



JOHANNES GUTENBERG
UNIVERSITÄT MAINZ

Data driven approaches to the hadronic LbL contribution to $(g-2)_\mu$

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Mainz, Germany

$(g-2)_\mu$: Quo Vadis?

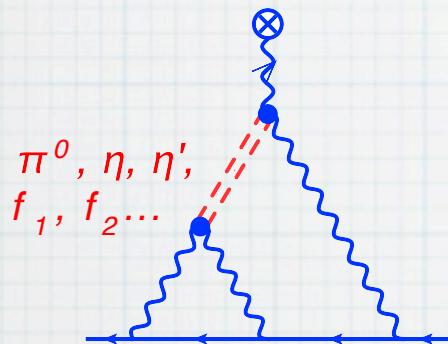
Institute for Nuclear Physics,
JGU Mainz, Germany

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Single meson LbL contribution to $(g-2)_\mu$: state of art

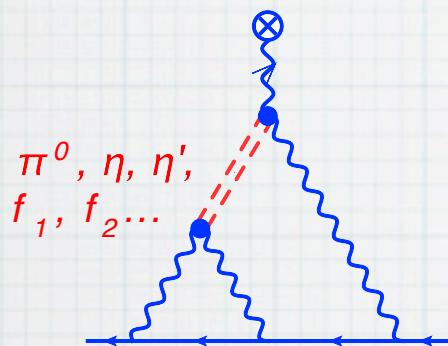
Hadronic LbL contribution to $(g-2)_\mu$ due to single meson exchanges:



	pseudo-scalars	axial-vectors	scalars	tensors
BPP	85 ± 13	2.5 ± 1.0	-7 ± 2	-
HKS	82.7 ± 6.4	1.7 ± 1.7	-	-
MV	114 ± 10	22 ± 5	-	-
KN	83 ± 12	-	-	-
J	93.9 ± 12.4	~ 7	-6.0 ± 1.2	-

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BPP (Bijnens, Pallante, Prades)

Extended Nambu-Jona-Lasigno model

HKS (Hayakawa, Kinoshita, Sanda)

Hidden local gauge symmetry model

MV (Melnikov, Vainshtein)

OPE and short-distance constraints

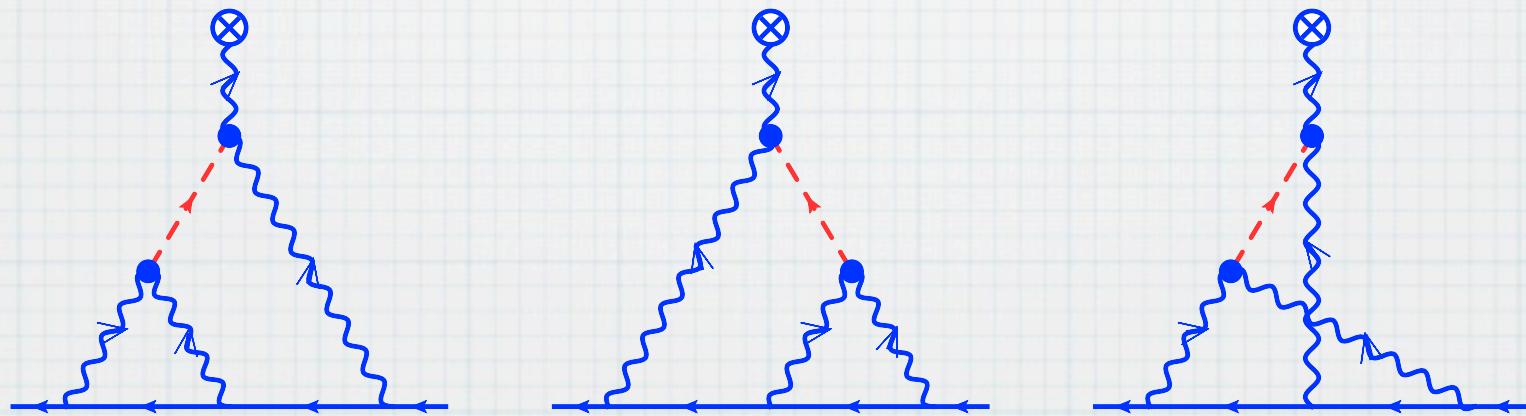
KN (Knecht, Nyffeler)

Large- N_c and data on meson form factors

J (Jegerlehner)

Large- N_c and data on form factors + off-shell effects
OPE and short-distance constraints

Heavy-meson contributions to the $(g-2)_\mu$



Projection technique

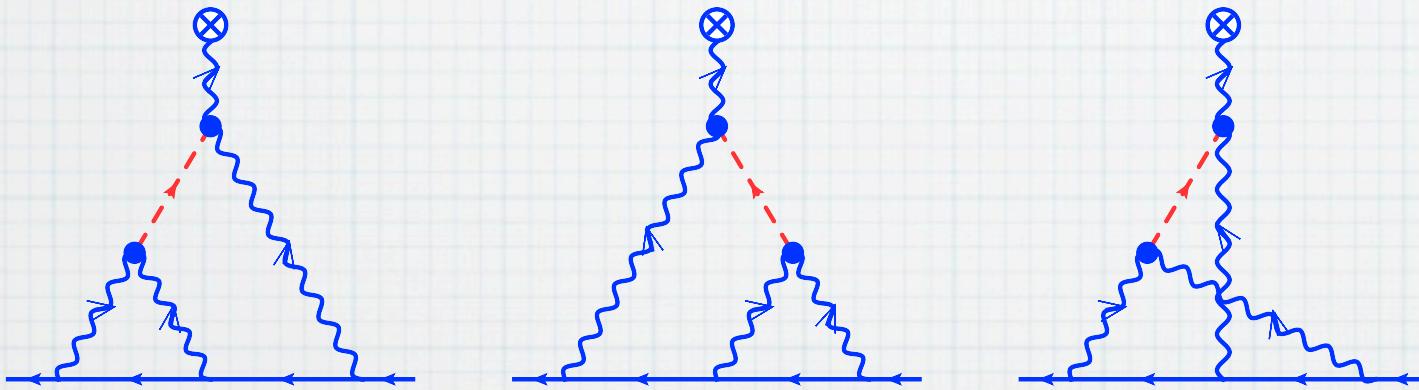
$$a_\mu = \lim_{k \rightarrow 0} \text{Tr} [(\not{p} + m) \Lambda_\nu(p', p) (\not{p}' + m) \Gamma^\nu(p', p)]$$

$$\begin{aligned} a_\mu^{LbL} = & \lim_{k \rightarrow 0} ie^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (k - q_1 - q_2)^2} \frac{1}{(p + q_1)^2 - m^2} \frac{1}{(p' - q_2)^2 - m^2} \\ & \times T^{\mu\nu\lambda\sigma}(q_1, k - q_1 - q_2, q_2) \Pi_{\mu\nu\lambda\sigma}(q_1, k - q_1 - q_2, q_2) \end{aligned}$$

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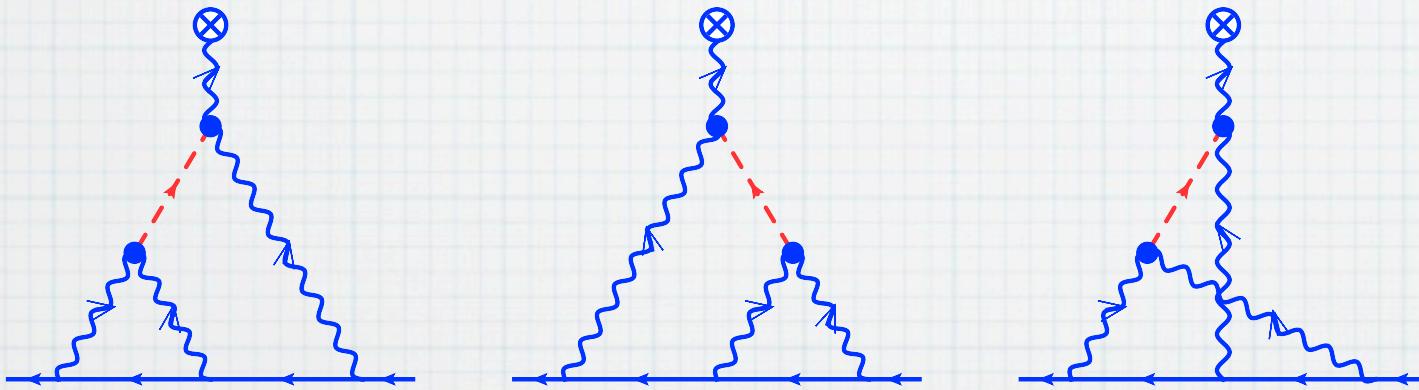
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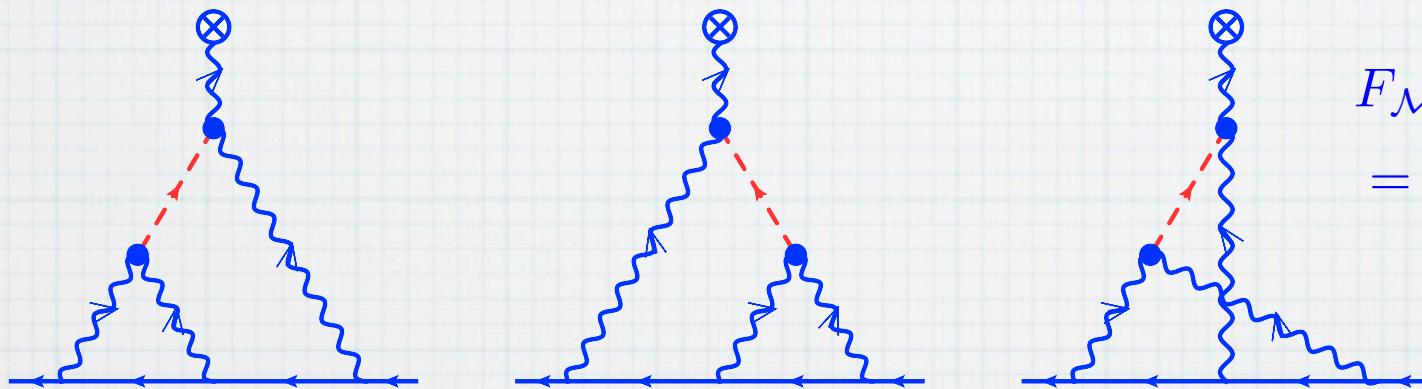


$$\begin{aligned} a_\mu^{LbL} = \lim_{k \rightarrow 0} -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} & \frac{1}{q_1^2 q_2^2 (k - q_1 - q_2)^2 [(p + q_1)^2 - m^2] [(p + k - q_2)^2 - m^2]} \\ & \times \left[\frac{F_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, (k - q_1 - q_2)^2) F_{\mathcal{M}\gamma^*\gamma^*}(k^2, q_2^2)}{(k - q_2)^2 - M^2} T_1(q_1, k - q_1 - q_2, q_2) \right. \\ & \left. + \frac{F_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\mathcal{M}\gamma^*\gamma^*}((k - q_1 - q_2)^2, k^2)}{(q_1 + q_2)^2 - M^2} T_2(q_1, k - q_1 - q_2, q_2) \right] \end{aligned}$$

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$$\begin{aligned} F_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, q_2^2, (q_1 + q_2)^2) \\ = F_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, q_2^2, M^2) \end{aligned}$$

constant FFs

$$\begin{aligned} a_\mu^{LbL} = \lim_{k \rightarrow 0} -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} & \frac{1}{q_1^2 q_2^2 (k - q_1 - q_2)^2 [(p + q_1)^2 - m^2] [(p + k - q_2)^2 - m^2]} \\ \times \left[\frac{F_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, (k - q_1 - q_2)^2) F_{\mathcal{M}\gamma^*\gamma^*}(k^2, q_2^2)}{(k - q_2)^2 - M^2} T_1(q_1, k - q_1 - q_2, q_2) \right. \\ \left. + \frac{F_{\mathcal{M}\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\mathcal{M}\gamma^*\gamma^*}((k - q_1 - q_2)^2, k^2)}{(q_1 + q_2)^2 - M^2} T_2(q_1, k - q_1 - q_2, q_2) \right] \end{aligned}$$

Axial-vector meson transition amplitude

Transition form factor

for 2γ decay widths $\Gamma_{\gamma\gamma}$ and dipole masses Λ_A entering the FF, we use the experimental results from the L3 Collaboration.

dipole parametrization
 $A \rightarrow \gamma\gamma$ transition FF:

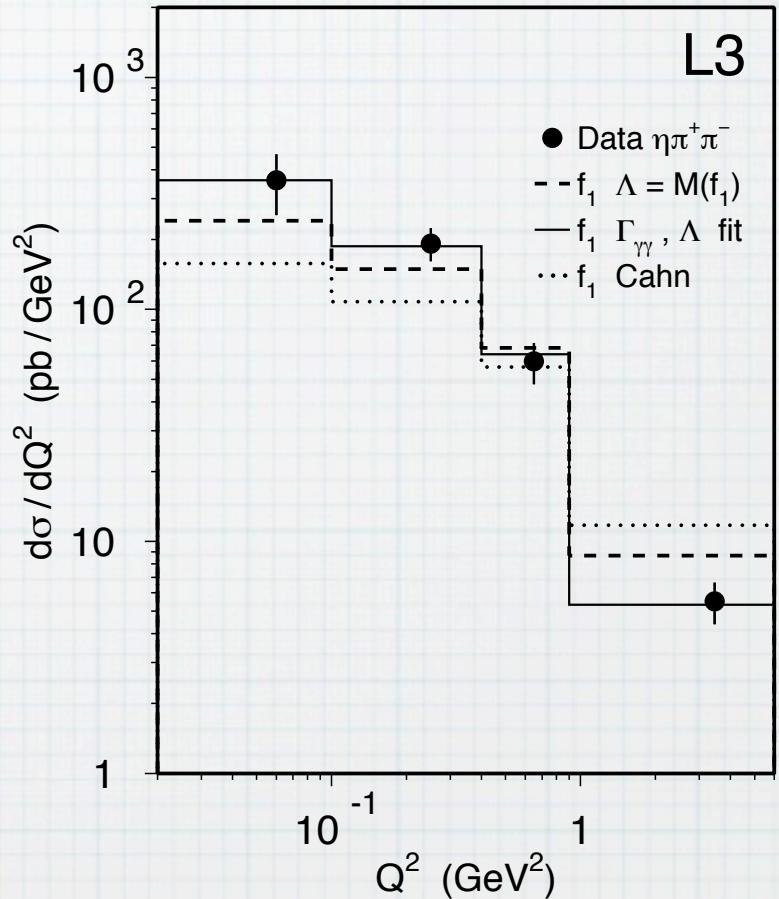
$$\frac{A(q_1^2, 0)}{A(0, 0)} = \frac{1}{(1 - q_1^2/\Lambda_A^2)^2}$$

normalization

$$[A(0, 0)]^2 = \frac{12}{\pi\alpha^2} \frac{1}{m_A^2} \Gamma_{\gamma\gamma}$$

	m_A [MeV]	$\tilde{\Gamma}_{\gamma\gamma}$ [keV]	Λ_A [MeV]
$f_1(1285)$	1281.8 ± 0.6	3.5 ± 0.8	1040 ± 78
$f_1(1420)$	1426.4 ± 0.9	3.2 ± 0.9	926 ± 78

L3 Collaboration



tensor & scalar mesons:
no direct phenomenological information is available

use simple dipole (monopole) parametrizations
estimates based on sum rules

Meson form factors and LbL sum rules

Meson form factors and LbL sum rules

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s + Q_1^2)} [\sigma_2 - \sigma_0]_{Q_2^2=0}$$

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s + Q_1^2)^2} \left[\sigma_{\parallel} + \sigma_{LT} + \frac{(s + Q_1^2)}{Q_1 Q_2} \tau_{TL}^a \right]_{Q_2^2=0}$$

Meson form factors and LbL sum rules

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s + Q_1^2)} [\sigma_2 - \sigma_0]_{Q_2^2=0}$$

at finite Q_1^2 the SRs imply information
on meson transition form-factors:

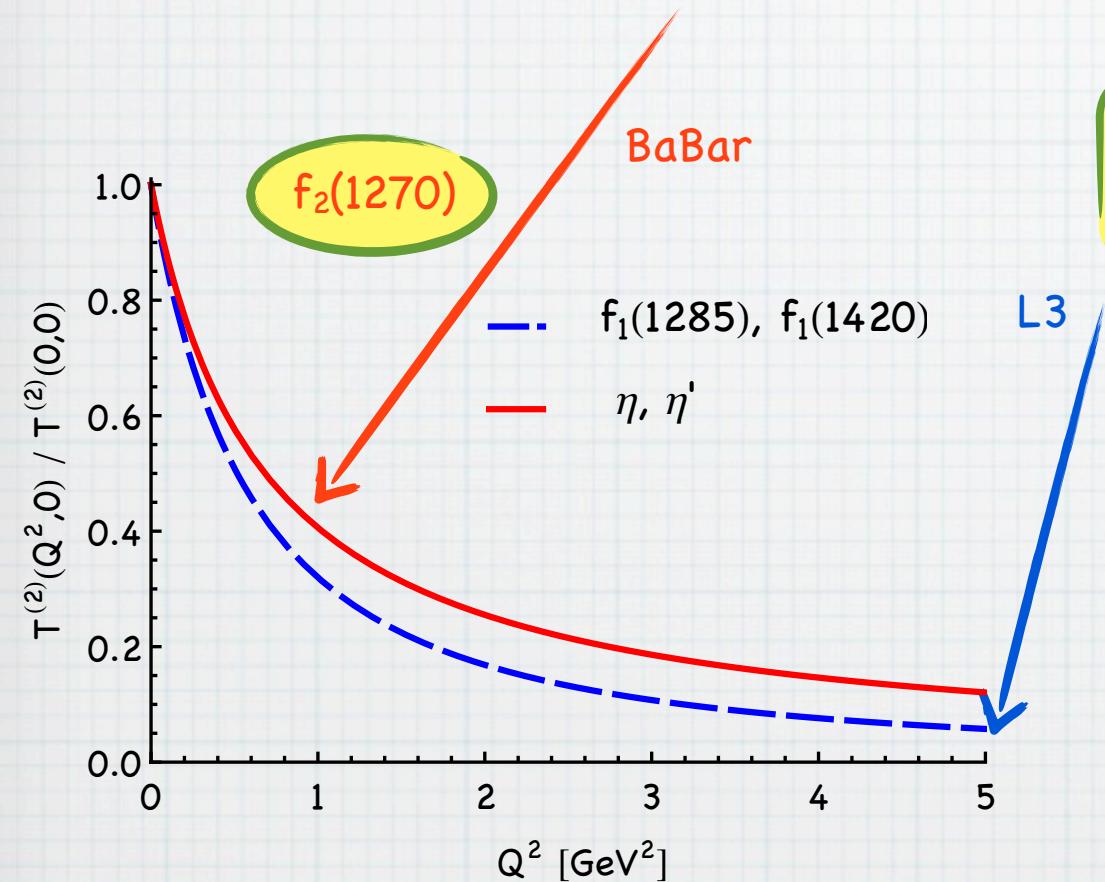
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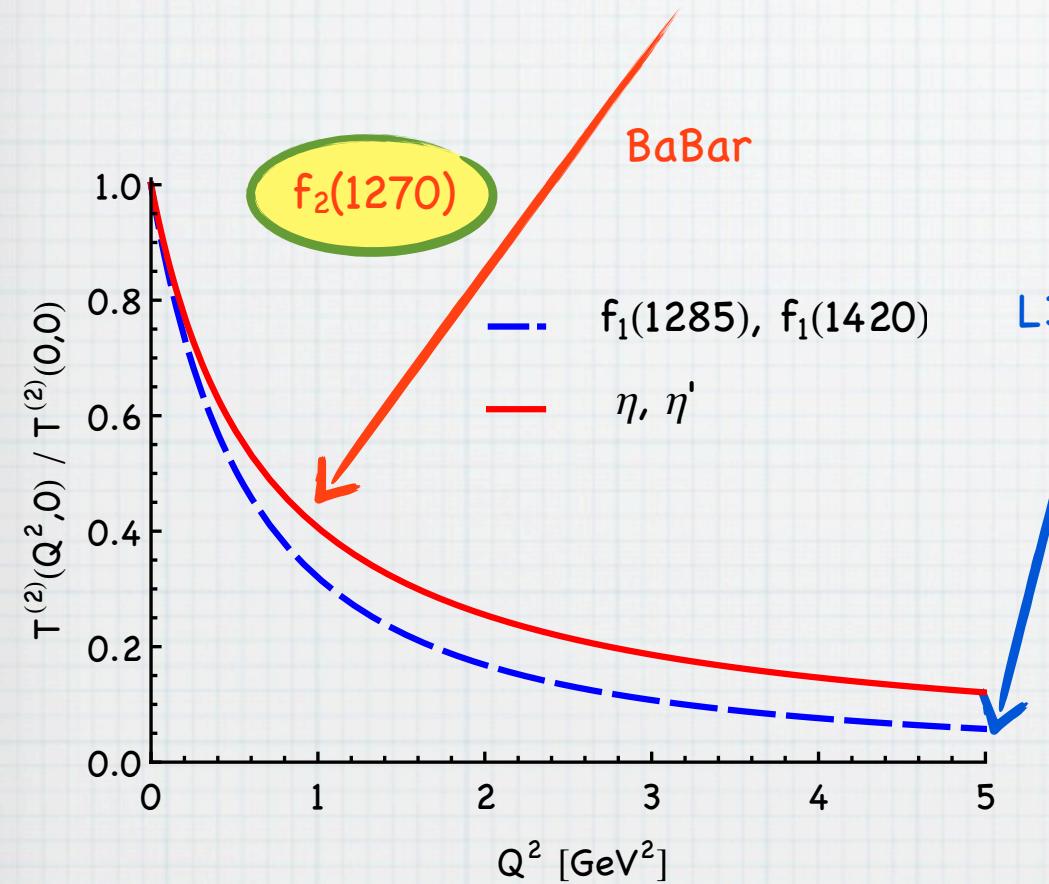


L3

Meson form factors and LbL sum rules

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$$\frac{T(q_1^2, 0)}{T(0, 0)} = \frac{1}{(1 - q_1^2/\Lambda_T^2)^2}$$

$$\Lambda_T = 1.5 \text{ GeV}$$

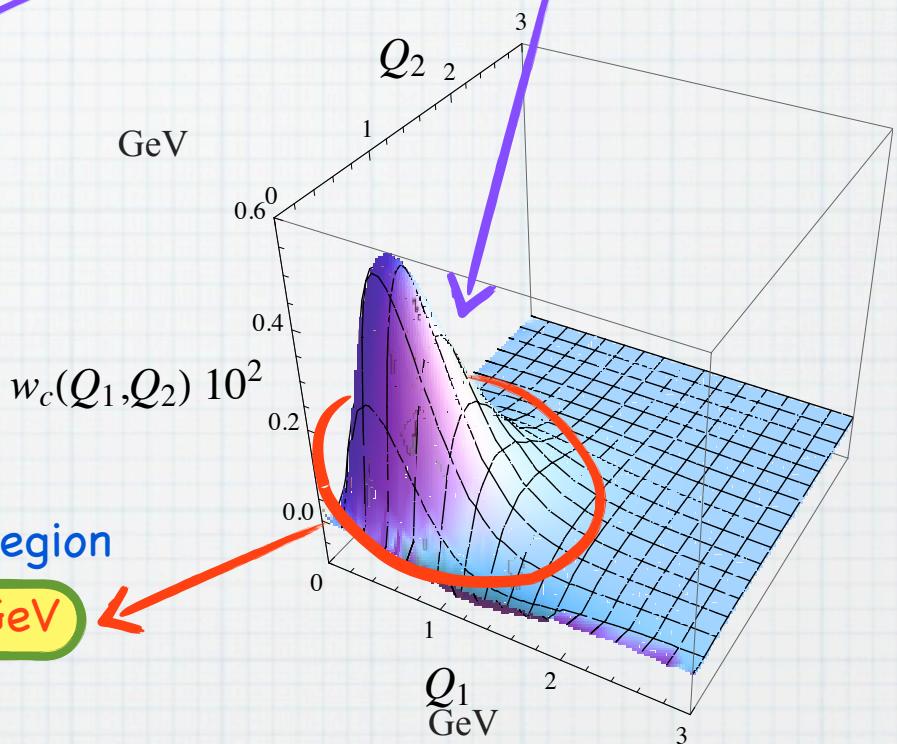
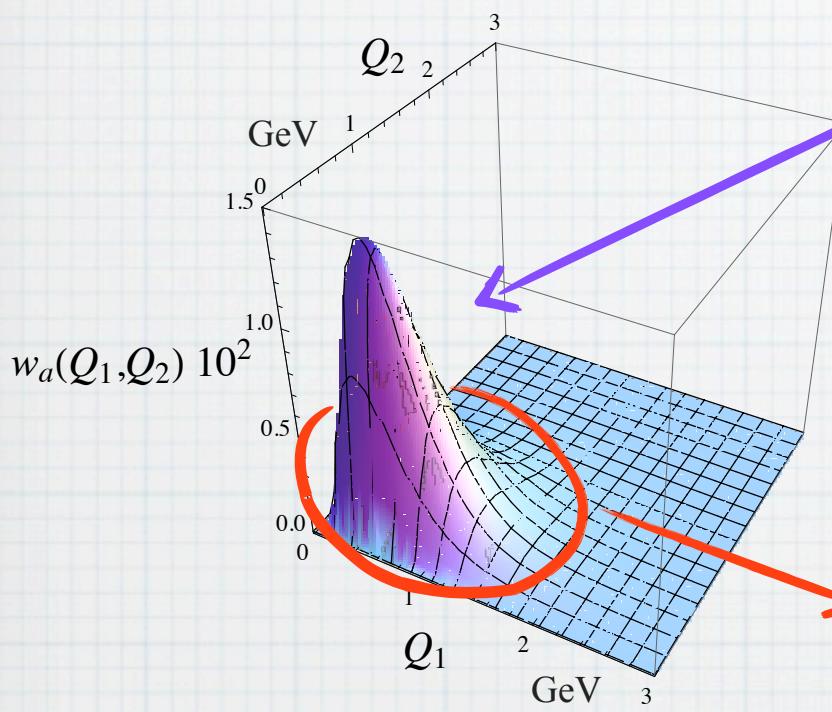
	M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]
$f_2(1270)$	1275.1 ± 1.2	3.03 ± 0.35
$f_2(1565)$	1562 ± 13	0.70 ± 0.14
$a_2(1320)$	1318.3 ± 0.6	1.00 ± 0.06
$a_2(1700)$	1732 ± 16	0.30 ± 0.05

Results: axial-vector mesons

$$a_\mu^{LbL} = \frac{\alpha}{(2\pi)^2} \frac{\Lambda_{A1}^6 \Lambda_{A2}^6 \tilde{\Gamma}_{\gamma\gamma}(A)}{m m_A^5} \int dQ_1 \int dQ_2 [2w_a(Q_1, Q_2) + w_c(Q_1, Q_2)]$$

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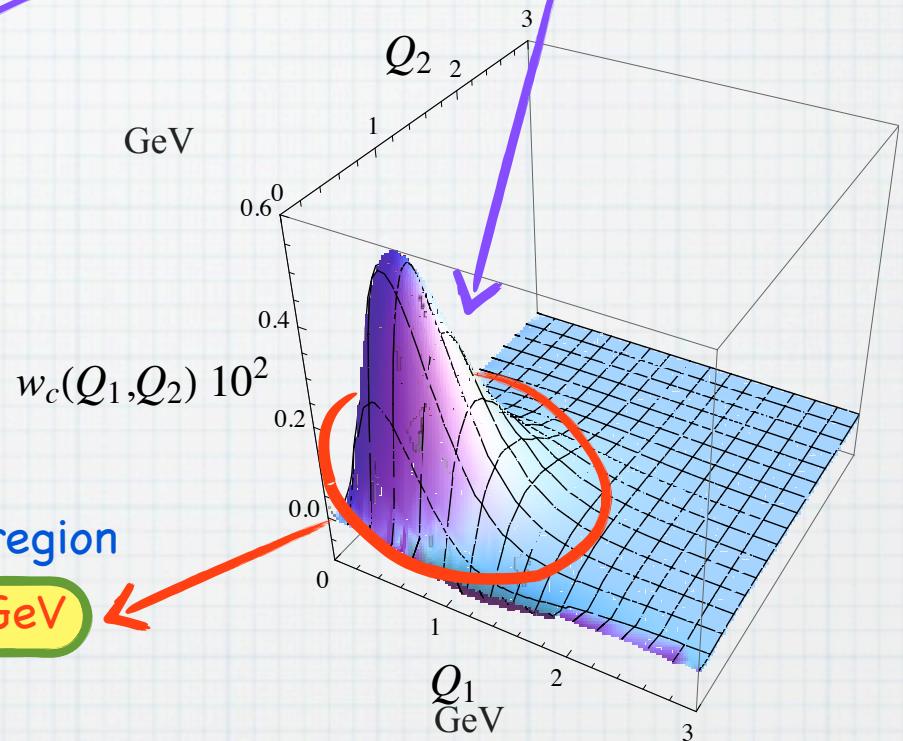
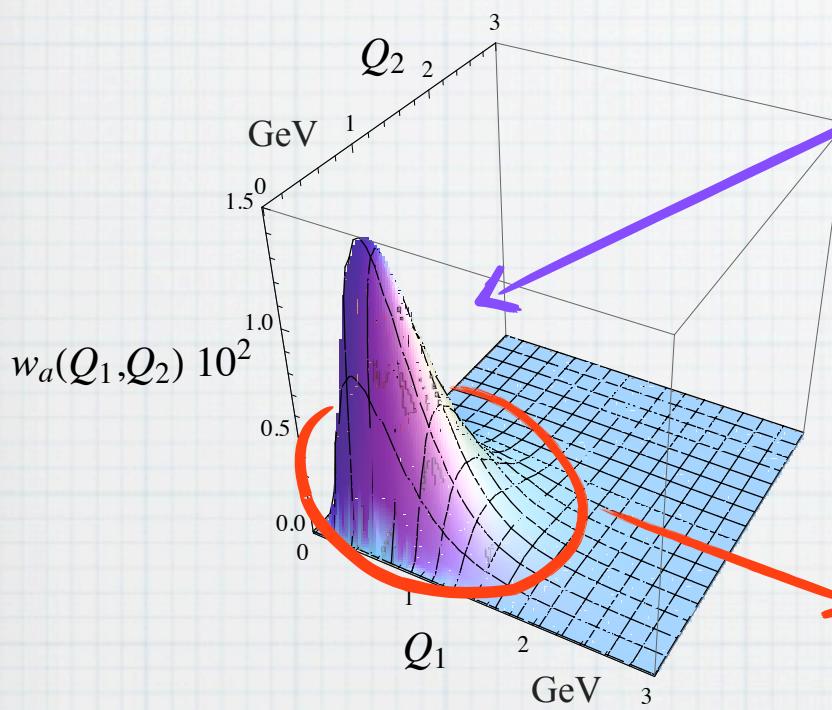


dominating region

$Q_1 \sim Q_2 \sim 1 \text{ GeV}$

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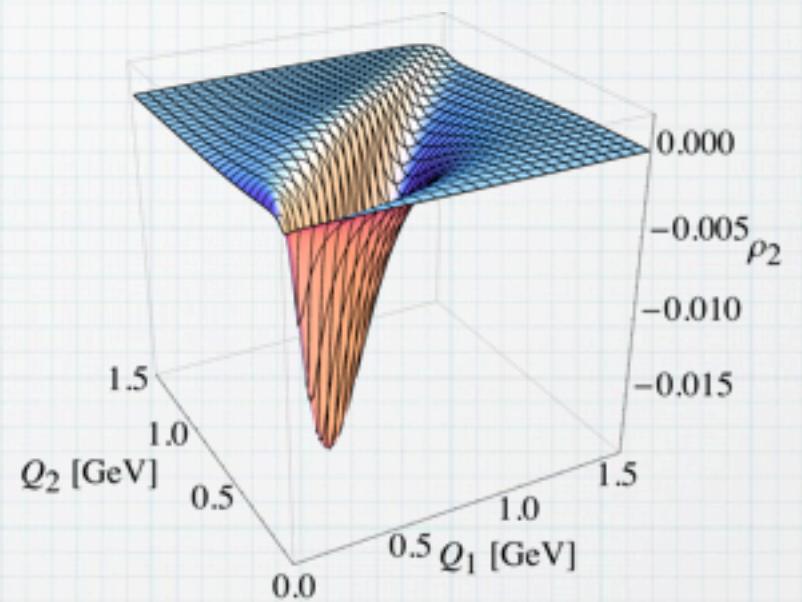
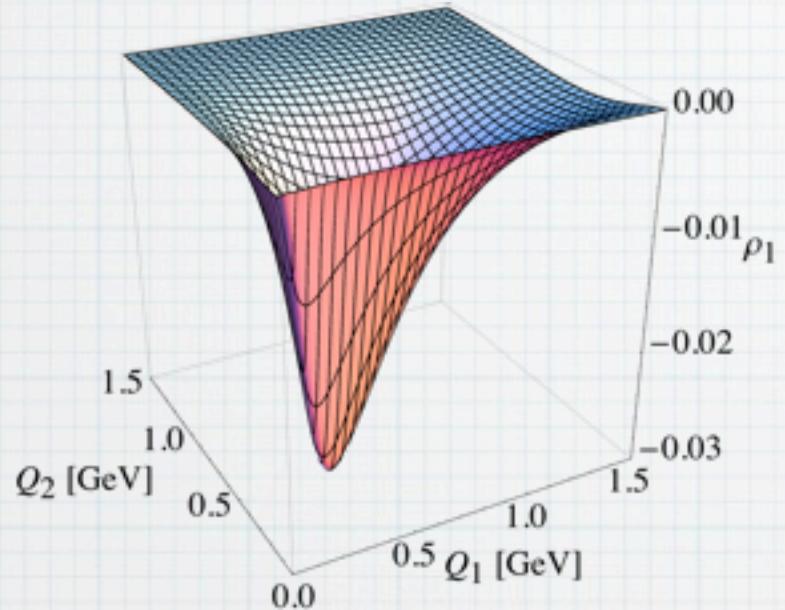


	m_A [MeV]	$\tilde{\Gamma}_{\gamma\gamma}$ [keV]	Λ_A [MeV]	$a_\mu^{LbL;A} \times 10^{10}$
$f_1(1285)$	1281.8 ± 0.6	3.5 ± 0.8	1040 ± 78	$0.50^{+0.20}_{-0.17}$
$f_1(1420)$	1426.4 ± 0.9	3.2 ± 0.9	926 ± 78	$0.14^{+0.07}_{-0.06}$

the contribution of the
axial-vector pole
to the $(g-2)_\mu$
V.P., M. Vanderhaeghen
(2014)

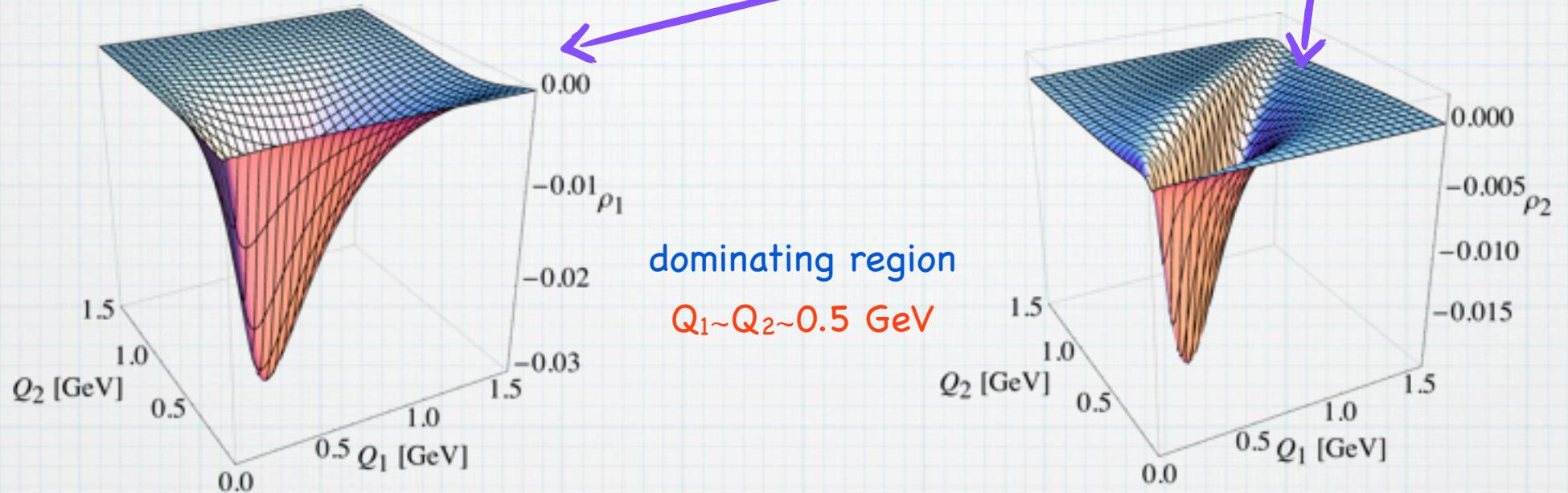
Results: scalar mesons

$$a_\mu^{LbL} = -\frac{4\alpha^3}{\pi^3} |F_{\mathcal{M}\gamma^*\gamma^*}(0,0)|^2 \int_0^\infty dQ_1 \int_0^\infty dQ_2 [2 \rho_1(Q_1, Q_2) + \rho_2(Q_1, Q_2)]$$



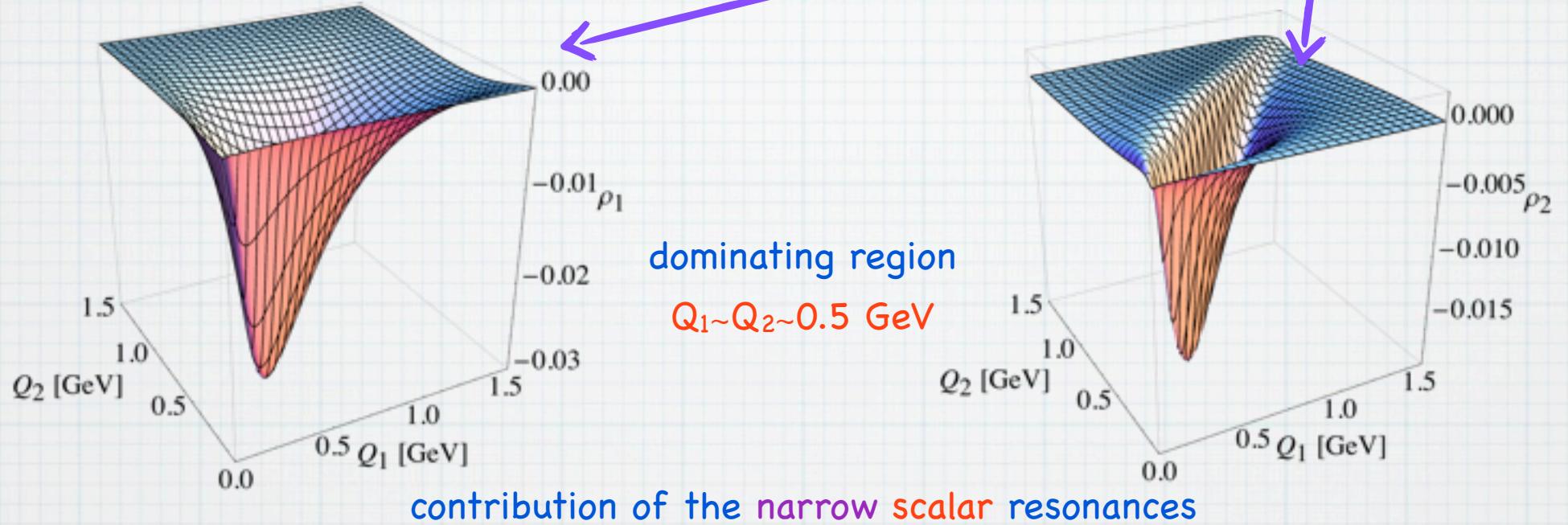
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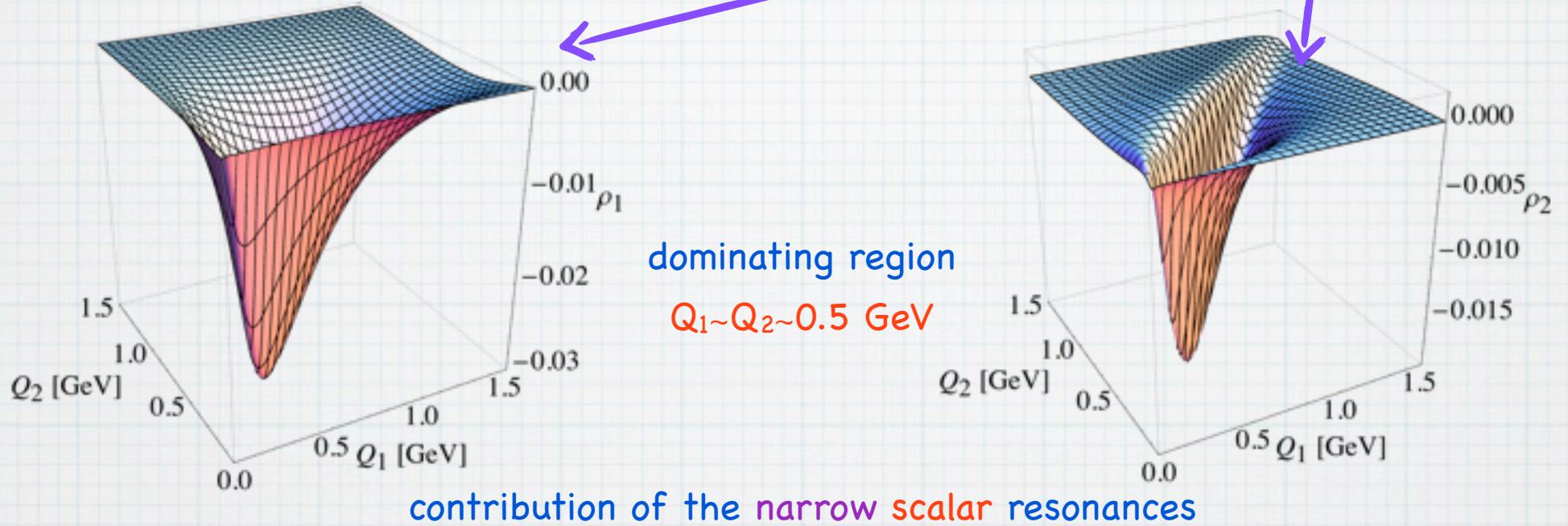
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	m_M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	a_μ ($\Lambda_{mono} = 1 \text{ GeV}$) $[10^{-11}]$	a_μ ($\Lambda_{mono} = 2 \text{ GeV}$) $[10^{-11}]$
$f_0(980)$	980 ± 10	0.29 ± 0.07	-0.19 ± 0.05	-0.61 ± 0.15
$f'_0(1370)$	$1200 - 1500$	3.8 ± 1.5	-0.54 ± 0.21	-1.84 ± 0.73
$a_0(980)$	980 ± 20	0.3 ± 0.1	-0.20 ± 0.07	-0.63 ± 0.21
Sum			-0.9 ± 0.2	-3.1 ± 0.8

Results: scalar mesons

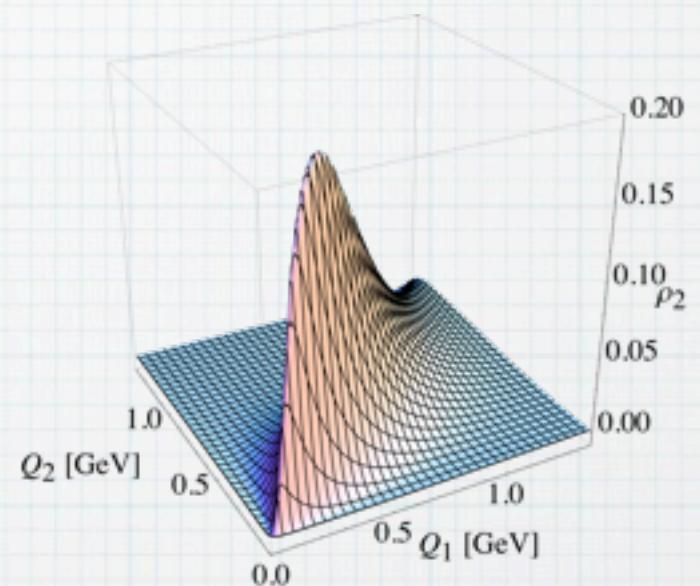
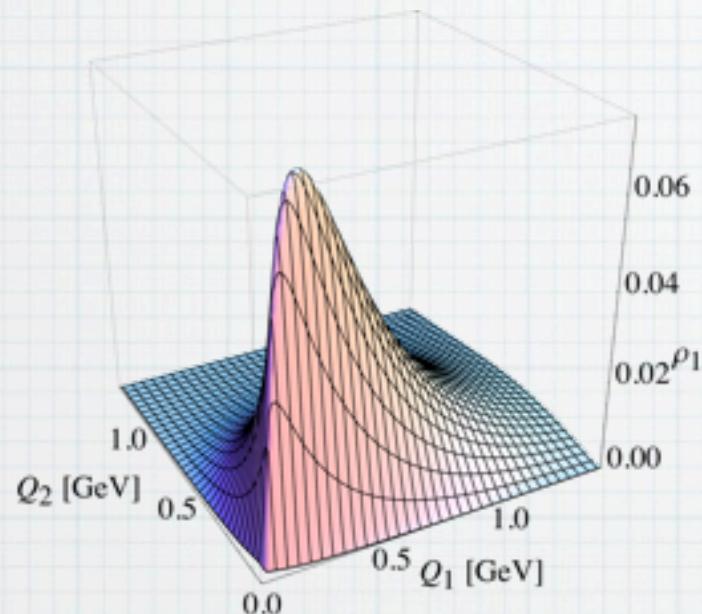
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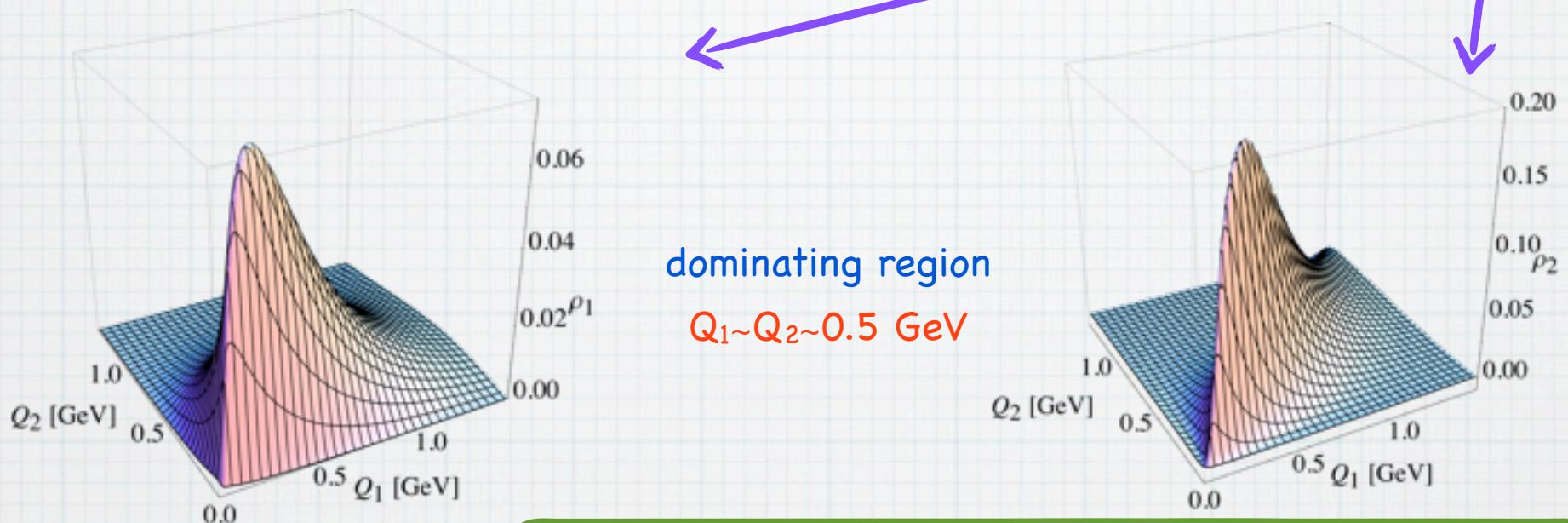
Results: tensor mesons

$$a_{\mu}^{LbL} = -\frac{20\alpha^3}{\pi^3} |F_{\mathcal{M}\gamma^*\gamma^*}(0,0)|^2 \int_0^\infty dQ_1 \int_0^\infty dQ_2 [2 \rho_1(Q_1, Q_2) + \rho_2(Q_1, Q_2)]$$



Results: tensor mesons

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contribution of the
narrow tensor resonances

	m_M [MeV]	$\Gamma_{\gamma\gamma}$ [keV]	a_μ ($\Lambda_{dip} = 1.5 \text{ GeV}$) [10^{-11}]
$f_2(1270)$	1275.1 ± 1.2	3.03 ± 0.35	0.79 ± 0.09
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$a_2(1700)$	1732 ± 16	0.30 ± 0.05	0.02 ± 0.003
Sum			1.1 ± 0.1

Summary

E821 measurement of $(g-2)_\mu$ (2009)

$$a_\mu^{\text{exp}} = (11\ 659\ 2089 \pm 63) \times 10^{-11}$$

	pseudo-scalars	axial-vectors	scalars	tensors
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new FNAL $(g-2)_\mu$ measurement (2015):

factor 4 precision improvement

$$\pm 16 \cdot 10^{-11}$$

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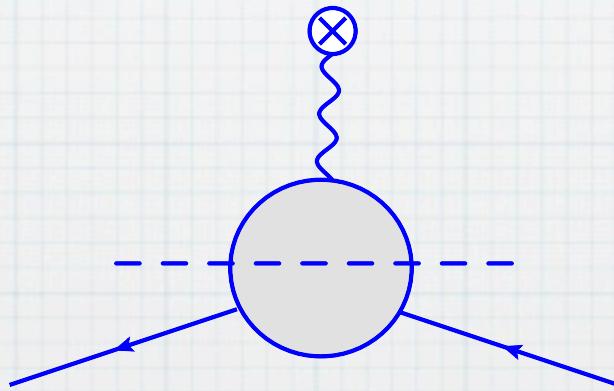
factor 4 precision improvement

$$\pm 16 \cdot 10^{-11}$$

data on meson form factors in a space-like region

decay width of the mesons

$F_2(t)$ and dispersion relations



a_μ from a dispersion relation

$$a_\mu = \lim_{k \rightarrow 0} F_2(k^2, (p+k)^2, p^2)$$

Pauli form factor
in the limit of static electromagnetic field

a_μ from a dispersion relation

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Pauli form factor
in the limit of static electromagnetic field

on-shell conditions
for a muon

$$\begin{aligned}(p+k)^2 &= m^2 \\ p^2 &= m^2\end{aligned}\quad \longrightarrow \quad (p \cdot k) = -k^2/2$$

a_μ from a dispersion relation

$$a_\mu = \lim_{k \rightarrow 0} F_2(k^2, (p+k)^2, p^2)$$

Pauli form factor
in the limit of static electromagnetic field

on-shell conditions
for a muon

$$\begin{aligned} (p+k)^2 &= m^2 \\ p^2 &= m^2 \end{aligned} \quad \longrightarrow \quad (p \cdot k) = -k^2/2$$

$$a_\mu = F_2(0)$$

a_μ from a dispersion relation

$$a_\mu = \lim_{k \rightarrow 0} F_2(k^2, (p+k)^2, p^2)$$

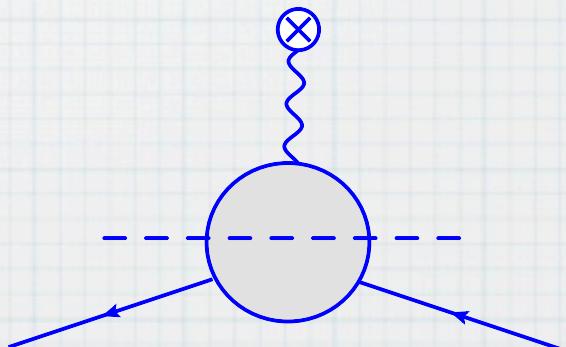
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$$a_\mu = F_2(0)$$

$$F_2(0) = \frac{1}{2\pi i} \int \frac{dk^2}{k^2} \text{Abs } F_2(k^2)$$

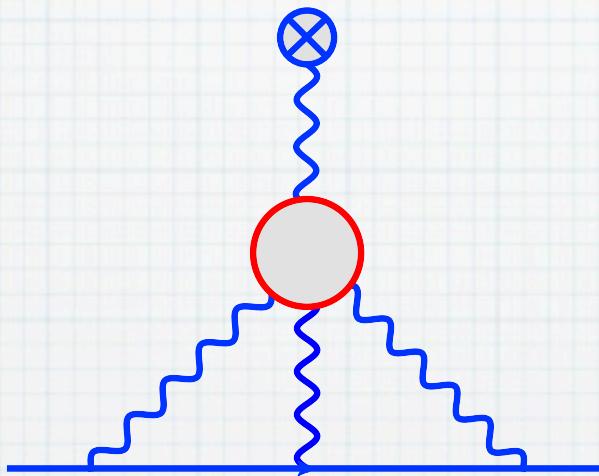


HLbL contribution to $(g-2)_\mu$

projection technique

$$F_2(k^2) = \text{Tr} [(\not{p} + m)\Lambda_\nu(p', p)(\not{p}' + m)\Gamma^\nu(p', p)]$$

$$\Gamma_{\mu}^{\text{LBLL}}(p', p)$$



$$F_2(k^2) = e^6 \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda} (-1)^{\lambda + \lambda_1 + \lambda_2 + \lambda_3} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} L_{\lambda_1 \lambda_2 \lambda_3 \lambda}(p, p', q_1, k - q_1 - q_2, q_2)$$

weighting functions (entire)

analytic structure



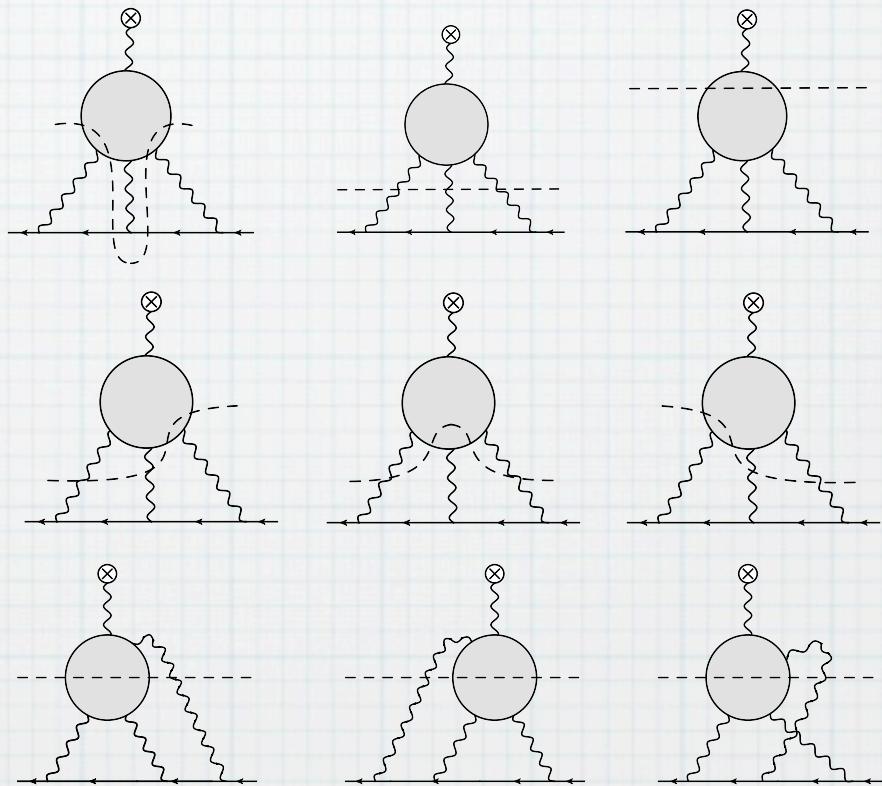
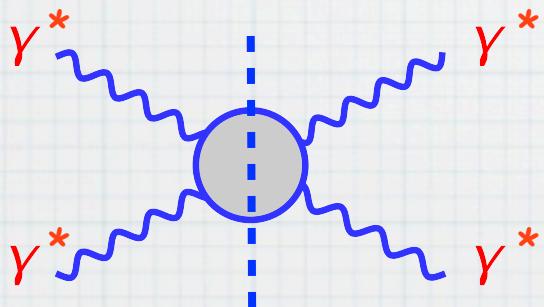
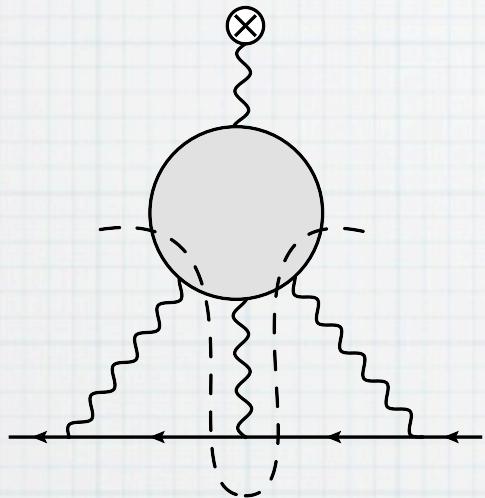
$$\times \boxed{\frac{\Pi_{\lambda_1 \lambda_2 \lambda_3 \lambda}(q_1, k - q_1 - q_2, q_2, k)}{q_1^2 q_2^2 (k - q_1 - q_2)^2 [(p + q_1)^2 - m^2] [(p + k - q_2)^2 - m^2]}}$$

Discontinuity

$$D_{2\mu H^+} = e^6 \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda} (-1)^{\lambda + \lambda_1 + \lambda_2 + \lambda_3} \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} (-2\pi i)^2 \delta((p + q_1)^2 - m^2) \delta((p + k - q_2)^2 - m^2)$$

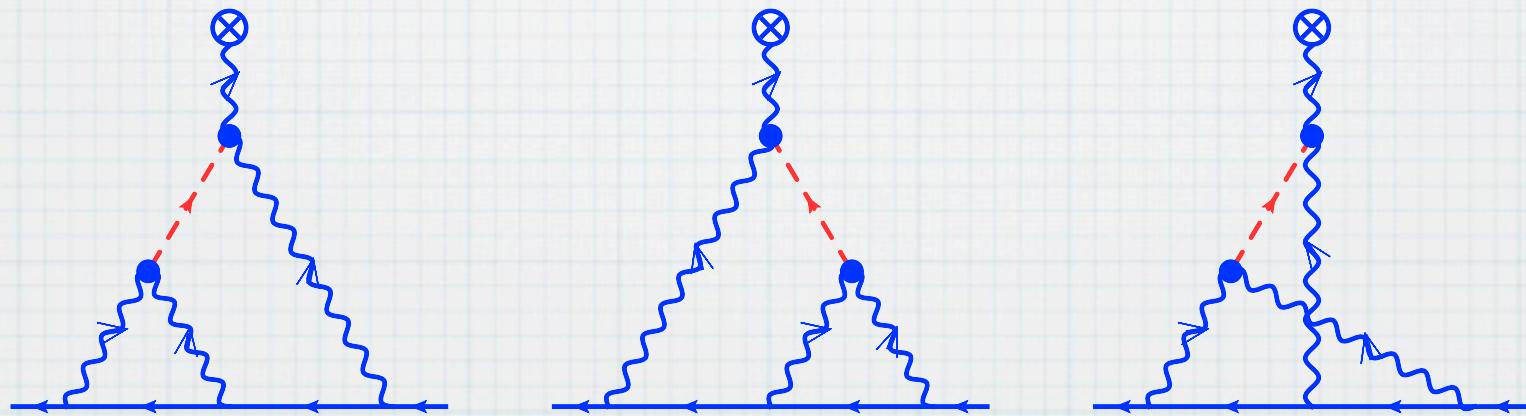
$$\times \frac{L_{\lambda_1 \lambda_2 \lambda_3 \lambda}(p, p', q_1, k - q_1 - q_2, q_2)}{q_1^2 q_2^2 (k - q_1 - q_2)^2}$$

$$\times \text{Disc}_{(q_1 + q_2)^2} \Pi_{\lambda_1 \lambda_2 \lambda_3 \lambda}(q_1, k - q_1 - q_2, q_2, k)$$



$$\text{Abs}F_2(k^2) = D_{3\gamma} + D_{H^-} + D_{2\mu H^+} + D_{\gamma H^+}^{(1)} + D_{\gamma H^+}^{(2)} + D_{\gamma H^+}^{(3)} + D_{2\gamma H^-}^{(1)} + D_{2\gamma H^-}^{(2)} + D_{2\gamma H^-}^{(3)}$$

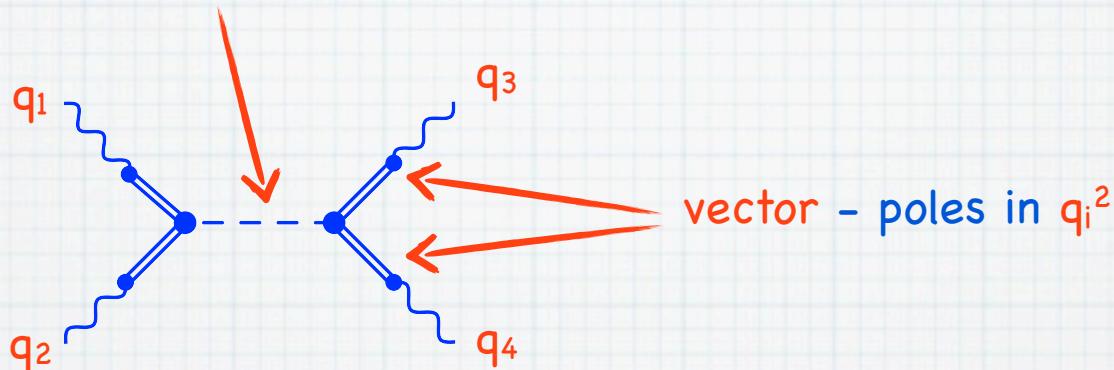
Meson pole contributions to $(g-2)_\mu$



Pole contributions

π^0 - pole in $(q_i+q_j)^2$

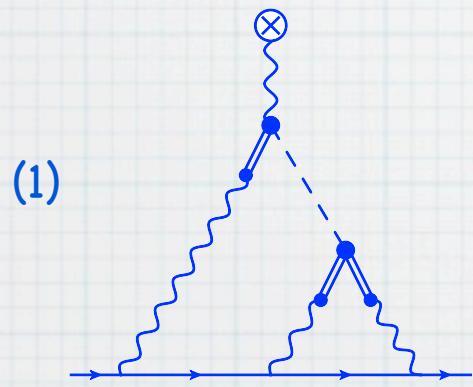
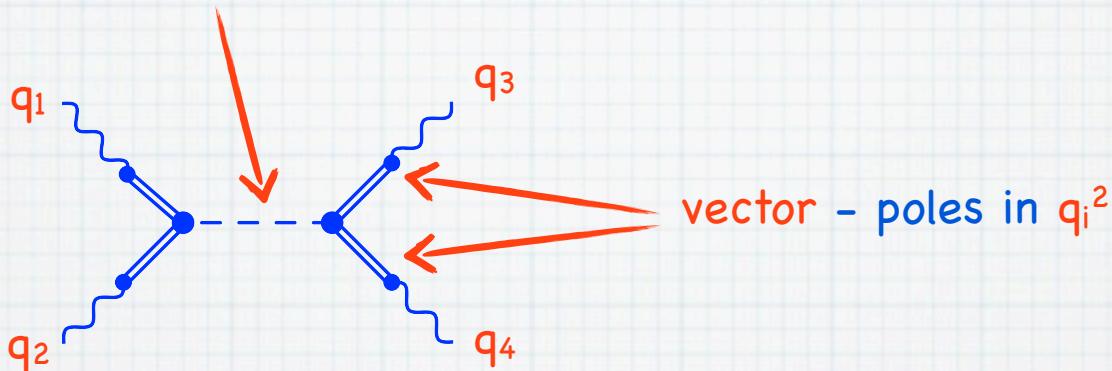
analytical structure of LbL amplitude



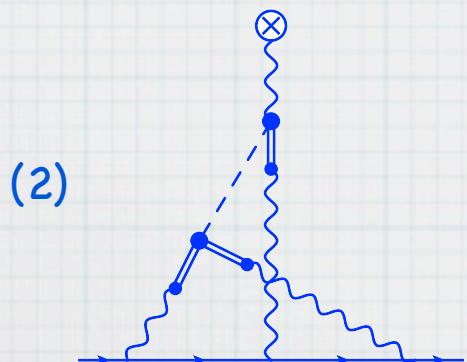
Pole contributions

π^0 - pole in $(q_i+q_j)^2$

analytical structure of LbL amplitude



$$\begin{aligned}
 F_2^{(1)}(t) = & e^6 \Lambda^6 |F_{P\gamma^*\gamma^*}(0, 0, M^2)|^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \\
 & \times \frac{1}{q_1^2} \frac{1}{q_1^2 - \Lambda^2} \frac{1}{q_2^2} \frac{1}{q_2^2 - \Lambda^2} \frac{1}{(k - q_1 - q_2)^2} \frac{1}{(k - q_1 - q_2)^2 - \Lambda^2} \\
 & \times \frac{1}{(p + q_1)^2 - m^2} \frac{1}{(p + k - q_2)^2 - m^2} \frac{1}{(k - q_1)^2 - M^2} T_1(q_1, q_2, p, k)
 \end{aligned}$$

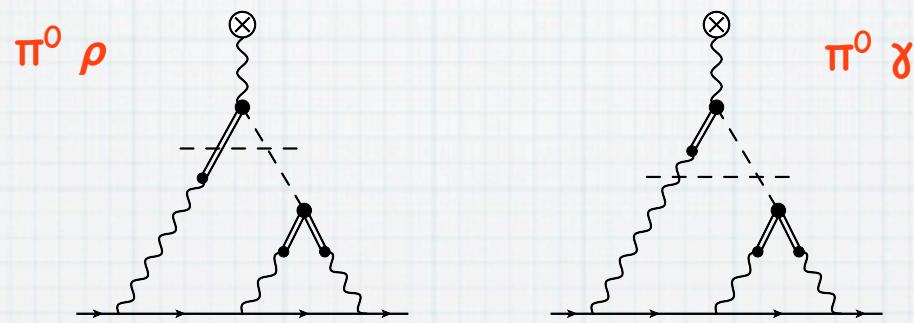


$$\begin{aligned}
 F_2^{(2)}(t) = & e^6 \Lambda^6 |F_{P\gamma^*\gamma^*}(0, 0, M^2)|^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \\
 & \times \frac{1}{q_1^2} \frac{1}{q_1^2 - \Lambda^2} \frac{1}{q_2^2} \frac{1}{q_2^2 - \Lambda^2} \frac{1}{(k - q_1 - q_2)^2} \frac{1}{(k - q_1 - q_2)^2 - \Lambda^2} \\
 & \times \frac{1}{(p + q_1)^2 - m^2} \frac{1}{(p + k - q_2)^2 - m^2} \frac{1}{(q_1 + q_2)^2 - M^2} T_2(q_1, q_2, p, k)
 \end{aligned}$$

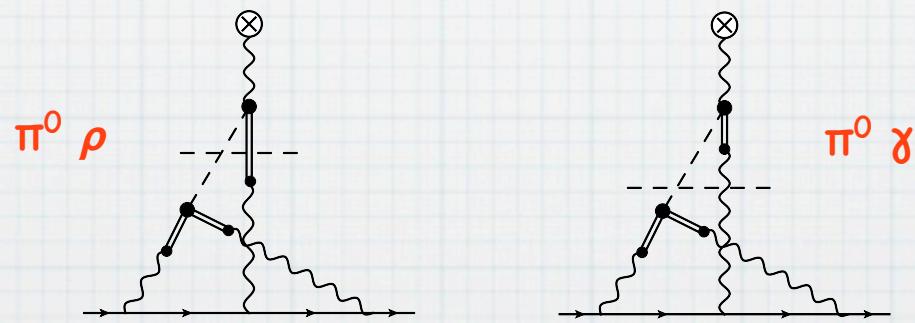
2-particle discontinuities

$$\text{Disc}^{(2)} \Gamma_i^\rho(t) = \text{Disc}_{\pi^0 \rho} \Gamma_i^\rho(t) + \text{Disc}_{\pi^0 \gamma} \Gamma_i^\rho(t)$$

(1)



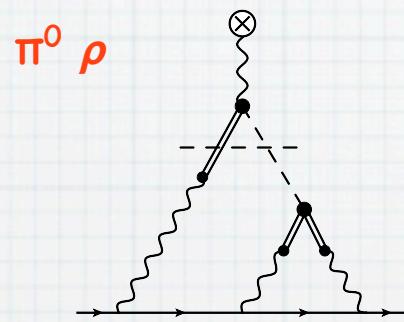
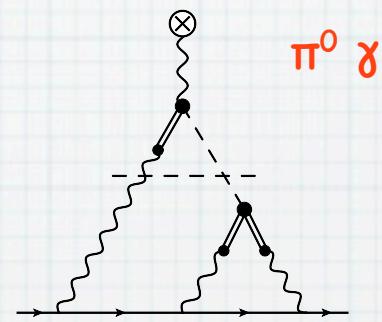
(2)



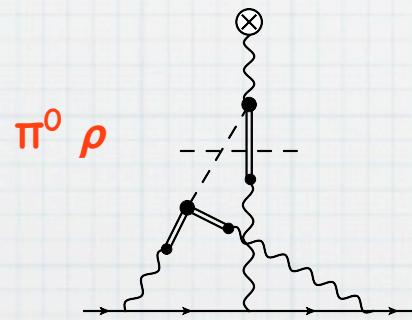
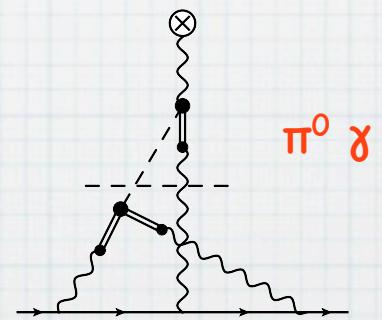
2-particle discontinuities

$$\text{Disc}^{(2)} \Gamma_i^\rho(t) = \text{Disc}_{\pi^0 \rho} \Gamma_i^\rho(t) + \text{Disc}_{\pi^0 \gamma} \Gamma_i^\rho(t)$$

(1)

 $\pi^0 \gamma$ 

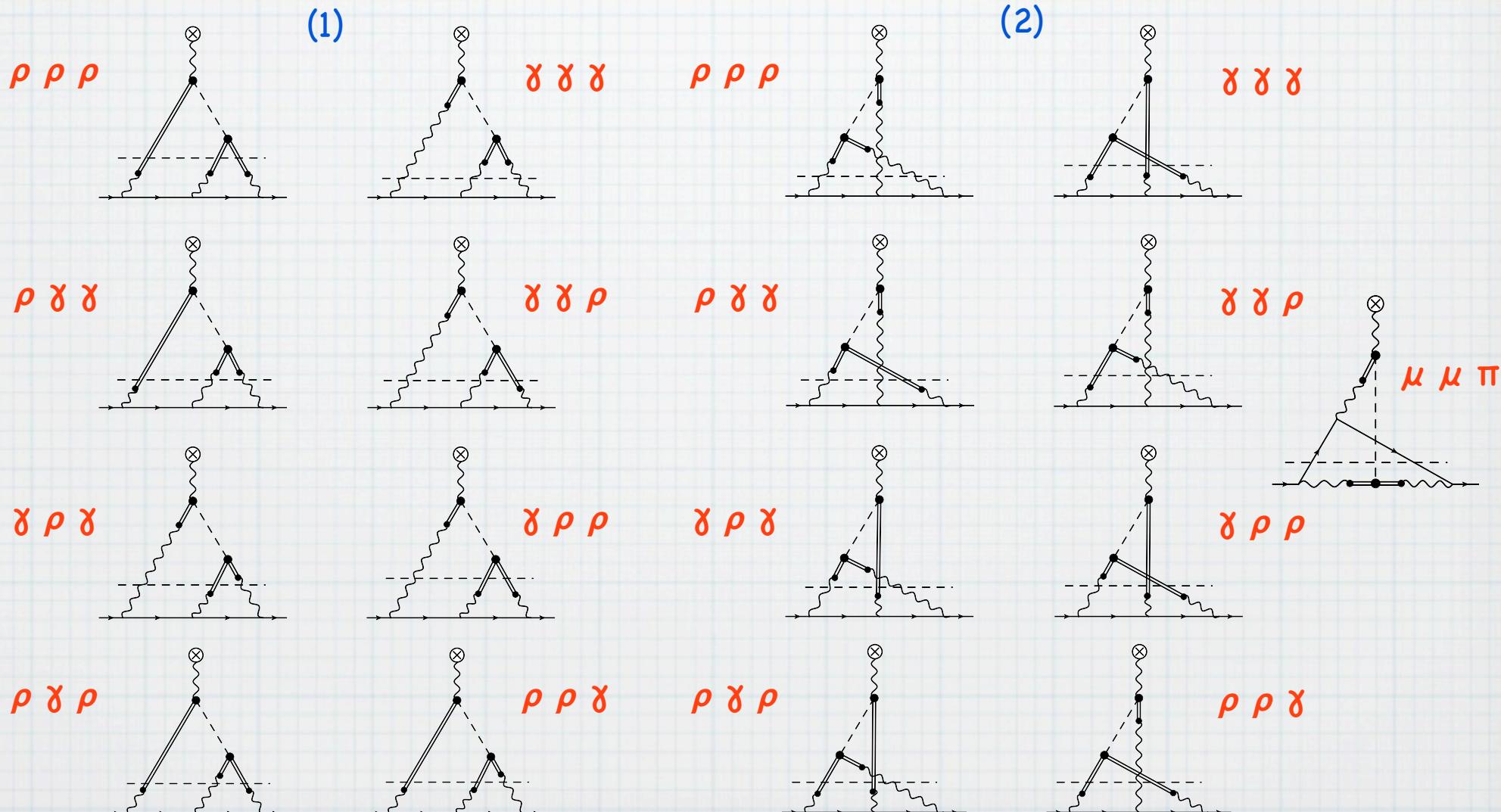
(2)

 $\pi^0 \gamma$ 

3-particle discontinuities

$$\begin{aligned} \text{Disc}^{(3)}\Gamma_1^\rho(t) = & \text{Disc}_{\gamma\gamma\gamma}\Gamma_1^\rho(t) + \text{Disc}_{\gamma\gamma\rho}\Gamma_1^\rho(t) + \text{Disc}_{\gamma\rho\gamma}\Gamma_1^\rho(t) \\ & + \text{Disc}_{\rho\gamma\gamma}\Gamma_1^\rho(t) + \text{Disc}_{\gamma\rho\rho}\Gamma_1^\rho(t) + \text{Disc}_{\rho\gamma\rho}\Gamma_1^\rho(t) \\ & + \text{Disc}_{\rho\rho\gamma}\Gamma_1^\rho(t) + \text{Disc}_{\rho\rho\rho}\Gamma_1^\rho(t) \end{aligned}$$

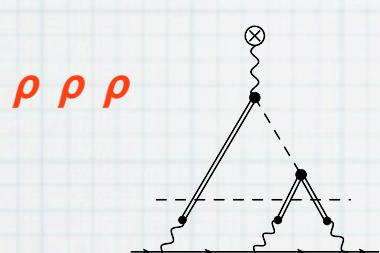
$$\begin{aligned} \text{Disc}^{(3)}\Gamma_2^\rho(t) = & \text{Disc}_{\gamma\gamma\gamma}\Gamma_2^\rho(t) + \text{Disc}_{\gamma\gamma\rho}\Gamma_2^\rho(t) + \text{Disc}_{\gamma\rho\gamma}\Gamma_2^\rho(t) \\ & + \text{Disc}_{\rho\gamma\gamma}\Gamma_2^\rho(t) + \text{Disc}_{\gamma\rho\rho}\Gamma_2^\rho(t) + \text{Disc}_{\rho\gamma\rho}\Gamma_2^\rho(t) \\ & + \text{Disc}_{\rho\rho\gamma}\Gamma_2^\rho(t) + \text{Disc}_{\rho\rho\rho}\Gamma_2^\rho(t) + \text{Disc}_{\mu\mu\pi}\Gamma_2^\rho(t) \end{aligned}$$



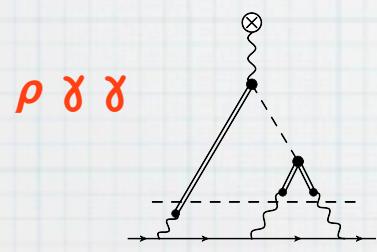
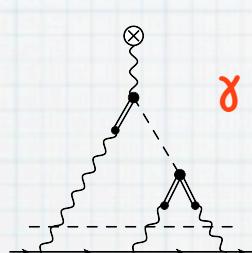
3-particle discontinuities

$$\begin{aligned} \text{Disc}^{(3)}\Gamma_1^\rho(t) = & \text{Disc}_{\gamma\gamma\gamma}\Gamma_1^\rho(t) + \text{Disc}_{\gamma\gamma\rho}\Gamma_1^\rho(t) + \text{Disc}_{\gamma\rho\gamma}\Gamma_1^\rho(t) \\ & + \text{Disc}_{\rho\gamma\gamma}\Gamma_1^\rho(t) + \text{Disc}_{\gamma\rho\rho}\Gamma_1^\rho(t) + \text{Disc}_{\rho\gamma\rho}\Gamma_1^\rho(t) \\ & + \text{Disc}_{\rho\rho\gamma}\Gamma_1^\rho(t) + \text{Disc}_{\rho\rho\rho}\Gamma_1^\rho(t) \end{aligned}$$

$$\begin{aligned} \text{Disc}^{(3)}\Gamma_2^\rho(t) = & \text{Disc}_{\gamma\gamma\gamma}\Gamma_2^\rho(t) + \text{Disc}_{\gamma\gamma\rho}\Gamma_2^\rho(t) + \text{Disc}_{\gamma\rho\gamma}\Gamma_2^\rho(t) \\ & + \text{Disc}_{\rho\gamma\gamma}\Gamma_2^\rho(t) + \text{Disc}_{\gamma\rho\rho}\Gamma_2^\rho(t) + \text{Disc}_{\rho\gamma\rho}\Gamma_2^\rho(t) \\ & + \text{Disc}_{\rho\rho\gamma}\Gamma_2^\rho(t) + \text{Disc}_{\rho\rho\rho}\Gamma_2^\rho(t) + \text{Disc}_{\mu\mu\pi}\Gamma_2^\rho(t) \end{aligned}$$



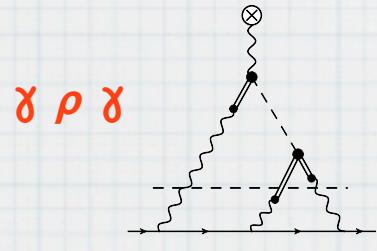
(1)



δ δ δ

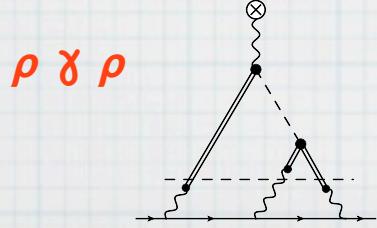
ρ ρ ρ

(2)



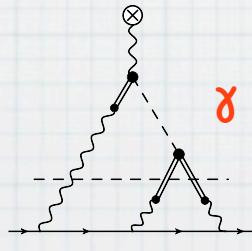
δ ρ δ

ρ δ δ

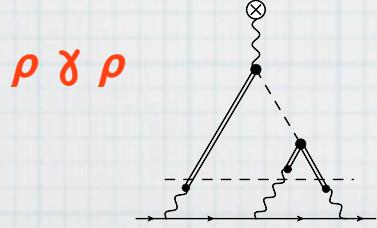


ρ ρ δ

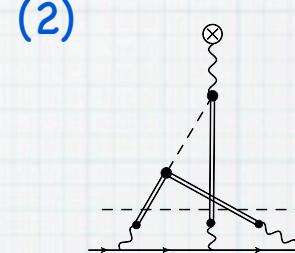
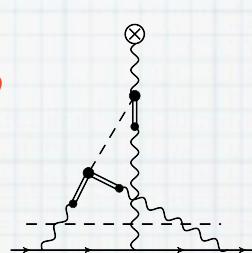
ρ δ ρ



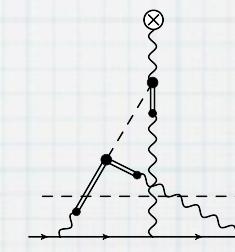
δ ρ ρ



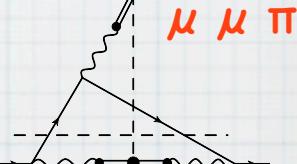
ρ ρ δ



δ δ δ



δ δ ρ



δ ρ ρ

ρ ρ δ

(g-2) $_{\mu}$: 2-particle cuts

rational fraction decomposition

$$\frac{1}{q_1^2} \frac{1}{q_1^2 - \Lambda^2} = \frac{1}{\Lambda^2} \left(\frac{1}{q_1^2 - \Lambda^2} - \frac{1}{q_1^2} \right)$$

$$F_2^{(i)}(t) = F_i^{\Lambda\Lambda\Lambda}(t) - F_i^{\Lambda\Lambda 0}(t) - F_i^{\Lambda 0\Lambda}(t) + F_i^{\Lambda 00}(t) - F_i^{0\Lambda\Lambda}(t) + F_i^{0\Lambda 0}(t) + F_i^{00\Lambda}(t) - F_i^{000}(t)$$

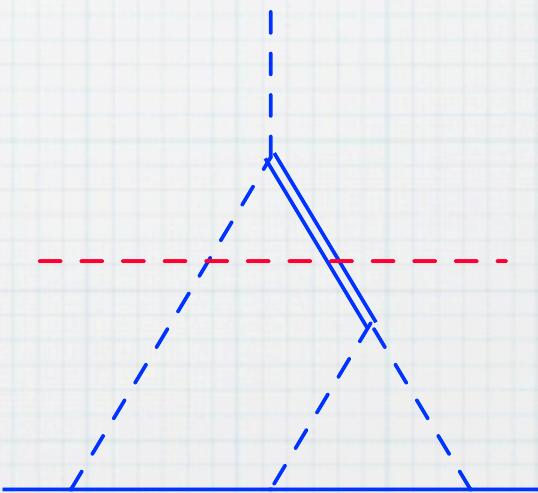
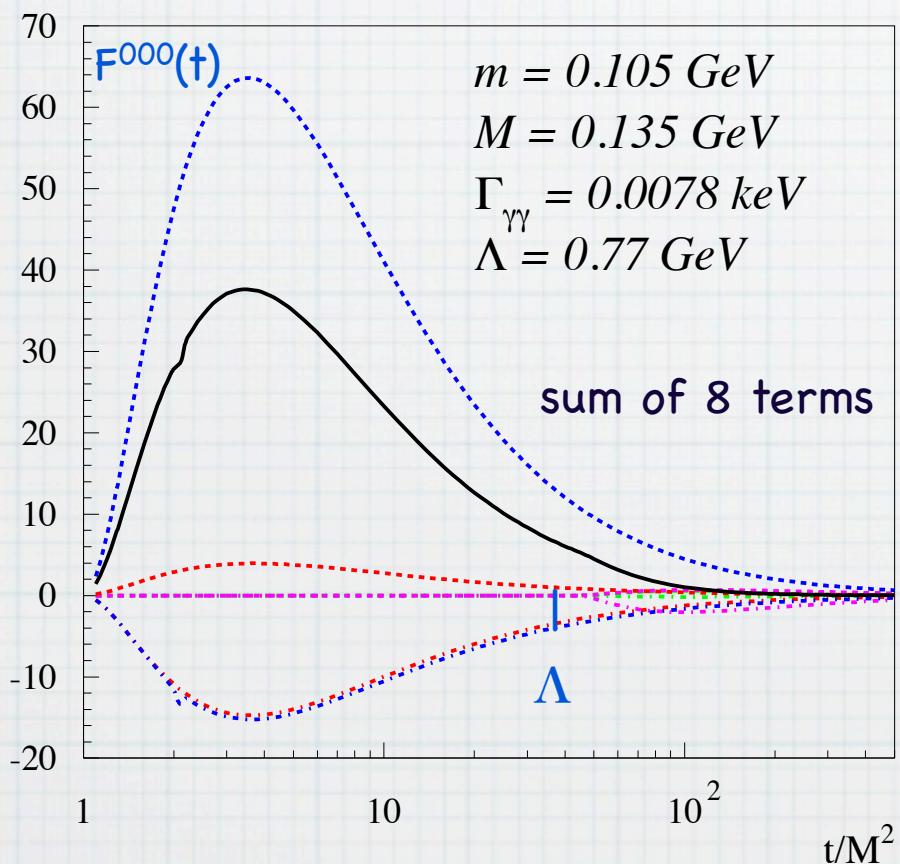
$$\begin{aligned} F_2^{(1)}(t) &= e^6 \Lambda^6 |F_{P\gamma^*\gamma^*}(0, 0, M^2)|^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \\ &\times \frac{1}{q_1^2} \frac{1}{q_1^2 - \Lambda^2} \frac{1}{q_2^2} \frac{1}{q_2^2 - \Lambda^2} \frac{1}{(k - q_1 - q_2)^2} \frac{1}{(k - q_1 - q_2)^2 - \Lambda^2} \\ &\times \frac{1}{(p + q_1)^2 - m^2} \frac{1}{(p + k - q_2)^2 - m^2} \frac{1}{(k - q_1)^2 - M^2} T_1(q_1, q_2, p, k) \end{aligned}$$

$$\begin{aligned} F_2^{(2)}(t) &= e^6 \Lambda^6 |F_{P\gamma^*\gamma^*}(0, 0, M^2)|^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \\ &\times \frac{1}{q_1^2} \frac{1}{q_1^2 - \Lambda^2} \frac{1}{q_2^2} \frac{1}{q_2^2 - \Lambda^2} \frac{1}{(k - q_1 - q_2)^2} \frac{1}{(k - q_1 - q_2)^2 - \Lambda^2} \\ &\times \frac{1}{(p + q_1)^2 - m^2} \frac{1}{(p + k - q_2)^2 - m^2} \frac{1}{(q_1 + q_2)^2 - M^2} T_2(q_1, q_2, p, k) \end{aligned}$$

(g-2) $_{\mu}$: 2-particle cuts

$$\text{Disc}_t^2 F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t) = -\frac{e^6 |F(0, 0, M^2)|^2}{8\pi} \beta_1 \int d\cos\theta_1 \frac{N^{(1)}(q_1^2, m^2, t_1, t, \cos\theta_1)}{q_1^2 + t_1 - t - t\beta_1\beta \cos\theta_1}$$

$\text{Im } F_2(t)/t$ (in $10^{-10} \text{ GeV}^{-2}$): $\pi\gamma$ cut, diagram a

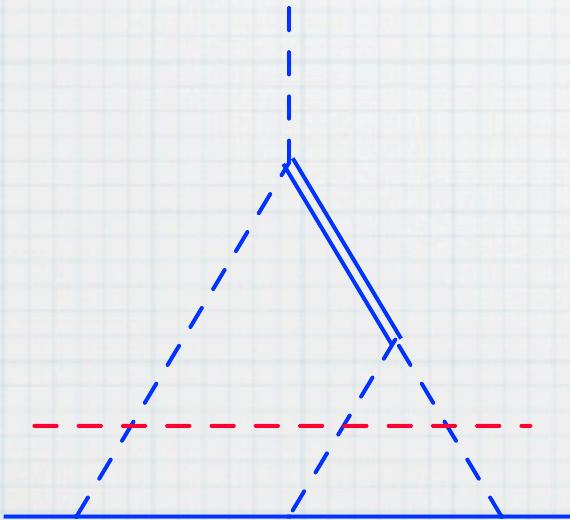


(g-2) $_{\mu}$: 3-particle cuts

$$\begin{aligned}
 \text{Disc}_t^3 F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t) &= \frac{i e^6 |F(0, 0, M^2)|^2}{(2\pi)^4 32 t} \int dt_1 \int dt_2 \frac{1}{t_1 - M^2} \\
 &\times \int_0^\pi d \cos \theta_1 \int_0^{2\pi} d \theta_2 \frac{2}{2m^2 - 2m_1^2 + q_1^2 + t_1 - t - t\beta_1 \beta \cos \theta_1} \\
 &\times \frac{2}{2m^2 - 2m_2^2 + q_2^2 - t + t_2 + t\beta_2 \beta (\sin \theta_1 \cos \theta_2 \sin \theta + \cos \theta_1 \cos \theta)} L(t_1, \dots) P(M^2, \dots)
 \end{aligned}$$

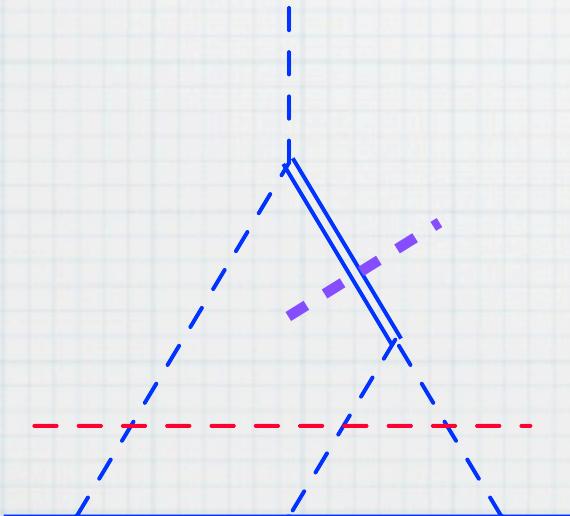
(g-2) $_{\mu}$: 3-particle cuts

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 \end{aligned}$$



(g-2) $_{\mu}$: 3-particle cuts

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 \text{Disc}_t^3 F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t) &= \frac{ie^6 |F(0, 0, M^2)|^2}{(2\pi)^4 32t} \int dt_1 \int dt_2 \frac{1}{t_1 - M^2} \\
 &\times \int_0^\pi d\cos\theta_1 \int_0^{2\pi} d\theta_2 \frac{2}{2m^2 - 2m_1^2 + q_1^2 + t_1 - t - t\beta_1\beta \cos\theta_1} \\
 &\times \frac{2}{2m^2 - 2m_2^2 + q_2^2 - t + t_2 + t\beta_2\beta (\sin\theta_1 \cos\theta_2 \sin\theta + \cos\theta_1 \cos\theta)} L(t_1, \dots) P(M^2, \dots)
 \end{aligned}$$

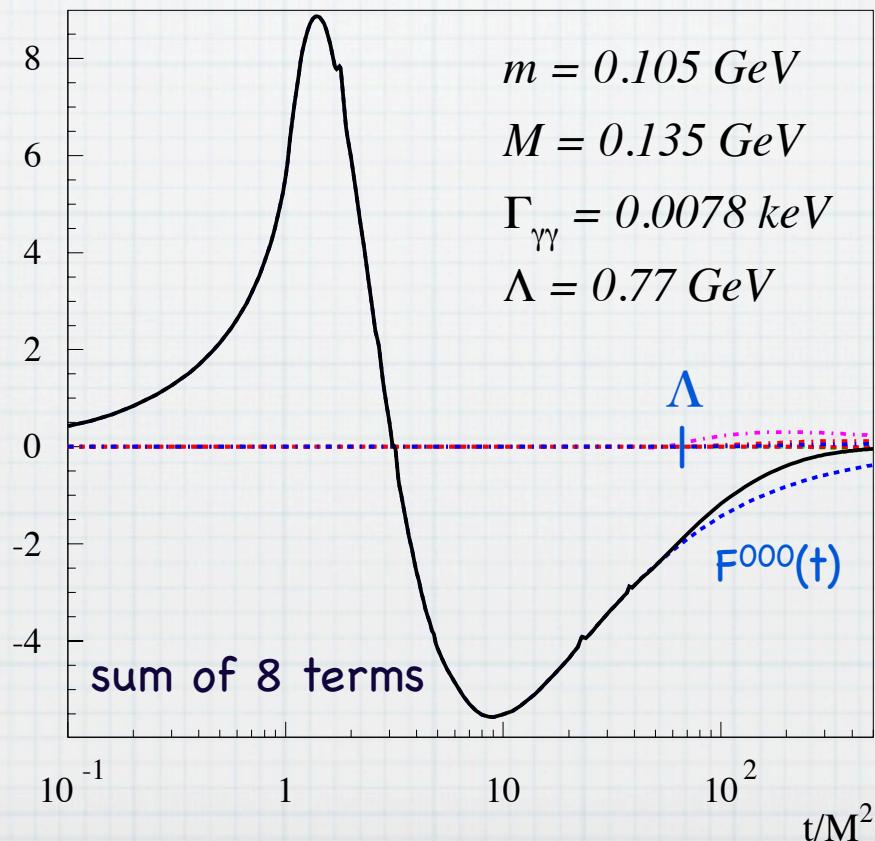


$$\Pi(q_1, q_2, k - q_1 - q_2, k) = \frac{P(q_1, q_2, k - q_1 - q_2, k)|_{(q_1 - k)^2 = M^2}}{(q_1 - k)^2 - M^2}$$

(g-2) $_{\mu}$: 3-particle cuts

$$\text{Disc}_t^3 F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t) = \frac{i e^6 |F(0, 0, M^2)|^2}{(2\pi)^4 32 t} \int dt_1 \int dt_2 \frac{1}{t_1 - M^2} \\ \times \int_0^\pi d \cos \theta_1 \int_0^{2\pi} d\theta_2 \frac{2}{2m^2 - 2m_1^2 + q_1^2 + t_1 - t - t\beta_1 \beta \cos \theta_1} \\ \times \frac{2}{2m^2 - 2m_2^2 + q_2^2 - t + t_2 + t\beta_2 \beta (\sin \theta_1 \cos \theta_2 \sin \theta + \cos \theta_1 \cos \theta)} L(t_1, \dots) P(M^2, \dots)$$

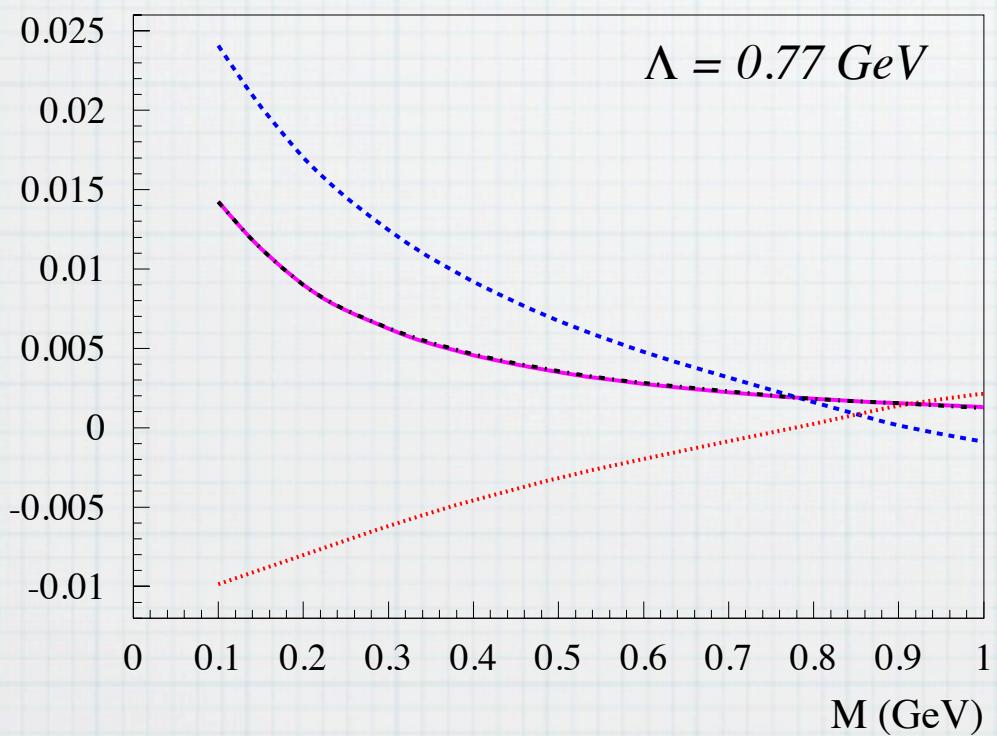
$\text{Im } F_2(t)/t$ (in $10^{-10} \text{ GeV}^{-2}$): 3γ cut, diagram a



Real parts

$$F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(0) = \frac{1}{2\pi i} \int_{(\Lambda_1+M)^2}^{\infty} \frac{dt}{t} \text{Disc}_t^{(2)} F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t) + \frac{1}{2\pi i} \int_{(\Lambda_1+\Lambda_2+\Lambda_3)^2}^{\infty} \frac{dt}{t} \text{Disc}_t^{(3)} F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t)$$

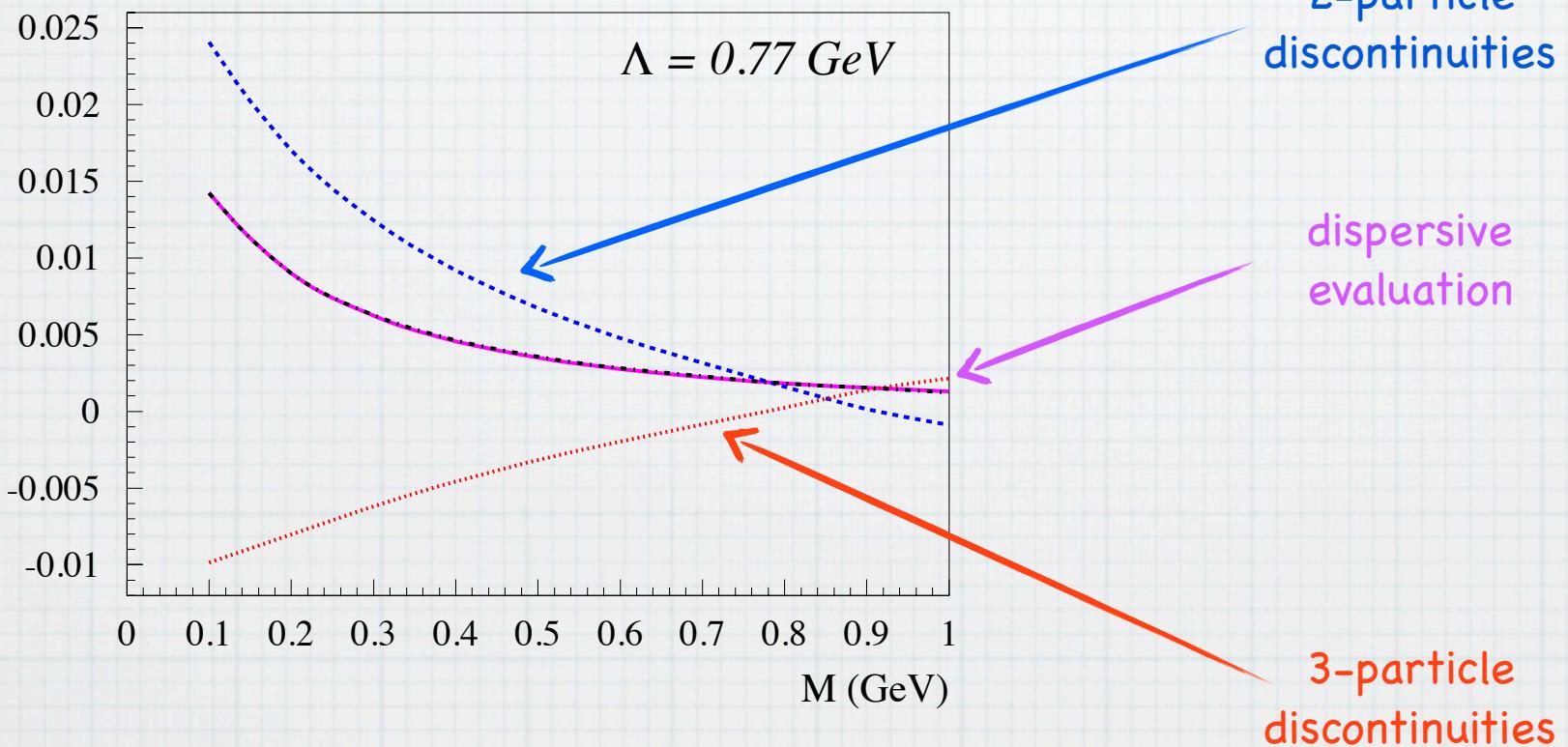
$a_\mu * M^3 / (\alpha \Gamma_{\gamma\gamma})$ (in GeV²): diagram a



Real parts

$$F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(0) = \frac{1}{2\pi i} \int_{(\Lambda_1+M)^2}^{\infty} \frac{dt}{t} \text{Disc}_t^{(2)} F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t) + \frac{1}{2\pi i} \int_{(\Lambda_1+\Lambda_2+\Lambda_3)^2}^{\infty} \frac{dt}{t} \text{Disc}_t^{(3)} F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t)$$

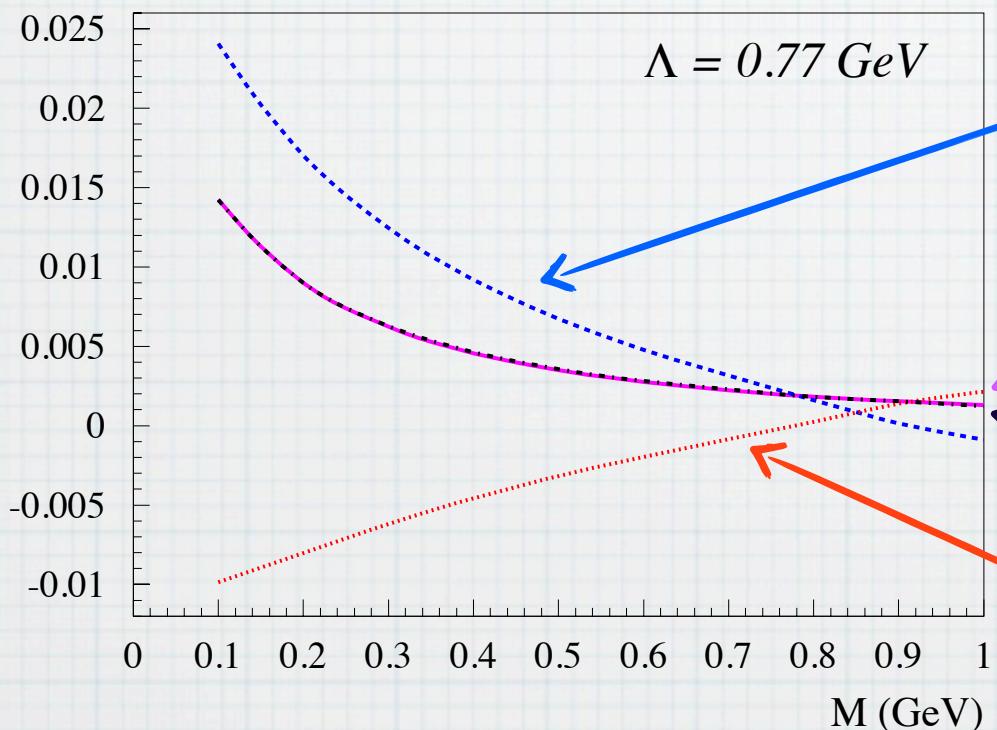
$a_\mu * M^3 / (\alpha \Gamma_{\gamma\gamma})$ (in GeV²): diagram a



Real parts

$$F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(0) = \frac{1}{2\pi i} \int_{(\Lambda_1+M)^2}^{\infty} \frac{dt}{t} \text{Disc}_t^{(2)} F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t) + \frac{1}{2\pi i} \int_{(\Lambda_1+\Lambda_2+\Lambda_3)^2}^{\infty} \frac{dt}{t} \text{Disc}_t^{(3)} F_1^{\Lambda_1 \Lambda_2 \Lambda_3}(t)$$

$a_\mu * M^3 / (\alpha \Gamma_{\gamma\gamma})$ (in GeV^2): diagram a



2-particle
discontinuities

dispersive
evaluation

Feynman integral
evaluation

3-particle
discontinuities

constant FF
Knecht, Nyffeler
(2001);
VP, Vanderhaeghen
(2014)

↓
pole
contribution!

Conclusions

total $(6.6 \sim 4.4) \pm 2.9 \times 10^{-11}$

axial mesons dominate the single meson channel
beyond pseudoscalar mesons

space-like data on single meson transition form-factors
is required to reduce the uncertainty

model independent implementation of data:

dispersive approaches

direct expression through
the physical phase-space integrals

allows different ways of direct implementation of data:
space-, time-like and on-shell data

freedom to choose a parametrization of hadronic matrix elements