Introduction to hadronic light-by-light scattering in the muon $g-2$

Andreas Nyffeler*

$(g-2)_\mu$: Quo vadis ?

SFB 1044: The Low-Energy Frontier of the Standard Model

Institut für Kernphysik, Johannes Gutenberg Universität Mainz, Germany

April 7-10, 2014

* Hospitality by the Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Switzerland, is gratefully acknowledged.

Outline

- Muon $g-2$: current status
- Hadronic light-by-light (HLbL) scattering in the muon $g-2$: basics, main results
- Off-shell versus on-shell form factors
  Resonance exchange versus resonance pole contribution
- Relevant momentum regions for pion-pole contribution
  Impact of form factor measurements: example KLOE-2
- One-particle intermediate states: resonance exchanges / poles
- Two-particle intermediate states (pion-loop)
- Dressed quark-loop
- Current status of HLbL, Conclusions and Outlook

Omissions in this talk

- Experiments to measure decays, form factors, cross-sections: only a few discussed / mentioned. Many more examples discussed in talk by Kupsc (and others).
- Lattice QCD and HLbL: only a few words. More details: see talk by Izubuchi at last week's MITP Workshop.
- Very new proposal: Dispersion relations approach to HLbL.
  - Colangelo et al. '14
    Dispersion relation for $\langle VVVV \rangle$ in $\gamma^* \gamma^* \to \gamma^* \gamma$ with external on-shell photon $k^2 = 0$, then inserted into $g - 2$.
    Considered so far: one-pion intermediate state (pion-pole), two pion intermediate state (pion-loop).
    See talk by Colangelo at MITP meeting last week and talk by Hoferichter.
  - Pauk, Vanderhaeghen '14
    Dispersion relation directly for $a_\mu = F_2(0)$ via absorptive part of $F_2(k^2)$.
    Considered so far: resonance pole contributions (pion-pole).
    See talks by Pauk at MITP meeting last week and at this meeting.

Muon $g-2$: current status

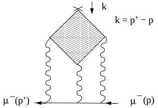
- Experimental value (world average dominated by BNL experiment '06):
  $a_\mu^{\text{exp}} = (116\ 592\ 089 \pm 63) \times 10^{-11}$
- Theory: total SM contribution (based on various recent papers):
  $a_\mu^{\text{SM}} = (116\ 591\ 795 \pm \underbrace{47}_{\text{HVP}} \pm \underbrace{40}_{\text{HLbL}} \pm \underbrace{1.8}_{\text{QED + EW}} \quad [\pm 62]) \times 10^{-11}$

  Hadronic contributions are largest source of error: vacuum polarization (HVP) and light-by-light (HLbL) scattering.

  $a_\mu^{\text{HLbL}} = (116 \pm 40) \times 10^{-11}$ (Nyffeler '09; Jegerlehner, Nyffeler '09)
  Often used: $a_\mu^{\text{had. LbyL}} = (105 \pm 26) \times 10^{-11}$ (Prades, de Rafael, Vainshtein '09; "Glasgow consensus")
- $\Rightarrow a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (294 \pm 88) \times 10^{-11}$ $\quad [3.3\ \sigma]$
- Other evaluations: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} \sim (250 - 400) \times 10^{-11}$ $\quad [2.9 - 4.9\ \sigma]$
  (Jegerlehner, Nyffeler '09; Davier et al. '10; Jegerlehner, Szafron '11; Hagiwara et al. '11; Aoyama et al. '12; Benayoun et al. '13)
- Discrepancy a sign of New Physics ?
- Note: Hadronic contributions need to be better controlled (by factor two or so) in order to fully profit from future muon $g-2$ experiments at Fermilab and J-PARC with $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$.
- Goal for HLbL (MITP Workshop): Controlled error of $\delta a_\mu^{\text{HLbL}} = 20 \times 10^{-11}$.

Hadronic light-by-light scattering in the muon $g-2$

$\mathcal{O}(\alpha^3)$ hadronic contribution to muon $g-2$: four-point function $\langle VVVV \rangle$ projected onto a_μ (external soft photon $k \to 0$).



Consider matrix element of light-quark electromagnetic current

$$j_\rho(x) = \frac{2}{3}(\bar{u}\gamma_\rho u)(x) - \frac{1}{3}(\bar{d}\gamma_\rho d)(x) - \frac{1}{3}(\bar{s}\gamma_\rho s)(x)$$

between muon states:

$$\begin{aligned}
\langle \mu^-(p')|(ie)j_\rho(0)|\mu^-(p)\rangle &= (-ie)\bar{\mathrm{u}}(p')\Gamma_\rho(p',p)\mathrm{u}(p) \\
&= \int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4} \frac{(-i)^3}{q_1^2\, q_2^2\, (q_1+q_2-k)^2} \frac{i}{(p'-q_1)^2-m^2} \frac{i}{(p'-q_1-q_2)^2-m^2} \\
&\quad \times (-ie)^3\, \bar{\mathrm{u}}(p')\gamma^\mu(\not{p}'-\not{q}_1+m)\gamma^\nu(\not{p}'-\not{q}_1-\not{q}_2+m)\gamma^\lambda \mathrm{u}(p) \\
&\quad \times (ie)^4\, \Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2)
\end{aligned}$$

with $k = p' - p$ and the fourth-rank light-quark hadronic tensor

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3) = \int d^4x_1 \int d^4x_2 \int d^4x_3\, e^{i(q_1\cdot x_1 + q_2\cdot x_2 + q_3\cdot x_3)} \langle\, \Omega \,|\, \mathrm{T}\{j_\mu(x_1)j_\nu(x_2)j_\lambda(x_3)j_\rho(0)\}\,|\,\Omega\,\rangle$$

Momentum conservation: $k = q_1 + q_2 + q_3$.

Projection onto $g-2$, Properties of $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$

Flavor diagonal current $j_\mu(x)$ is conserved and one has the Ward identities:

$$\{q_1^\mu; q_2^\nu; q_3^\lambda; k^\rho\} \, \Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3) \; = \; 0$$

$$\Rightarrow \Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2) = -k^\sigma \frac{\partial}{\partial k^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$$

Defining $\Gamma_\rho(p', p) = k^\sigma \Gamma_{\rho\sigma}(p', p)$ one finally obtains (Aldins et al. '70):

$$a_\mu = F_2(0) \; = \; \frac{1}{48m} \mathrm{tr}\left((\not{p} + m)[\gamma^\rho, \gamma^\sigma](\not{p} + m)\Gamma_{\rho\sigma}(p, p)\right)$$

i.e. one can formally take the limit $k_\mu \to 0$ inside the loop integrals. Problem reduces to calculation of two-point function with zero-momentum insertion. One also gets better UV convergence properties of individual Feynman diagrams (fermion-loop).

Properties of $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3)$ (Bijnens et al. '95):

- In general 138 Lorentz structures. But only 32 contribute to $g-2$.
- Using Ward identities, there are 43 gauge invariant structures.
- Bose symmetry relates some of them.
- All depend on $q_1^2, q_2^2, q_3^2, q_i \cdot q_j$, but before taking derivative and $k_\mu \to 0$, also on $k^2, k \cdot q_i$.
- Compare with HVP: one function, one variable.

HLbL in muon $g-2$

Current approach: use some hadronic model at low energies with exchanges and loops of resonances and some form of (dressed) "quark-loop" at high energies.

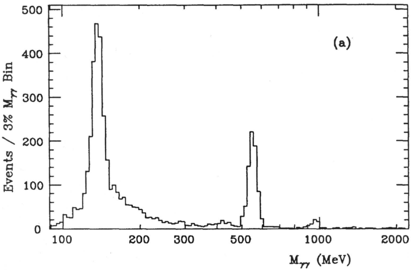
Problem: $\langle VVVV \rangle$ depends on several invariant momenta $\Rightarrow$ distinction between low and high energies is not as easy as for two-point function $\langle VV \rangle$ (HVP).

Note: one can always perform Wick rotation to Euclidean momenta, where effects of resonances are smoothed out, but there are mixed regions where Q_1^2 is small and Q_2^2 large and vice versa.

Data on $\gamma\gamma \to \gamma\gamma$

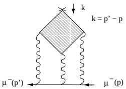
In any case, it is a good idea to look at actual data. Invariant $\gamma\gamma$ mass spectrum obtained with the Crystal Ball detector '88 via $e^+e^- \to e^+e^-\gamma^*\gamma^*$. One observes three spikes from the light pseudoscalars in the reaction

$$\gamma\gamma \to \pi^0, \eta, \eta' \to \gamma\gamma$$

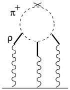


HLbL in the muon $g-2$ (continued)

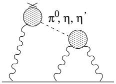
Classification of de Rafael '94: Chiral counting p^2 (ChPT) and large-N_C counting as guideline (all higher orders in p^2 and N_C contribute):



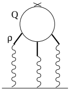
$=$



$+ \cdots +$



$+ \cdots +$ Exchanges of other resonances ($f_0, a_1, f_2 \ldots$) $+$



$+ \cdots$

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C
	pion-loop (dressed)	pseudoscalar exchanges		quark-loop (dressed)

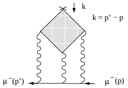
Relevant scales in HLbL ($\langle VVVV\rangle$ with off-shell photons): $0-2$ GeV, i.e. larger than m_μ !

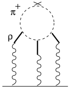
Constrain models using experimental data (form factors of hadrons with photons) and theory (ChPT at low energies; short-distance constraints from pQCD / OPE at high momenta).

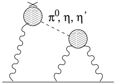
Issue: on-shell versus off-shell form factors. For instance pion-pole versus pion-exchange: How do we define the pion-pole contribution ? Is there a form factor at external vertex ?

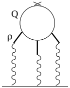
Going beyond models: Dispersion relations or Lattice QCD.

HLbL scattering: anno 2009



=


+ ··· +


+ ··· + Exchanges of other resonances ($f_0, a_1, f_2 \dots$) +


+ ···

Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN:	+116 (40)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02;
KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04;
2007 = Bijnens, Prades; Miller, de Rafael, Roberts (compilation);
PdRV = Prades, de Rafael, Vainshtein '09 (compilation, "Glasgow consensus");
N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09 (compilation)

- 2001: sign change in dominant pseudoscalar contribution: $a_\mu^{\mathrm{HLbL}} \sim 85 \times 10^{-11}$.
- 2004: MV $\Rightarrow$ enhanced pion-pole and axial-vector contributions. Estimate shifted upwards.
- 2010: (almost) consensus reached on central value $a_\mu^{\mathrm{HLbL}} \sim 110 \times 10^{-11}$, still discussion about error estimate. Conservative in N, JN: $\pm 40 \times 10^{-11}$, more progressive in PdRV: $\pm 26 \times 10^{-11}$.

Recall (in units of 10^{-11}): $\delta a_\mu(\mathrm{HVP}) \approx 45$; $\delta a_\mu(\exp\,[\mathrm{BNL}]) = 63$; $\delta a_\mu(\text{future exp}) = 16$

HLbL scattering: anno 2009 (continued)

- Evaluations of full HLbL scattering contribution:
  - Bijnens, Pallante, Prades '95, '96, '02
    Use mainly Extended Nambu-Jona-Lasinio (ENJL) model; but for some contributions also other models (in particular for pseudoscalars and pion loop)
  - Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, '02
    Use mainly Hidden Local Symmetry (HLS) model; often HLS = VMD

- Selected partial evaluations:
  - Knecht, Nyffeler '02: use large-N_C QCD for pion-pole (Lowest Meson Dominance (LMD), LMD+V)
  - Melnikov, Vainshtein '04: use large-N_C QCD, short-distance constraint from $\langle VVVV \rangle$ on pion-pole and axial-vector contribution, mixing of two axial-vector nonets

- Prades, de Rafael, Vainshtein '09: Analyzed results obtained by different groups with various model and suggested new estimates for some contributions (shifted central values, enlarged errors). No dressed light quark loops ! Assumed to be taken into account by short-distance constraint of MV '04 on pseudoscalar-pole contribution. Added errors in quadrature.

- Nyffeler '09; Jegerlehner, Nyffeler '09: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on pion-exchange with off-shell form factors. Combined with MV (for axial-vectors) + BPP (rest of contributions). Added errors linearly.

HLbL scattering: anno 2009 (continued)

Some selected results for the various contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops $+$subl. N_C	–	–	–	0 ± 10	–	–	–
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3 (c-quark)	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

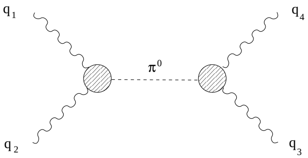
BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09;

- Pseudoscalar-exchanges dominate numerically. Other contributions not negligible. Cancellation between π, K-loops and quark loops !
- PdRV: Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV '04 on pseudoscalar-pole contribution. Added all errors in quadrature !
- N, JN: New evaluation of pseudoscalars. Took over most values from BPP, except axial vectors from MV. Added all errors linearly.

Off-shell versus on-shell form factors
Resonance exchange versus resonance pole contribution

Pion-pole in $\langle VVVV \rangle$ versus pion-exchange in $a_\mu^{\mathrm{LbyL};\pi^0}$

- To uniquely identify contribution of exchanged neutral pion π^0 in Green's function $\langle VVVV \rangle$, we need to pick out pion-pole:

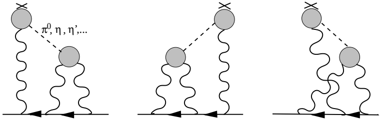


+ crossed diagrams

$$\lim_{(q_1+q_2)^2 \to m_\pi^2} ((q_1+q_2)^2 - m_\pi^2)\langle VVVV \rangle$$

Residue of pole: on-shell vertex function $\langle 0|VV|\pi\rangle \to$ on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

- But in contribution to muon $g-2$, we evaluate Feynman diagrams, integrating over photon momenta with exchanged off-shell pions.
For all the pseudoscalars:



Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ where $\mathrm{PS} = \pi^0, \eta, \eta', {\pi^0}', \ldots$

Off-shell form factors are either inserted "by hand" starting from constant, pointlike Wess-Zumino-Witten (WZW) form factor or using e.g. some resonance Lagrangian.

- Similar statements apply for exchanges (or loops) of other resonances.

Off-shell pion form factor from $\langle VVP \rangle$

- Following Bijnens, Pallante, Prades '96; Hayakawa, Kinoshita, Sanda '96; Hayakawa, Kinoshita '98, we can define off-shell form factor for π^0:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2 \cdot y)} \langle 0 | T\{j_\mu(x) j_\nu(y) P^3(0)\} | 0 \rangle$$

$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta \frac{i\langle \overline{\psi}\psi \rangle}{F_\pi} \frac{i}{(q_1+q_2)^2 - m_\pi^2} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) + \ldots$$

  Up to small mixing effects of P^3 with η and η' and neglecting exchanges of heavier states like $\pi^{0\prime}, \pi^{0\prime\prime}, \ldots$

  $$j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x), \quad \psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \hat{Q} = \mathrm{diag}(2,-1,-1)/3$$

  (light quark part of electromagnetic current)

  $$P^3 = \overline{\psi} i \gamma_5 \frac{\lambda^3}{2} \psi = \left(\overline{u} i \gamma_5 u - \overline{d} i \gamma_5 d\right)/2, \quad \langle \overline{\psi}\psi \rangle = \text{single flavor quark condensate}$$

  Bose symmetry: $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_2^2, q_1^2)$

- Note: for off-shell pions, instead of $P^3(x)$, we could use any other suitable interpolating field, like $(\partial^\mu A_\mu^3)(x)$ or even an elementary pion field $\pi^3(x)$! Off-shell form factor is therefore model dependent and not a physical quantity !

On-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ and transition form factor $F(Q^2)$

- On-shell $\pi^0\gamma^*\gamma^*$ form factor between an on-shell pion and two off-shell photons:

$$i \int d^4x \, e^{iq_1 \cdot x} \langle 0 | T\{j_\mu(x) j_\nu(0)\} | \pi^0(q_1 + q_2) \rangle = \varepsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$

  Relation to off-shell form factor:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$$

  Form factor for real photons is related to $\pi^0 \to \gamma\gamma$ decay width:

$$\mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(q_1^2 = 0, q_2^2 = 0) = \frac{4}{\pi\alpha^2 m_\pi^3} \Gamma_{\pi^0 \to \gamma\gamma}$$

  Often normalization with chiral anomaly is used:

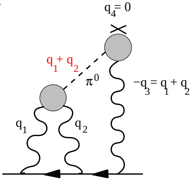
$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, 0) = -\frac{1}{4\pi^2 F_\pi}$$

- Pion-photon transition form factor:

$$F(Q^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, q_2^2 = 0), \qquad Q^2 \equiv -q_1^2$$

  Note that $q_2^2 = 0$, but $\vec{q}_2 \neq \vec{0}$ for on-shell photon !

Pion-exchange versus pion-pole contribution to $a_\mu^{\mathrm{LbyL};\pi^0}$



- Off-shell form factors have been used to evaluate the pion-exchange contribution in Bijnens, Pallante, Prades '96 and Hayakawa, Kinoshita, Sanda '96, '98. "Rediscovered" by Jegerlehner in '07, '08. Consider diagram:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \ \times \ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

- On the other hand, Knecht, Nyffeler '02 used on-shell form factors:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \ \times \ \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

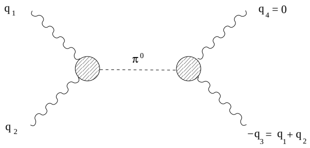
- But form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following (misleading ?) notation was used:
  $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$
  At external vertex identification with transition form factor was made.
- Melnikov, Vainshtein '04 had observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \ \times \ \mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the WZW term.
- Puzzle: it seems, we have two different prescriptions to define pion-pole contribution to a_μ, depending on whether we go on-shell with pion momentum before or after taking limit $q_4 \to 0$ (external soft photon). First prescription seems necessary for dispersive approach to a_μ in order to avoid the exceptional momentum $q_4 = 0$. But resulting expression for pion-pole contribution violates momentum conservation at external vertex.

QCD short-distance constraint on $\langle VVVV \rangle$ in $g-2$

- Melnikov, Vainshtein '04 found QCD short-distance constraint on whole 4-point function:



$$\langle \underbrace{VV}_{\text{OPE}}\, V|\gamma\rangle \overset{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

- From this they deduced for the LbyL scattering amplitude for finite $q_1^2, q_2^2, -q_3 = q_1 + q_2$ (Eq. (18) in MV '04, using our normalization for form factor; Minkowski space notation):

$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu}\tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$. For external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$. Except in $\tilde{f}_{\rho\sigma}$, $q_4 \to 0$ is understood.

- Expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$. No form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by constant WZW form factor $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0) \approx \mathcal{F}_{\pi^0\gamma\gamma}(0, 0)$! $\Rightarrow$ MV '04 consider the pion-pole contribution !

- If one then studies the behavior for large q_3^2, one obtains from the pion propagator an overall $1/q_3^2$ behavior (apart from $f_3^{\sigma\rho}$). According to MV '04 this agrees exactly with behavior of quark-loop in perturbative QCD for large momenta.

- From quark-hadron duality in large-N_C QCD it follows that the sum of all resonance exchanges has to match with the quark-loop ! But why should already the pion-pole contribution alone match with the quark-loop ?

Relevant momentum regions for pion-pole contribution
Impact of form factor measurements: example KLOE-2

Integral representation for pion-exchange contribution

Projection onto the muon $g-2$

(Knecht, Nyffeler '02 (pion-pole with two on-shell form factors); adapted in Jegerlehner '07, '08; Nyffeler '09; Jegerlehner, Nyffeler '09)

$$
\begin{aligned}
a_\mu^{\mathrm{LbL};\pi^0} &= -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\
&\times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, q_1^2, (q_1+q_2)^2) \;\; \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_2^2, q_2^2, 0)}{q_2^2 - m_\pi^2} \; T_1(q_1, q_2; p) \right. \\
&\left. + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \;\; \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2} \; T_2(q_1, q_2; p) \right]
\end{aligned}
$$

$$
\begin{aligned}
T_1(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_2)^2 q_1^2 - \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 \\
&\quad + 8(p\cdot q_2) q_1^2 q_2^2 - \frac{16}{3}(p\cdot q_2)(q_1\cdot q_2)^2 + \frac{16}{3} m_\mu^2 q_1^2 q_2^2 - \frac{16}{3} m_\mu^2 (q_1\cdot q_2)^2 \\
T_2(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_1)^2 q_2^2 + \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 \\
&\quad + \frac{8}{3}(p\cdot q_1) q_1^2 q_2^2 + \frac{8}{3} m_\mu^2 q_1^2 q_2^2 - \frac{8}{3} m_\mu^2 (q_1\cdot q_2)^2
\end{aligned}
$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon).

Relevant momentum regions in $a_\mu^{\mathrm{LbL};\pi^0}$

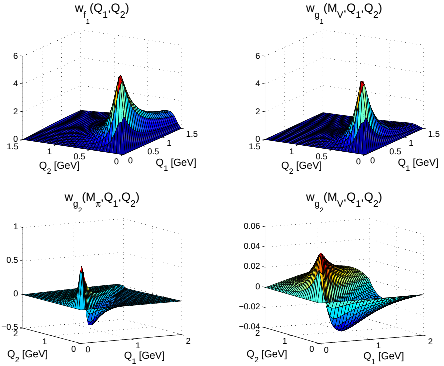
- In Knecht, Nyffeler '02, a 2-dimensional integral representation was derived for a certain class (VMD-like) of form factors (schematically):

$$a_\mu^{\mathrm{LbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\, f_i(Q_1, Q_2)$$

  with universal weight functions w_i. Dependence on form factors resides in the f_i.

- Expressions with on-shell form factors are maybe not valid as they stand. Maybe one needs to set form factor at external vertex to a constant to obtain pion-pole contribution (Melnikov, Vainshtein '04). Expressions valid for WZW and off-shell VMD form factors.

- Plot of weight functions w_i from Knecht, Nyffeler '02:



- $w_{f_1}(Q_1, Q_2)$ enters for WZW form factor. Tail leads to $\ln^2 \Lambda$ divergence for momentum cutoff Λ.

- $w_{g_1}(M_V, Q_1, Q_2)$ enters for VMD form factor.

- Relevant momentum regions around $0.25 - 1.25$ GeV. As long as form factors in different models lead to damping, expect comparable results for $a_\mu^{\mathrm{LbL};\pi^0}$, at level of 20%.

General form factors: 3-dimensional integral representation for $a_\mu^{\mathrm{LbL};\pi^0}$

- 2-loop integral $\rightarrow$ 8-dim. integral. Integration over 3 angles can be done easily
- 5 non-trivial integrations: 2 moduli: $|q_1|$, $|q_2|$, 3 angles: $p \cdot q_1$, $p \cdot q_2$, $q_1 \cdot q_2$ (recall $p^2 = m_\mu^2$).
- Observation: $p \cdot q_1$, $p \cdot q_2$ do not appear in the model-dependent off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$
- Can perform those two angular integrations by averaging expression for $a_\mu^{\mathrm{LbL};\pi^0}$ over the direction of p (Jegerlehner + Nyffeler '09)

Method of Gegenbauer polynomials (hyperspherical approach)

(Baker, Johnson, Willey '64, '67; Rosner '67; Levine, Roskies '74; Levine, Remiddi, Roskies '79)

Denote by $\hat{K}$ unit vector of four-momentum vector K in Euclidean space

Propagators in Euclidean space:

$$\frac{1}{(K-L)^2 + M^2} = \frac{Z_{KL}^M}{|K||L|} \sum_{n=0}^{\infty} \left(Z_{KL}^M\right)^n C_n(\hat{K} \cdot \hat{L})$$

$$Z_{KL}^M = \frac{K^2 + L^2 + M^2 - \sqrt{(K^2 + L^2 + M^2)^2 - 4K^2L^2}}{2|K||L|}$$

Use orthogonality conditions of Gegenbauer polynomials:

$$\int \mathrm{d}\Omega(\hat{K})\, C_n(\hat{Q}_1 \cdot \hat{K})\, C_m(\hat{K} \cdot \hat{Q}_2) = 2\pi^2 \frac{\delta_{nm}}{n+1} C_n(\hat{Q}_1 \cdot \hat{Q}_2)$$

$$\int \mathrm{d}\Omega(\hat{K})\, C_n(\hat{Q} \cdot \hat{K})\, C_m(\hat{K} \cdot \hat{Q}) = 2\pi^2\, \delta_{nm}$$

$\hat{Q}_1 \cdot \hat{K}$ = Cosine of angle between the four-dimensional vectors Q_1 and K

General form factors: 3-dim. integral representation for $a_\mu^{\mathrm{LbL};\pi^0}$ (continued)

Average over direction $\hat{P}$ (note: $P^2 = -m_\mu^2$):

$$\langle \cdots \rangle = \frac{1}{2\pi^2} \int \mathrm{d}\Omega(\hat{P}) \, \cdots$$

After reducing numerators in the functions T_i in $a_\mu^{\mathrm{LbL};\pi^0}$ against denominators of propagators, one is left with the following integrals, denoting propagators by $(4) \equiv (P + Q_1)^2 + m_\mu^2$, $(5) \equiv (P - Q_2)^2 + m_\mu^2$:

$$\begin{aligned}
\langle \frac{1}{(4)} \frac{1}{(5)} \rangle &= \frac{1}{m_\mu^2 R_{12}} \arctan\left(\frac{zx}{1 - zt}\right) \\
\langle (P \cdot Q_1) \frac{1}{(5)} \rangle &= -(Q_1 \cdot Q_2) \frac{(1 - R_{m2})^2}{8 m_\mu^2} \\
\langle (P \cdot Q_2) \frac{1}{(4)} \rangle &= (Q_1 \cdot Q_2) \frac{(1 - R_{m1})^2}{8 m_\mu^2} \\
\langle \frac{1}{(4)} \rangle &= -\frac{1 - R_{m1}}{2 m_\mu^2} \\
\langle \frac{1}{(5)} \rangle &= -\frac{1 - R_{m2}}{2 m_\mu^2}
\end{aligned}$$

$Q_1 \cdot Q_2 = Q_1 \, Q_2 \, \cos\theta$, $\quad t = \cos\theta \quad$ ($\theta =$ angle between Q_1 and Q_2), $\quad Q_i \equiv |Q_i|$

$R_{mi} = \sqrt{1 + 4m_\mu^2/Q_i^2}$, $\quad x = \sqrt{1 - t^2}$, $\quad R_{12} = Q_1 \, Q_2 \, x$, $\quad z = \frac{Q_1 Q_2}{4 m_\mu^2} (1 - R_{m1})(1 - R_{m2})$

General form factors: 3-dim. integral representation for $a_\mu^{\mathrm{LbL};\pi^0}$ (continued)

Integral representation for general off-shell form factors (Jegerlehner, Nyffeler '09):

$$
\begin{aligned}
a_\mu^{\mathrm{LbL};\pi^0} \quad = \quad & -\frac{2\alpha^3}{3\pi^2} \int_0^\infty \mathrm{d}Q_1 \mathrm{d}Q_2 \int_{-1}^{+1} \mathrm{d}t\, \sqrt{1-t^2}\, Q_1^3\, Q_2^3 \\
& \times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_2^2, -Q_1^2, -Q_3^2)\; \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_2^2, -Q_2^2, 0)}{(Q_2^2 + m_\pi^2)}\, I_1(Q_1, Q_2, t) \right. \\
& \left. + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_3^2, -Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_3^2, -Q_3^2, 0)}{(Q_3^2 + m_\pi^2)}\, I_2(Q_1, Q_2, t) \right]
\end{aligned}
$$

where $Q_3^2 = (Q_1 + Q_2)^2, \quad Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta, \quad t = \cos\theta$

$$
\begin{aligned}
I_1(Q_1, Q_2, t) = X(Q_1, Q_2, t) \Big(& 8\, P_1\, P_2\, (Q_1 \cdot Q_2) - 2\, P_1\, P_3\, (Q_2^4/m_\mu^2 - 2\, Q_2^2) - 2\, P_1\, (2 - Q_2^2/m_\mu^2 + 2\,(Q_1 \cdot Q_2)/m_\mu^2) \\
& + 4\, P_2\, P_3\, Q_1^2 - 4\, P_2 - 2\, P_3\, (4 + Q_1^2/m_\mu^2 - 2\, Q_2^2/m_\mu^2) + 2/m_\mu^2 \Big) \\
& - 2\, P_1\, P_2\, (1 + (1 - R_{m1})\,(Q_1 \cdot Q_2)/m_\mu^2) + P_1\, P_3\, (2 - (1 - R_{m1})\, Q_2^2/m_\mu^2) + P_1\, (1 - R_{m1})/m_\mu^2 \\
& + P_2\, P_3\, (2 + (1 - R_{m1})^2\, (Q_1 \cdot Q_2)/m_\mu^2) + 3\, P_3\, (1 - R_{m1})/m_\mu^2 \\
I_2(Q_1, Q_2, t) = X(Q_1, Q_2, t) \Big(& 4\, P_1\, P_2\, (Q_1 \cdot Q_2) + 2\, P_1\, P_3\, Q_2^2 - 2\, P_1 + 2\, P_2\, P_3\, Q_1^2 - 2\, P_2 - 4 P_3 - 4/m_\mu^2 \Big) \\
& - 2\, P_1\, P_2 - 3\, P_1\, (1 - R_{m2})/(2m_\mu^2) - 3\, P_2\, (1 - R_{m1})/(2m_\mu^2) - P_3\, (2 - R_{m1} - R_{m2})/(2m_\mu^2) \\
& + P_1\, P_3\, (2 + 3\,(1 - R_{m2})\, Q_2^2/(2m_\mu^2) + (1 - R_{m2})^2\, (Q_1 \cdot Q_2)/(2m_\mu^2)) \\
& + P_2\, P_3\, (2 + 3\,(1 - R_{m1})\, Q_1^2/(2m_\mu^2) + (1 - R_{m1})^2\, (Q_1 \cdot Q_2)/(2m_\mu^2))
\end{aligned}
$$

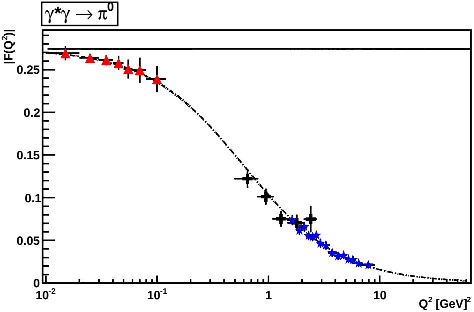
where $P_1^2 = 1/Q_1^2,\ P_2^2 = 1/Q_2^2,\ P_3^2 = 1/Q_3^2, \quad X(Q_1, Q_2, t) = \frac{1}{Q_1 Q_2 x} \arctan\left(\frac{zx}{1-zt}\right)$

Idea taken up by Dorokhov et al. '12 (for scalars) and Bijnens, Zahiri-Abyaneh '12-14 (for all contributions; results presented by Bijnens at MITP Workshop last week). See also: Talk by Jegerlehner at MITP Workshop and at this meeting (for scalars, axial vectors) and Pauk, Vanderhaeghen '14 (2D plot of integrand for axial vector pole contribution).

Impact of form factor measurements: example KLOE-2

On the possibility to measure the $\pi^0 \to \gamma\gamma$ decay width and the $\gamma^*\gamma \to \pi^0$ transition form factor with the KLOE-2 experiment

D. Babusci et al. '12 H. Czyż, F. Gonnella, S. Ivashyn, M. Mascolo, R. Messi, D. Moricciani, A. Nyffeler, G. Venanzoni and the KLOE-2 Collaboration '12



Simulation of KLOE-2 measurement of $F(Q^2)$ (red triangles). MC program EKHARA 2.0 (Czyż, Ivashyn '11) and detailed detector simulation.
Solid line: $F(0)$ given by chiral anomaly (WZW).
Dashed line: form factor according to on-shell LMD+V model (Knecht, Nyffeler '01).
CELLO (black crosses) and CLEO (blue stars) data at higher Q^2.

Within 1 year of data taking, collecting 5 fb^{-1}, KLOE-2 will be able to measure:

- $\Gamma_{\pi^0 \to \gamma\gamma}$ to 1% statistical precision.
- $\gamma^*\gamma \to \pi^0$ transition form factor $F(Q^2)$ in the region of very low, space-like momenta $0.01\ \mathrm{GeV}^2 \leq Q^2 \leq 0.1\ \mathrm{GeV}^2$ with a statistical precision of less than 6% in each bin. KLOE-2 can (almost) directly measure slope of form factor at origin (note: logarithmic scale in Q^2 in plot !).

Impact of form factor measurements: example KLOE-2 (continued)

- Error in $a_\mu^{\mathrm{LbL};\pi^0}$ related to model parameters determined by $\Gamma_{\pi^0\to\gamma\gamma}$ (normalization of form factor; not taken into account in most papers) and $F(Q^2)$ will be reduced as follows:
  - $\delta a_\mu^{\mathrm{LbL};\pi^0} \approx 4 \times 10^{-11}$ (with current data for $F(Q^2)$ + $\Gamma_{\pi^0\to\gamma\gamma}^{\mathrm{PDG}}$)
  - $\delta a_\mu^{\mathrm{LbL};\pi^0} \approx 2 \times 10^{-11}$ (+ $\Gamma_{\pi^0\to\gamma\gamma}^{\mathrm{PrimEx}}$)
  - $\delta a_\mu^{\mathrm{LbL};\pi^0} \approx (0.7 - 1.1) \times 10^{-11}$ (+ KLOE-2 data)
- Note that this error does not account for other potential uncertainties in $a_\mu^{\mathrm{LbL};\pi^0}$, e.g. related to choice of model, 2nd off-shell photon, off-shellness of pion.
- Simple models with few parameters, like VMD (two parameters: F_π, M_V), which are completely determined by the data on $\Gamma_{\pi^0\to\gamma\gamma}$ and $F(Q^2)$, can lead to very small errors in $a_\mu^{\mathrm{LbL};\pi^0}$. For illustration:

  $a_{\mu;\mathrm{VMD}}^{\mathrm{LbL};\pi^0} = (57.3 \pm 1.1) \times 10^{-11}$

  $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbL};\pi^0} = (72 \pm 12) \times 10^{-11}$ (off-shell LMD+V form factor, including all errors)
- But this might be misleading ! VMD and LMD+V give equally good fits to transition form factor $F(Q^2)$, but differ in doubly-off shell transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$. Results for $a_\mu^{\mathrm{LbL};\pi^0}$ differ by about 20% ! Reason: VMD form factor has wrong high-energy behavior $\Rightarrow$ too strong damping in $a_{\mu;\mathrm{VMD}}^{\mathrm{LbL};\pi^0}$: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{VMD}}(q^2, q^2) \sim 1/q^4$, for large q^2, i.e. falls off too fast compared to OPE prediction $\mathcal{F}_{\pi^0\gamma^*\gamma^*}^{\mathrm{OPE}}(q^2, q^2) \sim 1/q^2$ which is fulfilled by LMD+V $\Rightarrow$ Dispersive approach to not rely (or less) on models.

The VMD form factor

Vector Meson Dominance:

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = -\frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{q_1^2 - M_V^2} \frac{M_V^2}{q_2^2 - M_V^2}$$

on-shell = off-shell form factor !

Only two model parameters even for off-shell form factor: F_π and M_V

Note:

- VMD form factor factorizes $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = f(q_1^2) \times f(q_2^2)$. This might be a too simplifying assumption / representation.
- VMD form factor has wrong short-distance behavior: $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^4$, for large q^2, falls off too fast compared to OPE prediction $\mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^2$.

Transition form factor:

$$F^{\mathrm{VMD}}(Q^2) = -\frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{Q^2 + M_V^2}$$

One-particle intermediate states: resonance exchanges / poles

Note

- Within phenomenological approach with models, one particle exchanges / poles come from pseudoscalars, scalars, axial vectors and tensor mesons.
- Within dispersion relation approach only "stable" physical intermediate states are considered: one-pion, two-pion, three-pion, photons, muons ...

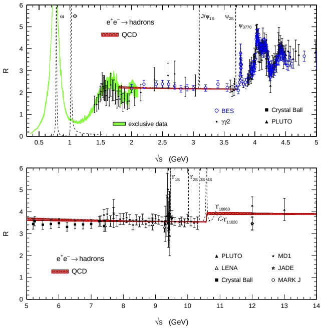
Large-N_C QCD approach: Minimal Hadronic Ansatz (MHA)

Moussallam, Stern '94; Moussallam '95, '97; Peris et al. '98; Knecht et al. '99; ...

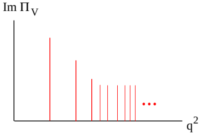
- In QCD, in leading order in N_C, in each channel of a Green's function an infinite tower of narrow resonances contributes $\Rightarrow$ only poles, no cuts (meromorphic functions).
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively. Interpolation works best for order parameters (Green's functions, LEC's) and integrals over Green's functions in Euclidean space. Not suited to describe shape of resonances in physical region.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD).

Example: 2-point function $\langle VV\rangle \to$ spectral function $\mathrm{Im}\Pi_V \sim \sigma(e^+e^- \to \text{hadrons})$

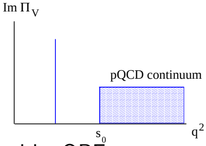
Real world (Davier et al., '03)



Large-N_C QCD ('t Hooft '74)



Minimal Hadronic Ansatz (MHA)



Scale s_0 fixed by OPE

$\Rightarrow a_\mu(HVP) = (5700 \pm 1900) \times 10^{-11}$

The LMD and LMD+V form factors

Knecht, Nyffeler, EPJC '01; Nyffeler '09

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in large-N_C QCD in chiral limit with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all leading (and some subleading) QCD short-distance constraint from OPE
- Reproduces Brodsky-Lepage (BL): $\lim_{Q^2\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$

  (OPE and BL cannot be fulfilled simultaneously with only one vector resonance, LMD)
- Normalized to decay width $\Gamma_{\pi^0\to\gamma\gamma}$

LMD form factors (off-shell, on-shell, transition form factor):

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^{0*}\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2 + q_2^2 + q_3^2 - c_V}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}, \qquad q_3^2 = (q_1 + q_2)^2$$

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^{0}\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2 + q_2^2 - \bar{c}_V}{(q_1^2 - M_V^2)(q_2^2 - M_V^2)}, \qquad \bar{c}_V = c_V - m_\pi^2$$

$$F^{\mathrm{LMD}}(Q^2) = -\frac{F_\pi}{3M_V^2} \frac{Q^2 + \bar{c}_V}{Q^2 + M_V^2}$$

Note that the LMD transition form factor does not fall off like $1/Q^2$ for large Q^2.

The LMD+V form factor

Off-shell LMD+V form factor:

$$\begin{aligned}
\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) &= \frac{F_\pi}{3}\,\frac{q_1^2\, q_2^2\,(q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)} \\
P_H^V(q_1^2, q_2^2, q_3^2) &= h_1\,(q_1^2 + q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2 + q_2^2)\, q_3^2 + h_4\, q_3^4 \\
&\quad + h_5\,(q_1^2 + q_2^2) + h_6\, q_3^2 + h_7 \\
q_3^2 &= (q_1 + q_2)^2
\end{aligned}$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV, Free parameters: h_i

On-shell LMD+V form factor:

$$\begin{aligned}
\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) &= \frac{F_\pi}{3}\,\frac{q_1^2\, q_2^2\,(q_1^2 + q_2^2) + h_1\,(q_1^2 + q_2^2)^2 + \bar{h}_2\, q_1^2\, q_2^2 + \bar{h}_5\,(q_1^2 + q_2^2) + \bar{h}_7}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)} \\
\bar{h}_2 &= h_2 + m_\pi^2, \qquad \bar{h}_5 = h_5 + h_3 m_\pi^2, \qquad \bar{h}_7 = h_7 + h_6 m_\pi^2 + h_4 m_\pi^4
\end{aligned}$$

Transition form factor:

$$F^{\mathrm{LMD+V}}(Q^2) = \frac{F_\pi}{3}\,\frac{1}{M_{V_1}^2 M_{V_2}^2}\,\frac{h_1 Q^4 - \bar{h}_5 Q^2 + \bar{h}_7}{(Q^2 + M_{V_1}^2)(Q^2 + M_{V_2}^2)}$$

- $h_1 = 0$ in order to reproduce Brodsky-Lepage behavior.
- Can treat h_1 as free parameter to fit the BABAR data, but the form factor does then not vanish for $Q^2 \to \infty$, if $h_1 \neq 0$.

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0 \text{ GeV}^2$ (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63 \text{ GeV}^2$ (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26 \text{ GeV}^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83 \text{ GeV}^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (or normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to BABAR data: $h_1 = (-0.17 \pm 0.02) \text{ GeV}^2$, $h_5 = (6.51 \pm 0.20) \text{ GeV}^4 - h_3 m_\pi^2$ with $\chi^2/\text{dof} = 15.0/15 = 1.0$. Result for $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ not affected (shifts compensate).

h_3, h_4, h_6 are unknown / less constrained:

- New short-distance constraint $\Rightarrow h_1 + h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$ $(*)$
  χ = quark condensate magnetic susceptibility of QCD (Ioffe, Smilga '84).
  LMD ansatz for $\langle VT \rangle \Rightarrow \chi^{\mathrm{LMD}} = -2/M_V^2 = -3.3 \text{ GeV}^{-2}$ (Balitsky, Yung '83)
  Close to $\chi(\mu = 1 \text{ GeV}) = -(3.15 \pm 0.30) \text{ GeV}^{-2}$ (Ball et al. '03)
  Assume large-N_C (LMD/LMD+V) framework is self-consistent
  $\Rightarrow \chi = -(3.3 \pm 1.1) \text{ GeV}^{-2}$
  $\Rightarrow$ vary $h_3 = (0 \pm 10) \text{ GeV}^2$ and determine h_4 from relation $(*)$ and vice versa
  Lattice QCD: $\chi_u^{\overline{\mathrm{MS}}}(\mu = 2 \text{ GeV}) = -(2.08 \pm 0.08) \text{ GeV}^{-2}$ (Bali et al. '12).
- Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to h_6
  Assume that LMD/LMD+V estimates of low-energy constants from chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$ are self-consistent. Assume 100% error on estimate for the relevant, presumably small low-energy constant $\Rightarrow h_6 = (5 \pm 5) \text{ GeV}^4$

Limitations of MHA approach

- Cannot fulfill all short-distance constraints on Green's functions (or form factors) with finite number of resonances (Bijnens et al. '03) $\Rightarrow$ needs to choose among constraints and make compromises.
- Beyond LMD: many parameters in ansatz, not all of them can be fixed from experimental or short-distance constraints.
- Most of the work done in chiral limit, assuming octet symmetry.
- Difficult to treat terms subleading in N_C: width of resonances, loops with resonances (pion loop in HLbL).

Generalization: Resonance Chiral Theory (RχT)

- Gasser, Leutwyler, '84; Donoghue et al. '89; Ecker et al. NPB '89: use of resonance Lagrangian with vector mesons, axial vector mesons, heavy scalars ($f_0(980)$) and heavy pseudoscalars to estimate low-energy constants (LEC) in ChPT at order p^4 when integrating out the resonances at tree level (explains success of vector meson dominance).
- Ecker et al. PLB '89: Imposing short-distance constraints on resonance Lagrangian leads to unique estimates for LEC's at order p^4, at least for vector mesons in different representations (vector field, tensor field, gauged vector fields, massive vector fields).
  Note: Imposing short-distance constraints on resonance Lagrangian does not uniquely determine LEC's at order p^6 (Moussallam, Stern '94; Knecht, Nyffeler '01).
- Advantage of resonance Lagrangian: can easily identify in which Green's functions (processes) the parameters in Lagrangian enter.
  Problem: in general many terms in Lagrangian allowed by chiral symmetry, not all can be determined from short-distance constraints.
- RχT Lagrangian for odd intrinsic parity sector which fulfills various QCD short-distance constraints to fix parameters in the Lagrangian: Pallante, Petronzio '93; Prades '94; Ananthanarayan, Moussallam '02; Ruiz-Femenia, Pich, Portoles '03; Kampf, Moussallam '09; Kampf, Novotny '11; Roig, Sanz Cillero '13.
  RχT with two vector resonances: Mateu, Portoles '07; Czyz et al. '12.
- Attempts to go beyond leading order in N_C: Loops with resonances, including renormalization (Pich et al.).

Pseudoscalar pole / exchange in large-N_C QCD

Model for $\mathcal{F}_{P^{(*)}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
VMD	57	83
LMD (on-shell) [KN]	73	–
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10\ \mathrm{GeV}^2$) [KN]	63(10)	88(12)
LMD+V (on-shell, constant 2nd FF) [MV]	77(7)	114(10)
LMD+V (off-shell) [N]	72(12)	99(16)
LMD+P (off-shell) [KaNo]	65.8(1.2)	–
LMD+P (off-shell) [RGL]	66.5(1.9)	104.3(5.2)
LMD+P (on-shell) [RGL]	57.5(0.5)	82.7(2.8)

KN = Knecht, Nyffeler '02
MV = Melnikov, Vainshtein '04
N = Nyffeler '09
KaNo = Kampf, Novotny '11 (RχT)
RGL = Roig, Guevara, Lopez Castro '14 (RχT)

Note: KN, MV, N use VMD FF for η, η'

Other recent partial evaluations (mostly pseudoscalars)

- Nonlocal chiral quark model (off-shell) [Dorokhov et al.]
  2008: $a_\mu^{\mathrm{LbyL};\pi^0} = 65(2) \times 10^{-11}$
  2011: $a_\mu^{\mathrm{LbyL};\pi^0} = 50.1(3.7) \times 10^{-11}, \quad a_\mu^{\mathrm{LbyL;PS}} = 58.5(8.7) \times 10^{-11}$
  2012: $a_\mu^{\mathrm{LbyL};\pi^0+\sigma} = 54.0(3.3) \times 10^{-11}, \quad a_\mu^{\mathrm{LbyL};a_0+f_0} \sim 0.1 \times 10^{-11}$
  $a_\mu^{\mathrm{LbyL;PS+S}} = 62.5(8.3) \times 10^{-11}$
  Strong damping for off-shell form factors. Positive and small contribution from scalar $\sigma(600)$, differs from other estimates (BPP '96, '02; Blokland, Czarnecki, Melnikov '02).

- Holographic (AdS/QCD) model 1 (off-shell ?) [Hong, Kim '09]
  $a_\mu^{\mathrm{LbyL};\pi^0} = 69 \times 10^{-11}, \quad a_\mu^{\mathrm{LbyL;PS}} = 107 \times 10^{-11}$

- Holographic (AdS/QCD) model 2 (off-shell) [Cappiello, Cata, D'Ambrosio '10]
  $a_\mu^{\mathrm{LbyL};\pi^0} = 65.4(2.5) \times 10^{-11}$
  Used AdS/QCD to fix parameters in ansatz by D'Ambrosio et al. '98.

- Padé approximants (on-shell, no constant FF at external vertex)
  $a_\mu^{\mathrm{LbyL};\pi^0} = 54(5) \times 10^{-11}$ [Masjuan '12 (using on-shell LMD+V FF)]
  $a_\mu^{\mathrm{LbyL};\pi^0} = 64.9(5.6) \times 10^{-11}, \quad a_\mu^{\mathrm{LbyL;PS}} = 89(7) \times 10^{-11}$
  [Escribano, Masjuan, Sanchez-Puertas '13]
  Fix parameters in Padé approximants from data on transition form factors.

More on single-resonance poles and exchanges

- Pauk, Vanderhaeghen '14:

$$\begin{aligned} a_\mu(f_0(980), f_0', a_0) &= [(-0.9 \pm 0.2) \text{ to } (-3.1 \pm 0.8)] \times 10^{-11} \\ a_\mu(f_1, f_1') &= (6.4 \pm 2.0) \times 10^{-11} \\ a_\mu(f_2, f_2', a_2, a_2') &= (1.1 \pm 0.1) \times 10^{-11} \end{aligned}$$

  Meson pole approximation. Using monopole form factor (VMD) for scalars and dipole form factors (strong damping !) for axial vectors and tensors:

$$\begin{aligned} \frac{F^{\rm mon}_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, q_2^2)}{F_{\mathcal{M}\gamma^*\gamma^*}(0,0)} &= \frac{1}{(1 - q_1^2/\Lambda^2_{\rm mon})} \frac{1}{(1 - q_2^2/\Lambda^2_{\rm mon})} \\ \frac{F^{\rm dip}_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, q_2^2)}{F_{\mathcal{M}\gamma^*\gamma^*}(0,0)} &= \frac{1}{(1 - q_1^2/\Lambda^2_{\rm dip})^2} \frac{1}{(1 - q_2^2/\Lambda^2_{\rm dip})^2} \end{aligned}$$

  First calculation of tensor meson contribution to HLbL. For the tensor mesons, forward sum rules for $\gamma\gamma$ scattering (Pascalutsa, Vanderhaeghen, '10; Pascalutsa, Pauk, Vanderhaeghen '12) used to constrain the model. Sum rules link transition form factors of pseudoscalars, axial vectors and tensor mesons $\Rightarrow \Lambda_{\rm dip} = 1.5$ GeV.

- Jegerlehner (Talk at MITP Workshop; Opening talk at this meeting):

$$\begin{aligned} a_\mu(f_0(980), f_0', a_0) &= (-(6.01 - 6.31) \pm 1.20) \times 10^{-11} \\ a_\mu(f_1, f_1', a_1) &= (7.55 - 7.58) \pm 2.71) \times 10^{-11} \end{aligned}$$

  Meson exchange contributions. Models inspired from Melnikov, Vainshtein '04, but correctly implement Landau-Yang theorem.

- Both calculations give contribution for axial vectors, which is substantially smaller than Melnikov, Vainshtein '04: $a_\mu(f_1, f_1', a_1) = (22 \pm 5) \times 10^{-11}$.

Two-particle intermediate states (pion-loop)

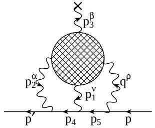
Dressed pion-loop

1. ENJL/VMD versus HLS

Model	$a_\mu^{\pi-\text{loop}} \times 10^{11}$
scalar QED (no FF)	-45
HLS	-4.5
ENJL	-19
full VMD	-15

Strong damping if form factors are introduced, very model dependent: compare ENJL (BPP '96) versus HLS (HKS '96). See also discussion in Melnikov, Vainshtein '04.

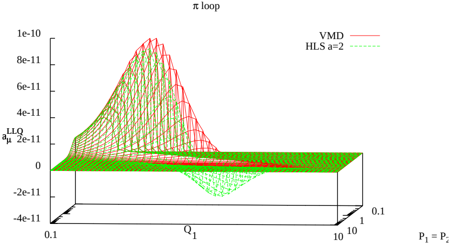
Origin: different behavior of integrands in contribution to $g-2$ (Zahiri Abyaneh '12; Bijnens, Zahiri Abyaneh '12; Talks by Bijnens at MesonNet 2013, Prague; MITP Workshop last week)



One can do 5 of the 8 integrations in the 2-loop integral for $g-2$ analytically, using the hyperspherical approach / Gegenbauer polynomials (Jegerlehner, Nyffeler '09; taken up in Bijnens, Zahiri Abyaneh '12):

$$a_\mu^{\rm X} = \int dl_{P_1}\, dl_{P_2}\, a_\mu^{\rm XLL} = \int dl_{P_1}\, dl_{P_2}\, dl_Q\, a_\mu^{\rm XLLQ}, \quad \text{with} \quad l_P = \ln(P/\text{GeV})$$

Contribution of type X at given scale P_1, P_2, Q is directly proportional to volume under surface when $a_\mu^{\rm XLL}$ and $a_\mu^{\rm XLLQ}$ are plotted versus the energies on a logarithmic scale.



Momentum distribution of the full VMD and HLS pion-loop contribution for $P_1 = P_2$.

HLS: Integrand changes from positive to negative at high momenta. Leads to cancellation and therefore smaller absolute value. Usual HLS model ($a = 2$) known to not fullfill certain QCD short-distance constraints.

Dressed pion-loop (continued)

2. Role of pion polarizability and a_1 resonance

- Engel, Patel, Ramsey-Musolf '12: ChPT analysis of LbyL up to order p^6 in limit $p_1, p_2, q \ll m_\pi$. Identified potentially large contributions from pion polarizability ($L_9 + L_{10}$ in ChPT) which are not fully reproduced in ENJL / HLS models used by BPP '96 and HKS '96. Pure ChPT approach is not predictive. Loops not finite, would need new a_μ counterterm (Knecht et al. '02).

- Engel, Ph.D. Thesis '13; Engel, Ramsey-Musolf '13: tried to include a_1 resonance explicitly in EFT. Problem: contribution to $g-2$ in general not finite (loops with resonances) $\Rightarrow$ Form factor approach with a_1 that reproduces pion polarizability at low energies, has correct QCD scaling at high energies and generates a finite result in a_μ:

$$
\begin{aligned}
\mathcal{L}_I &= -\frac{e^2}{4} F_{\mu\nu}\pi^+ \left(\frac{1}{D^2 + M_A^2}\right) F^{\mu\nu}\pi^- + \mathrm{h.c.} + \cdots \\
\mathcal{L}_{II} &= -\frac{e^2}{2M_A^2}\pi^+\pi^- \left[\left(\frac{M_V^2}{\partial^2 + M_V^2}\right) F^{\mu\nu}\right]^2 + \cdots
\end{aligned}
$$

$a_\mu^{\pi-\mathrm{loop}} \times 10^{11}$:

Model	(a)	(b)
I	-11	-34
II	-40	-71

Second and third columns in Table correspond to different values for the polarizability LECs, $(\alpha_9^r + \alpha_{10}^r)$: (a) $(1.32 \pm 1.4) \times 10^{-3}$ (from radiative pion decay $\pi^+ \to e^+\nu_e\gamma$) and (b) $(3.1 \pm 0.9) \times 10^{-3}$ (from radiative pion photoproduction $\gamma p \to \gamma' \pi^+ n$).

Potentially large results (absolute value): $a_\mu^{\pi-\mathrm{loop}} \sim -(11 - 71) \times 10^{-11}$. Variation of 60×10^{-11} ! Uncertainty underestimated in earlier calculations ?

Dressed pion-loop (continued)

- Issue taken up in Zahiri Abyaneh '12; Bijnens, Zahiri Abyaneh '12; Bijnens, Relefors (to be published); Talk by Bijnens at MITP Workshop.

  Tried various ways to include a_1, but again no finite result for $g-2$ achieved. With a cutoff of 1 GeV:

  $$a_\mu^{\pi-\text{loop}} = (-20 \pm 5) \times 10^{-11} \qquad \text{(preliminary)}$$

  Very close to old result $a_\mu^{\pi-\text{loop}} = (-19 \pm 13) \times 10^{-11}$ in BPP '96, '02.

- Maybe only model-independent approach with dispersion relations can give a reliable prediction for pion-loop. But to include the effects of the axial vector meson a_1 and the matching with QCD short-distances, the region $1-2$ GeV should be covered as well.

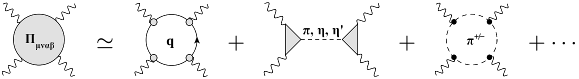
Dressed quark-loop

Dressed quark-loop

Dyson-Schwinger equation (DSE) approach [Fischer, Goecke, Williams '11, '13]

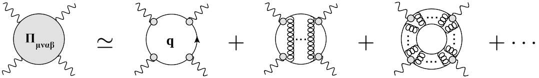
Claim: no double-counting between quark-loop and pseudoscalar exchanges (or exchanges of other resonances)

Had. LbyL in Effective Field Theory (hadronic) picture:

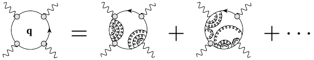


Quarks here may have different interpretation than below !

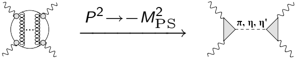
Had. LbyL using functional methods (all propagators and vertices fully dressed):



Expansion of quark-loop in terms of planar diagrams (rainbow-ladder approx.):



Pole representation of ladder-exchange contribution:

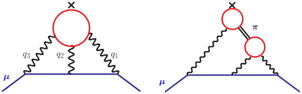


Truncate DSE using well tested model for dressed quark-gluon vertex (Maris, Tandy '99).

Large contribution from quark-loop (even after recent correction), in contrast to all other approaches, where coupling of (constituent) quarks to photons is dressed by form factors ($\rho - \gamma$-mixing, VMD).

Dressed quark-loop (continued)

- Dyson-Schwinger equation approach [Fischer, Goecke, Williams '11, '13]
  $a_\mu^{\mathrm{LbyL};\pi^0} = 57.5(6.9) \times 10^{-11}$ (off-shell), $a_\mu^{\mathrm{LbyL;PS}} = 81(2) \times 10^{-11}$
  $a_\mu^{\mathrm{LbyL;quark-loop}} = 107(2) \times 10^{-11}$, $a_\mu^{\mathrm{had.\ LbyL}} = 188(4) \times 10^{-11}$
  Error for PS, quark-loop and total only from numerics. Quark-loop: some parts are missing. Not yet all contributions to HLbL calculated. Systematic error ?
  Note: numerical error in quark-loop in earlier paper (GFW PRD83 '11):
  $a_\mu^{\mathrm{LbyL;quark-loop}} = 136(59) \times 10^{-11}$, $a_\mu^{\mathrm{had.\ LbyL}} = 217(91) \times 10^{-11}$
- Constituent quark loop [Boughezal, Melnikov '11]
  $a_\mu^{\mathrm{had.\ LbyL}} = (118 - 148) \times 10^{-11}$
  Consider ratio of had. VP and had. LbyL with pQCD corrections. Paper was reaction to earlier results using DSE yielding large values for the quark-loop and the total.
- Constituent Chiral Quark Model [Greynat, de Rafael '12]
  $a_\mu^{\mathrm{LbyL;CQloop}} = 82(6) \times 10^{-11}$
  $a_\mu^{\mathrm{LbyL};\pi^0} = 68(3) \times 10^{-11}$ (off-shell)
  $a_\mu^{\mathrm{had.\ LbyL}} = 150(3) \times 10^{-11}$



  Error only reflects variation of constituent quark mass $M_Q = 240 \pm 10$ MeV, fixed to reproduce had. VP in $g-2$. Determinations from other quantities give larger value for $M_Q \sim 300$ MeV and thus smaller value for quark-loop. 20%-30% systematic error estimated. Not yet all contributions calculated.
- Padé approximants [Masjuan, Vanderhaeghen '12]
  $a_\mu^{\mathrm{had.\ LbyL}} = (76(4) - 125(7)) \times 10^{-11}$
  Quark-loop with running mass $M(Q) \sim (180 - 220)$ MeV, where the average momentum $\langle Q \rangle \sim (300 - 400)$ MeV is fixed from relevant momenta in 2-dim. integral representation for pion-pole in Knecht, Nyffeler '02.

Current status of HLbL, Outlook, Conclusions

HLbL @ NLO

Golangelo, Hoferichter, Nyffeler, Passera, Stoffer '14

Recently, a surprisingly large NNLO HVP contribution was obtained by Kurz et al. '14.

$$a_\mu^{\text{HVP, LO}} = (6907.5 \pm 47.2) \times 10^{-11}$$
$$a_\mu^{\text{HVP, NLO}} = (-100.3 \pm 2.2) \times 10^{-11}$$
$$a_\mu^{\text{HVP, NNLO}} = (12.4 \pm 0.1) \times 10^{-11}$$

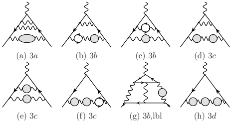
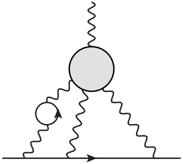


Figure 1: Sample NNLO Feynman diagrams contributing to $a_\mu^{\rm had}$.

Enhancement because of large $\ln(m_\mu/m_e)$, prefactors π^2 in QED LbL.

Could there be a similarly large effect in HLbL @ NLO ? We calculated the potentially large contribution from an additional electron loop (using simple VMD model for pion-pole to model full HLbL)



$$a_\mu^{\pi^0\text{-pole, NLO}} = 1.5 \cdot 10^{-11}$$

2.6% effect of $a_\mu^{\pi^0\text{-pole}} = 57.2 \cdot 10^{-11}$, very close to renormalization group arguments $3 \times \frac{\alpha}{\pi} \times \frac{2}{3} \log \frac{m_\mu}{m_e} \approx 2.5\%$.

Estimating the not yet calculated diagrams with HLbL with additional radiative corrections to the muon line or internal HLbL by comparing with HLbL insertions with muon loop in QED, which are suppressed by factor 4, we obtain the estimate:

$$a_\mu^{\text{HLbL, NLO}} = (3 \pm 2) \cdot 10^{-11},$$

Negligible with current precision goal for HLbL !

HLbL scattering: anno 2014

- Recent partial evaluations:

$$a_\mu^{\mathrm{LbyL};\pi^0} \sim (50-69)\times 10^{-11}$$
$$a_\mu^{\mathrm{LbyL;PS}} \sim (59-107)\times 10^{-11}$$

  Most evaluations agree at level of 15%, but some estimates are quite low or high.
- New estimates for axial vectors: $a_\mu^{\mathrm{LbyL;axial}} \sim (6-8)\times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14)
- First estimate for tensor mesons: $a_\mu^{\mathrm{LbyL;tensor}} = (1.1 \pm 0.1)\times 10^{-11}$ (Pauk, Vanderhaeghen '14)
- Open problem: Dressed pion-loop
  Potentially important effect from pion polarizability and a_1 resonance (Engel, Patel, Ramsey-Musolf '12; Engel '13; Engel, Ramsey-Musolf '13):

$$a_\mu^{\mathrm{LbyL};\pi-\mathrm{loop}} = -(11-71)\times 10^{-11}$$

  Large negative contribution, no damping seen, in contrast to BPP '96, HKS '96. Hopefully new dispersive approaches can help to settle the issue beyond the use of models (Colangelo et al. '14; Pauk, Vanderhaeghen '14).
- Open problem: Dressed quark-loop
  Dyson-Schwinger equation (DSE) approach (Fischer, Goecke, Williams '11, '13):

$$a_\mu^{\mathrm{LbyL;quark-loop}} = 107\times 10^{-11} \qquad \text{(still incomplete !)}$$

  Large contribution, no damping seen, in contrast to BPP '96, HKS '96.

HLbL scattering: anno 2014 (continued)

- If we take those newer estimates of the pion-loop and quark-loop seriously and combine the extreme estimates:

$$a_\mu^{\mathrm{HLbL}} = (64 - 202) \times 10^{-11}$$

or: $$a_\mu^{\mathrm{HLbL}} = (133 \pm 69) \times 10^{-11}$$

$\Rightarrow$ We do not understand HLbL scattering at all !?

- Option 1: Wait for final result from Lattice QCD . . .

  One idea: put QCD + QED on the lattice !
  Blum et al. '05, '08, '09; Chowdhury '09; Blum, Hayakawa, Izubuchi '12 + poster at Lattice 2013 (private communication by Blum):

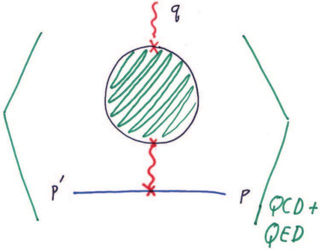
$$F_2(0.18 \text{ GeV}^2) = (127 \pm 29) \times 10^{-11} \qquad \text{(result } 4.4\sigma \text{ from zero)}$$

$$F_2(0.11 \text{ GeV}^2) = (-15 \pm 39) \times 10^{-11} \qquad \text{(result consistent with zero)}$$

$$a_\mu^{\mathrm{HLbL;models}} = F_2(0) = (116 \pm 40) \times 10^{-11} \qquad \text{(Jegerlehner, Nyffeler '09)}$$

  For $m_\mu = 190$ MeV, $m_\pi = 329$ MeV. Still large statistical errors, systematic errors not yet under control, still quenched QED, potentially large "disconnected" contributions missing ! Status report in last week's MITP workshop by Izubuchi.

Non-perturbative approach: QCD+QED on the lattice



Attach one photon by hand

Correlation of hadronic loop and muon line

[Hayakawa, *et al.* hep-lat/0509016;

Chowdhury *et al.* (2008);

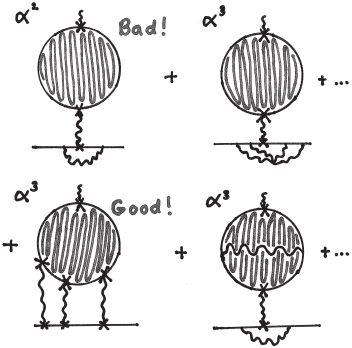
Chowdhury Ph. D. thesis (2009)]

(Poster by Blum at Lattice 2013)

HLbL on the lattice (continued)

Formally expand in α

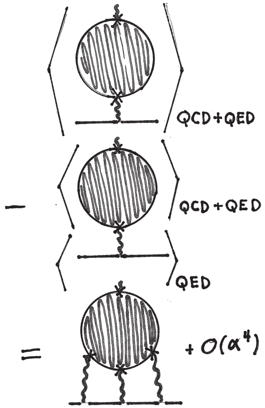
The leading and next-to-leading contributions in α to magnetic part of correlation function come from



(Poster by Blum at Lattice 2013)

HLbL on the lattice (continued)

Subtraction of lowest order piece



Subtraction term is product of separate averages of the loop and line

Gauge configurations identical in both, so two are highly correlated

In PT, correlation function and subtraction have same contributions except the light-by-light term which is absent in the subtraction

(Poster by Blum at Lattice 2013)

Conclusions and Outlook

Option 2: Maybe non-Lattice theorists and experimentalists can still do some work in the coming years, as far as HLbL scattering in muon $g-2$ is concerned !

- Despite recent developments concerning the pion-loop and quark-loop, we think that the estimate

$$a_\mu^{\mathrm{HLbL}} = (116 \pm 40) \times 10^{-11} \qquad \text{(Nyffeler '09; Jegerlehner, Nyffeler '09)}$$

  still gives a fair description of the current situation.

- Only a concerted effort of different methods in theory (models, dispersion relations, Schwinger-Dyson equations, Lattice QCD) in close collaboration with experiment will lead to a reliable error estimate of

$$\delta a_\mu^{\mathrm{HLbL}} = 20 \times 10^{-11}$$

  as proposed in the MITP Workshop to match in a few years precision of new muon $g-2$ experiments $\delta a_\mu^{\text{future exp}} = 16 \times 10^{-11}$.

- Error estimates for individual contributions: a small error does not necessarily imply that the estimate is "better", maybe the model used is too simple !
  Small error of $\pm 26 \times 10^{-11}$ in Prades, de Rafael, Vainshtein '09 ("Glasgow consensus") from adding errors of individual contributions in quadrature might be misleading.

- Do we all agree on the definition of the pion-pole contribution ? What about momentum conservation at external vertex with soft photon $k_\mu \to 0$? On-shell-pion form factor: $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$. How to recover the non-pole contributions ?

Conclusions and Outlook (continued)

- Needed: more experimental constraints from resonance decays, form factors, cross-sections, at small and intermediate momenta $|q| \leq 2$ GeV (time-like and space-like).
  In addition to many processes mentioned at this workshop, have a look at:
  - Need more information on pion-polarizability, e.g. from radiative pion decay $\pi^+ \to e^+ \nu_e \gamma$, radiative pion photoproduction $\gamma p \to \gamma' \pi^+ n$, the hadronic Primakoff process $\pi A \to \pi' \gamma A$ (with some heavy nucleus A) or $\gamma A \to \pi^+ \pi^- A$.
  - Properties of the a_1 resonance should be better determined, e.g. its decay modes $a_1 \to \rho\pi$ and $a_1 \to \pi\gamma$. Important for pion-loop and axial-vector exchange contribution.

- Need more theoretical constraints on form factors and $\langle VVVV \rangle$ at low energies from ChPT and short-distance constraints from OPE and pQCD. More short-distance constraints will also be useful for other approaches using resonances Lagrangians or dispersion relations.
  Short-distance analysis of full 4-point $\langle VVVV \rangle$ (Knecht, Nyffeler, work in progress).
  Also useful to constrain models: sum rules for the (on-shell) hadronic light-by-light scattering (Pascalutsa, Pauk, Vanderhaeghen '12)
- Study relevant momentum regions in HLbL (as model independent as possible) (Knecht, Nyffeler, work in progress).
- Note: only Bijnens, Pallante, Prades '96, '02 and Hayakawa, Kinoshita, Sanda '96, '98, '02 are "full" calculations so far ! But the models used have their deficiencies.
  Dispersive approach can help to make some progress, but how far can we go ?
  Helpful: one consistent (as much as possible) hadronic (resonance) model to calculate all contributions ! Along the lines of HLS model (Benayoun et al.) or Resonance Chiral Theory which fit as much as possible data in resonance region and fulfill as much as possible QCD short-distance constraints.

Backup

Experimental constraints on (on-shell) $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

• Any hadronic model of the form factor has to reproduce the $\pi^0 \to \gamma\gamma$ decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}[1 + \mathcal{O}(m_q)]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small), see also Kampf, Moussallam '09. Leads to normalization:

$$\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = -\frac{N_C}{12\pi^2 F_\pi}$$

For $F_\pi = 92.4$ MeV, this reproduces very well the decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.49)$ eV (PDG 2010, 6.3% precision). More recently the PrimEx Collaboration (Larin et al. '11) presented the measurement $\Gamma(\pi^0 \to \gamma\gamma) = (7.82 \pm 0.23)$ eV (2.8% precision).

Note: Uncertainty in neutral pion contribution to HLbL originating from $\Gamma(\pi^0 \to \gamma\gamma)$ has not been taken into account in most evaluations ! F_π is used without any error attached to it.

• Information on the $\pi^0 - \gamma$ transition form factor with one on-shell and one off-shell photon from the process $e^+e^- \to e^+e^-\pi^0$

Brodsky-Lepage '79-'81 predict the following behavior:

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$$

Maybe with slightly different prefactor !

Data from CELLO '90 and CLEO '08 fairly well confirm this behavior, although $Q^2 \leq 9~\mathrm{GeV}^2$ maybe not yet large enough. Data from BABAR '09 in range $4~\mathrm{GeV}^2 \leq Q^2 \leq 40~\mathrm{GeV}^2$ do not show this fall-off. But data from BELLE '12 in same range seem to fall off.

Theory: QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

Knecht, Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

• When the space-time arguments of all three currents approach each other one obtains with the Operator Product Expansion (OPE), up to corrections $\mathcal{O}(\alpha_s)$:

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3} \frac{1}{\lambda^2} \frac{q_1^2 + q_2^2 + (q_1 + q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

• When the space-time arguments of the two vector currents in $\langle VVP \rangle$ approach each other the OPE leads to Green's function $\langle AP \rangle$ and one obtains:

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, (\lambda q_1)^2, (q_2 - \lambda q_1)^2) = \frac{2F_0}{3} \frac{1}{\lambda^2} \frac{1}{q_1^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

As pointed out in Melnikov, Vainshtein '04, higher twist corrections have been worked out in Shuryak, Vainshtein '82, Novikov et al. '84 (in chiral limit):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, (\lambda q_1)^2, (\lambda q_1)^2) = \frac{2F_0}{3} \left\{ \frac{1}{\lambda^2 q_1^2} + \frac{8}{9} \frac{\delta^2}{\lambda^4 q_1^4} + \mathcal{O}\left(\frac{1}{\lambda^6}\right) \right\}$$

δ^2 parametrizes the relevant higher-twist matrix element.

The sum-rule estimate in Novikov et al. '84 yielded $\delta^2 = (0.2 \pm 0.02)\ \mathrm{GeV}^2$

Short-distance constraint on form factor at external vertex

• When the space-time argument of one of the vector currents approaches the argument of the pseudoscalar density in $\langle VVP \rangle$ one obtains (Knecht, Nyffeler, EPJC '01):

$$\langle V \underbrace{VP}_{\text{OPE}} \rangle \to \langle VT \rangle \qquad \text{Vector-Tensor two-point function}$$

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + q_2)^2, (\lambda q_1)^2, q_2^2) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(q_2^2) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

The vector-tensor two-point function Π_{VT} is defined by:

$$\delta^{ab}(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = \int d^4x e^{ip\cdot x}\langle 0|T\{V_\mu^a(x)(\overline{\psi}\,\sigma_{\rho\sigma}\frac{\lambda^b}{2}\,\psi)(0)\}|0\rangle, \quad \sigma_{\rho\sigma} = \frac{i}{2}[\gamma_\rho, \gamma_\sigma]$$

$$(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = (p_\rho\eta_{\mu\sigma} - p_\sigma\eta_{\mu\rho})\,\Pi_{\mathrm{VT}}(p^2), \quad \text{CVC, parity invariance}$$

At the external vertex in light-by-light scattering the following limit is relevant (soft photon $q_2 \to 0$)

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(0) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Short-distance constraint at the external vertex (cont.)

Ioffe, Smilga '84 defined the quark condensate magnetic susceptibility χ of QCD in the presence of a constant external electromagnetic field

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e\, e_q\, \chi\, \langle\overline{\psi}\psi\rangle_0\, F_{\mu\nu}, \qquad e_u = 2/3,\ e_d = -1/3$$

Belyaev, Kogan '84 then showed that

$$\Pi_{\mathrm{VT}}(0) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\,\chi$$

$\Rightarrow$ Short-distance constraint on off-shell form factor at external vertex (Nyffeler '09):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2,(\lambda q_1)^2,0) = \frac{F_0}{3}\,\chi\ + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad (*)$$

- Note that there is no falloff in OPE in $(*)$, unless χ vanishes !
  A constituent quark model for the form factor would lead to a $1/q_1^2$ fall-off.
- Corrections of $\mathcal{O}(\alpha_s)$ in OPE $\Rightarrow$ χ depends on renormalization scale μ. Not obvious, what is the "correct" scale μ in HLbL.
- Unfortunately there is no agreement in the literature what the value of $\chi(\mu)$ should be !
  Range of values from $\chi(\mu \sim 0.5\ \mathrm{GeV}) \approx -9\ \mathrm{GeV}^{-2}$ (Ioffe, Smilga '84; Vainshtein '03, ..., Narison '08) to $\chi(\mu \sim 1\ \mathrm{GeV}) \approx -3\ \mathrm{GeV}^{-2}$ (Balitsky, Yung '83; Ball et al. '03; ...; Ioffe '09). Running with μ cannot explain such a difference. Recent result from Lattice QCD: $\chi_u^{\overline{\mathrm{MS}}}(\mu = 2\ \mathrm{GeV}) = -(2.08 \pm 0.08)\ \mathrm{GeV}^{-2}$ (Bali et al. '12).

Our estimate for pseudoscalar-exchange contribution

Nyffeler '09; Jegerlehner, Nyffeler '09

- π^0
  - Estimate with off-shell form factor $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma^*}$ which obeys new short-distance constraint at external vertex:

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

  - Largest uncertainty from $h_6 = (5 \pm 5)$ GeV^4 $\Rightarrow \pm 6.4 \times 10^{-11}$ in $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$

    If we would vary $h_6 = (0 \pm 10)$ GeV^4 $\Rightarrow \pm 12 \times 10^{-11}$!
  - Varying $\chi = -(3.3 \pm 1.1)$ GeV^{-2} $\Rightarrow \pm 2.1 \times 10^{-11}$

    Value of χ not so important, but range does not include Vainshtein's estimate $\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$ GeV^{-2}
  - Varying $h_3 = (0 \pm 10)$ GeV^2 $\Rightarrow \pm 2.5 \times 10^{-11}$ (h_4 via $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$)
  - With h_1, h_5 from fit to BABAR data: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = 71.8 \times 10^{-11}$ $\rightarrow$ result unchanged !

- η, η'
  - Short-distance analysis of LMD+V form factor performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
  - Simplified approach: VMD normalized to decay width $\Gamma(\mathrm{PS} \rightarrow \gamma\gamma)$.

$$\mathcal{F}^{\mathrm{VMD}}_{\mathrm{PS}^*\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \mathrm{PS} = \eta, \eta'$$

  - $\Rightarrow a_\mu^{\mathrm{LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5 \times 10^{-11}$

    Not taking pole-approximation as done in Melnikov, Vainshtein '04 !

    Note: VMD form factor has too strong damping at large momenta $\rightarrow$ values might be a bit too small !

- Estimate for sum of all light pseudoscalars:

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

Relevant momentum regions in $a_\mu^{\mathrm{LbyL;PS}}$

Result for pseudoscalar exchange contribution $a_\mu^{\mathrm{LbyL;PS}} \times 10^{11}$ for off-shell LMD+V and VMD form factors obtained with momentum cutoff Λ in 3-dimensional integral representation of Jegerlehner, Nyffeler '09 (integration over square). In brackets, relative contribution of the total obtained with $\Lambda = 20$ GeV.

Λ [GeV]	π^0 LMD+V ($h_3 = 0$)	π^0 LMD+V ($h_4 = 0$)	π^0 VMD	η VMD	η' VMD
0.25	14.8 (20.6%)	14.8 (20.3%)	14.4 (25.2%)	1.76 (12.1%)	0.99 (7.9%)
0.5	38.6 (53.8%)	38.8 (53.2%)	36.6 (64.2%)	6.90 (47.5%)	4.52 (36.1%)
0.75	51.9 (72.2%)	52.2 (71.7%)	47.7 (83.8%)	10.7 (73.4%)	7.83 (62.5%)
1.0	58.7 (81.7%)	59.2 (81.4%)	52.6 (92.3%)	12.6 (86.6%)	9.90 (79.1%)
1.5	64.9 (90.2%)	65.6 (90.1%)	55.8 (97.8%)	14.0 (96.1%)	11.7 (93.2%)
2.0	67.5 (93.9%)	68.3 (93.8%)	56.5 (99.2%)	14.3 (98.6%)	12.2 (97.4%)
5.0	71.0 (98.8%)	71.9 (98.8%)	56.9 (99.9%)	14.5 (99.9%)	12.5 (99.9%)
20.0	71.9 (100%)	72.8 (100%)	57.0 (100%)	14.5 (100%)	12.5 (100%)

π^0:

- Although weight functions plotted earlier are not applicable to off-shell LMD+V form factor, region below $\Lambda = 1$ GeV gives the bulk of the result: 82% for LMD+V, 92% for VMD.
- No damping from off-shell LMD+V form factor at external vertex since $\chi \neq 0$ (new short-distance constraint). Note: VMD falls off too fast, compared to OPE.

η, η':

- Mass of intermediate pseudoscalar is higher than pion mass $\rightarrow$ expect a stronger suppression from propagator.
- Peak of relevant weight functions shifted to higher values of Q_i. For η', vector meson mass is also higher $M_V = 859$ MeV. Saturation effect and the suppression from the VMD form factor only fully set in around $\Lambda = 1.5$ GeV: 96% of total for η, 93% for η'.

Minimal Hadronic Ansatz for $\langle VV \rangle$ and HVP contribution to $g-2$

Adapted from de Rafael

Consider Adler function $\mathcal{A}(Q^2) \equiv -Q^2 \frac{\partial \Pi_V(Q^2)}{\partial Q^2}$

$$\mathcal{A}(Q^2)\Big|_{\mathrm{MHA}} = \left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right) e^2 \left\{ 2 f_V^2 M_V^2 \frac{Q^2}{(Q^2+M_V^2)^2} + \frac{N_C}{16\pi^2}\frac{4}{3}\frac{Q^2}{Q^2+s_0}(1+\ldots)\right\}$$

Chiral loops (two-pion states) subleading in $1/N_C$.

No $1/Q^2$ term in the OPE $\Rightarrow$ fixes s_0: $\quad 2 f_V^2 M_V^2 = \frac{N_C}{16\pi^2}\frac{4}{3} s_0 \left(1 + \frac{3}{8}\frac{\alpha_s(s_0)}{\pi} + \ldots\right)$

General relation:

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi}\int_0^1 \frac{dx}{x}(1-x)\left(1-\frac{x}{2}\right)\mathcal{A}\left(\frac{x^2}{1-x} m_\mu^2\right)$$

$$a_\mu^{\mathrm{HVP}}\Big|_{\mathrm{MHA}} = (5700 \pm 1900) \times 10^{-11} \qquad \text{(33\% systematic error from } 1/N_C\text{)}$$

Of course, this error for HVP cannot compete with evaluations based on data on $\sigma(e^+e^- \to \text{hadrons})$ with $\pm 45 \times 10^{-11}$. Imposing further theoretical and experiment constraints on the MHA for the relevant Green's functions can maybe bring down the error to 10-15%. Would be almost enough for HLbL.