mn}N_i^{mn}) Z(J_i) \prod_{j=1}^{11} Y_{C^j} \prod_{p=1}^{N} \int d\tau_p \mathcal{V}_p^{(0)}$$

where we used

$$Z(X) = \Big[Q, \Theta(X)\Big] = \Big[Q, X\Big]\delta(X)$$

Picture Raising Operators

$$Y_{C^i} = C^i_\alpha \theta^\alpha \delta(C^i_\alpha \lambda^\alpha)$$

Picture Lowering Operators

$$\boxed{\Big\{Q, b_X\Big\} = Z(X)T}$$

# Viewed from String Field Theory perspective

$$S = \int d^{10}x_0 d^{16}\theta_0 \mathcal{D}\lambda_0 \mu(\lambda_0, \theta_0) \Big( \Phi^{(1)} Q^{(1)} \Phi^{(1)} + \dots \Big)$$

Properties:

1) BRST invariance
2) SUSY
3) Saturation of zero modes
(bosonic and fermionic zero modes)

The result is the following

$$\mathcal{D}\lambda_0 = d\lambda^{\alpha_1} \wedge \dots \wedge d\lambda^{\alpha_{11}} \epsilon_{\alpha_1 \dots \alpha_{16}} (\gamma^m \gamma^n \gamma^p \gamma_{mnp})^{[\alpha_{12} \dots \alpha_{16}](\beta_1 \dots \beta_3)} \frac{\partial}{\partial \lambda^{\beta_1}} \dots \frac{\partial}{\partial \lambda^{\beta_3}}$$

$$\mu(\lambda_0, \theta_0) = \prod_{i=1}^{11} (C^i_\alpha \theta^\alpha) \delta(C^i_\alpha \lambda^\alpha)$$

# Picture Changing Operators and Supergeometry

$$\mu(\lambda_0, \theta_0) = \prod_{i=1}^{11} (C^i_\alpha \theta^\alpha) \delta(C^i_\alpha \lambda^\alpha)$$

1) -- BRST invariant
2) -- Any change of the gauge paramenters $C^i_\alpha$: $\delta_{C_\alpha} \mu = \{Q, \theta^\alpha C_\beta \theta^\beta \delta'(C_\gamma \lambda^\gamma)\}$
3) -- It changes the picture, but
   what is the picture in this context?

Given a vertex operator $\mathcal{U}^{(n)} = \lambda^{\alpha_1} \dots \lambda^{\alpha_n} A_{\alpha_1 \dots \alpha_n}(x, \theta)$

$$(and \ \ \{Q, \theta^\alpha\} = \lambda^\alpha)$$

Using the usual "dictionary" $Q \leftrightarrow d$ and $\lambda^\alpha \leftrightarrow d\theta^\alpha$ we have

$$\boxed{\mathcal{U}^{(n)} = d\theta^{\alpha_1} \dots d\theta^{\alpha_n} A_{\alpha_1 \dots \alpha_n}(x, \theta)}$$

# Definition of PCO

On a supermanifold $\mathcal{M}^{(n|m)}$

Cartan Calculus

$$d = d\theta^\alpha D_\alpha + (dx^m + \theta\gamma^m d\theta)\partial_m$$

Even/Odd Vector fields:

$$v = v^\alpha D_\alpha + v^m \partial_m$$

with $v^\alpha$ odd/even, $v^m$ even/odd

Even $\quad \iota_v \,, \qquad \iota_v^2 = 0 \,, \qquad \mathcal{L}_v = d\iota_v + \iota_v d$

Odd $\quad \iota_{\tilde{v}} \,, \qquad \iota_{\tilde{v}}^2 \neq 0 \,, \qquad \mathcal{L}_{\tilde{v}} = d\iota_{\tilde{v}} - \iota_{\tilde{v}} d$

## Finally,

$$\delta(\iota_{\tilde{v}}) = \int_{-\infty}^{\infty} dt\, e^{it\iota_{\tilde{v}}}$$

$$\Gamma_{\tilde{v}} = \Big[d, \Theta(\iota_{\tilde{v}})\Big]$$

# 3. Application to Topological Theories with Target Supermanifolds

# An application: Super-Chern-Simon theory in superfields

Using topological strings (or particles) moving on supermanifolds, we can construct the target space theory by usual Witten's type string field theory

$$S = \frac{1}{2}\langle \Psi, Q\,\Psi \rangle + \frac{1}{3}\langle \Psi, \Psi\Psi \rangle\,,$$

but we need to define the integration measure

$$\langle \Psi_1, \Psi_2 \rangle = \int \mu(x, \lambda, c, \theta)\,(\Psi_1 \Psi_2)$$

and the string field is defined as follows

$$\Psi = C + \boxed{\lambda^\alpha A_\alpha + c^m A_m} + \lambda^\alpha \lambda^\beta A^*_{\alpha\beta} + \lambda^\alpha c^m A^*_{\alpha m} + c^m c^n A^*_{[mn]}$$
$$+ \lambda^\alpha \lambda^\beta \lambda^\gamma C^*_{\alpha\beta\gamma} + \lambda^\alpha \lambda^\beta c^m C^*_{(\alpha\beta)m} + \lambda^\alpha c^m c^n C^*_{\alpha[mn]} + c^m c^n c^r C^*_{[mnr]}\,.$$

target space ghost

physical fields

# The measure is therefore

$$\langle F(x,\theta,\lambda,c)\rangle = \int d\mu_0 F(x,\theta,\lambda,c)\,,$$

$$d\mu_0 = \boxed{\Big(\epsilon^m{}_{nr} c^n c^r \gamma_m^{\alpha\beta}\partial_{\lambda^\alpha}\partial_{\lambda^\beta} - u(\theta\ \not{c}\,\partial_\lambda) + w\,\theta^2\Big)\delta^2(\lambda)}\, d^3x\, d^2\theta\, d^2\lambda\, d^3c$$

integral form

and from Witten's string field theory we get finally the SuperChern-Simon theory

$$S = \int d^3x \int d^2\theta \gamma^{m\alpha\beta} \mathrm{tr} X_{m\alpha\beta}\,,$$

$$X_{MNR} = \frac{1}{2} A_{[M} D_N A_{R\}} + \frac{1}{3} A_{[M} A_N A_{R\}} + A^*_{[MN} D_{R\}} C + C^*_{[MNR\}} C^2$$

# Embedding of a Kähler manifold into a SCY

A Calabi-Yau manifold is characterized by two ingredients: Kahler 2-form K and a holomorphic 3-form Ω, this can be embedded into a superCY by defining the two integral forms

$$\hat{K} = K_{m\bar{n}}\, \mathrm{d}z^m \wedge \mathrm{d}z^{\bar{n}} + \theta^\mu \theta^{\bar{\nu}}\, \delta(d\theta^\mu \wedge)\, \delta(d\theta^{\bar{\nu}} \wedge)$$

$$\hat{\Omega} = \Omega_{mnp}\, \theta^\mu \theta^\nu \theta^\rho \theta^\sigma\, \mathrm{d}z^m \wedge \mathrm{d}z^n \wedge \mathrm{d}z^p \wedge \\ \delta(\mathrm{d}\theta^\mu \wedge)\, \delta(\mathrm{d}\theta^\nu \wedge)\, \delta(\mathrm{d}\theta^\rho \wedge)\, \delta(\mathrm{d}\theta^\sigma \wedge)\,.$$

they are closed and $\hat{K} \wedge \hat{\Omega} = 0$ (the supermanifold ha dimension (3|4))

$$\frac{1}{35 \cdot (4!)^2} \int_{\hat{\mathcal{M}}} \underbrace{\hat{K} \wedge \cdots \wedge \hat{K}}_{7} = \int_{\hat{\mathcal{M}}} K \wedge K \wedge K \wedge \hat{U} = \int_{\mathcal{M}} K \wedge K \wedge K\,.$$

$$\int_{\hat{\mathcal{M}}} \hat{\Omega} \wedge \overline{\hat{\Omega}} = \int_{\hat{\mathcal{M}}} \Omega \wedge \overline{\Omega} \wedge \hat{U} = \int_{\mathcal{M}} \Omega \wedge \overline{\Omega}\,.$$

Extensive analysis is presented in two papers **arXiv:0712.2600** and **hep-th/0607243**

# 4. Application to Susy and Sugra

# Entropy Current

A convenient formulation for fluid dynamics is the co-moving coordinate formalism (Lagrangian approach) where a fluid in a d-dimensional spacetime is described by (d-1)-coordinates $\phi^I(x^I, t)$

One can define an entropy current

$$J^\mu = \frac{1}{d!}\,\epsilon^{\mu,\nu_1,\ldots,\nu_d}\epsilon_{I_1,\ldots,I_d}\partial_{\nu_1}\phi^{I_1}\ldots\partial_{\nu_d}\phi^{I_d}$$

1. It is conserved off-shell
2. Its Hodge dual is conserved on-shell (no dissipative fluids)
3. It is a1-form

$$d\,J^{(d)} = 0$$

see the papers Dubovsky, Hui, Nicolis, Son, or by Loganayangam for a applications and discussions

# Supersymmetric Description

We add to the theory some fermionic coordinates (the number depends upon the dimension of the spacetime, and we introduce the supersymmetric invariant quantity

$$\Pi = d\phi + \theta\, d\theta\,,$$

Invariant under the susy transformation

$$\delta\phi = -\epsilon\theta\,, \quad \delta\theta = \epsilon\,,$$

1. It is conserved off-shell
2. Its Hodge dual is conserved on-shell (no dissipative fluids)
3. It is a1-form
4. It is supersymmetric
5. It reduces to the bosonic expression

but $$d\Pi = d\theta \wedge d\theta\,,$$

The solution is found in the space of integral forms

$$\boxed{J = \Pi \wedge \delta(d\theta)}$$

it is easy to check that it is closed off-shell

$$dJ = d\Pi \wedge \delta(d\theta) = d\theta \wedge d\theta \wedge \delta(d\theta) = 0\,.$$

In generic d+1 dimensions, by suitably adjusting the number of fermions, we have the formula

$$\boxed{J^{(d|m)} = \frac{1}{d!}\epsilon_{I_1 \dots I_d}\Pi^{I_1} \wedge \cdots \wedge \Pi^{I_d} \bigwedge_{\alpha=1}^{m} \delta(d\theta^\alpha)}$$

it suggests a relation between the entropy and sub-supermanifold, the form $J^{(d|m)}$ is related to that submanifold

# Supergravity

The differential forms $\Pi^{I}$ and $d\theta^{\alpha}$ are the flat reduction of components of supervielbein. Therefore, it is natural to generalize the construction to supergravity

$$\boxed{J_{(3|4)} = \frac{1}{3!}\,\epsilon_{I_1\dots I_3} V^{I_1}\wedge V^{I_2}\wedge V^{I_3} \bigwedge_{\beta=1}^{4} \delta(\Psi^{\beta})}$$

Vielbein Gravitinos

to prove the conservation, we observe

$$\begin{aligned}
\nabla J_{(3|4)} &= \frac{\mathrm{i}}{4}\,\epsilon_{I_1 I_2 I_3}\,\bar\Psi\wedge\gamma^{I_1}\Psi\wedge V^{I_2}\wedge V^{I_3}\bigwedge_{\beta=1}^{4}\delta(\Psi^{\beta}) + \\
&+ \frac{2}{3}\,\epsilon_{I_1 I_2 I_3} V^{I_1}\wedge V^{I_2}\wedge V^{I_3}\wedge\left[\sum_{\beta=1}^{4}\delta'(\Psi^{\beta})\wedge\nabla\Psi^{\beta}\bigwedge_{\alpha\neq\beta}\delta(\Psi^{\alpha})\right] = 0
\end{aligned}$$

using the sugra equations

$$\begin{aligned}
dV^{a} + \omega^{a}{}_{b}\wedge V^{b} - \frac{i}{2}\,\bar\Psi\gamma^{a}\Psi &= 0 \\
\nabla\Psi = d\Psi - \frac{1}{4}\gamma^{ab}\omega_{ab}\Psi &= \rho_{ab}V^{a}\wedge V^{b} + \mathrm{L}_{a}\,\gamma_{5}\,\gamma^{ab}\Psi\wedge V^{b} + \left[(\mathrm{Re}\,S) + \mathrm{i}\gamma^{5}(\mathrm{Im}\,S)\right]\gamma^{a}\,\Psi\wedge V_{a}
\end{aligned}$$

# Extended Supegravity

by increasing the number of fermions extending the supersymmetry, we can try to generalize the previous formulas and we found the natural generalization

$$\boxed{J_{(3|4\mathcal{N})} = \frac{1}{3!}\,\epsilon_{I_1 \dots I_3} V^{I_1} \wedge V^{I_2} \wedge V^{I_3} \wedge \bigwedge_{A=1}^{\mathcal{N}} \bigwedge_{\alpha=1}^{2} \delta(\Psi_{A\alpha}) \wedge \delta(\Psi^{A\alpha})\,,}$$

by superdiffeomorphisms the current transforms as a Berezinian

# 5. Mathematical Aspects

# Čech cohomology of $\mathbb{P}^{1|1}$

- $\mathbb{P}^{1|1}$ is the simplest non-trivial example of super projective space. It is defined as usual as an algebraic variety by quotient of the complex superspace $\mathbf{C}^{2|1}$ with respect to a complex number different from zero. It can be cover by two patches

Transition functions from one patch to another

$$U_0 = \{[z_0; z_1] \in \mathbb{P}^1 : z_0 \neq 0\},$$
$$U_1 = \{[z_0; z_1] \in \mathbb{P}^1 : z_1 \neq 0\}.$$

$$\boxed{\Phi^*(\gamma) = \frac{1}{\widetilde{\gamma}}, \qquad \Phi^*(\psi) = \frac{\widetilde{\psi}}{\widetilde{\gamma}}.}$$

The change of patch reflects upon the following transformation

$$\Phi^* \delta^n(d\widetilde{\psi}) = \gamma^{n+1}\delta^n\,(d\psi) - \gamma^n \psi\, d\gamma\, \delta^{n+1}(d\psi).$$

# Results

For non-negative integer

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong \mathbb{C}^{4n+4},$$

$$\check{H}^0(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong 0.$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n|0}) \cong \mathbb{C}^{4n}$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{-n|1}) \cong 0,$$

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}.$$

Notice that $\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0})$ and $\check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1})$ have the same dimension.

$$\check{H}^1(\mathbb{P}^{1|1}, \Omega^{n+1|0}) \times \check{H}^0(\mathbb{P}^{1|1}, \Omega^{-n|1}) \to \check{H}^1(\mathbb{P}^{1|1}, \Omega^{1|1}) \cong \mathbb{C}$$

Defined in terms of the this pairing

$$\langle d\gamma\ (d\psi)^n, \delta^n(d\psi)\rangle = (-1)^n n!\, d\gamma\ \delta(d\psi),$$

$$\langle (d\psi)^{n+1}, d\gamma\ \delta^{n+1}(d\psi)\rangle = -\ (-1)^n (n+1)!\, d\gamma\ \delta(d\psi),$$

$$\langle d\gamma\ (d\psi)^n, d\gamma\ \delta^{n+1}(d\psi)\rangle = \langle\ (d\psi)^{n+1}, \delta^n(d\psi)\rangle = 0.$$

# Super de Rham cohomology

1. $d$ behaves as a differential on functions;
2. $d^2 = 0$;
3. $d$ commutes with $\delta$ and its derivatives, and so $d(\delta^{(k)}(d\psi)) = 0$.

*For $n \geq 0$, the holomorphic de Rham cohomology groups of $\mathbb{P}^{1|1}$ are as follows:*

$$H^{n|0}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H^{-n|1}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong \begin{cases} 0, & n > 0, \\ \mathbb{C}, & n = 0. \end{cases}$$

$$H^{1|1}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol}) \cong 0.$$

In this computation $H^{0|1}_{\mathrm{DR}}(\mathbb{P}^{1|1}, \mathrm{hol})$ is generated by the constant sheaf

$$\boxed{\psi\,\delta(d\psi)}$$

# Future Directions

- Integral Forms are a natural generalization of differential forms for supermanifolds
- Deformations of Super-Calabi-Yau and embedding of manifolds into supermanifold (applications to twistor string theory and Grassmannians)
- Supergravity with RR fields and geometrical formulation
- String Field Theory with Target Space Supersymmetry