

# Resurgence and Non-Perturbative Physics

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## Lecture 1

- ▶ motivation
- ▶ divergence of perturbation theory in QM
- ▶ basics of Borel summation
- ▶ the Bogomolny/Zinn-Justin cancellation mechanism

## Motivation

Resurgence:

- ‘new’ idea in mathematics
- goal: explore implications for physics

- ▶ unification of perturbation theory and non-perturbative physics
- ▶ applications to QM, QFT, Strings, ...
- ▶ consistent non-perturbative definition of asymptotically free QFT
- ▶ insight into localization
- ▶ analytic continuation of path integrals
- ▶ exponentially improved (‘exact’) semi-classical analysis

# Perturbation theory

- perturbation theory generally produces a divergent series
- semiclassical (WKB) expansions are generally divergent
- there is a lot of interesting physics encoded in these facts
- **perturbation theory has nontrivial ‘hidden’ structure**
- perturbation theory and non-perturbative physics are intricately entwined
- “resurgence” describes these inter-relations
- general mathematical approach to *instanton calculus*

Perturbation theory generally produces a divergent series

*Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever ... That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.*

*N. Abel, 1802 – 1829*

*The series is divergent; therefore we may be able to do something with it*

*O. Heaviside, 1850 – 1925*

# Perturbation theory works

QED perturbation theory:

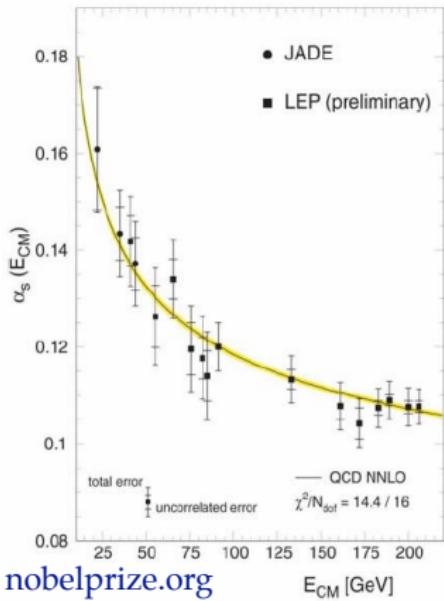
$$\frac{1}{2} (g - 2) = \frac{1}{2} \left( \frac{\alpha}{\pi} \right) - (0.32848...) \left( \frac{\alpha}{\pi} \right)^2 + (1.18124...) \left( \frac{\alpha}{\pi} \right)^3 - (1.7283(35)) \left( \frac{\alpha}{\pi} \right)^4 + \dots$$

$$\left[ \frac{1}{2} (g - 2) \right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

$$\left[ \frac{1}{2} (g - 2) \right]_{\text{theory}} = 0.001\,159\,652\,184\,42$$

QCD: asymptotic freedom

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2} \left( \frac{11}{3} N_C - \frac{4}{3} \frac{N_F}{2} \right)$$



# Resurgence

resurgence = unification of perturbation theory and non-perturbative physics

- cures inconsistencies in perturbative OR non-perturbative analyses
- series expansion  $\longrightarrow$  *trans-series* expansion
- trans-series well-defined under analytic continuation of parameter
- philosophical shift:  
*view semiclassical expansions as potentially exact*
- applications: ODEs, PDEs, QM, QFT, String Theory, ...

## Resurgent Trans-Series

- trans-series expansion:

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} a_{n,k,l} g^{2n} \left[ \exp\left(-\frac{S}{g^2}\right) \right]^k \left[ \log\left(-\frac{1}{g^2}\right) \right]^l$$

- J. Écalle (1980): set of functions with these trans-monomial elements is closed under:

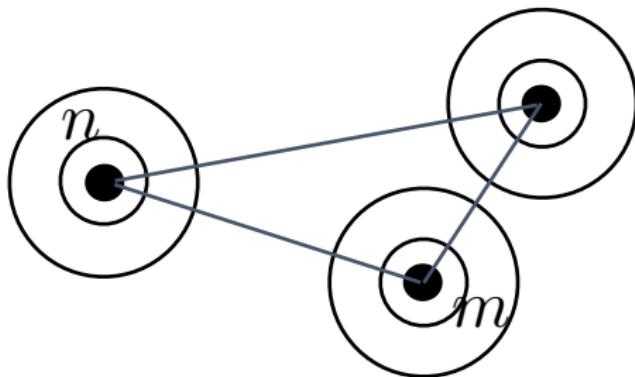
(Borel transform)+(analytic continuation)+(Laplace transform)

- “any reasonable function” has a trans-series expansion
- differential equations, iterated maps, ...
- trans-series expansion coefficients are highly correlated
- exponentially improved asymptotic expansions

# Resurgence

*resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities*

*J. Écalle, 1980*



# Divergence of perturbation theory in quantum mechanics

e.g. ground state energy:

$$E = \sum_{n=0}^{\infty} c_n (\text{coupling})^n$$

- ▶ cubic oscillator:  $c_n \sim -\frac{(60)^{n+1/2}}{(2\pi)^{3/2}} \Gamma(n + \frac{1}{2})$
- ▶ quartic oscillator:  $c_n \sim (-1)^{n+1} \frac{3^n \sqrt{6}}{\pi^{3/2}} \Gamma(n + \frac{1}{2})$
- ▶ Zeeman:  $c_n \sim (-1)^n \left(\frac{4}{\pi}\right)^{5/2} \frac{1}{\pi^{2n}} (2n + \frac{1}{2})!$
- ▶ Stark:  $c_n \sim -\frac{4}{\pi} \left(\frac{3}{2}\right)^{2n+1} (2n)!$
- ▶ periodic Sine-Gordon potential:  $c_n \sim n!$
- ▶ double-well:  $c_n \sim 3^n n!$

note generic factorial growth of perturbative coefficients

## Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n (x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \rightarrow 0 \quad , \quad N \rightarrow \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

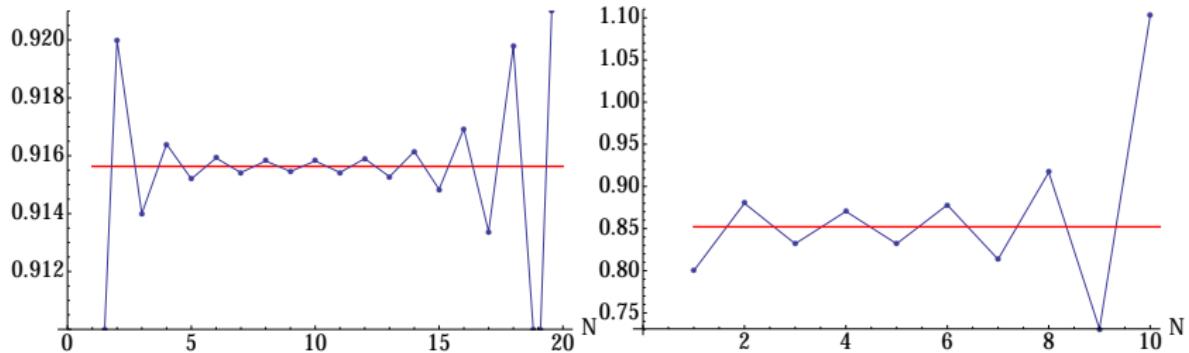
$$|R_N(x)| \ll |x - x_0|^N \quad , \quad x \rightarrow x_0 \quad , \quad N \quad \text{fixed}$$

→ “optimal truncation”:

truncate just before least term ( $x$  dependent!)

# Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty} (-1)^n n! x^n \sim \frac{1}{x} e^{\frac{1}{x}} E_1 \left( \frac{1}{x} \right)$$



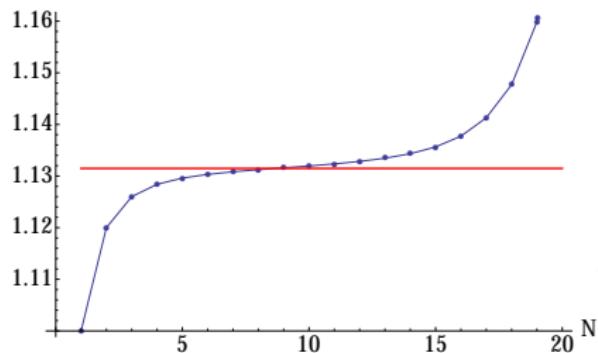
$$x = 0.1$$

$$x = 0.2$$

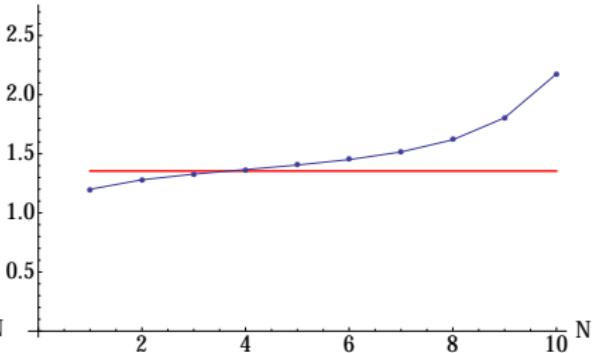
optimal order depends on  $x$ :  $N \approx \frac{1}{x}$

## Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty} n! x^n \sim \frac{1}{x} e^{-\frac{1}{x}} Ei\left(\frac{1}{x}\right)$$



$$x = 0.1$$



$$x = 0.2$$

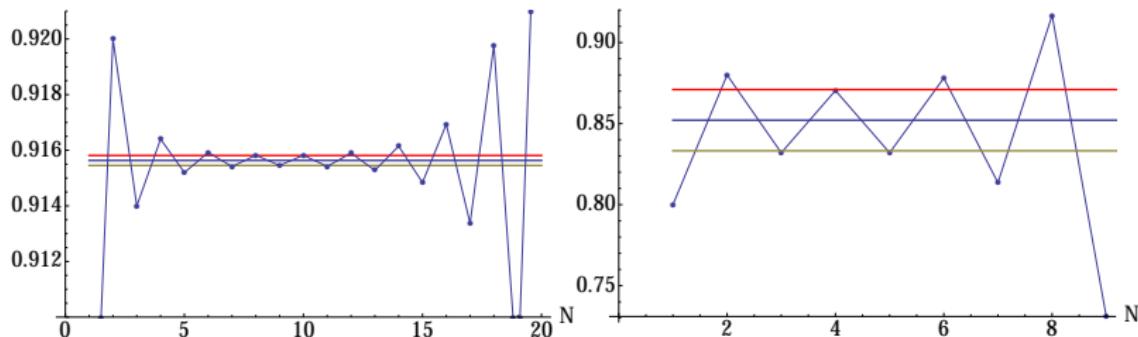
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## Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty} (-1)^n n! x^n \sim \frac{1}{x} e^{\frac{1}{x}} E_1\left(\frac{1}{x}\right)$$

optimal truncation: error term is exponentially small

$$|R_N(x)|_{N \approx 1/x} \approx N! x^N \Big|_{N \approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$



$$x = 0.1$$

$$x = 0.2$$

# Divergence of perturbation theory in quantum mechanics

typical large order growth:

$$c_n \sim (\pm 1)^n \beta^n \Gamma(\gamma n + \delta)$$

Related to factorial growth of number of Feynman diagrams

$$J = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2 - \frac{g}{4}x^4} dx = \sum_{n=0}^{\infty} J_n g^n$$

$$\Rightarrow J_n \sim (-1)^n \frac{(n-1)!}{4^n}$$

## Borel summation: basic idea

example: exponential integral function  
(<http://dlmf.nist.gov/6.2>)

$$\sum_{n=0}^{\infty} (-1)^n n! g^n = \frac{1}{g} e^{\frac{1}{g}} E_1 \left( \frac{1}{g} \right)$$

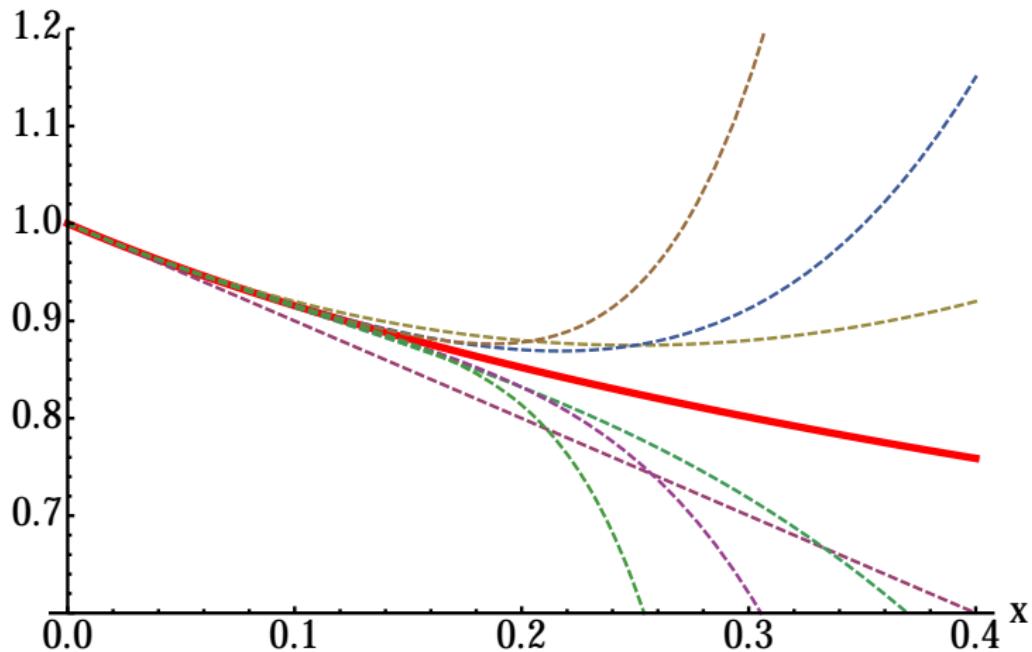
write  $n! = \int_0^\infty dt e^{-t} t^n$

$$\sum_{n=0}^{\infty} (-1)^n n! g^n = \int_0^\infty dt e^{-t} \frac{1}{1+gt} = \frac{1}{g} \int_0^\infty dt e^{-t/g} \frac{1}{1+t}$$

integral convergent for all  $g > 0$ : “Borel sum” of the series

## Borel Summation: basic idea

$$\sum_{n=0}^{\infty} (-1)^n n! x^n = \int_0^{\infty} dt e^{-t} \frac{1}{1 + x t}$$



## Borel summation: basic idea

example: non-alternating series:

$$\sum_{n=0}^{\infty} n! g^n = \frac{1}{g} e^{-\frac{1}{g}} Ei\left(\frac{1}{g}\right)$$

write  $n! = \int_0^\infty dt e^{-t} t^n$

$$\sum_{n=0}^{\infty} n! g^n = \int_0^\infty dt e^{-t} \frac{1}{1 - g t} = \frac{1}{g} \int_0^\infty dt e^{-t/g} \frac{1}{1 - t} \quad ???$$

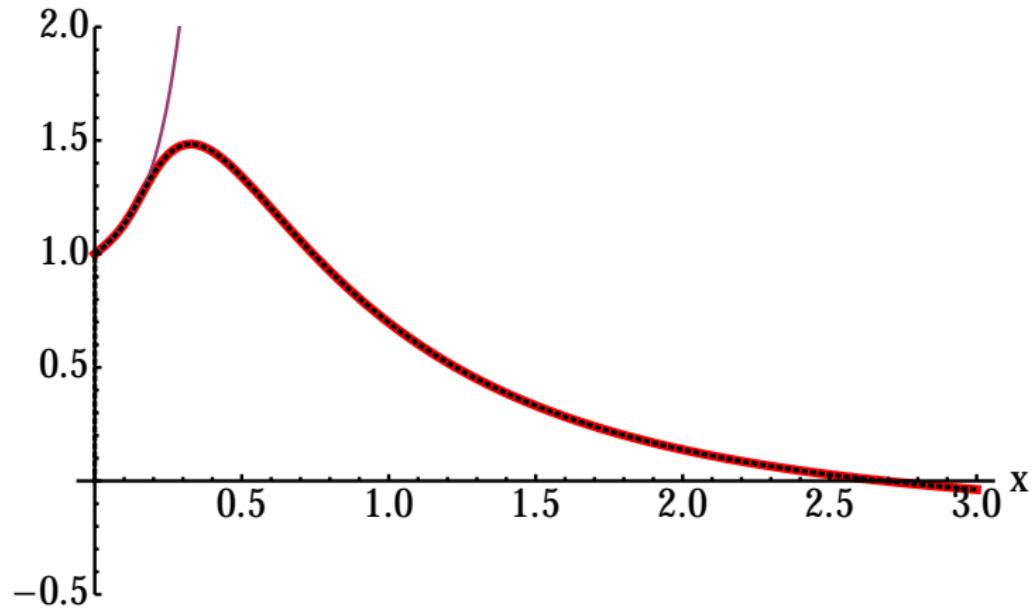
pole on the Borel axis!

⇒ non-perturbative imaginary part

$$\pm \frac{i\pi}{g} e^{-\frac{1}{g}}$$

## Borel Summation: Basic Idea

$$\text{Borel} \quad \Rightarrow \quad \mathcal{R}e \left[ \sum_{n=0}^{\infty} n! x^n \right] = \mathcal{P} \int_0^{\infty} dt e^{-t} \frac{1}{1 - x t} = \frac{1}{x} e^{-\frac{1}{x}} Ei \left( \frac{1}{x} \right)$$



## Borel summation

Borel transform of series  $f(g) \sim \sum_{n=0}^{\infty} c_n g^n$ :

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has finite radius of convergence.

Borel resummation of original asymptotic series:

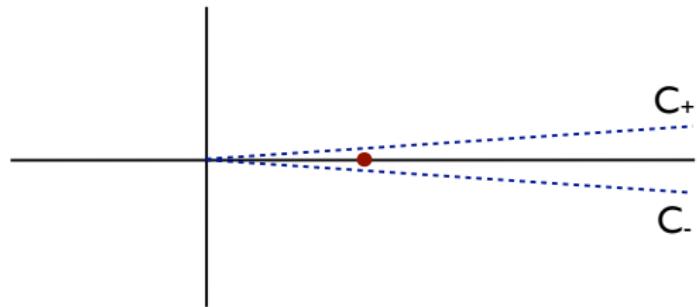
$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

warning:  $\mathcal{B}[f](t)$  may have singularities in (Borel)  $t$  plane

## Borel singularities

avoid singularities on  $\mathbb{R}^+$ : lateral Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g} \int_0^{e^{i\theta}\infty} \mathcal{B}[f](t) e^{-t/g} dt$$



go above/below the singularity:  $\theta = 0^\pm$

→ non-perturbative ambiguity:  $\pm \text{Im}[\mathcal{S}_0 f(g)]$

challenge: use physical input to resolve ambiguity

# Divergence of perturbation theory in quantum mechanics

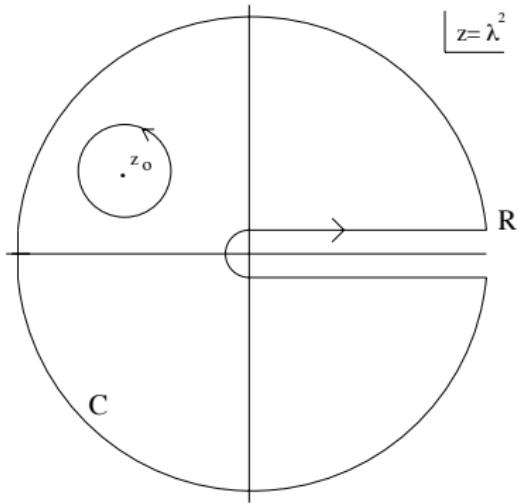
e.g. ground state energy:

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- ▶ double-well:  $c_n \sim 3^n n!$

note generic factorial growth of perturbative coefficients

# Borel Summation and Dispersion Relations



$$\begin{aligned} E(z_0) &= \frac{1}{2\pi i} \oint_C dz \frac{E(z)}{z - z_0} \\ &= \frac{1}{\pi} \int_0^R dz \frac{\text{Im } E(z)}{z - z_0} \\ &= \sum_{n=0}^{\infty} z_0^n \left( \frac{1}{\pi} \int_0^R dz \frac{\text{Im } E(z)}{z^{n+1}} \right) \end{aligned}$$

$$\text{WKB} \Rightarrow \text{Im } E(z) \sim \frac{a}{\sqrt{z}} e^{-b/z} \quad , \quad z \rightarrow 0$$

$$\Rightarrow c_n \sim \frac{a}{\pi} \int_0^{\infty} dz \frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi} \frac{\Gamma(n + \frac{1}{2})}{b^{n+1/2}}$$

# Divergence of perturbation theory

an important part of the story ...

*The majority of nontrivial theories are seemingly unstable at some phase of the coupling constant, which leads to the asymptotic nature of the perturbative series*

A. Vainshtein (1964)

## Borel summation: existence theorem (Nevanlinna & Sokal)

$f(z)$  analytic in circle  $C_R = \{z : |z - \frac{R}{2}| < \frac{R}{2}\}$

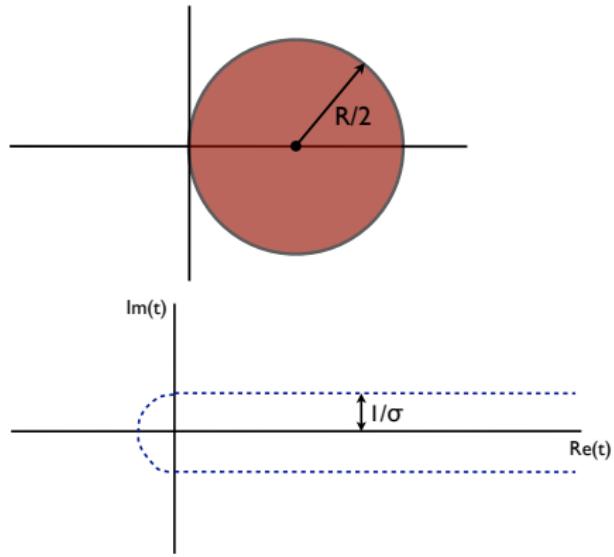
$$f(z) = \sum_{n=0}^{N-1} a_n z^n + R_N(z) \quad , \quad |R_N(z)| \leq A \sigma^N N! |z|^N$$

Borel transform

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$$

analytic continuation to  
 $S_\sigma = \{t : |t - \mathbb{R}^+| < 1/\sigma\}$

$$f(z) = \frac{1}{z} \int_0^\infty e^{-t/z} B(t) dt$$



## Borel summation in practice

$$f(g) \sim \sum_{n=0}^{\infty} c_n g^n \quad , \quad c_n \sim \beta^n \Gamma(\gamma n + \delta)$$

- **alternating series:** real Borel sum

$$f(g) \sim \frac{1}{\gamma} \int_0^\infty \frac{dt}{t} \left( \frac{1}{1+t} \right) \left( \frac{t}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{t}{\beta g} \right)^{1/\gamma} \right]$$

- **nonalternating series:** ambiguous imaginary part

$$\operatorname{Re} f(-g) \sim \frac{1}{\gamma} \mathcal{P} \int_0^\infty \frac{dt}{t} \left( \frac{1}{1-t} \right) \left( \frac{t}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{t}{\beta g} \right)^{1/\gamma} \right]$$

$$\operatorname{Im} f(-g) \sim \pm \frac{\pi}{\gamma} \left( \frac{1}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{1}{\beta g} \right)^{1/\gamma} \right]$$

## Borel summation in practice

direct quantitative correspondence between:

rate of growth  $\leftrightarrow$  Borel poles  $\leftrightarrow$  non-perturbative exponent

non-alternating factorial growth:  $c_n \sim \beta^n \Gamma(\gamma n + \delta)$

positive Borel singularity:  $t_c = \left( \frac{1}{\beta g} \right)^{1/\gamma}$

non-perturbative exponent:  $\pm i \frac{\pi}{\gamma} \left( \frac{1}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{1}{\beta g} \right)^{1/\gamma} \right]$

## recall: Divergence of perturbation theory in QM

e.g. ground state energy:

$$E = \sum_{n=0}^{\infty} c_n (\text{coupling})^n$$

- cubic oscillator:  $c_n \sim -\frac{(60)^{n+1/2}}{(2\pi)^{3/2}} \Gamma(n + \frac{1}{2})$
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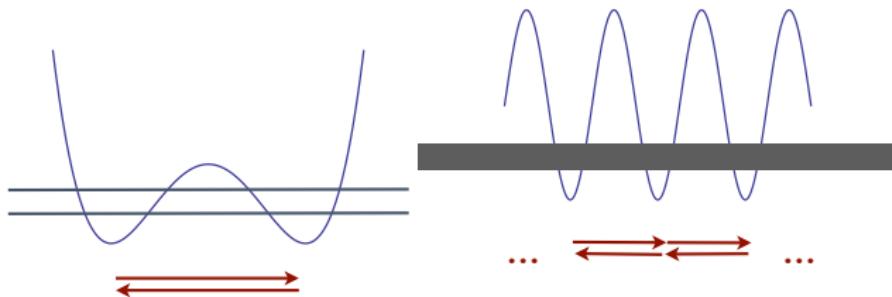
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- cubic oscillator:  $c_n \sim -\frac{(60)^{n+1/2}}{(2\pi)^{3/2}} \Gamma(n + \frac{1}{2})$  unstable
- quartic oscillator:  $c_n \sim (-1)^{n+1} \frac{3^n \sqrt{6}}{\pi^{3/2}} \Gamma(n + \frac{1}{2})$  stable
- Zeeman:  $c_n \sim (-1)^n \left(\frac{4}{\pi}\right)^{5/2} \frac{1}{\pi^{2n}} \left(2n + \frac{1}{2}\right)!$  stable
- Stark:  $c_n \sim -\frac{4}{\pi} \left(\frac{3}{2}\right)^{2n+1} (2n)!$  unstable
- periodic Sine-Gordon potential:  $c_n \sim n!$  stable ???
- double-well:  $c_n \sim 3^n n!$  stable ???

## Bogomolny/Zinn-Justin mechanism in QM



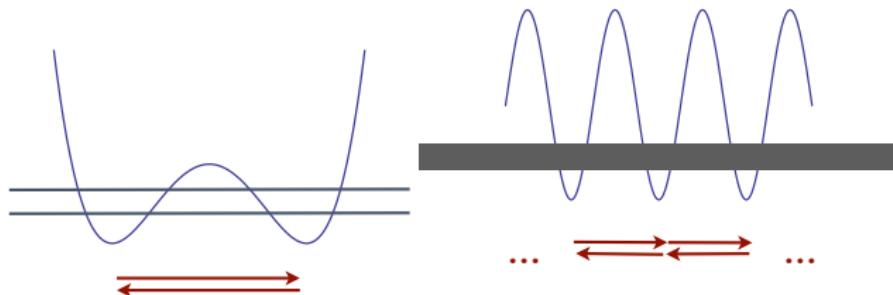
- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect:  $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable:  $c_n \sim \frac{n!}{(2S)^n}$

- ▶ stable systems
- ▶ ambiguous imaginary part
- ▶  $\pm i e^{-\frac{2S}{g^2}}$ , a 2-instanton effect

## Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel summable:  
ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete:  
 $\mathcal{I}$  and  $\bar{\mathcal{I}}$  attract
- regularize both by analytic continuation of coupling  
⇒ ambiguous, imaginary non-perturbative terms cancel!

## Bogomolny/Zinn-Justin mechanism in QM

e.g., double-well:  $V(x) = x^2(1 - g x)^2$

$$E_0 \sim \sum_n c_n g^{2n}$$

- perturbation theory:

$$c_n \sim -3^n n! \quad \rightarrow \quad \text{Im } E_0 \sim \mp \pi e^{-\frac{1}{3g^2}}$$

- non-perturbative instanton gas:

$$\text{Im } E_0 \sim \pm \pi e^{-2\frac{1}{6g^2}}$$

- BZJ cancellation  $\Rightarrow E_0$  is real and unambiguous

“resurgence”  $\Rightarrow$  cancellation to all orders

## Bogomolny/Zinn-Justin mechanism in QM

- double-well potential:  $V(x) = \frac{1}{2}x^2(1-gx)^2$
- instanton solution:  $g x_0(t) = 1/(1+e^{-t})$
- classical Euclidean action:  $S_0 = \frac{1}{6g^2}$

approximate  $\mathcal{I}\bar{\mathcal{I}}$  soln. :  $x_{cl}(t) = \begin{cases} x_0(R+t) & , \quad t > 0 \\ x_0(R-t) & , \quad t < 0 \end{cases}$

effective interaction potential:  $U_{int}(t_1, t_2) = -\frac{2}{g^2} e^{-|t_1-t_2|}$

$$Z_{int} = a^2 \int dt_1 \int dt_2 e^{-U_{int}(t_1, t_2)} \quad \left( a \equiv \frac{1}{g\sqrt{\pi}} e^{-\frac{1}{6g^2}} \right)$$
$$\stackrel{T \rightarrow \infty}{\sim} \frac{1}{2} T^2 a^2 + T a^2 \int_0^\infty dt \left( \exp \left[ \frac{2}{g^2} e^{-t} \right] - 1 \right) + \dots$$

- as  $g^2 \rightarrow 0$ , dominated by  $t \rightarrow 0$  ???

## Bogomolny/Zinn-Justin mechanism in QM

$$Z_{\text{int}} \underset{T \rightarrow \infty}{\sim} \frac{1}{2} T^2 a^2 + T a^2 \int_0^\infty dt \left( \exp \left[ \frac{2}{g^2} e^{-t} \right] - 1 \right) + \dots$$

BZJ idea: analytically continue  $g^2 \rightarrow -g^2$

$\Rightarrow$  dominated by finite  $t \Rightarrow$  stable instanton gas

$$\int_0^\infty dt \left( \exp \left[ -\frac{2}{g^2} e^{-t} \right] - 1 \right) \sim -\gamma_E + \ln \left( \frac{g^2}{2} \right) + Ei \left( -\frac{2}{g^2} \right)$$

- ambiguous imaginary part (from log) when  $-g^2 \rightarrow g^2$
- recall  $Z \sim e^{-E_0 T} \Rightarrow$  imaginary  $E_0$  from instanton gas

BZJ cancellation: cancels against ambiguous imaginary part from analytic continuation of Borel summation of perturbation theory

# Bogomolny/Zinn-Justin mechanism in SUSY QM

Balitsky/Yung: SUSY double-well

$$V_{\text{bosonic}} = W^2 - W' = \frac{1}{2} (1 + g x^2)^2 - 1$$

- ground state perturbatively zero (very convergent!)
- SUSY broken non-perturbatively (single-instanton)

$\mathcal{I}\bar{\mathcal{I}}$  interaction involves bosonic and fermionic zero modes

$$Z_1 = \frac{T}{\sqrt{\pi}} \frac{2}{\pi g^2} \int dt e^{-\frac{1}{3g^2}} \left( e^{\left( -2t + \frac{2}{g^2} e^{-2t} \right)} - 1 \right)$$

# Trans-series for Energy Eigenvalues

- perturbation theory:  $E_{\text{pert. theory}}^{(N)}(g^2) = \sum_{k=0}^{\infty} g^{2k} E_k^{(N)}$
- non-Borel-summable: incomplete
- all non-perturbative multi-instanton terms:  
“trans-series”

$$E^{(N)}(g^2) = E_{\text{pert. theory}}^{(N)}(g^2) + \sum_{k=1}^{\infty} \sum_{l=1}^{k-1} \sum_{p=0}^{\infty} \underbrace{\left( \frac{1}{g^{2N+1}} \exp \left[ -\frac{c}{g^2} \right] \right)^k}_{\text{k-instanton}} \underbrace{\left( \ln \left[ \pm \frac{1}{g^2} \right] \right)^l}_{\text{quasi-zero-mode}} \underbrace{c_{k,l,p} g^{2p}}_{\text{perturbative fluctuations}}$$

precisely of Écalle’s trans-series form !

## Decoding of Trans-series

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q} g^{2n} \left[ \exp\left(-\frac{S}{g^2}\right) \right]^k \left[ \ln\left(-\frac{1}{g^2}\right) \right]^q$$

- perturbative fluctuations about vacuum:  $\sum_{n=0}^{\infty} c_{n,0,0} g^{2n}$
- divergent (non-Borel-summable):  $c_{n,0,0} \sim \alpha \frac{n!}{(2S)^n}$   
⇒ ambiguous imaginary non-pert energy  $\sim \pm i \pi \alpha e^{-2S/g^2}$
- but  $c_{0,2,1} = -\alpha$ : BZJ cancellation !

pert flucs about instanton:  $e^{-S/g^2} (1 + a_1 g^2 + a_2 g^4 + \dots)$

divergent:

$$a_n \sim \frac{n!}{(2S)^n} (a \ln n + b) \Rightarrow \pm i \pi e^{-3S/g^2} \left( a \ln \frac{1}{g^2} + b \right)$$

- 3-instanton:  $e^{-3S/g^2} \left[ \frac{a}{2} \left( \ln \left( -\frac{1}{g^2} \right) \right)^2 + b \ln \left( -\frac{1}{g^2} \right) + c \right]$

resurgence: *ad infinitum*, also sub-leading large-order terms

## Lecture 2

- ▶ divergence of perturbation theory in QFT
- ▶ Euler-Heisenberg effective actions
- ▶ curing the IR renormalon puzzle in  $\mathbb{CP}^{N-1}$  models

# Divergence of perturbation theory in QFT

- Hurst (1952):  $\phi^4$  perturbation theory is divergent:
  - (i) factorial growth of number of diagrams
  - (ii) explicit lower bounds on diagrams

*If it be granted that the perturbation expansion does not lead to a convergent series in the coupling constant for all theories which can be renormalized, at least, then a reconciliation is needed between this and the excellent agreement found in electrodynamics between experimental results and low-order calculations. It is suggested that this agreement is due to the fact that the S-matrix expansion is to be interpreted as an asymptotic expansion in the fine-structure constant ...*

*C. A. Hurst, 1952*

## Dyson's argument (QED)

- Dyson (1952): *physical argument* for divergence of QED perturbation theory

$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots$$

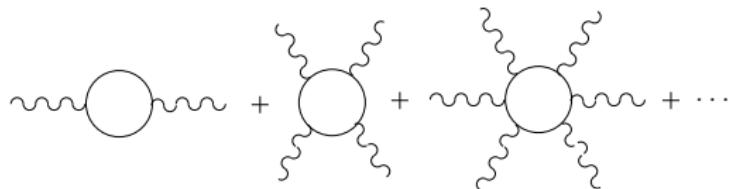
*Thus [for  $e^2 < 0$ ] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.*

*F. J. Dyson, 1952*

- suggests perturbative expansion cannot be convergent

# Euler-Heisenberg Effective Action (1935)

review: [hep-th/0406216](#)



- 1-loop QED effective action in uniform background emag field
- e.g., constant  $B$  field:

$$S = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left( \coth s - \frac{1}{s} - \frac{s}{3} \right) \exp \left[ -\frac{m^2 s}{eB} \right]$$

$$S = -\frac{e^2 B^2}{2\pi^2} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left( \frac{2eB}{m^2} \right)^{2n+2}$$

## Euler-Heisenberg Effective Action

- e.g., constant  $B$  field: characteristic factorial divergence

$$c_n = (-1)^{n+1} \frac{\Gamma(2n+2)}{8} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2n+4}}$$

- recall Borel summation:

$$f(g) \sim \sum_{n=0}^{\infty} c_n g^n \quad , \quad c_n \sim \beta^n \Gamma(\gamma n + \delta)$$

$$\rightarrow f(g) \sim \frac{1}{\gamma} \int_0^\infty \frac{ds}{s} \left( \frac{1}{1+s} \right) \left( \frac{s}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{s}{\beta g} \right)^{1/\gamma} \right]$$

- reconstruct correct Borel transform:

$$\sum_{k=1}^{\infty} \frac{s}{k^2 \pi^2 (s^2 + k^2 \pi^2)} = -\frac{1}{2s^2} \left( \coth s - \frac{1}{s} - \frac{s}{3} \right)$$

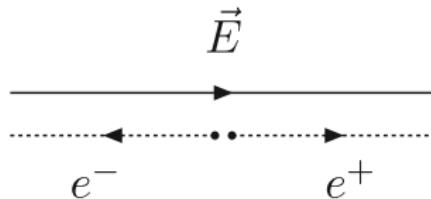
## Euler-Heisenberg Effective Action

$B$  field: QFT analogue of Zeeman effect

$E$  field: QFT analogue of Stark effect

$B^2 \rightarrow -E^2$ : series becomes non-alternating

$$\text{Borel summation} \Rightarrow \text{Im } S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp \left[ -\frac{k m^2 \pi}{e E} \right]$$



Schwinger pair production from vacuum:

$\text{Im } S \rightarrow$  physical pair production rate

- suggests Euler-Heisenberg series must be divergent

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$

$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature:  $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in  $dS_d$  with d even

# Euler-Heisenberg and Matrix Models, Large N, Strings, ...

- scalar QED Euler-Heisenberg in self-dual background ( $F = \pm \tilde{F}$ ):

$$\begin{aligned} S &= \frac{F^2}{16\pi^2} \int_0^\infty \frac{dt}{t} e^{-t} \left( \frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3} \right) \\ &= -\frac{m^4}{16\pi^2} \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+2)} \left( \frac{2F}{m^2} \right)^{2n+2} \end{aligned}$$

- “electric” self-dual ( $F \rightarrow i F$ ):

$$\text{Im } S = \frac{m^2 F}{32\pi^3} \sum_{k=1}^{\infty} \left( \frac{2\pi}{k} + \frac{2F}{k^2 m^2} \right) e^{-\pi k m^2 / F}$$

## Euler-Heisenberg and Matrix Models, Large N, Strings, ...

- scalar QED EH in self-dual background ( $F = \pm \tilde{F}$ ):

$$S = \frac{F^2}{16\pi^2} \int_0^\infty \frac{dt}{t} e^{-t} \left( \frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3} \right)$$

- Gaussian matrix model:  $\lambda = g N$

$$\mathcal{F} = -\frac{1}{4} \int_0^\infty \frac{dt}{t} e^{-2\lambda t/g} \left( \frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3} \right)$$

- $c = 1$  String:  $\lambda = g N$

$$\mathcal{F} = \frac{1}{4} \int_0^\infty \frac{dt}{t} e^{-2\lambda t/g} \left( \frac{1}{\sin^2(t)} - \frac{1}{s^2} - \frac{1}{3} \right)$$

- Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4} \sum_{m \in \mathbb{Z}} \int_0^\infty \frac{dt}{t} e^{-2(\lambda + 2\pi i m)t/g} \left( \frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3} \right)$$

## Euler-Heisenberg and Matrix Models, Large N, Strings, ...

- similar structure arises in more general topological string theories and matrix models
- resurgence and Borel-Écalle summation provide a natural framework for combining perturbative genus expansions with non-perturbative information
- Mariño, Schiappa, Pasquetti, Aniceto, Vonk, ...

key problem: analytic continuation of functional integrals

# Resurgence and Analytic Continuation

one view (of many) of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

- “alien calculus” (Écalle)
- median resummation: encodes intricate combinatorics of cancellations in trans-series

# Asymptotic Expansions & Analytic Continuation

Stirling expansion for  $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$  is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \cdots$$

- functional relation:  $\psi(1+z) = \psi(z) + \frac{1}{z}$

$$\text{formal series} \quad \Rightarrow \quad \operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2}$$

- reflection formula:  $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

reflection formula

$$\Rightarrow \quad \operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

“raw” asymptotics inconsistent with analytic continuation

- resurgence fixes this

$$\operatorname{Re} \psi(1+iy) \sim \ln y + 2 \sum_{n=0}^{\infty} \frac{(2n+1)!}{(2\pi y)^{2n+2}} \zeta(2n+2) - i\pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

# Asymptotic Expansions & Analytic Continuation

- this example arises in many QFT and String Theory computations:
- Euler-Heisenberg, de Sitter, exact S-matrices, Chern-Simons partition functions, matrix models, ...

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\frac{n^2 \pi^2}{L^2} - \lambda} &= -\frac{L^2}{2} \left( \frac{\cot(L\sqrt{\lambda})}{L\sqrt{\lambda}} - \frac{1}{L^2 \lambda} \right) \\ &= \frac{L}{2\pi\sqrt{\lambda}} \left( \psi\left(1 + \frac{L\sqrt{\lambda}}{\pi}\right) - \psi\left(1 - \frac{L\sqrt{\lambda}}{\pi}\right) \right)\end{aligned}$$

# Divergence of derivative expansion

(GD, T. Hall, hep-th/9902064)

- time-dependent  $E$  field:  $E(t) = E \operatorname{sech}^2(t/\tau)$

$$S = -\frac{m^4}{8\pi^{3/2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(m\lambda)^{2j}} \sum_{k=2}^{\infty} (-1)^k \left(\frac{2E}{m^2}\right)^{2k} \frac{\Gamma(2k+j)\Gamma(2k+j-2)\mathcal{B}_{2k+2j}}{j!(2k)!\Gamma(2k+j+\frac{1}{2})}$$

- Borel sum perturbative expansion: large  $k$  ( $j$  fixed):

$$c_k^{(j)} \sim 2 \frac{\Gamma(2k+3j-\frac{1}{2})}{(2\pi)^{2j+2k+2}}$$

$$\operatorname{Im} S^{(j)} \sim \exp\left[-\frac{m^2\pi}{E}\right] \frac{1}{j!} \left(\frac{m^4\pi}{4\tau^2 E^3}\right)^j$$

- resum derivative expansion

$$\operatorname{Im} S \sim \exp\left[-\frac{m^2\pi}{E} \left(1 - \frac{1}{4} \left(\frac{m}{E\tau}\right)^2\right)\right]$$

## Divergence of derivative expansion

- Borel sum derivative expansion: large  $j$  ( $k$  fixed):

$$c_j^{(k)} \sim 2^{\frac{9}{2}-2k} \frac{\Gamma(2j+4k-\frac{5}{2})}{(2\pi)^{2j+2k}}$$

$$\text{Im } S^{(k)} \sim \frac{(2\pi E\tau^2)^{2k}}{(2k)!} e^{-2\pi m\tau}$$

- resum perturbative expansion:

$$\text{Im } S \sim \exp \left[ -2\pi m\tau \left( 1 - \frac{E\tau}{m} \right) \right]$$

- compare:

$$\text{Im } S \sim \exp \left[ -\frac{m^2\pi}{E} \left( 1 - \frac{1}{4} \left( \frac{m}{E\tau} \right)^2 \right) \right]$$

- different limits of full:  $\text{Im } S \sim \exp \left[ -\frac{m^2\pi}{E} f \left( \frac{m}{E\tau} \right) \right]$
- derivative expansion must be divergent

## Renormalons

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

QFT: new physical effects occur, due to running of coupling constant with momentum

faster source of divergence: “renormalons”

⇒ leading non-perturbative exponentials

non-alternating factorial growth:  $c_n \sim \beta^n \Gamma(\gamma n + \delta)$

positive Borel pole:  $t_c = \left( \frac{1}{\beta g} \right)^{1/\gamma}$

non-perturbative exponential:  $\pm i \frac{\pi}{\gamma} \left( \frac{1}{\beta g} \right)^{\delta/\gamma} \exp \left[ - \left( \frac{1}{\beta g} \right)^{1/\gamma} \right]$

# UV and IR Renormalons

e.g. QED with  $N_f$  massless flavors

- Adler function  $D(Q^2) = -4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2}$
- bubble-chains, momentum  $k \rightarrow$  interpolating expression

$$D(Q^2) = Q^2 \int_0^\infty d(k^2) \frac{k^2 \alpha_s(k^2)}{(k^2 + Q^2)^3}$$

- running coupling  $\alpha_s(k^2)$ :

$$\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 - \frac{\beta_0 \alpha_s(Q^2)}{4\pi} \ln(Q^2/k^2)}$$

$\beta_0$ : first beta-function coefficient

- expand  $\alpha_s(k^2)$  in series in powers of  $\alpha_s(Q^2)$ :

$$D(Q^2) = \alpha_s(Q^2) \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha_s(Q^2)}{4\pi} \right)^n Q^2 \int_0^\infty d(k^2) \frac{k^2 (\ln(Q^2/k^2))^n}{(k^2 + Q^2)^3}$$

## UV and IR Renormalons

$$D(Q^2) = \alpha_s(Q^2) \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha_s(Q^2)}{4\pi} \right)^n Q^2 \int_0^{\infty} d(k^2) \frac{k^2 (\ln(Q^2/k^2))^n}{(k^2 + Q^2)^3}$$

- IR low momentum: split at  $k^2 = Q^2$  ( $y \equiv 2 \ln(Q^2/k^2)$ )

$$\begin{aligned} Q^2 \int_0^{Q^2} d(k^2) \frac{k^2 (\ln(Q^2/k^2))^n}{(k^2 + Q^2)^3} &= \frac{1}{2} \frac{1}{2^n} \int_0^{\infty} dy \frac{e^{-y} y^n}{(1 + e^{-y/2})^3} \\ &\sim \frac{n!}{2} \left( \frac{1}{2^n} - \frac{2}{3^n} + O\left(\frac{1}{4^n}\right) \right) \end{aligned}$$

- UV high momentum: ( $\bar{y} \equiv \ln(k^2/Q^2)$ )

$$\begin{aligned} Q^2 \int_{Q^2}^{\infty} d(k^2) \frac{k^2 (\ln(Q^2/k^2))^n}{(k^2 + Q^2)^3} &= (-1)^n \int_0^{\infty} d\bar{y} \frac{e^{-\bar{y}} \bar{y}^n}{(1 + e^{-\bar{y}})^3} \\ &\sim (-1)^n n! \left( 1 - \frac{3}{2^{n+1}} + O\left(\frac{1}{3^n}\right) \right) \end{aligned}$$

## UV and IR Renormalons

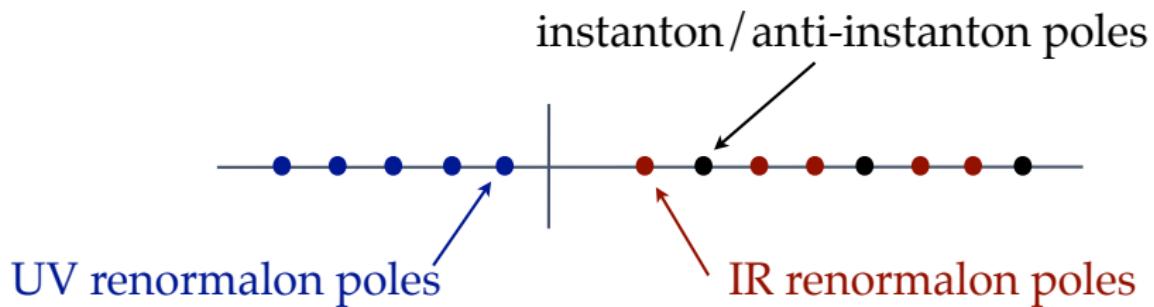
renormalon poles:

$$t_n^{IR} = +\frac{4\pi}{\beta_0} n \quad , \quad n = 2, 3, 4, \dots$$

$$t_n^{UV} = -\frac{4\pi}{\beta_0} n \quad , \quad n = 1, 2, 3, \dots$$

Borel poles due to renormalons are closer to the origin:

“dominant effect”



# IR Renormalon Puzzle in Asymptotically Free QFT

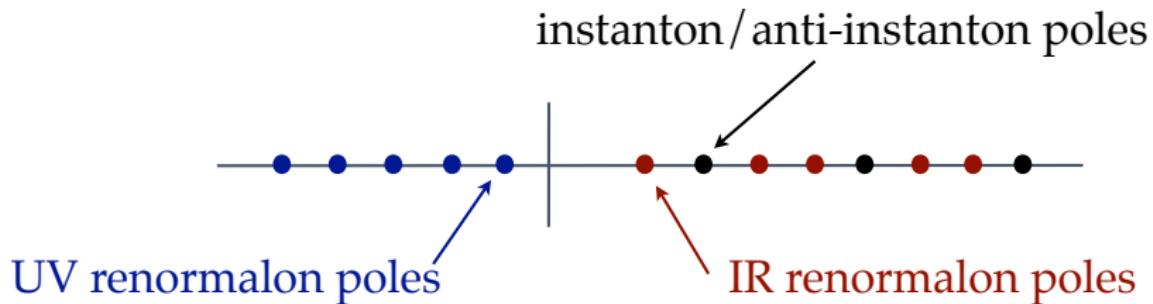
Asymptotically free QFT's: e.g. Yang-Mills or  $\mathbb{CP}^{N-1}$

- (i) degenerate classical vacua
- (ii) non Borel summable perturbation theory, due to infrared (IR) renormalons, associated with IR-momentum behaviour of certain bubble-chain diagrams
  - ▶ IR renormalons  $\Rightarrow$  perturbation theory is ill-defined and incomplete
  - ▶ for  $\mathbb{CP}^{N-1}$  on  $\mathbb{R}^2$ , or YM on  $\mathbb{R}^4$ , BZJ mechanism does not work, because Borel poles are in ‘wrong’ locations

# IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory:  $\longrightarrow \pm i e^{-2S/\beta_0}$

instantons on  $\mathbb{R}^2$  or  $\mathbb{R}^4$ :  $\longrightarrow \pm i e^{-2S}$



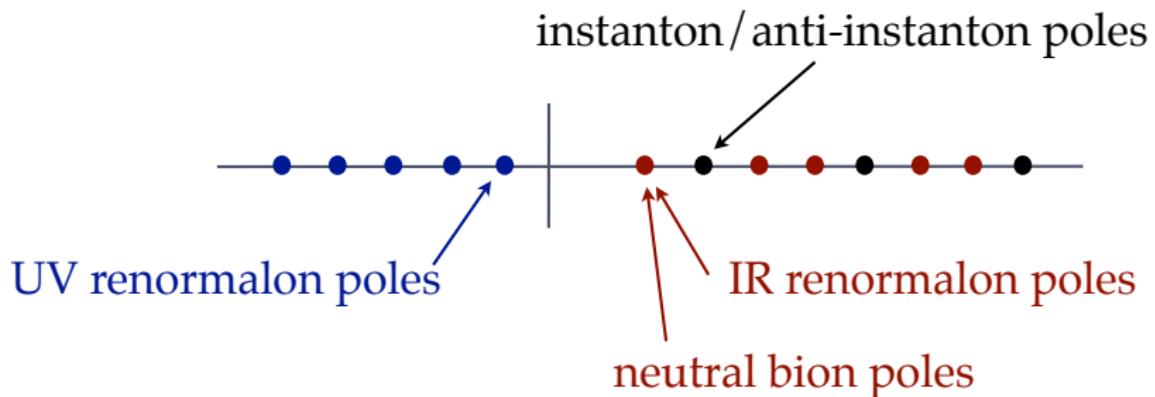
appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

# IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, GD, Ünsal, [1206.1890](#) [1210.2423](#))

- scale modulus of instantons
- spatial compactification and principle of continuity

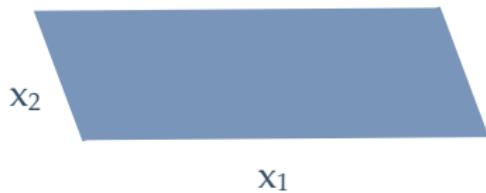


cancellation occurs !

# Topological Molecules in Spatially Compactified Theories

$\mathbb{CP}^{N-1}$ : regulate scale modulus problem with (spatial) compactification

$$\mathbb{R}^2 \rightarrow S^1 \times \mathbb{R}^1$$



Euclidean time

instantons fractionalize

# Topological Molecules in Spatially Compactified Theories

temporal compactification: information only about deconfined phase

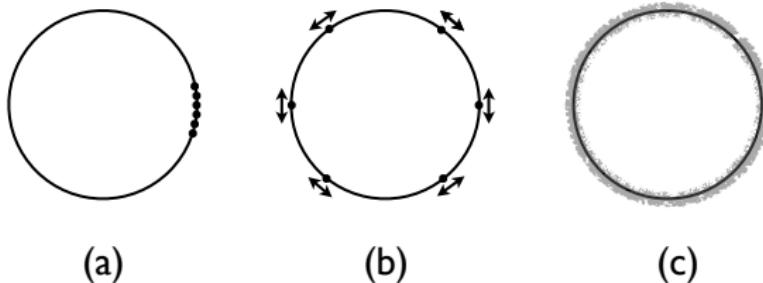
$$\begin{array}{ccc} \mathbb{R}^1 & \xrightarrow{\hspace{10cm}} & \mathbb{R}^2 \\ \text{high T} & & \text{low T} \\ \mathbb{R}^1 & \xrightarrow{\hspace{10cm}} & \mathbb{R}^2 \end{array}$$

spatial compactification: semi-classical small  $L$  regime  
continuously connected to large  $L$ :

*principle of continuity*

$$\begin{array}{ccc} \mathbb{S}_L^1 \times \mathbb{R}^1 & \xrightarrow{\hspace{10cm}} & \mathbb{R}^2 \\ \mathbb{R}^1 & \xrightarrow{\hspace{10cm}} & \mathbb{R}^2 \\ \text{"continuity"} & & \end{array}$$

## Weak Coupling Non-Trivial Holonomy and Center Symmetry



- ▶ (a) **Weak coupling trivial holonomy:** Semi-classical OK, but disconnected from strong coupling regime
- ▶ (b) **Weak coupling non-trivial holonomy:** Semi-classical analysis OK, and continuously connected to strong coupling regime
- ▶ (c) **Strong coupling non-trivial holonomy:** Weak-coupling semi-classical analysis not OK

# Topological Molecules in Spatially Compactified Theories

- ▶ monopole-instantons,  $\mathcal{M}_i$ , or kink-instantons  $\mathcal{K}_i$ ,  
 $i = 1, 2, \dots, N$ .
- ▶ Charged bions (correlated kink-anti-kink events):  
 $\mathcal{B}_{ij} = [\mathcal{M}_i \bar{\mathcal{M}}_j]$ , or  $\mathcal{B}_{ij} = [\mathcal{K}_i \bar{\mathcal{K}}_j]$ , with  $i \neq j$
- ▶ Neutral bions:  $\mathcal{B}_{ii} = [\mathcal{M}_i \bar{\mathcal{M}}_i]$ , and  $\mathcal{B}_{ii} = [\mathcal{K}_i \bar{\mathcal{K}}_i]$
- ▶ Neutral bion-anti-bion molecular events such as  $[\mathcal{B}_{ij} \mathcal{B}_{ji}]$ ,  
 $[\mathcal{B}_{ij} \mathcal{B}_{jk} \mathcal{B}_{ki}]$ , etc ...

hierarchy of scales:

$$\begin{array}{ccccccccc} r_k & \ll & r_b & \ll & d_{k-k} & \ll & d_{b-b}, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L & \ll & L \log\left(\frac{1}{g^2}\right) & \ll & L e^{S_0} & \ll & L e^{2S_0}. \end{array}$$

## Graded Resurgence Triangle

- ▶ Perturbation theory is independent of topological  $\Theta$ -angle  
⇒ ambiguity due to non-Borel summability of perturbation theory is also independent of  $\Theta$ .
- ▶ ⇒ non-Borel summability of large orders of perturbation theory can *never* be cancelled by non-perturbative configurations with non-vanishing topological charge. Can *only* be cancelled by topological configurations with zero topological charge, or equivalently, without any  $\Theta$ -angle dependence

# Graded Resurgence Triangle

saddle points labelled by:  $[n, m]$

$$n = n_{\text{instanton}} + n_{\text{anti-instanton}} \quad , \quad m = n_{\text{instanton}} - n_{\text{anti-instanton}}$$

$[0, 0]$

$[1, 1]$

$[1, -1]$

$[2, 2]$

$[2, 0]$

$[2, -2]$

$[3, 3]$

$[3, 1]$

$[3, -1]$

$[3, -3]$

$[4, 4]$

$[4, 2]$

$[4, 0]$

$[4, -2]$

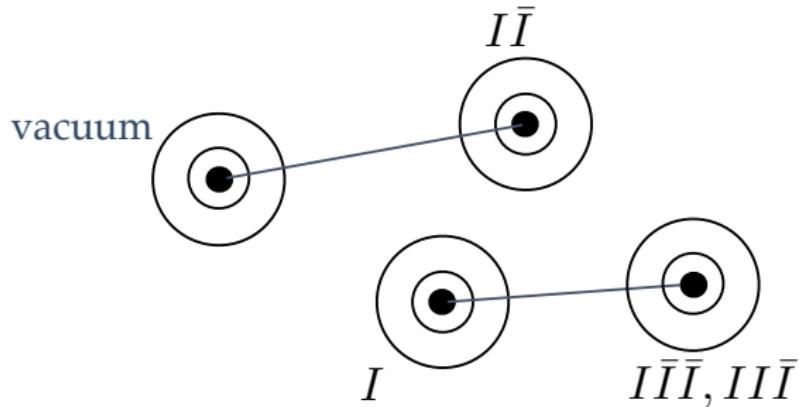
$[4, -4]$

$\ddots$

$\vdots$

$\ddots$

# Graded Resurgence Triangle



$$\begin{aligned} & (1 + a_1 g^2 + a_2 g^4 + \dots + \dots) + e^{-S/g^2} (1 + b_1 g^2 + b_2 g^4 + \dots + \dots) \\ & + e^{-2S/g^2} (1 + c_1 g^2 + c_2 g^4 + \dots + \dots) + e^{-3S/g^2} (1 + d_1 g^2 + d_2 g^4 + \dots + \dots) \\ & + \dots + (\text{log terms}) \end{aligned}$$

## $\mathbb{C}\mathbb{P}^{N-1}$ Model

$\mathbb{C}\mathbb{P}^{N-1}$  model: two-dim. sigma model analog of Yang-Mills

- ▶ asymptotically free:  $\beta_0 = N$  (independent of  $N_f$ )
- ▶ instantons, theta vacua, fermion zero modes, ...
- ▶ divergent perturbation theory (non-Borel summable)
- ▶ renormalons (both UV and IR)
- ▶ large- $N$  analysis
- ▶ non-perturbative mass gap:  $m_g = \Lambda = \mu e^{-4\pi/(g^2 N)}$
- ▶ couple to fermions, SUSY, ...
- ▶ ‘unstable’ finite action non-self-dual classical solutions  
(path integral saddle points)

## Basics of $\mathbb{CP}^{N-1}$ Model

- ▶ classical bosonic action:  $S = \frac{2}{g^2} \int d^2x (D_\mu n)^\dagger D_\mu n$
- ▶  $n = N$ -component column vector with  $n^\dagger n = 1$
- ▶ local  $U(1)$  symmetry , global  $U(N)$  symmetry
- ▶  $D_\mu = \partial_\mu + iA_\mu$ , with abelian gauge field  $A_\mu = i n^\dagger \partial_\mu n$
- ▶ target space  $\mathcal{M}_{N,1} \equiv \mathbb{CP}^{N-1} = \frac{U(N)}{U(N-1) \times U(1)}$
- ▶  $N^2 - 1 - (N-1)^2 = 2(N-1)$  real fields
- ▶ topological charge

$$Q = -\frac{i}{2\pi} \int d^2x \epsilon_{\mu\nu} \partial_\mu \left( n^\dagger \partial_\nu n \right) = \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} \partial_\mu A_\nu$$

- ▶ couple to fermions

$$S_{\text{fermion}} = \frac{2}{g^2} \int [ -i\bar{\psi} \gamma_\mu D_\mu \psi + \frac{1}{4} ((\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_3\psi)^2 - (\bar{\psi}\gamma_\mu\psi)^2) ]$$

# Instantons in $\mathbb{CP}^{N-1}$ Model

Bogomolny factorization:

$$(D_\mu n)^\dagger D_\mu n = |(D_\mu \pm i\epsilon_{\mu\nu} D_\nu) n|^2 \mp i \epsilon_{\mu\nu} \partial_\mu (n^\dagger \partial_\nu n)$$

self-dual instanton equations

$$D_\mu n = \mp i \epsilon_{\mu\nu} D_\nu n$$

homogeneous fields:  $n \equiv \frac{v}{|v|}$

instanton:  $v = v(z)$  , anti-instanton:  $v = v(\bar{z})$

e.g., simplest instanton for  $\mathbb{CP}^1$  on  $\mathbb{R}^2$ :

$$v = \begin{pmatrix} 1 \\ (z - b)/a \end{pmatrix} \Rightarrow Q = \frac{1}{\pi} \int d^2x \frac{|a|^2}{(|a|^2 + |z - b|^2)^2} = 1$$

## “Center Symmetry” in $\mathbb{CP}^{N-1}$

$2(N - 1)$  angular fields:

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos \frac{\theta_1}{2} \\ e^{i\varphi_2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \\ e^{i\varphi_3} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_3}{2} \\ \vdots \\ e^{i\varphi_N} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \sin \frac{\theta_3}{2} \dots \sin \frac{\theta_{N-1}}{2} \end{pmatrix}$$

order parameter:

$$\Omega(x_1) = \begin{pmatrix} e^{i[\varphi_1(x_1, 0) - \varphi_1(x_1, L)]} & 0 & \dots & 0 \\ 0 & e^{i[\varphi_2(x_1, 0) - \varphi_2(x_1, L)]} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{i[\varphi_N(x_1, 0) - \varphi_N(x_1, L)]} \end{pmatrix}$$

twisted b.c.'s:  $\mathbb{Z}_N : \Omega \longrightarrow e^{i \frac{2\pi k}{N}} \Omega$

## “Center Symmetry” in $\mathbb{CP}^{N-1}$

$$V_-[\Omega] = \frac{2}{\pi\beta^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1 + (-1)^n N_f) (|\text{tr } \Omega^n| - 1) \quad (\text{thermal})$$

$$V_+[\Omega] = (N_f - 1) \frac{2}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (|\text{tr } \Omega^n| - 1) \quad (\text{spatial})$$

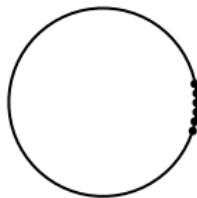
minima:

$$\Omega_0^{\text{thermal}} = e^{i \frac{2\pi k}{N}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad (\text{thermal})$$

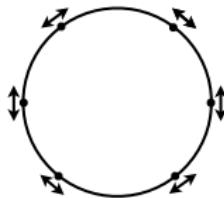
$$\Omega_0^{\text{spatial}} = \begin{pmatrix} 1 & & & \\ & e^{i \frac{2\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i \frac{2\pi(N-1)}{N}} \end{pmatrix} \quad (\text{spatial})$$

## “Center Symmetry” in $\mathbb{C}\mathbb{P}^{N-1}$

- ▶  $N_f > 1$ : repulsive interaction between eigenvalues of holonomy  $\Omega$ : center symmetry preserved
- ▶  $N_f = 1$ :  $\mathcal{N} = (2, 2)$  SUSY  $\mathbb{C}\mathbb{P}^{N-1}$ : perturbative potential vanishes to all orders (SUSY). Non-perturbatively induced potential stabilizes center-symmetry
- ▶  $N_f = 0$ : deformed  $\mathbb{C}\mathbb{P}^{N-1}$ , or integrating out heavy fermions



(a)



(b)



(c)

# Fractionalized Instantons in $\mathbb{CP}^{N-1}$

- untwisted instanton on  $\mathbb{R}^1 \times \mathbb{S}_L^1$ :

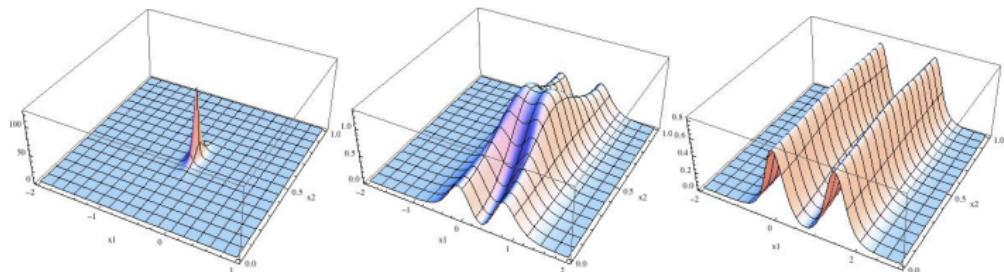
$$v = \begin{pmatrix} 1 \\ \lambda_1 + \lambda_2 e^{-\frac{2\pi}{L}z} \end{pmatrix}$$

- *spatial* twist:

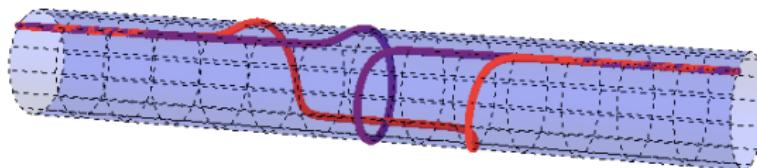
$$v_{\text{twisted}} = \begin{pmatrix} 1 \\ \left(\lambda_1 + \lambda_2 e^{-\frac{2\pi}{L}z}\right) e^{\frac{2\pi}{L}\mu_2 z} \end{pmatrix}$$

- twisted boundary condition  $\Rightarrow$  factor  $e^{\frac{2\pi i}{L}\mu_2 x_2}$
  - but, non-holomorphic:  $\Rightarrow$  factor  $e^{\frac{2\pi}{L}\mu_2 z}$
- $\Rightarrow$  twist in  $x_2$  also prescribes dependence in non-compact direction  $x_1$

# Fractionalized Instantons in $\mathbb{CP}^{N-1}$



**Figure:**  $Q = 1$  instanton in  $\mathbb{CP}^1$ , ( $N = 2$ ), in weak coupling center-symmetric background. Small circle: instanton splits into two  $Q = \frac{1}{2}$  instantons.



**Figure:** Wilson loop for small  $Q = 1$  instanton (purple). Large instanton (red) splits into two separate kink-instantons. Each wraps half-way around the cylinder.

# Fractionalized Instantons in $\mathbb{CP}^{N-1}$

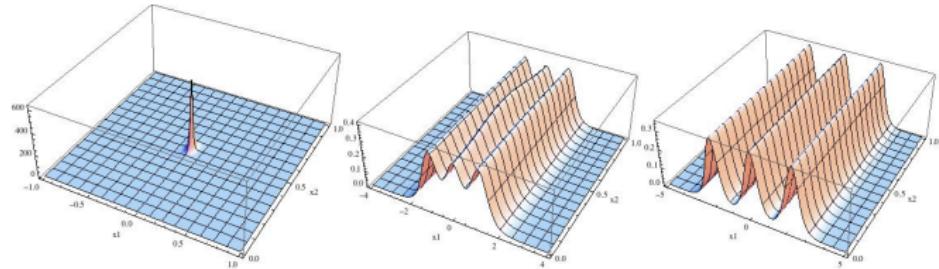


Figure:  $Q = 1$  instanton in  $\mathbb{CP}^2$ , ( $N = 3$ ), in weak coupling center-symmetric background. Small circle: instanton splits into three  $Q = \frac{1}{3}$  instantons.

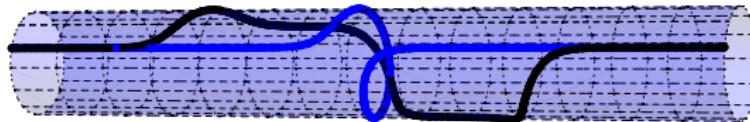


Figure: Wilson loop for small  $Q = 1$  instanton (blue). Large instanton (black) splits into three separate kink-instantons. Each wraps one-third-way around the cylinder.

# Fractionalized Instantons in $\mathbb{CP}^{N-1}$

- fundamental fractionalized instantons with  $Q = \frac{1}{N}$
- bosonic zero modes:

$$2N \xrightarrow{\text{short-distance}} 2 + 1 + (2N - 3) = (\mathbf{a}_I \in \mathbb{R}^2) + (\rho \in \mathbb{R}^+) + (\text{orient.})$$

$$2N \xrightarrow{\text{long-distance}} N[1 + 1] = N[(a \in \mathbb{R}) + (\phi \in U(1))]$$

# Fractionalized Bions in $\mathbb{CP}^{N-1}$

- bions: topological molecules of instantons/anti-instantons
- characterized by (extended) Cartan matrix (as in YM)
- “orientation” dependence of  $\mathcal{I}\bar{\mathcal{I}}$  interaction:
- charged bions:  $\hat{A}_{ij} < 0$ ; repulsive bosonic interaction

$$\mathcal{B}_{ij} = [\mathcal{K}_i \bar{\mathcal{K}}_j] \sim e^{-S_i(\varphi) - S_j(\varphi)} e^{i\sigma(\alpha_i - \alpha_j)}$$

- neutral bions:  $\hat{A}_{ii} > 0$ ; attractive bosonic interaction

$$\Re \mathcal{B}_{ii} = \Re [\mathcal{K}_i \bar{\mathcal{K}}_i] \sim e^{-2S_i(\varphi)}$$

# Fractionalized Bions in $\mathbb{CP}^{N-1}$

- charged bions:

$$\mathcal{A}_{ij} = \mathcal{A}_i \mathcal{A}_j \left( \frac{\alpha_i \cdot \alpha_j}{2} \right)^{2N_f} \left( \frac{g^2}{2L} \right)^{2N_f} 2 \int_0^\infty d\tau e^{-V_{\text{eff}}^{ij}(\tau)}$$

where ( $\xi \equiv \frac{2\pi}{NL}$ )

$$V_{\text{eff}}^{ij}(\tau) = -8\xi \frac{\alpha_i \cdot \alpha_j}{g^2} e^{-\xi\tau} + 2N_f \xi \tau$$

- characteristic scale dominating the integral:

$$\tau^* = \frac{1}{\xi} \log \left( \frac{4\pi}{g^2 N N_f} \right) \quad , \quad r_b = r_k \log \left( \frac{4\pi}{g^2 N N_f} \right) \quad N_f \geq 1$$

- quasi-zero mode integral:

$$I(g^2) = \int_0^\infty d\tau \exp \left[ - \left( \frac{4\xi}{g^2} e^{-\xi\tau} + 2N_f \xi \tau \right) \right] = \left( \frac{g^2}{4\xi} \right)^{2N_f} \int_0^{\frac{4\xi}{g^2}} du e^{-u} u^{2N_f - 1}$$

$$\xrightarrow[g^2 \ll 1]{} \left( \frac{g^2}{4\xi} \right)^{2N_f} \Gamma(2N_f) = \left( \frac{g^2 N}{8\pi} \right)^{2N_f} \Gamma(2N_f)$$

## Fractionalized Bions in $\mathbb{CP}^{N-1}$

- neutral bions:

$$\tilde{I}(g^2) = \int_0^\infty d\tau \exp \left[ - \left( -\frac{8\xi}{g^2} e^{-\xi\tau} + 2N_f \xi \tau \right) \right]$$

- both bosonic and fermionic zero mode induced interactions are attractive (as in gauge theory)
- semi-classical  $[\mathcal{K}_i \bar{\mathcal{K}}_i]$  configuration seems meaningless

N.B.  $[\mathcal{K}_i \bar{\mathcal{K}}_i]$  has same quantum nos. as pert. vacuum

- generalized BZJ-prescription: deform the contour of integration, or equivalently, rotate  $g^2 \rightarrow g^2 e^{i\theta}$

$$\tilde{I}(g^2, N_f) \rightarrow I(-g^2, N_f) = \left( -\frac{g^2 N}{8\pi} \right)^{2N_f} \Gamma(2N_f)$$

- $N_f = 0$ : ambiguous result:

$$\tilde{I}(g^2, N_f = 0) = \left( \log \left( -\frac{g^2 N}{8\pi} \right) - \gamma \right) = I(g^2) \pm i\pi$$

# Fractionalized Bions in $\mathbb{CP}^{N-1}$

- neutral bions: same ambiguity as in bosonic QM (Bogomolny)
- kink-anti-kink amplitude is two-fold ambiguous:

$$[\mathcal{K}_i \bar{\mathcal{K}}_i]_{\theta=0^\pm} = \left( \log \left( \frac{g^2 N}{8\pi} \right) - \gamma \right) 2\mathcal{A}_i^2 e^{-2S_0} \pm i\pi 2\mathcal{A}_i^2 e^{-2S_0}$$

# Perturbation Theory in Twisted $\mathbb{CP}^{N-1}$

- small radius limit: effective QM Hamiltonian

$$H_{\alpha_k}^{\text{zero}} = \frac{g^2}{2} P_\theta^2 + \frac{\xi^2}{2g^2} \sin^2 \theta + \frac{g^2}{2 \sin^2 \theta} P_\phi^2, \quad \xi = \frac{2\pi}{N}, \quad (\text{set } L = 1)$$

- Born-Oppenheimer approximation: drop high  $\phi$ -sector modes  
effective Mathieu equation:

$$\psi'' + \left( p + \frac{\xi^2}{2g^2} \cos(2g\theta) \right) \psi = 0, \quad p = 2E - \frac{\xi^2}{2g^2}$$

- Stone-Reeve (Bender-Wu methods):

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \quad a_q \sim -\frac{2}{\pi} \left( \frac{1}{4\xi} \right)^q q! \left( 1 - \frac{5}{2q} + O(q^{-2}) \right)$$

- non-Borel summable!

# Perturbation Theory in Twisted $\mathbb{CP}^{N-1}$

- Stone-Reeve (Bender-Wu methods):

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \quad a_q \sim -\frac{2}{\pi} \left( \frac{1}{4\xi} \right)^q q! \left( 1 - \frac{5}{2q} + O(q^{-2}) \right)$$

- lateral Borel summation  $\Rightarrow$

$$\begin{aligned} \mathcal{S}_{0^\pm} \mathcal{E}(g^2) &= \frac{1}{g^2} \int_{C_\pm} dt B\mathcal{E}(t) e^{-t/g^2} = \Re \mathcal{S}\mathcal{E}(g^2) \mp i \frac{8\xi}{g^2} e^{-\frac{4\xi}{g^2}} \\ &= \Re \mathbb{B}_0 \mp i \frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \end{aligned}$$

# BZJ cancellation in Twisted $\mathbb{CP}^{N-1}$

- perturbative sector: lateral Borel-Écalle summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_{\pm}} dt B\mathcal{E}(t) e^{-t/g^2} = \text{Re } B\mathcal{E}(g^2) \mp i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

- non-perturbative sector: bion-bion amplitudes

$$[\mathcal{K}_i \bar{\mathcal{K}}_i]_{\pm} = \left( \ln \left( \frac{g^2 N}{8\pi} \right) - \gamma \right) \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \pm i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

exact cancellation !

application of resurgence to nontrivial QFT

# Graded Resurgence Triangle and Extended SUSY

extended SUSY: no superpotential; no bions; no condensates

$[0, 0]$

$[1, 1]$        $[1, -1]$

$[2, 2]$        $\emptyset$        $[2, -2]$

$[3, 3]$        $\emptyset$        $\emptyset$        $[3, -3]$

$[4, 4]$        $\emptyset$        $\emptyset$        $\emptyset$        $[4, -4]$

$\ddots$

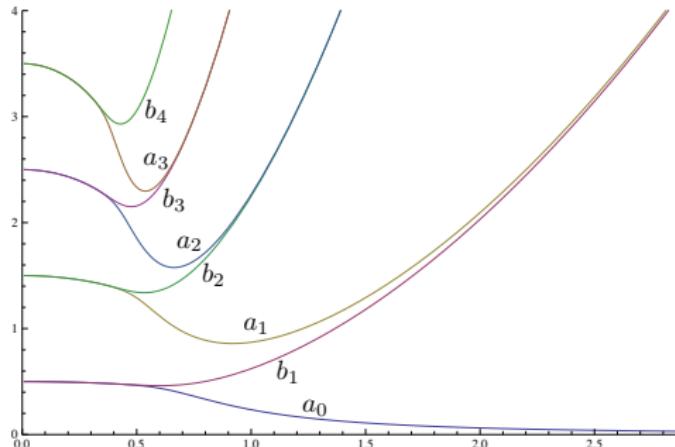
$\vdots$

$\ddots$

no cancellations can occur

$\Rightarrow$  perturbative expansions must be Borel summable !

# Microscopic Origin of Mass Gap in Twisted $\mathbb{CP}^{N-1}$



$$h = \frac{\xi}{2g^2} = \frac{2\pi}{N(2g^2)}$$

$$\Delta E_n = \frac{g^2}{2} (b_{n+1}(h^2) - a_n(h^2))$$

$$= \frac{g^2}{2} \left( \frac{2^{4n+5}}{n!} \left( \frac{2}{\pi} \right)^{\frac{1}{2}} h^{n+\frac{3}{2}} e^{-4h} \left( 1 - \frac{6n^2 + 14n + 7}{32h} + O\left(\frac{1}{h^2}\right) \right) \right)$$

$$m_g = \frac{C}{Ng^2} \left( 1 - \frac{7Ng^2}{32\pi} + O((Ng^2)^2) \right) e^{-\frac{4\pi}{Ng^2}} \sim e^{-S_I/N}$$

## Lecture 3

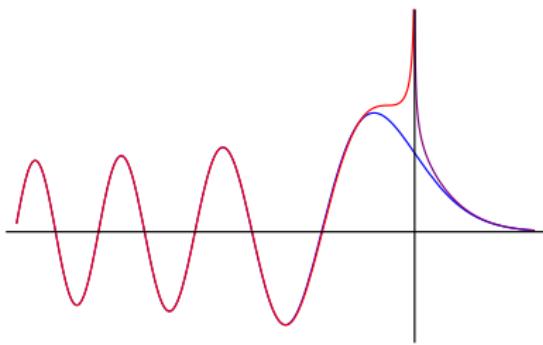
- ▶ Stokes Phenomenon
- ▶ Uniform WKB and origin of trans-series structure
- ▶ all non-perturbative orders from perturbation theory

# The Stokes Phenomenon

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* By G. G. STOKES, M.A., D.C.L., Sec. R.S., Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.

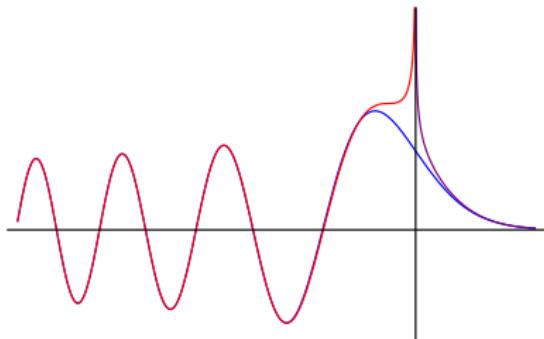
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[Read May 11, 1857.]



# The Stokes Phenomenon

- supernumerary rainbows



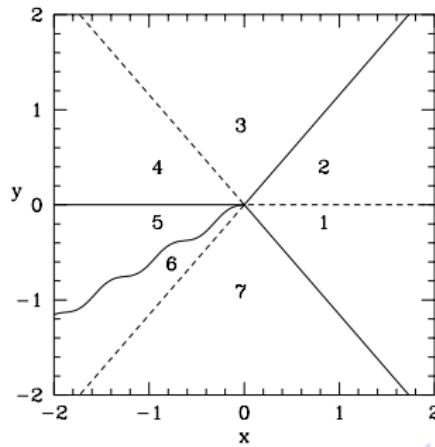
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\frac{1}{3}t^3 + xt)} dt \sim \begin{cases} \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}, & x \rightarrow +\infty \\ \frac{\sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}(-x)^{1/4}}, & x \rightarrow -\infty \end{cases}$$

# The Stokes Phenomenon

- Stokes:
- how to reconcile two exponentials in one region with one exponential in another region?
  - how to reconstruct an analytic solution from non-analytic approximations?

Stokes line: real exponentials

anti-Stokes line: imaginary exponentials



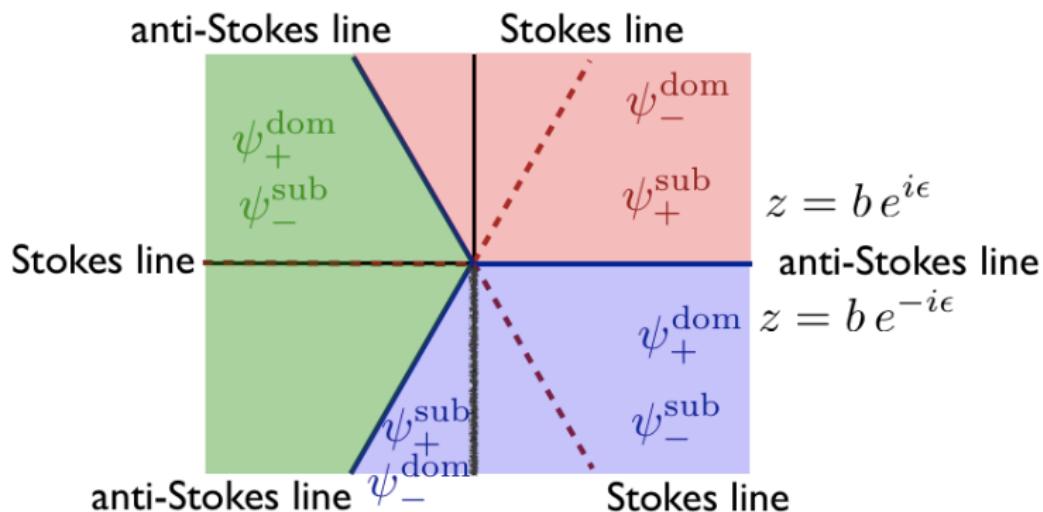
# The Stokes Phenomenon

$$\hbar^2 \psi'' + Q^2 \psi = 0$$

Stokes phenomenon: suppose  $Q$  has a simple zero at  $z=0$

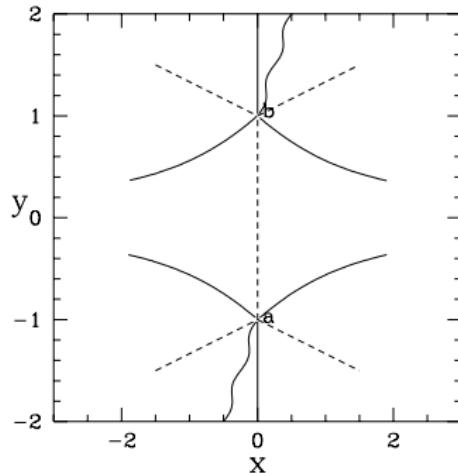
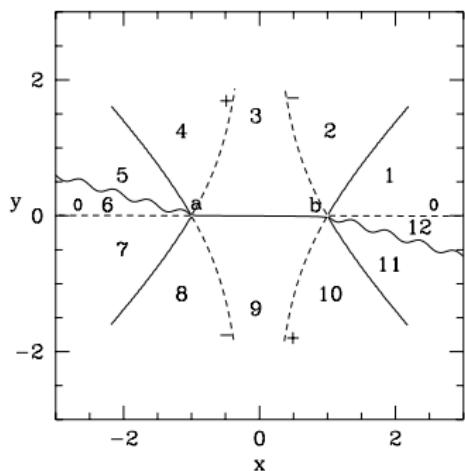
$$\psi_{\pm} \sim \frac{e^{\pm iz^{3/2}}}{z^{1/4}}$$

$$\psi_{\pm} \sim e^{\pm ib^{3/2}(\cos(\frac{3\epsilon}{2})+i\sin(\frac{3\epsilon}{2}))}$$



# The Stokes Phenomenon

- Stokes:
- how to reconcile two exponentials in one region with one exponential in another region?
  - how to reconstruct an analytic solution from non-analytic approximations?



# The Stokes Phenomenon

- Stokes:
- how to reconcile two exponentials in one region with one exponential in another region?
  - how to reconstruct an analytic solution from non-analytic approximations?

different exponentials turn on/off crossing between sectors

- universal smooth behavior (Stokes, Berry)

*The inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficients changed. The range during which the inferior term remains in a mist decreases indefinitely as the [asymptotic parameter] increases indefinitely.*

G. G. Stokes, 1902

- intricate monodromy behaviour

## Universal large-order WKB

- Liouville-Green (WKB) approximation:

$$\hbar^2 \psi'' + Q(x) \psi(x) = 0 \quad \rightarrow \quad \psi_{\pm}(x) \sim \frac{e^{\pm i/\hbar \int^x \sqrt{Q}}}{Q^{1/4}} (1 + \dots)$$

- all-orders expansion: Dingle's universal large-order form:

$$\psi_{\pm}(S) \sim \frac{1}{\sqrt{S'}} e^{\pm i S/\hbar} \sum_{n=0}^{\infty} n! \left( \frac{\pm i \hbar}{2 S} \right)^n$$

- “singulant” variable:  $S = \int^x \sqrt{Q}$
- exponential asymptotics of special functions, and of wavefunctions

# Resurgence and Analytic Continuation

resurgence can be viewed as a method for making asymptotic expansions consistent with global analytic continuation properties

e.g.: asymptotics of special functions

# Resurgence: Exponential Asymptotics of Special Functions

- zero-dimensional partition functions

$$\begin{aligned} Z_1(\lambda) &= \int_{-\infty}^{\infty} dx e^{-\frac{1}{2\lambda} \sinh^2(\sqrt{\lambda}x)} = \frac{1}{\sqrt{\lambda}} e^{\frac{1}{4\lambda}} K_0\left(\frac{1}{4\lambda}\right) \\ &\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^n (2\lambda)^n \frac{\Gamma(n + \frac{1}{2})^2}{n! \Gamma\left(\frac{1}{2}\right)^2} \quad \text{Borel-summable} \end{aligned}$$

$$\begin{aligned} Z_2(\lambda) &= \int_0^{\pi/\sqrt{\lambda}} dx e^{-\frac{1}{2\lambda} \sin^2(\sqrt{\lambda}x)} = \frac{\pi}{\sqrt{\lambda}} e^{-\frac{1}{4\lambda}} I_0\left(\frac{1}{4\lambda}\right) \\ &\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (2\lambda)^n \frac{\Gamma(n + \frac{1}{2})^2}{n! \Gamma\left(\frac{1}{2}\right)^2} \quad \text{non-Borel-summable} \end{aligned}$$

- connection formula:  $K_0(e^{\pm i\pi} |z|) = K_0(|z|) \mp i\pi I_0(|z|)$

# Resurgence: Exponential Asymptotics of Special Functions

- Borel summation

$$Z_1(\lambda) = \sqrt{\frac{\pi}{2}} \frac{1}{2\lambda} \int_0^\infty dt e^{-\frac{t}{2\lambda}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -t\right)$$

- lateral Borel summation

$$\begin{aligned} & Z_1(e^{i\pi}\lambda) - Z_1(e^{-i\pi}\lambda) \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{2\lambda} \int_1^\infty dt e^{-\frac{t}{2\lambda}} \left[ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; t - i\varepsilon\right) - {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; t + i\varepsilon\right) \right] \\ &= -(2i) \sqrt{\frac{\pi}{2}} \frac{1}{2\lambda} e^{-\frac{1}{2\lambda}} \int_0^\infty dt e^{-\frac{t}{2\lambda}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; -t\right) \\ &= -2i e^{-\frac{1}{2\lambda}} Z_1(\lambda) \end{aligned}$$

- connection formula:  $Z_1(e^{\pm i\pi}\lambda) = Z_2(\lambda) \mp i e^{-\frac{1}{2\lambda}} Z_1(\lambda)$

# Resurgence in Nonlinear ODEs

what changes going from linear to nonlinear ODE's ?

- Painlevé functions are generalization of special functions to nonlinear ODE's: many physical applications: statistical physics, optics, QFT, strings, ...
- resurgent trans-series are the natural language for their asymptotics

see: Mariño, Schiappa, Aniceto, Pasquetti, Vonk, ...

# Resurgence in Nonlinear ODEs

- physical example: Painlevé I:
  - (i) all-genus solution of  $c=0$  2d gravity
  - (ii) double-scaling limit of quartic matrix model
- perturbative amplitudes generated by series solution of Painlevé I:  $\mathcal{F}''(z) = u(z)$ , where

$$u^2(z) - \frac{1}{6}u'' = z$$

- non-perturbative results (Shenker, David, ...):

$$\mathcal{F}^{(1)}(z) = \frac{i}{8\sqrt{\pi} 3^{3/4} z^{5/8}} \exp \left[ -\frac{8\sqrt{3}}{5} z^{5/4} \right] \left( 1 - \frac{37}{64\sqrt{3}} \frac{1}{z^{5/4}} + \dots \right)$$

- resurgence framework ...

## Resurgence in Nonlinear ODEs: e.g. Painlevé I

Painlevé I:  $u^2(z) - \frac{1}{6}u'' = z$

►  $u = \sqrt{z} w(z)$

►  $\xi = z^\alpha$ , matching powers  $\Rightarrow \alpha = \frac{5}{4}$ :

$$\frac{d^2w}{d\xi^2} + \frac{1}{\xi} \frac{dw}{d\xi} - \frac{4}{25} \frac{w}{\xi^2} = \frac{96}{25}(w^2 - 1)$$

- ansatz:  $w \sim \frac{e^{-\xi}}{\sqrt{\xi}} \sum_{n=0}^{\infty} \frac{a_n^{(0)}}{\xi^{2n}}$   $\Rightarrow a^{(0)} = \left\{ 1, -\frac{1}{48}, -\frac{49}{4608}, \dots \right\}$
- large-order behavior non-Borel-summable:

$$a_n^{(0)} \sim -\Gamma\left(2n - \frac{1}{2}\right) \left(\frac{8\sqrt{3}}{5}\right)^{-2n+1/2} \frac{3^{1/4}}{2\pi^{3/2}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

- imaginary part:  $\text{Im} \sim \pm \frac{e^{-A\xi}}{\sqrt{\xi}}$  ,  $A \equiv \frac{8\sqrt{3}}{5}$

## Resurgence in Nonlinear ODEs: e.g. Painlevé I

$$\text{Painlevé I: } \frac{d^2w}{d\xi^2} + \frac{1}{\xi} \frac{dw}{d\xi} - \frac{4}{25} \frac{w}{\xi^2} = \frac{96}{25}(w^2 - 1)$$

- ▶ perturbative ansatz:  $w \sim \frac{e^{-\xi}}{\sqrt{\xi}} \sum_{n=0}^{\infty} \frac{a_n^{(0)}}{\xi^{2n}}$
- ▶ non-perturbative term:  $\text{Im } \sim \pm \frac{e^{-A\xi}}{\sqrt{\xi}}$  ,  $A \equiv \frac{8\sqrt{3}}{5}$
- ▶ nonlinearity  $\Rightarrow$  also need  $e^{\mp l A \xi}$  terms,  $l \in \mathbb{Z}^+$
- ▶ *double trans-series ansatz:*

$$w \sim \frac{e^{-\xi}}{\sqrt{\xi}} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \sigma_1^l \sigma_2^k e^{-(l-k)A\xi} \mathcal{F}_{(l,k)} \left( \frac{1}{\xi} \right)$$

- ▶ resurgence  $\Rightarrow$   $\mathcal{F}_{(l,k)} \left( \frac{1}{\xi} \right)$  fluctuations entwined
- ▶ full resurgent details still being investigated (sectors and analytic continuation)

# Resurgence in Nonlinear ODEs: e.g. Painlevé II

Painlevé II:

$$u'' - 2u^3(z) + 2z u(z) = 0$$

perturbative solution is non-Borel-summable

⇒ trans-series solution(s)

- ▶ Tracy-Widom law for statistics of max. eigenvalue for Gaussian random matrices
- ▶ double-scaling limit in 2d Yang-Mills
- ▶ double-scaling limit in unitary matrix models
- ▶ all-genus solution of 2d supergravity

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

- origin of trans-series structure (GD, Ünsal, [1306.4405](#), [1401.5202](#))

$$-g^4 \frac{d^2}{dy^2} \psi(y) + V(y) \psi(y) = g^2 E \psi(y)$$

where

$$V_{\text{DW}}(y) = y^2(1+y)^2 , \quad V_{\text{SG}}(y) = \sin^2(y)$$

- weak coupling: degenerate harmonic classical vacua
  - non-perturbative effects:  $g^2 \leftrightarrow \hbar \Rightarrow \exp\left(-\frac{c}{g^2}\right)$
  - approximately harmonic
- ⇒ uniform WKB with parabolic cylinder functions

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

$$\frac{d^2\psi}{dx^2} + \frac{p^2(x)}{\hbar^2} \psi(x) = 0$$

- uniform WKB: “comparison functions”

uniform approxs. are smooth at turning points ( $p = 0$ )

$$\psi = \frac{1}{\sqrt{S'(x)}} \phi(S(x)) \quad \Rightarrow \quad \frac{d^2\phi}{dS^2} + \frac{P^2(S)}{\hbar^2} \phi(S) = 0$$

- ▶  $P^2(S) = \text{constant} \rightarrow \text{usual WKB: } \psi(x) = \frac{e^{i \int p}}{\sqrt{p(x)}}$
- ▶  $P^2(S) = S \rightarrow \text{uniform } \psi(x) = \frac{(\int^x p)^{1/6}}{\sqrt{p(x)}} Ai\left(\frac{3}{2} \left(\int^x p\right)^{2/3}\right)$
- ▶  $P^2(S) = S^2 \rightarrow \text{uniform } \psi = \frac{(\int^x p)^{1/4}}{\sqrt{p(x)}} D_{\frac{(E-1)}{2}} \left((2 \int^x p)^{1/2}\right)$

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

- uniform WKB ansatz ( $\nu$  a parameter)

$$\psi(y) = \frac{D_\nu \left( \frac{1}{g} u(y) \right)}{\sqrt{u'(y)}}$$

- nonlinear equation for  $u(y)$ :

$$V(y) - \frac{1}{4} u^2 (u')^2 - g^2 E + g^2 \left( \nu + \frac{1}{2} \right) (u')^2 + \frac{g^4}{2} \sqrt{u'} \left( \frac{u''}{(u')^{3/2}} \right)' = 0$$

- perturbative expansion  $\rightarrow u(y)$  and energy:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

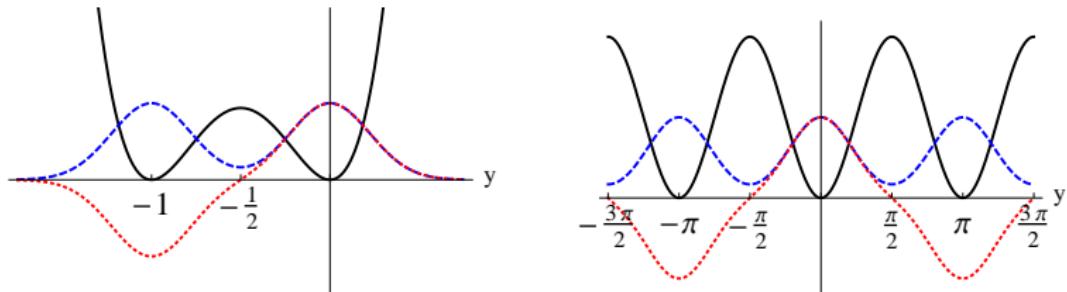
- $\nu = N$ : Rayleigh-Schrödinger perturbation theory:

$$E(\nu = N, g^2) \equiv E_{\text{pert. theory}}^{(N)}(g^2)$$

- not Borel summable !

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

- global analysis  $\Rightarrow$  boundary conditions:



- midpoint  $\sim \frac{1}{g}$ ; non-Borel summability  $\Rightarrow g^2 \rightarrow e^{\pm i\epsilon} g^2$

$$D_\nu(z) \sim z^\nu e^{-z^2/4} (1 + \dots) + e^{\pm i\pi\nu} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} z^{-1-\nu} e^{z^2/4} (1 + \dots)$$

→ exact quantization condition

$$\frac{1}{\Gamma(-\nu)} \left( \frac{e^{\pm i\pi} 2}{g^2} \right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi g^2}} \mathcal{F}(\nu, g^2)$$

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

- exact quantization condition

$$\frac{1}{\Gamma(-\nu)} \left( \frac{e^{\pm i\pi} 2}{g^2} \right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi g^2}} \mathcal{F}(\nu, g^2)$$

- expand  $\nu = N + \delta\nu$ :

$$LHS = -N! \left( \frac{e^{\pm i\pi} 2}{g^2} \right)^{-N} \left\{ \delta\nu - \left[ \gamma + \ln \left( \frac{e^{\pm i\pi} 2}{g^2} \right) - h_N \right] (\delta\nu)^2 + \dots \right\}$$

$\Rightarrow$   $\nu$  is only exponentially close to  $N$  (here  $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi g^2}}$ ):

$$\begin{aligned} \nu &= N + \frac{\left( \frac{2}{g^2} \right)^N \mathcal{F}(N, g^2)}{N!} \xi \\ &\quad - \frac{\left( \frac{2}{g^2} \right)^{2N}}{(N!)^2} \left[ \mathcal{F} \frac{\partial \mathcal{F}}{\partial N} + \left( \ln \left( \frac{e^{\pm i\pi} 2}{g^2} \right) - \psi(N+1) \right) \mathcal{F}^2 \right] \xi^2 + O(\xi^3) \end{aligned}$$

- insert:  $E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu) \Rightarrow$  trans-series!

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

conclusion:

for QM problems with degenerate harmonic vacua, the trans-series form of the exact expressions for energy eigenvalues arises from the (resurgent) analytic continuation properties of the parabolic cylinder functions

generic and universal

Zinn-Justin/Jentschura conjecture: generate entire trans-series from

- (i) perturbative expansion  $E = E(\nu, g^2)$  ( $\nu = \nu(E, g^2)$ )
- (ii) single-instanton fluctuation function  $\mathcal{F}(E, g^2)$
- (iii) rule connecting neighbouring vacua (parity, Bloch, ...)

uniform WKB approach explains why this is the case

# Connecting Perturbative/Non-Perturbative Sectors

(GD,Unsal,  
1401.5202)

Zinn-Justin/Jentschura:  $\mathcal{F}(E, g) \sim \exp[-A(E, g)/2]$

- perturbative function:  $(B \equiv \nu + \frac{1}{2})$

$$\begin{aligned} B_{\text{DW}}(E, g) = & E + g \left( 3E^2 + \frac{1}{4} \right) + g^2 \left( 35E^3 + \frac{25}{4}E \right) \\ & + g^3 \left( \frac{1155}{2}E^4 + \frac{735}{4}E^2 + \frac{175}{32} \right) + \dots \end{aligned}$$

- non-perturbative function:

$$\begin{aligned} A_{\text{DW}}(E, g) = & \frac{1}{3g} + g \left( 17E^2 + \frac{19}{12} \right) + g^2 \left( 227E^3 + \frac{187E}{4} \right) \\ & + g^3 \left( \frac{47431}{12}E^4 + \frac{34121}{24}E^2 + \frac{28829}{576} \right) + \dots \end{aligned}$$

- uniform WKB  $\rightarrow E = E(B, g)$

## Connecting Perturbative/Non-Perturbative Sectors

- perturbative function:

$$\begin{aligned} E_{\text{DW}}(B, g) = & B - g \left( 3B^2 + \frac{1}{4} \right) - g^2 \left( 17B^3 + \frac{19}{4}B \right) - \\ & g^3 \left( \frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32} \right) - g^4 \left( \frac{10689}{4}B^5 + \frac{23405}{8}B^3 + \frac{22709}{64}B \right) - \dots \end{aligned}$$

- non-perturbative function ( $\mathcal{F} \sim \exp[-A/2]$ ):

$$\begin{aligned} A_{\text{DW}}(B, g) = & \frac{1}{3g} + g \left( 17B^2 + \frac{19}{12} \right) + g^2 \left( 125B^3 + \frac{153B}{4} \right) + \\ & g^3 \left( \frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576} \right) + g^4 \left( \frac{87549}{4}B^5 + \frac{50715}{2}B^3 + \frac{217663}{64}B \right) \end{aligned}$$

- simple relation:

$$\frac{\partial E_{\text{DW}}}{\partial B} = -6Bg - 3g^2 \frac{\partial A_{\text{DW}}}{\partial g}$$

## Connecting Perturbative/Non-Perturbative Sectors

- similar relations for Sine-Gordon, Fokker-Planck (SUSY DW) and  $O(d)$  AHO, ...
- general expression:

$$\frac{\partial E}{\partial B} = -\frac{g}{2S} \left( 2B + g \frac{\partial A}{\partial g} \right)$$

- reason: consistency with resurgent trans-series structure at higher non-perturbative order
- implication: non-perturbative function  $A(B, g)$  completely determined by perturbative expression  $E(B, g)$

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

$$\begin{aligned} f(g^2) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} a_{n,k,l} g^{2n} \left[ \exp\left(-\frac{S}{g^2}\right) \right]^k \left[ \log\left(-\frac{1}{g^2}\right) \right]^l \\ &= E_{\text{pert}}(g^2) + e^{-S/g^2} f_1(g^2) \\ &\quad + e^{-2S/g^2} \left( f_2(g^2) + \ln\left(-\frac{1}{g^2}\right) \tilde{f}_2(g^2) \right) \\ &\quad + e^{-3S/g^2} \left( f_3(g^2) + \ln\left(-\frac{1}{g^2}\right) \tilde{f}_3(g^2) + \ln^2\left(-\frac{1}{g^2}\right) \bar{f}_3(g^2) \right) \\ &\quad + \dots \end{aligned}$$

uniform WKB  $\Rightarrow$

- (i) all  $f_i$  come from a single function  $\mathcal{F}$
- (ii) moreover can be deduced immediately from  $E(N, g^2)$

# Uniform WKB and Resurgent Trans-Series for Eigenvalues

Zinn-Justin/Jentschura: generate entire trans-series from

- (i) perturbative expansion  $E = E(\nu, g^2)$
- (ii) single-instanton fluctuation function  $\mathcal{F}(\nu, g^2)$
- (iii) rule connecting neighbouring vacua (parity, Bloch, ...)

Dunne/Ünsal: **perturbation theory generates everything!**

$$\mathcal{F}(\nu, g^2) = \exp \left[ S \int_0^{g^2} \frac{dg^2}{g^4} \left( \frac{\partial E}{\partial \nu} - 1 + \frac{(\nu + \frac{1}{2}) g^2}{S} \right) \right]$$

dramatic implication: **all orders of the multi-instanton trans-series are encoded in perturbation theory of the fluctuations about the perturbative vacuum !!!**

why ? turn to path integrals ....

## Lecture 4

- ▶ Darboux's theorem and resurgent steepest descents analysis
- ▶ QM resurgence in terms of saddles
- ▶ analytic continuation and complex saddles
- ▶ non-perturbative physics without instantons

# Analytic Continuation of Path Integrals

*The shortest path between two truths in the real domain  
passes through the complex domain*

*Jacques Hadamard, 1865 - 1963*

## Analytic Continuation of Path Integrals: Darboux Theorem

- zero dimensions: all-orders steepest descents of contour integrals (Berry/Howls: *hyperasymptotics*)

$$I^{(n)}(k) = \int_{C_n} dz e^{-k f(z)}$$

- separate out fluctuations:

$$I^{(n)}(k) = \frac{1}{\sqrt{k}} e^{-k f_n} T^{(n)}(k) \quad , \quad T^{(n)}(k) \equiv \sqrt{k} \int_{C_n} dz e^{-k(f(z)-f_n)}$$

- asymptotic expansion of fluctuations about the saddle  $n$ :

$$T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^r}$$

# Analytic Continuation of Path Integrals: Darboux Theorem

- singulant variable:  $u \equiv k(f(z) - f_n)$

M. V. Berry and C. J. Howls

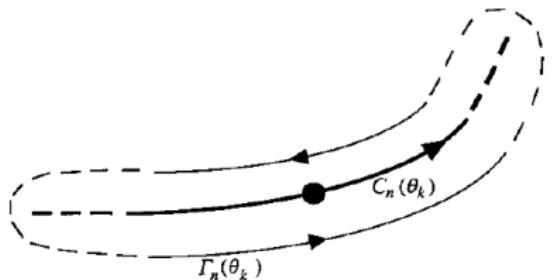
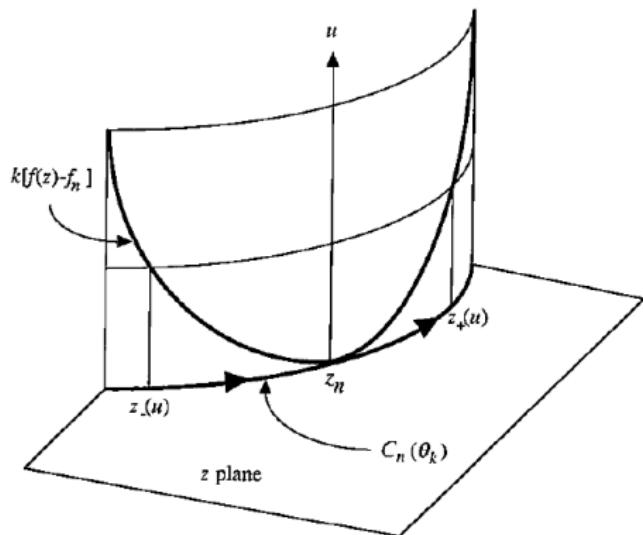


Figure 1. Double-valued mapping (equation (4)) from  $z$  to  $u$ .

. Steepest path  $C_n(\theta_k)$  through saddle  $n$ , and loop  $\Gamma_n(\theta_k)$  encloses

# Analytic Continuation of Path Integrals: Darboux Theorem

- singulant variable:

$$u \equiv k(f(z) - f_n)$$

- noting double-valuedness

$$\begin{aligned} T^{(n)}(k) &= \int_0^\infty du \frac{e^{-u}}{\sqrt{k}} \left( \frac{1}{f'(z_+(u))} - \frac{1}{f'(z_-(u))} \right) \\ &= \frac{1}{2\pi i} \int_0^\infty du \frac{e^{-u}}{\sqrt{u}} \oint_{\Gamma_n} dz \frac{\sqrt{f(z) - f_n}}{f(z) - f_n - u/k} \end{aligned}$$

- now expand in  $\frac{1}{k} \Rightarrow$  fluctuation coefficients:

$$T_r^{(n)} = \frac{(r - \frac{1}{2})!}{2\pi i} \oint_{\Gamma_n} dz \frac{1}{(f(z) - f_n)^{r+1/2}}$$

- universal factorial divergence of fluctuations (Darboux)

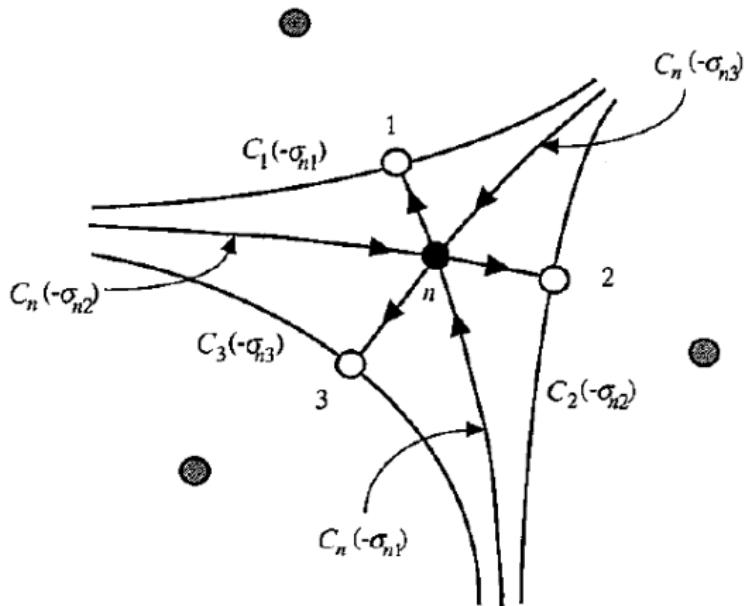
# Analytic Continuation of Path Integrals: Darboux Theorem

- deforming contours:

$$\oint_{\Gamma_n} dz(\dots) = \sum_{m \text{ adjacent}} (-1)^{\gamma_{nm}} \int_{C_m} dz (\dots)$$

# Analytic Continuation of Path Integrals: Darboux Theorem

deforming contours:



# Analytic Continuation of Path Integrals: Darboux Theorem

- deforming contours:

$$\oint_{\Gamma_n} dz(\dots) = \sum_{m \text{ adjacent}} (-1)^{\gamma_{nm}} \int_{C_m} dz (\dots)$$

- new singulant variables along each contour  $C_m$ :

$$T^{(n)}(k) = \frac{1}{2\pi i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - v/(k F_{nm})} T^{(m)} \left( \frac{v}{F_{nm}} \right)$$

- exact resurgent relation between fluctuations about  $n^{\text{th}}$  saddle and about neighboring saddles  $m$

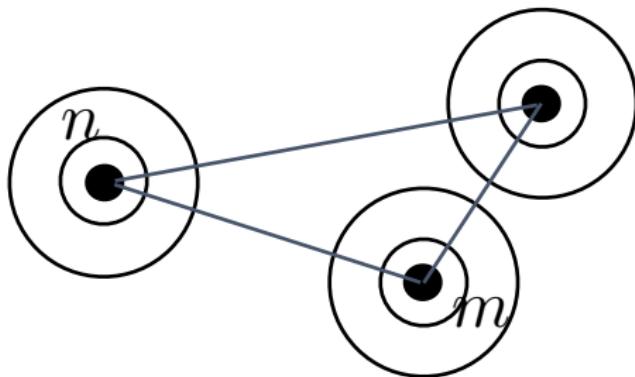
expand fluctuations  $T^{(n)}(k) = \sum_r \frac{T_r^{(n)}}{k^r} \Rightarrow$

$$T_r^{(n)} = \frac{(r-1)!}{2\pi i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[ T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right]$$

# Resurgence

*resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities*

*J. Écalle, 1980*

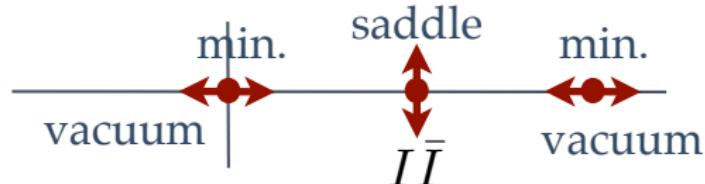


## Analytic Continuation of Path Integrals: Darboux Theorem

zero dim. partition function for periodic potential  
 $V(z) = \sin^2(z)$ :

$$I(k) = \int_0^\pi dz e^{-k \sin^2(z)}$$

two saddle points:  $z_0 = 0$  and  $z_1 = \frac{\pi}{2}$ .



## Analytic Continuation of Path Integrals: Darboux Theorem

- zero dim. partition function for periodic potential

$$V(z) = \sin^2(z):$$

$$I(k) = \int_0^\pi dz e^{-k \sin^2(z)}$$

- two saddle points:  $z_0 = 0$  and  $z_1 = \frac{\pi}{2}$ .

$$\begin{aligned} I^{(0)}(k) &= \frac{1}{\sqrt{k}} T^{(0)}(k) \quad , \quad T^{(0)}(k) &= \sqrt{k} \int_0^\infty \frac{du}{\sqrt{u}} \frac{e^{-k u}}{\sqrt{1-u}} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)^2}{\sqrt{\pi} \Gamma(n+1)} \frac{1}{k^n} \end{aligned}$$

- factorially divergent, as expected, and non-alternating

$$\begin{aligned} I^{(1)}(k) &= \frac{e^{-k}}{\sqrt{k}} T^{(1)}(k) \quad , \quad T^{(1)}(k) &= i \sqrt{k} \int_0^\infty \frac{du}{\sqrt{u}} \frac{e^{-k u}}{\sqrt{1+u}} \\ &= i \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(n + \frac{1}{2}\right)^2}{\sqrt{\pi} \Gamma(n+1)} \frac{1}{k^n} \end{aligned}$$

## Analytic Continuation of Path Integrals: Darboux Theorem

- large order behavior about saddle  $z_0$ :

$$\begin{aligned} T_r^{(0)} &= \frac{\Gamma\left(r + \frac{1}{2}\right)^2}{\sqrt{\pi} \Gamma(r+1)} \\ &\sim \frac{(r-1)!}{\sqrt{\pi}} \left(1 - \frac{1}{4r} + \frac{1}{32r^2} + \frac{1}{128r^3} + \dots\right) \\ &\sim \frac{(r-1)!}{\sqrt{\pi}} \left(1 - \frac{1/4}{(r-1)} + \frac{9/32}{(r-1)(r-2)} - \frac{75/128}{(r-1)(r-2)(r-3)} + \dots\right) \end{aligned}$$

- low order coefficients about saddle  $z_1$ :

$$T^{(1)}(k) \sim i\sqrt{\pi} \left(1 - \frac{1}{4k} + \frac{9}{32k^2} - \frac{75}{128k^3} + \dots\right)$$

- fluctuations about the two saddles are explicitly related
- resurgence at work!

## Resurgence in Path Integrals: “Functional Darboux Theorem”

- periodic potential:  $V(x) = \frac{1}{g^2} \sin^2(gx)$

- vacuum saddle point

$$c_n \sim n! \left( 1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\text{Im } E \sim \pi e^{-2\frac{1}{2g^2}} \left( 1 - \frac{5}{2}g^2 - \frac{13}{8}g^4 - \dots \right)$$

- double-well potential:  $V(x) = x^2(1-gx)^2$

- vacuum saddle point

$$c_n \sim 3^n n! \left( 1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\text{Im } E \sim \pi e^{-2\frac{1}{6g^2}} \left( 1 - \frac{53}{6}g^2 - \frac{1277}{72}g^4 - \dots \right)$$

## Resurgence in Path Integrals: “Functional Darboux Theorem”

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle

“functional Darboux theorem”

# Resurgence from path integral perspective

- semiclassical expansion of path integral

$$\mathcal{Z}(g^2) = \int \mathcal{D}\phi e^{-S[\phi]} \approx \sum_{\text{saddles } k} F_k(g^2) e^{-\frac{1}{g^2} S_k}$$

**Resurgence:** asymptotic expansions around different saddles of path integral influence one another

- in principle exact

# Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$Z = \int dx e^{-S(x)}$$

- critical points (saddle points):  $\partial S/\partial z = 0$
- steepest descent contour:  $\text{Im } S(z) = \text{constant}$
- contour flow-time parameter  $t$ :

$$\frac{d}{dt} \text{Im } S(z) = \frac{1}{2i} \left( \frac{\partial S}{\partial z} \dot{z} - \frac{\partial \bar{S}}{\partial \bar{z}} \dot{\bar{z}} \right) , \quad \frac{d}{dt} \text{Re } S(z) = \frac{1}{2} \left( \frac{\partial S}{\partial z} \dot{z} + \frac{\partial \bar{S}}{\partial \bar{z}} \dot{\bar{z}} \right)$$

- flow along a steepest decent path:

$$\dot{z} = \frac{\partial \bar{S}}{\partial \bar{z}} \quad \Rightarrow \frac{d}{dt} \text{Im } S(z) = 0 , \quad \frac{d}{dt} \text{Re } S(z) = \left| \frac{\partial S}{\partial z} \right|^2 > 0$$

- monotonic in real part

$$Z = e^{-S_{\text{imag}}(x)} \int_{\Gamma} dz e^{-S_{\text{real}}(z)}$$

# Analytic Continuation of Path Integrals: Lefschetz Thimbles

functional version: path integral

$$\int \mathcal{D}A e^{-\frac{1}{g^2}(S_{\text{real}}[A] + i S_{\text{imag}}[A])} \sim \sum_{\text{thimbles } k} e^{-\frac{i}{g^2} S_{\text{imag}}[A]} \int_{\Gamma_k} \mathcal{D}A e^{-\frac{1}{g^2} S_{\text{real}}[A]}$$

thimble = functional [configurational] steepest descents contour

remaining path integral has real measure: amenable to

- (i) Monte Carlo
- (ii) semiclassical expansion (resurgent relations between thimbles)

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

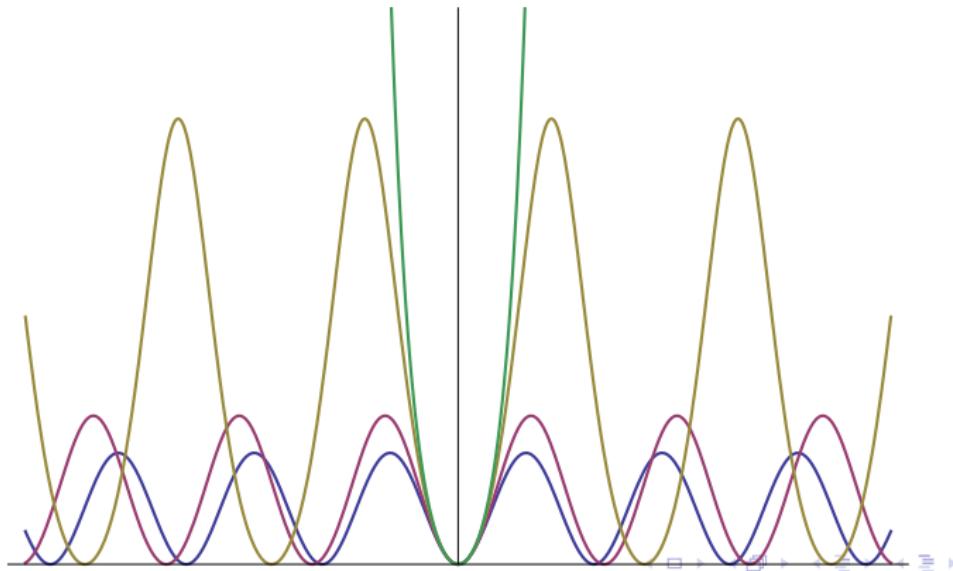
# Path integrals with complex saddles: “ghost instantons”

- elliptic potential:

(Basar, GD, Ünsal, [arXiv:1308.1108](https://arxiv.org/abs/1308.1108))

$$V(z|m) = \text{sd}^2(x|m)$$

interpolates between Sine-Gordon ( $m = 0$ ) and Sinh-Gordon ( $m = 1$ )



## Path integrals with complex saddles: zero dim. prototype

$$V(z|m) = \frac{1}{g^2} \operatorname{sd}^2(g z|m)$$

- duality property:

$$V(z|m)|_{g^2} = V(z|1-m)|_{-g^2}$$

- perturbative series  $\sum_n a_n(m)g^{2n}$  satisfies duality:

$$a_n(m) = (-1)^n a_n(1-m)$$

d=0 partition function:

$$\mathcal{Z}(g^2|m) = \frac{1}{g\sqrt{\pi}} \int_{-\mathbb{K}}^{\mathbb{K}} dz e^{-\frac{1}{g^2} \operatorname{sd}^2(z|m)}$$

## Path integrals with complex saddles: zero dim. prototype

$$\begin{aligned}\mathcal{Z}(g^2|0)\Big|_{\text{pert}} &= 1 + \frac{g^2}{4} + \frac{9g^4}{32} + \frac{75g^6}{128} + \frac{3675g^8}{2048} + \frac{59535g^{10}}{8192} + \dots \\ \mathcal{Z}(g^2|1)\Big|_{\text{pert}} &= 1 - \frac{g^2}{4} + \frac{9g^4}{32} - \frac{75g^6}{128} + \frac{3675g^8}{2048} - \frac{59535g^{10}}{8192} + \dots \\ \mathcal{Z}\left(g^2\left|\frac{1}{4}\right.\right)\Big|_{\text{pert}} &= 1 + \frac{g^2}{8} + \frac{9g^4}{64} + \frac{105g^6}{512} + \frac{1995g^8}{4096} + \frac{48195g^{10}}{32768} + \dots \\ \mathcal{Z}\left(g^2\left|\frac{3}{4}\right.\right)\Big|_{\text{pert}} &= 1 - \frac{g^2}{8} + \frac{9g^4}{64} - \frac{105g^6}{512} + \frac{1995g^8}{4096} - \frac{48195g^{10}}{32768} + \dots \\ \mathcal{Z}\left(g^2\left|\frac{1}{2}\right.\right)\Big|_{\text{pert}} &= 1 + 0g^2 + \frac{3g^4}{32} + 0g^6 + \frac{315g^8}{2048} + 0g^{10} + \dots\end{aligned}$$

- duality relation:  $\mathcal{Z}(g^2|m) = \mathcal{Z}(-g^2|1-m)$

non-alternating for  $m < \frac{1}{2}$       alternating for  $m > \frac{1}{2}$

**puzzles:** Borel summable? “instantons” ?

# Path integrals with complex saddles: zero dim. prototype

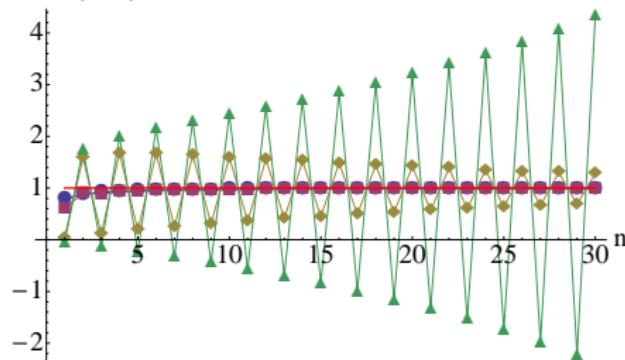
$$\mathcal{Z}(g^2|m) = \frac{2}{g\sqrt{\pi}} \int_0^{\mathbb{K}} dz e^{-\frac{1}{g^2} \text{sd}^2(z|m)}$$

- large-order behavior about 0 from saddle point  $B = \mathbb{K}$ :

$$S_B = \frac{1}{1-m} \quad \Rightarrow \quad a_n \sim \frac{(n-1)!}{\pi S_B^{n+1/2}}$$

- compare with actual series:

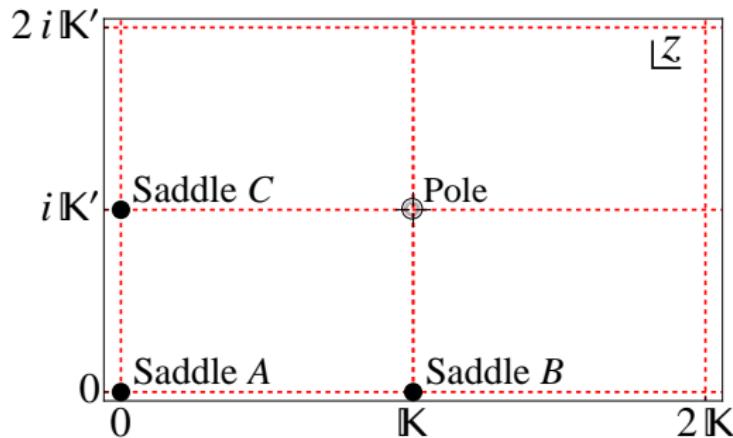
naive ratio ( $d=0$ )



disaster !

## Path integrals with complex saddles: zero dim. prototype

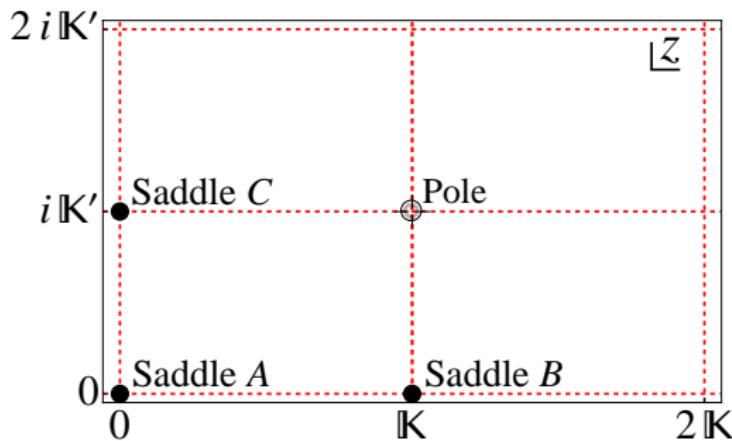
resolution: another saddle off the integration path!



$$S_C = -1/m \quad \Rightarrow \quad a_n \sim \frac{(n-1)!}{\pi} (S_B^{n+1/2} + S_C^{n+1/2})$$

## Path integrals with complex saddles: zero dim. prototype

resolution: another saddle off the integration path!

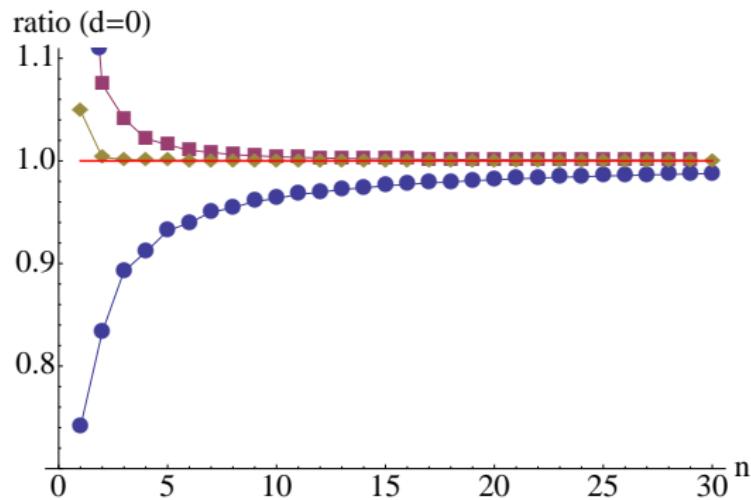


$$S_C = -1/m \quad \Rightarrow \quad a_n \sim \frac{(n-1)!}{\pi} (S_B^{n+1/2} + (-1)^n |S_C|^{n+1/2})$$

# Path integrals with complex saddles: zero dim. prototype

$$a_n \sim \frac{(n-1)!}{\pi} (S_B^{n+1/2} + (-1)^n |S_C|^{n+1/2})$$

⇒ improved asymptotics:



conclusion: perturbation series feels *all* saddles, both real and complex

# Path integrals with complex saddles: zero dim. prototype

the bigger picture:

- associated with each critical point  $z_i$ , there is a unique integration cycle  $\mathcal{J}_i$ , called a *Lefschetz thimble*, along which the phase remains stationary
- around each saddle there is a contribution of the form:

$$\mathcal{I}^{(k)}(\xi|m) = \frac{1}{\sqrt{\pi}} \sqrt{\xi} \int_{\mathcal{J}_k} dz e^{-\xi s d^2(z|m)}$$

- expansions around different saddles are connected via

exact resurgence relation:

$$\mathcal{I}^{(A)}\left(\frac{1}{g^2}|m\right) = \frac{2}{2\pi i} \sum_{k \in \{B,C\}} \int_0^\infty \frac{dv}{v} \frac{1}{1 - g^2 v} \mathcal{I}^{(k)}(v|m)$$

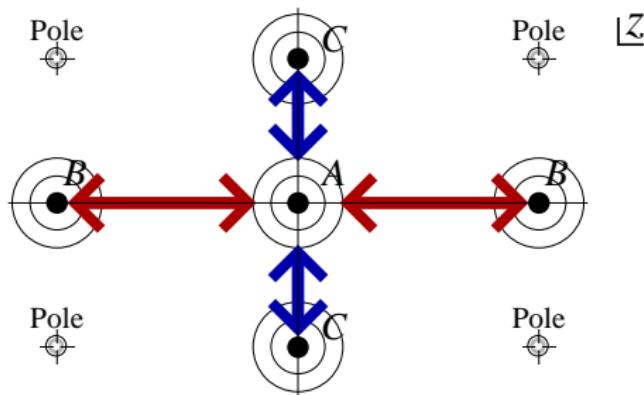
## Path integrals with complex saddles: zero dim. prototype

- most general expansion is a **three-term trans-series**

$$\mathcal{Z}_{\mathfrak{C}}(g^2|m) \equiv \sigma_A \Phi_A(g^2) + \sigma_B e^{-S_B/g^2} \Phi_B(g^2) + \sigma_C e^{-S_C/g^2} \Phi_C(g^2)$$

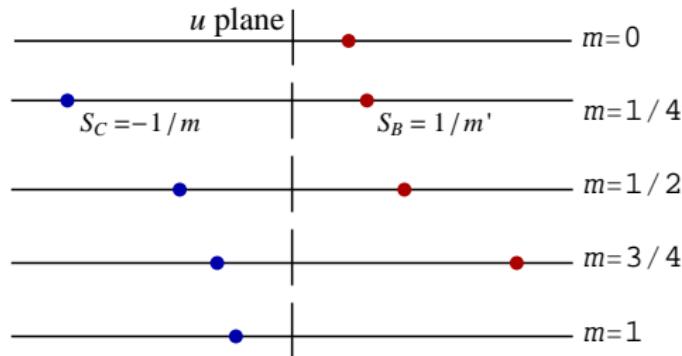
- coefficients of perturbative expansions are connected

$$a_n^{(A)}(m) = \sum_{j=0} \frac{(n-j-1)!}{\pi} \left( \frac{a_j^{(B)}(m)}{S_B^{n-j}} + \frac{a_j^{(C)}(m)}{S_C^{n-j}} \right)$$



# Path integrals with complex saddles: zero dim. prototype

view from the Borel plane:



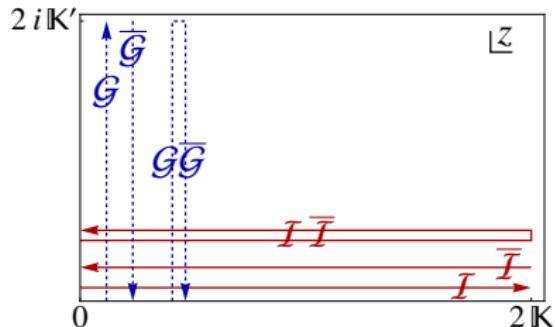
- ‘distance’ in Borel plane,  $\Delta S = S_i - S_j$  (“relative action”) controls divergence of perturbation series  $\Phi_j$
- $m > 1/2$ : closest singularity on  $\mathbb{R}^- \Leftrightarrow$  alternating series  $\Phi_A$
- mimics structure of both UV and IR renormalons

# Ghost Instantons: Quantum Mechanical Path Integrals

quantum mechanics: ordinary integral  $\longrightarrow$  path integral

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}\phi e^{-S[\phi]} = \int \mathcal{D}\phi e^{-\int d\tau \left( \frac{1}{4}\dot{\phi}^2 + \frac{1}{g^2} \text{sd}^2(g\phi|m) \right)}$$

- find *real* and *ghost* instantons



- actions:

$$\frac{S_I(m)}{g^2} = \frac{2 \sin^{-1}(\sqrt{m})}{g^2 \sqrt{mm'}} \geq \frac{2}{g^2} \quad , \quad \frac{S_{\bar{I}}(m)}{g^2} = \frac{2 \sin^{-1}(\sqrt{m'})}{g^2 \sqrt{mm'}} \leq -\frac{2}{g^2}$$

## Ghost Instantons: Quantum Mechanical Path Integrals

$$E^{(0)}(g^2|0) = 1 - \frac{g^2}{4} - \frac{g^4}{16} - \frac{3g^6}{64} - \frac{53g^8}{1024} - \frac{297g^{10}}{4096} - \dots$$

$$E^{(0)}(g^2|1) = 1 + \frac{g^2}{4} - \frac{g^4}{16} + \frac{3g^6}{64} - \frac{53g^8}{1024} - \frac{297g^{10}}{4096} - \dots$$

$$E^{(0)}\left(g^2 \middle| \frac{1}{4}\right) = 1 - \frac{g^2}{8} - \frac{11g^4}{128} - \frac{3g^6}{128} - \frac{889g^8}{32768} - \frac{225g^{10}}{8192} - \dots$$

$$E^{(0)}\left(g^2 \middle| \frac{3}{4}\right) = 1 + \frac{g^2}{8} - \frac{11g^4}{128} + \frac{3g^6}{128} - \frac{889g^8}{32768} + \frac{225g^{10}}{8192} - \dots$$

$$E^{(0)}\left(g^2 \middle| \frac{1}{2}\right) = 1 + 0g^2 - \frac{3g^4}{32} + 0g^6 - \frac{39g^8}{2048} + 0g^{10} - \dots$$

- duality relation:  $E^{(0)}(g^2|m) = E^{(0)}(-g^2|1-m)$

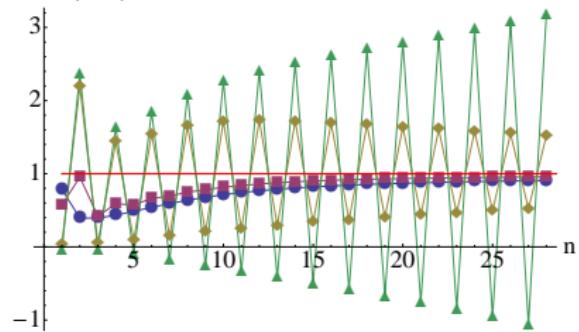
non-alternating for  $m < \frac{1}{2}$       alternating for  $m > \frac{1}{2}$

- very similar to zero-dimensional prototype!

# Ghost Instantons: Quantum Mechanical Path Integrals

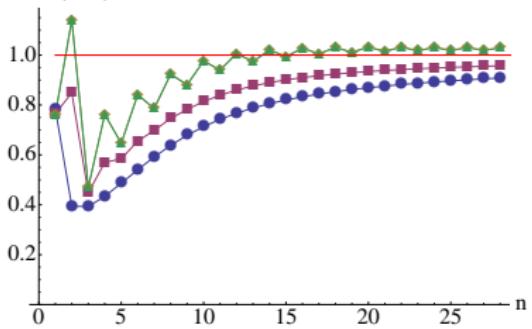
- large order growth of QM perturbation theory

naive ratio ( $d=1$ )



*without ghost instantons*

ratio ( $d=1$ )



*with ghost instantons*

$$a_n(m) \sim -\frac{16}{\pi} n! \left( \frac{1}{(S_{I\bar{I}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{G\bar{G}}(m)|^{n+1}} \right)$$

# Ghost Instantons: Quantum Mechanical Path Integrals

the bigger picture:

- vacuum “talks to” the topologically trivial sector:

$$\dots \leftrightarrow [\mathcal{G}^2 \bar{\mathcal{G}}^2] \leftrightarrow [\mathcal{G} \bar{\mathcal{G}}] \leftrightarrow \text{pert.vac} \leftrightarrow [\mathcal{I} \bar{\mathcal{I}}] \leftrightarrow [\mathcal{I}^2 \bar{\mathcal{I}}^2] \leftrightarrow \dots$$

- QM trans-series:

$$\mathcal{Z}(g^2|m) = \begin{cases} \Phi_0(g^2) + [\mathcal{I} \bar{\mathcal{I}}]_- \Phi_{[\mathcal{I} \bar{\mathcal{I}}]}(g^2) + [\mathcal{I}^2 \bar{\mathcal{I}}^2]_- \Phi_{[\mathcal{I}^2 \bar{\mathcal{I}}^2]}(g^2) + \dots & -\pi < \arg(g^2) < 0 \\ \Phi_0(g^2) + [\mathcal{I} \bar{\mathcal{I}}]_+ \Phi_{[\mathcal{I} \bar{\mathcal{I}}]}(g^2) + [\mathcal{I}^2 \bar{\mathcal{I}}^2]_+ \Phi_{[\mathcal{I}^2 \bar{\mathcal{I}}^2]}(g^2) + \dots & 0 < \arg(g^2) < \pi \end{cases}$$

- ambiguities cancel ad-infinitum (resurgence!)

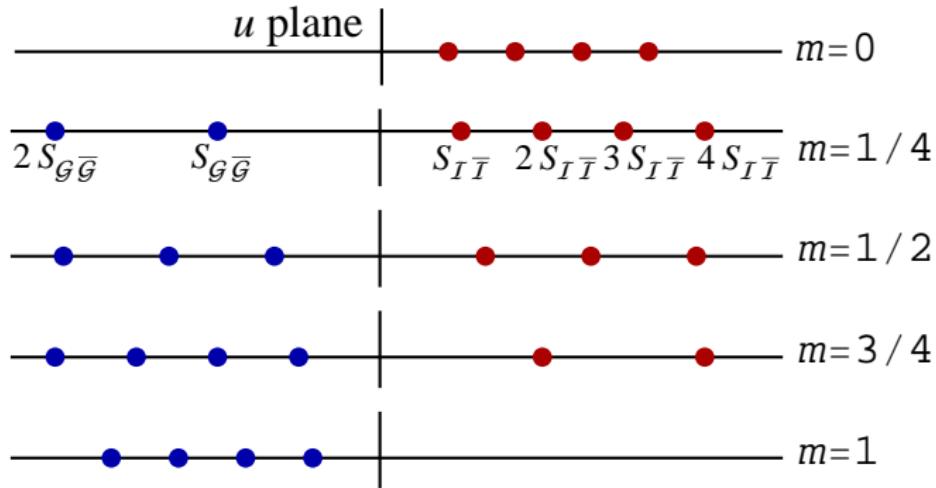
$$\text{Im} (\mathcal{S}_{0^\pm} \Phi_0 + [\mathcal{I} \bar{\mathcal{I}}]_{0^\pm} \text{Re} \mathcal{S}_0 \Phi_{[\mathcal{I} \bar{\mathcal{I}}]}) = 0 \quad \text{up to } \mathcal{O}(e^{-4S_I})$$

- Similar structure for one instanton, etc.. sector

$$\dots \leftrightarrow [\mathcal{I} \mathcal{G}^2 \bar{\mathcal{G}}^2] \leftrightarrow [\mathcal{I} \mathcal{G} \bar{\mathcal{G}}] \leftrightarrow [\mathcal{I}] \leftrightarrow [\mathcal{I}^2 \bar{\mathcal{I}}] \leftrightarrow [\mathcal{I}^3 \bar{\mathcal{I}}^2] \leftrightarrow \dots$$

# Ghost Instantons: Quantum Mechanical Path Integrals

Borel plane structure:



Mimics IR and UV renormalon structure of asymptotically free QFT

# Non-perturbative Physics Without Instantons

e.g, 2d Principal Chiral Model:

(Cherman, Dorigoni, GD, Ünsal, [1308.0127](#))

$$S_b = \frac{N}{2\lambda} \int d^2x \operatorname{tr} \partial_\mu U \partial^\mu U^\dagger, \quad U \in SU(N),$$

- non-Borel-summable perturbation theory due to IR renomalons
- but, the theory has no instantons !

resolution: there exist non-BPS saddle point solutions to the second-order classical Euclidean equations of motion:  
“unitons”

$$\partial_\mu \left( U^\dagger \partial_\mu U \right) = 0$$

# Non-perturbative Physics Without Instantons: Principal Chiral Model

- “unitons”:  $U \equiv (\mathbf{1} - 2\mathbb{P})$

$$\partial_\mu \left( U^\dagger \partial_\mu U \right) = 0 \quad \rightarrow \quad [\mathbb{P}, \partial_\mu^2 \mathbb{P}] = 0$$

- general solutions to  $\mathbb{CP}^{N-1}$  model (Din/Zakrzewski)
- simplest untions: from  $\mathbb{CP}^{N-1}$  instantons
- “fractons”: twisted & fractionalized solutions in PCM

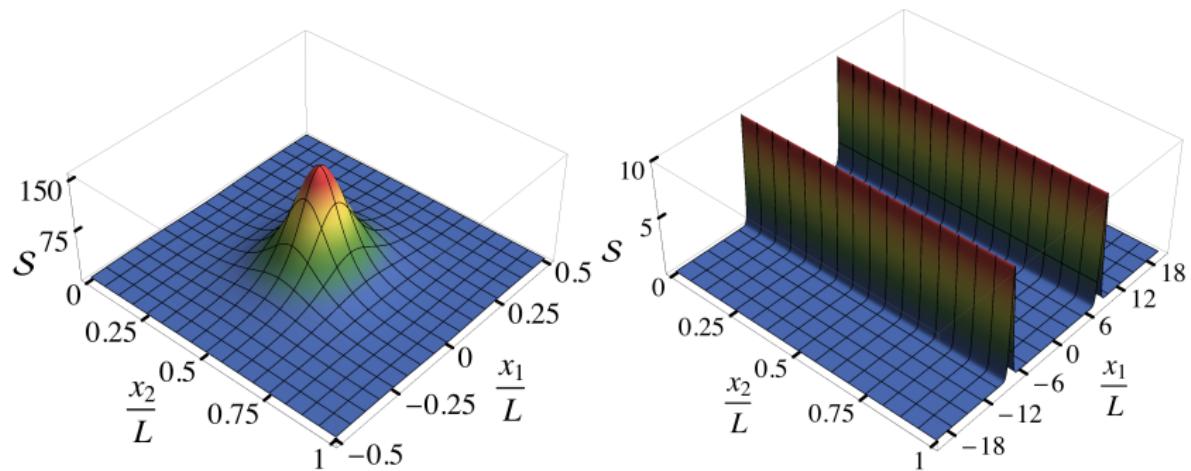
$N$  fundamental fractons,  $U(z, \bar{z}) = e^{i\pi/N}(1 - 2\mathbb{P})$ ,

$$\mathbb{P}_{ij} = \frac{v_i v_j^\dagger}{v^\dagger \cdot v}$$

$$S_F = \frac{8\pi}{g^2 N}$$

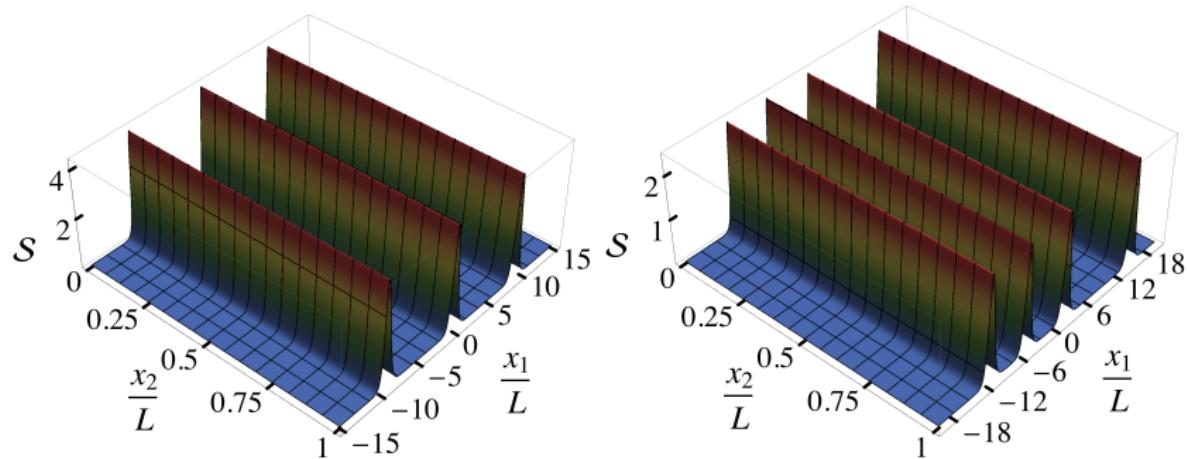
# Non-perturbative Physics Without Instantons: Principal Chiral Model

$SU(2)$  fractons



# Non-perturbative Physics Without Instantons: Principal Chiral Model

$SU(3)$  and  $SU(4)$  fractons



# Non-perturbative Physics Without Instantons: Principal Chiral Model

$$\mathcal{F}_i \sim e^{-\frac{8\pi(\mu_{i+1} - \mu_i)}{g^2}} \sim e^{-\frac{8\pi}{g^2 N}}, \quad \mathcal{U} = \prod_{i=1}^N \mathcal{F}_i$$

- perturbation theory: IR renormalon singularities on positive Borel axis

$$t_k^+ = 8\pi k/N = k[g^2 S_U]/\beta_0, \quad k \in \mathbb{Z}^+$$

- ambiguous lateral Borel sum:

$$\mathcal{S}_{0^\pm} \mathcal{E}(g^2) = \Re \mathbb{B}_0 \mp i \frac{32\pi}{g^2 N} e^{-\frac{16\pi}{g^2 N}}$$

- non-perturbative fracton/anti-fracton amplitude:

$$[\mathbb{F}_i \bar{\mathbb{F}}_i]_{\theta=0^\pm} = \left[ \log \left( \frac{g^2 N}{16\pi} \right) - \gamma \right] \frac{16}{g^2 N} e^{-\frac{16\pi}{g^2 N}} \pm i \frac{32\pi}{g^2 N} e^{-\frac{16\pi}{g^2 N}}$$

# Non-perturbative Physics Without Instantons

Yang-Mills,  $\mathbb{CP}^{N-1}$ , PCM, ... all have non-BPS solutions with finite action

- “unstable”: negative modes of fluctuation operator
- what do these mean ?

**resurgence:** ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

# Non-perturbative Physics Without Instantons: Yang-Mills

4d Yang-Mills:  $S_{YM} = \frac{1}{2} \int d^4x \text{tr} (F_{\mu\nu} F_{\mu\nu})$

- Bogomolny factorization:

$$S_{YM} = \frac{1}{4} \int d^4x \text{tr} \left\{ \left( F_{\mu\nu} \mp \tilde{F}_{\mu\nu} \right)^2 \pm 2F_{\mu\nu}\tilde{F}_{\mu\nu} \right\}$$

- classical equations of motion:

$$D_\mu F_{\mu\nu} = 0 \quad \longrightarrow \quad F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$$

- “instantons”: minima of classical action
- non-BPS finite action saddle-points: Sibner, Sibner, Uhlenbeck ( $SU(2)$ ): locally  $m$  instantons &  $m$  anti-instantons,  $m \in \mathbb{Z} \geq 2$
- ansatz constructions for  $SU(n)$ ,  $n \geq 3$
- solutions ‘unstable’ : negative modes

# Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$

(Dabrowski, GD, [arXiv:1306.0921](#))

Non-self-dual Solutions in  $\mathbb{CP}^{N-1}$

$$S = \int d^2x \left[ \frac{1}{2} \left| D_\mu n \pm i\epsilon_{\mu\nu} D_\nu n \right|^2 \mp i\epsilon_{\mu\nu} (D_\nu n)^\dagger D_\mu n \right]$$

- rank-1 projector representation:

$$\mathbb{P} \equiv n n^\dagger$$

$$\mathbb{P}^2 = \mathbb{P} = \mathbb{P}^\dagger, \text{Tr } \mathbb{P} = 1$$

$$\text{action} \quad S = 2 \int d^2x \text{Tr} [\partial_z \mathbb{P} \partial_{\bar{z}} \mathbb{P}]$$

$$\text{charge} \quad Q = 2 \int d^2x \text{Tr} \left[ \mathbb{P} \partial_{\bar{z}} \mathbb{P} \partial_z \mathbb{P} - \mathbb{P} \partial_z \mathbb{P} \partial_{\bar{z}} \mathbb{P} \right]$$

# Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$

- first-order instanton equations:

$$D_\mu n = \pm i\epsilon_{\mu\nu} D_\nu n$$

$$\partial_{\bar{z}} \mathbb{P} \mathbb{P} = 0 \quad (\text{instanton}) \quad , \quad \partial_z \mathbb{P} \mathbb{P} = 0 \quad (\text{anti-instanton})$$

- solution: holomorphic projector  $\mathbb{P} = \frac{\omega\omega^\dagger}{\omega^\dagger\omega}$ , with  $\omega = \omega(z)$   
second-order classical equations:

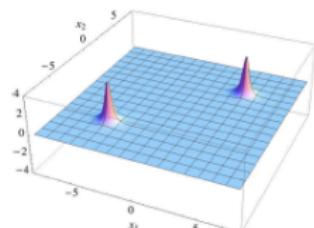
$$D_\mu D_\mu n - (n^\dagger \cdot D_\mu D_\mu n) n = 0 \quad \text{or} \quad [\partial_z \partial_{\bar{z}} \mathbb{P}, \mathbb{P}] = 0$$

- non-BPS solutions generated from instantons:

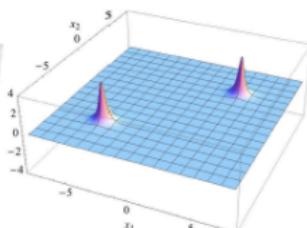
$$Z_+ : \omega \rightarrow Z_+ \omega \equiv \partial_z \omega - \frac{(\omega^\dagger \partial_z \omega)}{\omega^\dagger \omega} \omega \quad , \quad Z_+ : n \rightarrow Z_+ n \equiv \frac{Z_+ \omega}{|Z_+ \omega|}$$

$$\omega_{(0)} \xrightarrow{Z_+} \omega_{(1)} \xrightarrow{Z_+} \cdots \xrightarrow{Z_+} \omega_{(k)} \xrightarrow{Z_+} \cdots \omega_{(N-1)} \xrightarrow{Z_+} 0$$

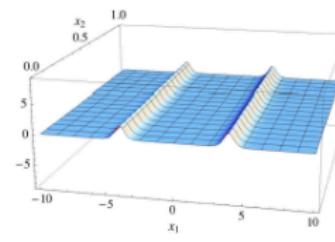
# Non-perturbative Physics Without Instantons: $\mathbb{C}\mathbb{P}^{N-1}$



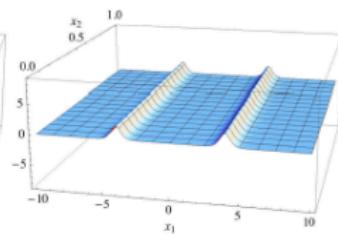
Action Density of  $\omega_{(0)}$ :  $(S_{(0)} = 2)$



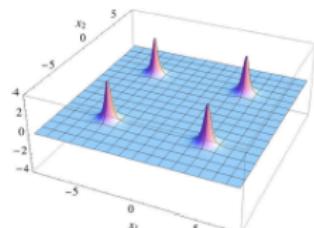
Charge Density of  $\omega_{(0)}$ : ( $Q_{(0)} = 2$ )



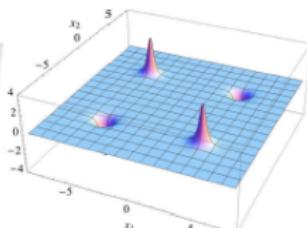
Action Density of  $\omega_{(g)}$ : ( $S_{(g)} = \frac{2}{3}$ )



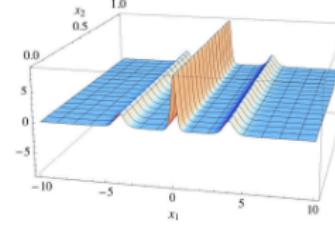
Charge Density of  $\omega_{(g)}$ : ( $Q_{(g)} = \frac{2}{3}$ )



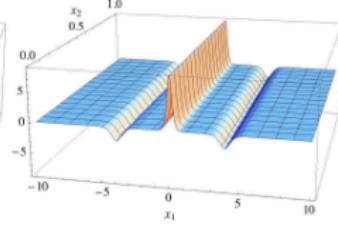
Action Density of  $\omega_{(1)}$ : ( $S_{(1)} = 4$ )



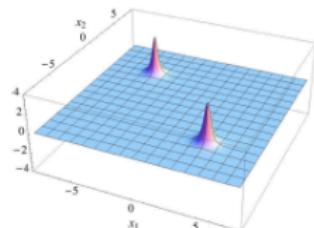
Charge Density of  $\omega_{(1)}$ : ( $Q_{(1)} = 0$ )



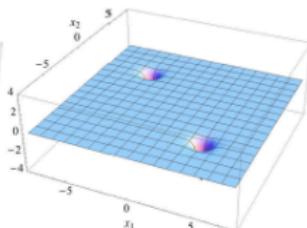
Action Density of  $\omega_{(1)}$ : ( $S_{(1)} = \frac{4}{3}$ )



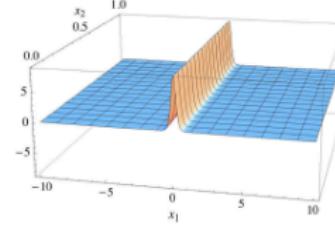
Charge Density of  $\omega_{(1)}$ : ( $Q_{(1)} = 0$ )



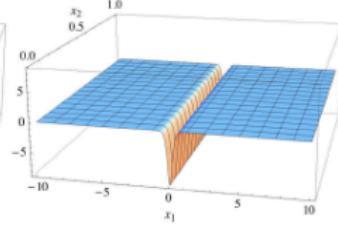
Action Density of  $\omega_{(1)}$ : ( $S_{(2)} = 2$ )



Charge Density of  $\omega_{(2)}$ : ( $Q_{(2)} = -2$ )



Action Density of  $\omega_{(2)}$ : ( $S_{(2)} = \frac{2}{3}$ )



Charge Density of  $\omega_{(2)}$ : ( $Q_{(2)} = -\frac{2}{3}$ )

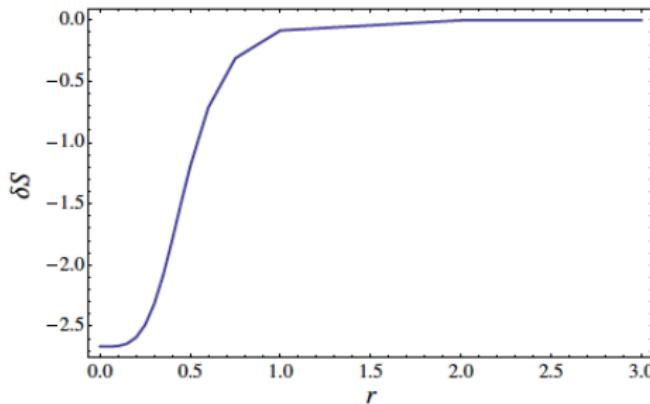
# Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$

- non-BPS solutions are ‘unstable’: e.g.

$$n \rightarrow \tilde{n} = n\sqrt{1 - \phi^\dagger \phi} + \phi \quad , \quad \phi = D_z n \quad ; \quad \phi^\dagger \cdot n = 0$$

- change in action is manifestly negative:

$$\delta S = - \int d^2x \left( \text{Tr} \left[ (D_z n)^\dagger D_z n (D_{\bar{z}} n)^\dagger D_{\bar{z}} n \right] + \text{Tr} \left[ (D_{\bar{z}} n)^\dagger D_z n (D_z n)^\dagger D_{\bar{z}} n \right] \right)$$



# Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$

physical origin of negative modes:

- single  $\mathbb{CP}^{N-1}$  instanton:  $2N$  parameters: i.e.  $2N$  zero modes
- $Q = 2$   $\mathbb{CP}^{N-1}$  instanton:  $4N$  parameters: i.e.  $4N$  zero modes
- mapped non-BPS solution also has  $4N$  parameters: i.e.  $4N$  zero modes
- but, “looks like” 2 instantons and 2 anti-instantons  $\Rightarrow 8N$  zero modes  
 $\Rightarrow 4N$  zero modes are lifted at finite separation  
some become negative modes

# Conclusions

- Resurgence systematically unifies perturbative and non-perturbative world
- there is extra ‘magic’ in perturbation theory
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- basic property of steepest descents expansions
- basic property of complex differential equations
- trans-series: sectors are inter-related
- resurgence triangle: network of connections
- moral: consider all saddles, including non-BPS
- resurgence required for analytic continuation

# Open Problems

- Resurgence in Chern-Simons theories, Euler-Heisenberg, dS/AdS, exact S-matrices, matrix models, topological strings, integrability, localization, ...
- nonlinear differential equations
- natural path integral construction
- analytic continuation of path integrals
- ODE/IM correspondence
- relating strong- and weak-coupling expansions: dualities
- relation to SUSY and extended SUSY
- ...

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