One-loop Integral Coefficients from Generalized Unitarity

William Kilgore
Brookhaven National Laboratory

LoopFest VII
University at Buffalo
May 16, 2008

Based upon arXiv:0711.5015

2 Directions in Radiative Corrections

There are two main focuses for the development of new techniques in the field of radiative corrections:

1. More Loops

   The goal is high precision for important processes whose observables have direct physical interpretations.

2. More Legs

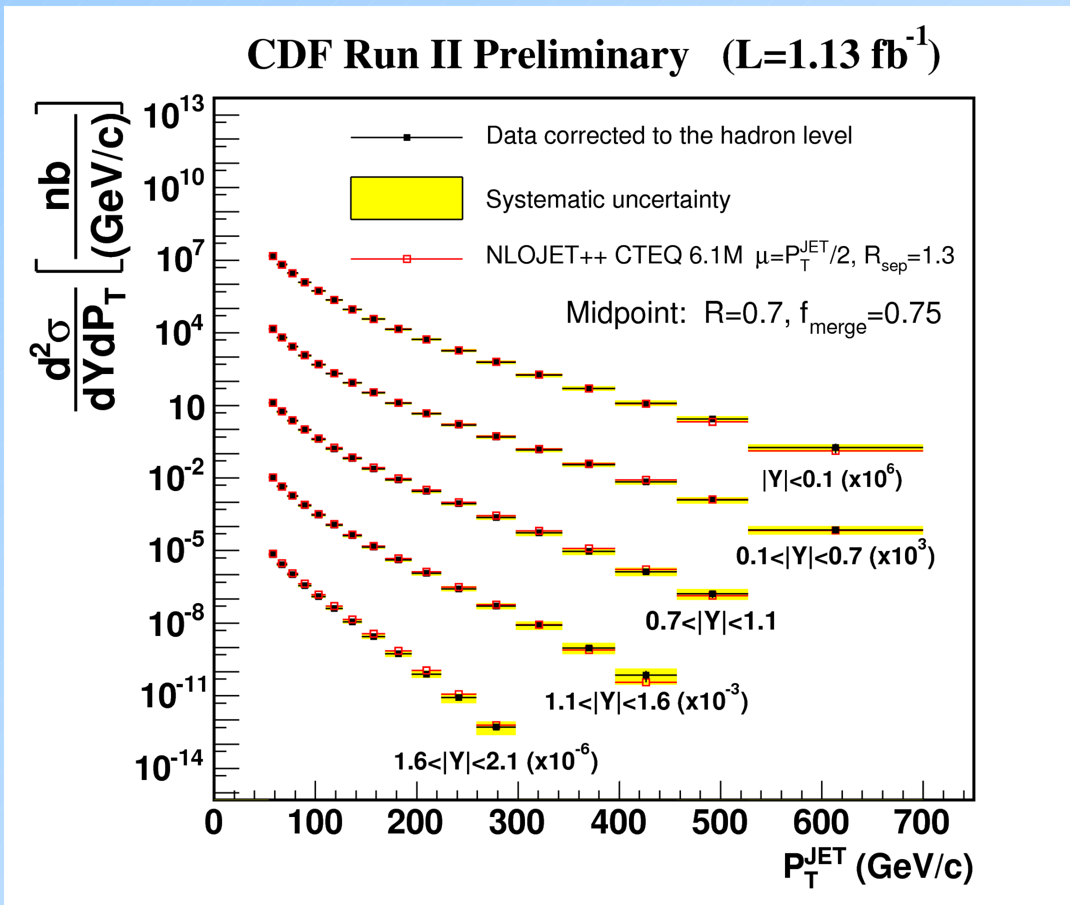
   The goal is to obtain reliable calculations for multi-particle signals and backgrounds at LHC.

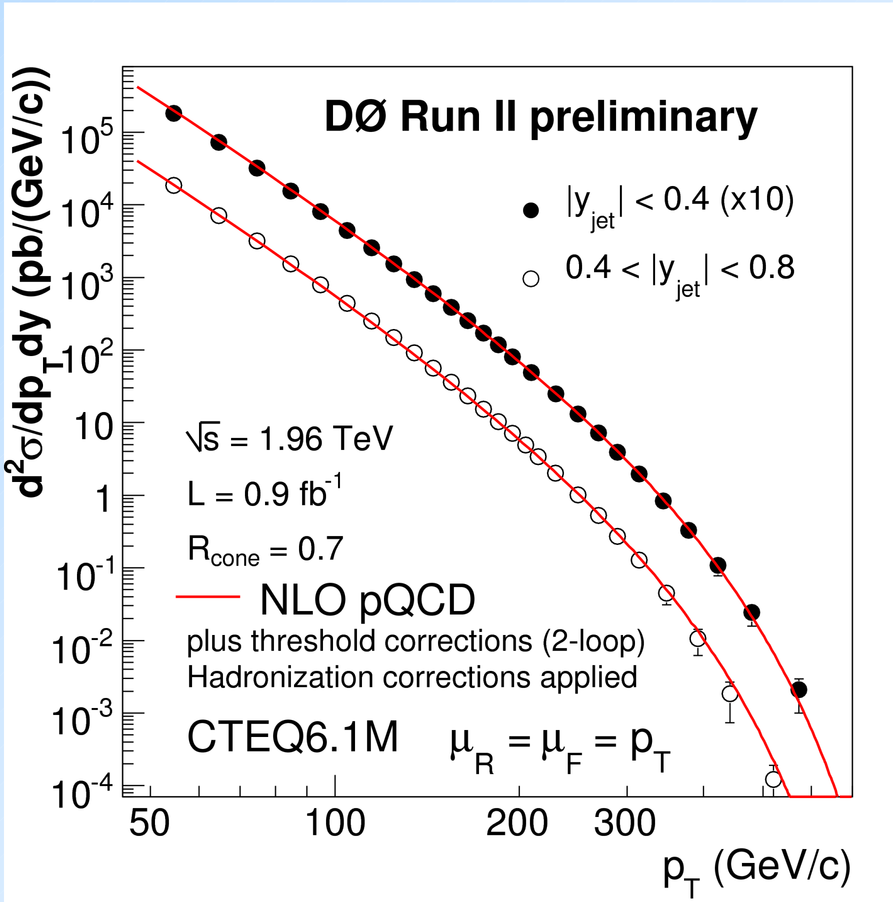
More Legs

A reliable calculation of any hadronic process requires one-loop corrections. Tree-level fixed-order calculations are (at best) of qualitative value.

Showering Monte-Carlo calculations are invaluable to modern experiments, but must be tuned to the experimental configuration. No one expects Pythia to accurately describe even Standard Model events at the LHC on day one. Only Next-to-Leading Order (or higher) calculations can expect to get the absolute normalization right.

Reliability of NLO





The NLO calculation of inclusive jet production is highly accurate and requires no tuning.

Everybody Wants NLO Calculations

An experimenter's wishlist

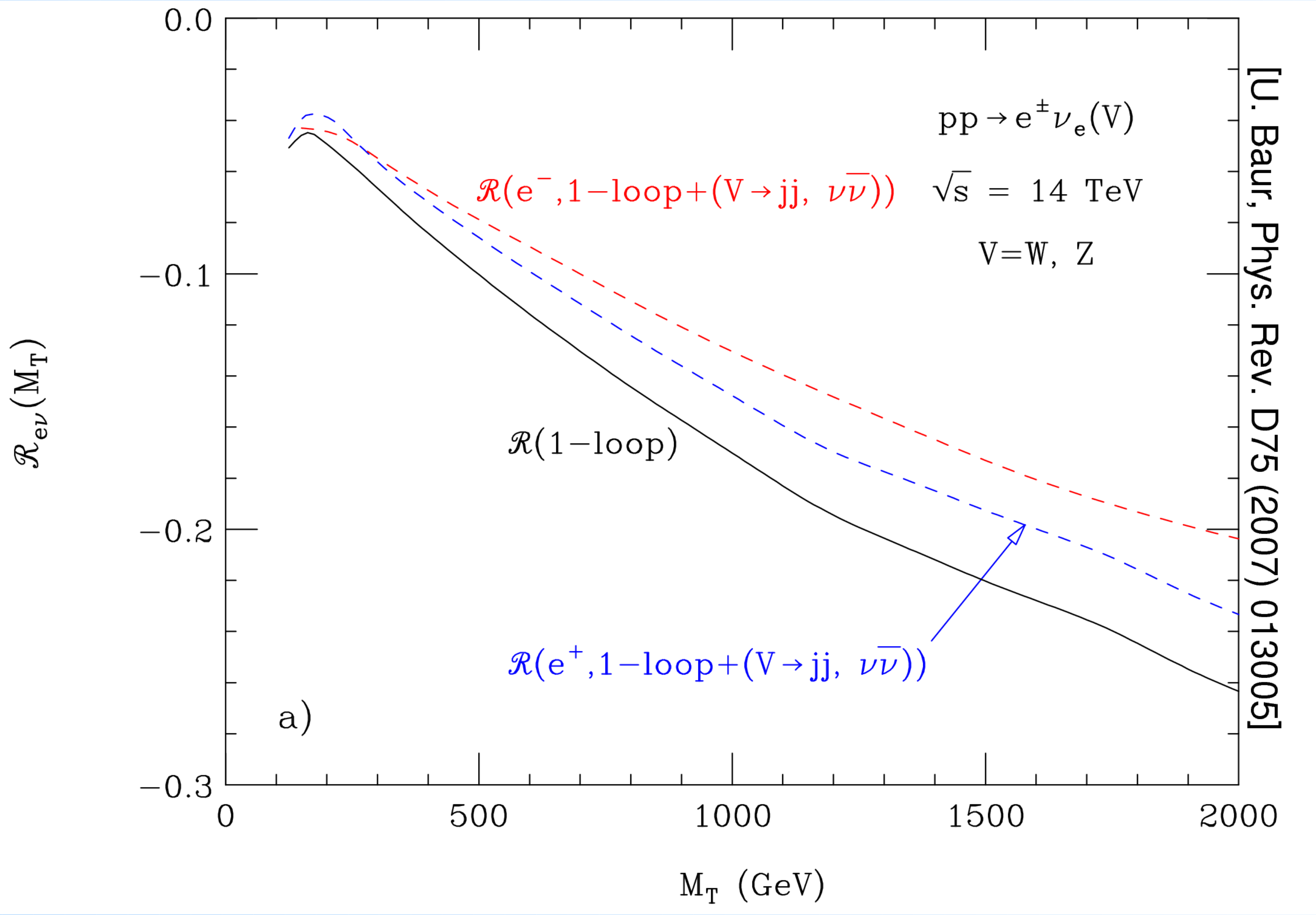
- Hadron collider cross-sections one would like to know at NLO

Run II Monte Carlo Workshop, April 2001

Single boson	Diboson	Triboson	Heavy flavour
$W + \leq 5j$	$WW + \leq 5j$	$WWW + \leq 3j$	$t\bar{t} + \leq 3j$
$W + b\bar{b} + \leq 3j$	$WW + b\bar{b} + \leq 3j$	$WWW + b\bar{b} + \leq 3j$	$t\bar{t} + \gamma + \leq 2j$
$W + c\bar{c} + \leq 3j$	$WW + c\bar{c} + \leq 3j$	$WWW + \gamma\gamma + \leq 3j$	$t\bar{t} + W + \leq 2j$
$Z + \leq 5j$	$ZZ + \leq 5j$	$Z\gamma\gamma + \leq 3j$	$t\bar{t} + Z + \leq 2j$
$Z + b\bar{b} + \leq 3j$	$ZZ + b\bar{b} + \leq 3j$	$WZZ + \leq 3j$	$t\bar{t} + H + \leq 2j$
$Z + c\bar{c} + \leq 3j$	$ZZ + c\bar{c} + \leq 3j$	$ZZZ + \leq 3j$	$t\bar{b} + \leq 2j$
$\gamma + \leq 5j$	$\gamma\gamma + \leq 5j$		$b\bar{b} + \leq 3j$
$\gamma + b\bar{b} + \leq 3j$	$\gamma\gamma + b\bar{b} + \leq 3j$		
$\gamma + c\bar{c} + \leq 3j$	$\gamma\gamma + c\bar{c} + \leq 3j$		
	$WZ + \leq 5j$		
	$WZ + b\bar{b} + \leq 3j$		
	$WZ + c\bar{c} + \leq 3j$		
	$W\gamma + \leq 3j$		
	$Z\gamma + \leq 3j$		

Electroweak Corrections

At all hadron colliders, NLO QCD is required. At the LHC, electroweak radiative corrections can also be quite large.



A Need for Automation

Using standard techniques, there is no hope of fulfilling the wish list anytime soon. Each calculation takes a year or so of hand crafting. There is a great need for automating the process.

The question is how to proceed. We've seen a number of approaches described at this conference.

The approach that I am taking is in line with the "Unitarity Bootstrap" idea [See talks by Bern, Forde]. It relies on the fact that any one-loop amplitude can be written as a sum of loop integral functions and other rational functions.

$$A_n^{1-loop} = \mathscr{R}_n + \frac{\mu^{2\varepsilon}}{(4\pi)^{2-\varepsilon}} \left(\sum_m a_m A_0^{(m)} + \sum_i b_i B_0^{(i)} + \sum_j c_j C_0^{(j)} + \sum_k d_k D_0^{(k)} \right)$$

The Unitarity Bootstrap

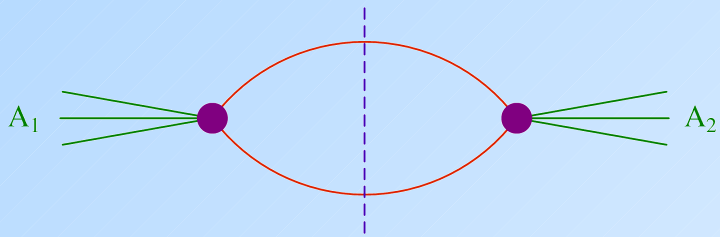
$$A_n^{1-loop} = \mathscr{R}_n + \frac{\mu^{2\varepsilon}}{(4\pi)^{2-\varepsilon}} \left(\sum_m a_m A_0^{(m)} + \sum_i b_i B_0^{(i)} + \sum_j c_j C_0^{(j)} + \sum_k d_k D_0^{(k)} \right)$$

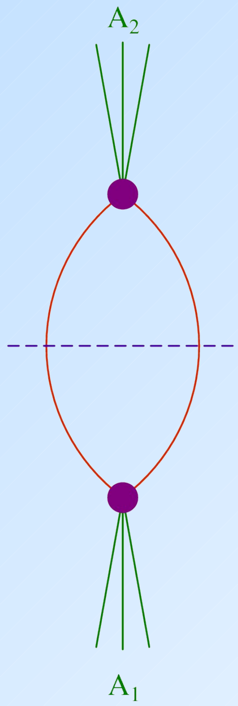
The loop-integral terms can be constructed using the techniques of generalized unitarity, while the rational terms can be derived using on-shell recursion relations. A key simplification is that the unitarity cuts can be made in D=4 dimensions. The finite terms due to ε dependence can all be put into $\mathscr{R}_n$.

Many things go into the bootstrap. For the rest of this talk, I will focus on the use of generalized unitarity to determine reduction coefficients. I ignore, for now, the problems of computing the rational terms and dealing with Gram-singular configurations.

Standard Unitarity Methods

Standard unitarity methods at NLO, worked out by Bern et al., were based upon performing all possible two-particle cuts and identifying the various contributions. Box and triangle contributions would show up in multiple channels, and one had to be careful to avoid double-counting.



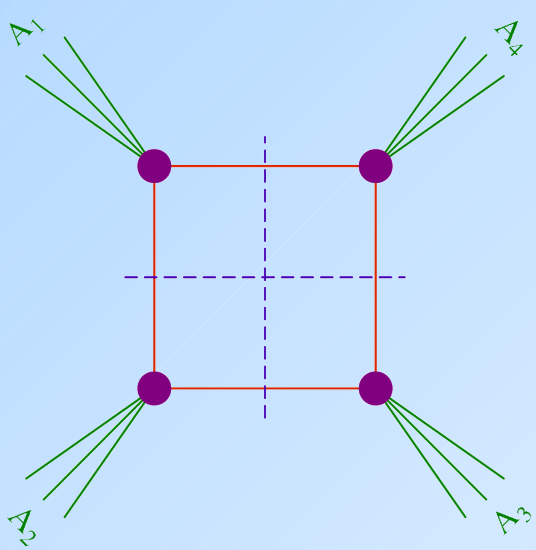


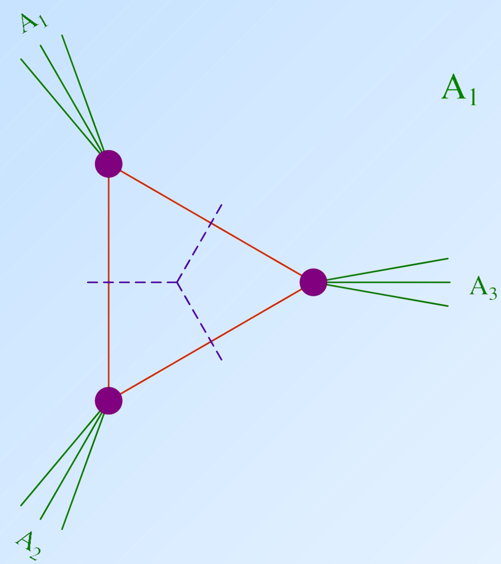
The rational terms were constructed from knowledge of soft and collinear limits.

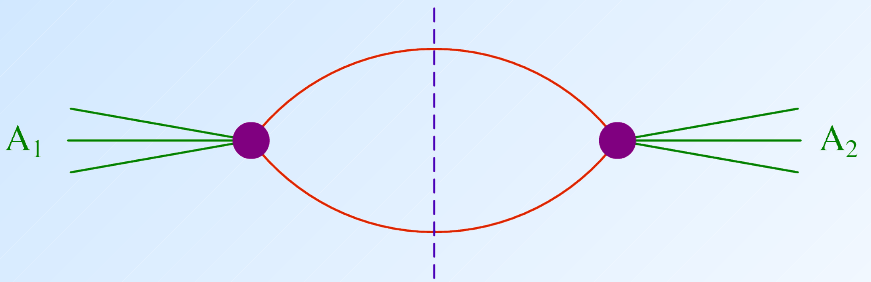
Generalized Unitarity

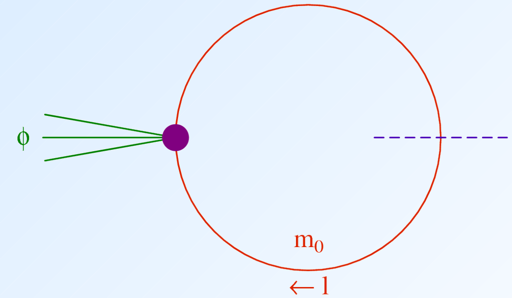
Generalized Unitarity uses multiple cuts to isolate the contribution of individual loop integral functions [Britto,Forde]. Forde sketched how D=4 unitarity works for QCD amplitudes.

The method he described adapts readily to loops with arbitrary internal masses, but there are minor complications at every step.

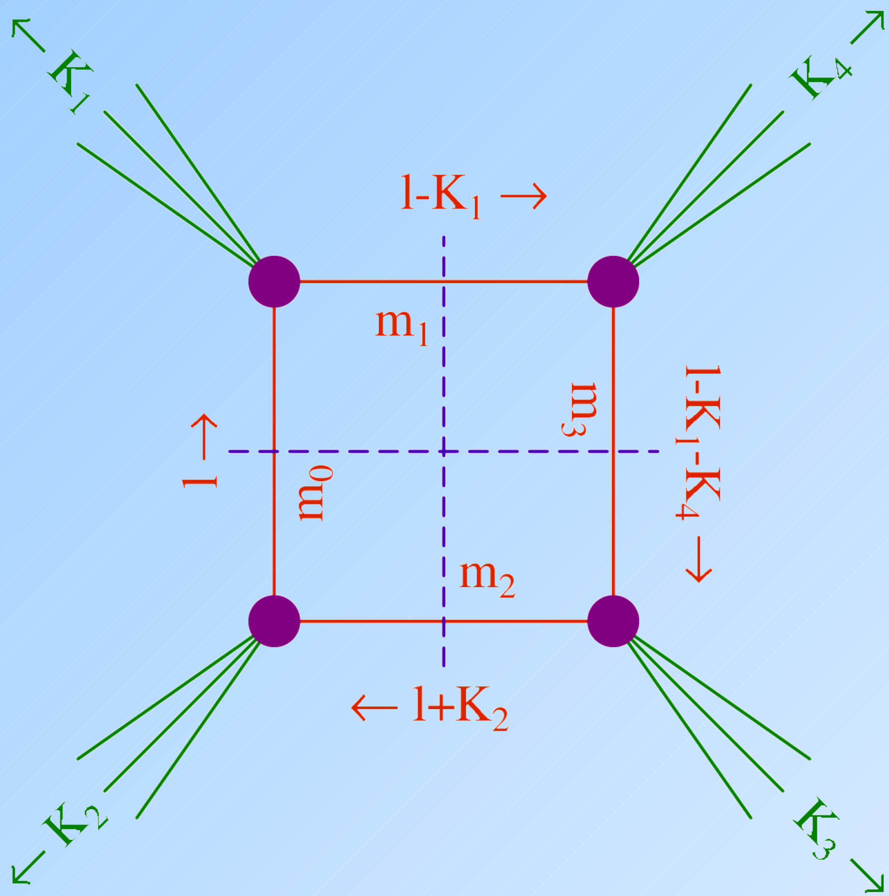








Generalized Unitarity - Boxes



The simplest and most direct application of generalized unitarity to the determination reduction coefficients is the determination of the box coefficient from quadruple cuts (Britto, Cachazo and Feng).

The four cuts place all four loop momenta, ℓ, $\ell - K_1$, $\ell + K_2$ and $\ell - K_1 - K_4$ on-shell.

The only way to do so is for ℓ to be complex valued!

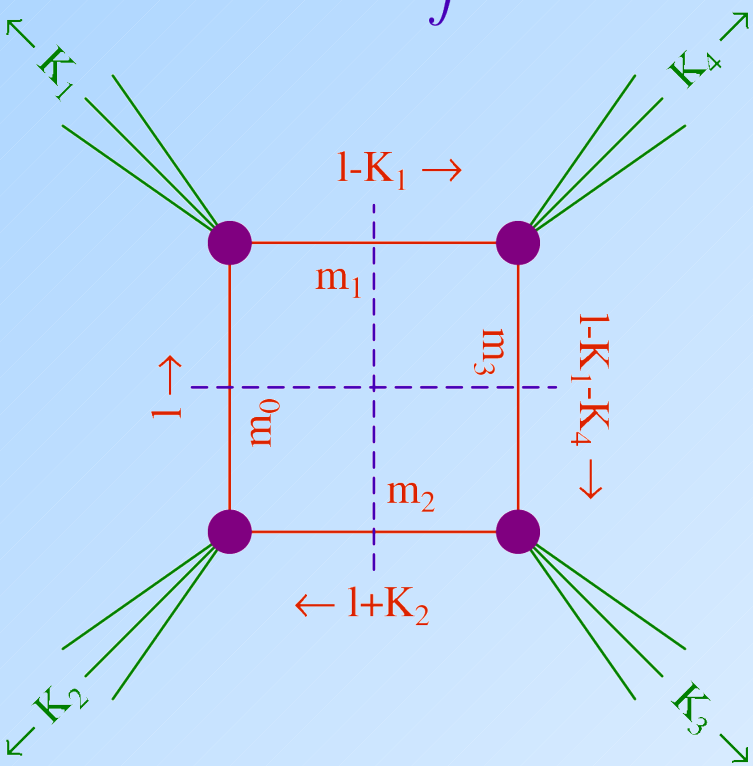
$$\ell^\mu = y\,\chi^\mu + w\,\psi^\mu + t\,\frac{\langle\chi^-|\gamma^\mu|\psi^-\rangle}{2} + \frac{\zeta}{t}\,\frac{\langle\psi^-|\gamma^\mu|\chi^-\rangle}{2}$$

Generalized Unitarity: Quadruple Cuts

Applying the cuts to the loop integral means replacing propagators with δ function constraints

$$\int d^4\ell \frac{N(\ell,K_1,K_2,\ldots)}{(\ell^2-m_0^2)((\ell-K_1)^2-m_1^2)((\ell+K_2)^2-m_2^2)((\ell-K_1-K_4)^2-m_3^2)} \longrightarrow$$

$$(-2i\pi)^4\int d^4\ell\ N(\ell,K_1,K_2,\ldots)\,\delta(\ell^2-m_0^2)\cdots\delta((\ell-K_1-K_4)^2-m_3^2)$$

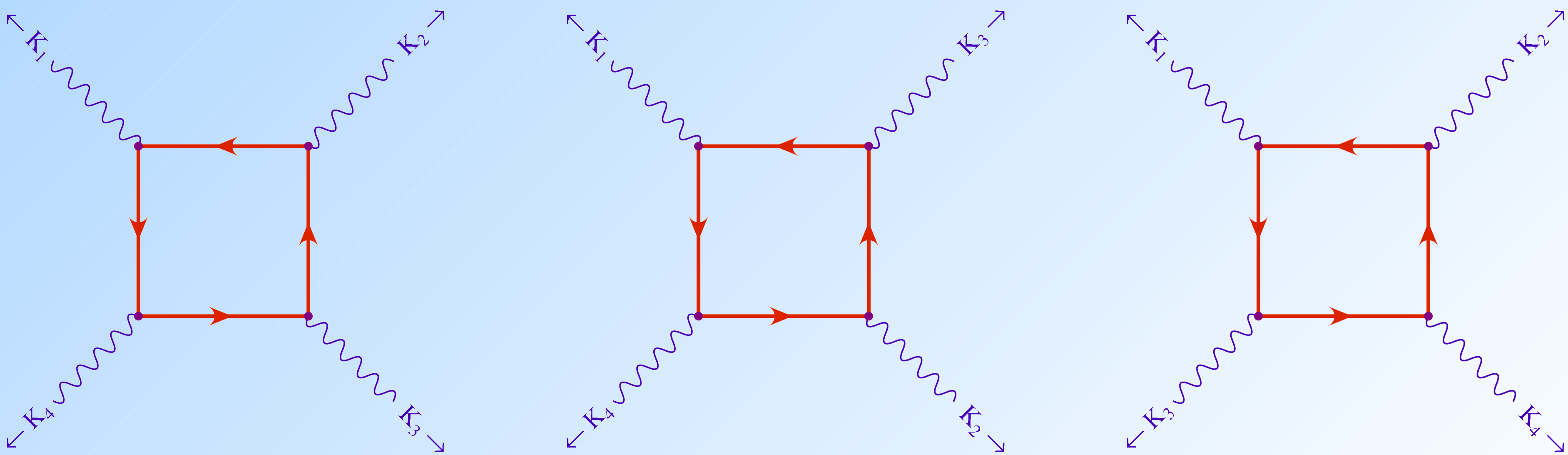


The four δ function constraints fix ℓ so that the numerator N evaluates to be the product of the four on-shell tree-level amplitudes that appear at the corners of the box! (BCF)

Generalized Unitarity for all Loop Integral Coefficients

Applying generalized unitarity to lower point functions is not quite as simple as is the case of the box integrals, but is still very clean and efficient. I will demonstrate by sketching the calculation of light-by-light scattering.

There are three independent one-loop Feynman diagrams:



Light-by-Light Box Terms

The box terms are determined as described above. The first task is to determine the loop momentum. It is convenient to form the basis momenta from two adjacent external momenta (Pittau et al.)

$$\ell^\mu = y K_1^{\flat\,\mu} + w K_2^{\flat\,\mu} + t \frac{\left\langle K_1^{\flat -} \left| \gamma^\mu \right| K_2^{\flat -} \right\rangle}{2} + \frac{\zeta}{t} \frac{\left\langle K_2^{\flat -} \left| \gamma^\mu \right| K_1^{\flat -} \right\rangle}{2}$$

$$K_1^\flat = \frac{K_1 - \frac{S_1}{\gamma_{12}} K_2}{1 - \frac{S_1 S_2}{\gamma_{12}^2}} \qquad K_2^\flat = \frac{K_2 - \frac{S_2}{\gamma_{12}} K_1}{1 - \frac{S_1 S_2}{\gamma_{12}^2}}$$

$$\gamma_{12} = 2 K_1^\flat \cdot K_2^\flat = K_1 \cdot K_2 \pm \sqrt{\Delta_2(K_1, K_2)}$$

Light-by-Light Box Terms (cont.)

y, w, t and ζ are fixed from the δ-function constraints.

$$y = \frac{S_2(\gamma_{12} - S_1) + (\gamma_{12} - S_2)\, m_0^2 - \gamma_{12} m_2^2 + S_2 m_1^2}{\gamma_{12}^2 - S_1 S_2}$$

$$w = \frac{S_1(\gamma_{12} - S_2) + (\gamma_{12} - S_1)\, m_0^2 - \gamma_{12} m_1^2 + S_2 m_2^2}{\gamma_{12}^2 - S_1 S_2}$$

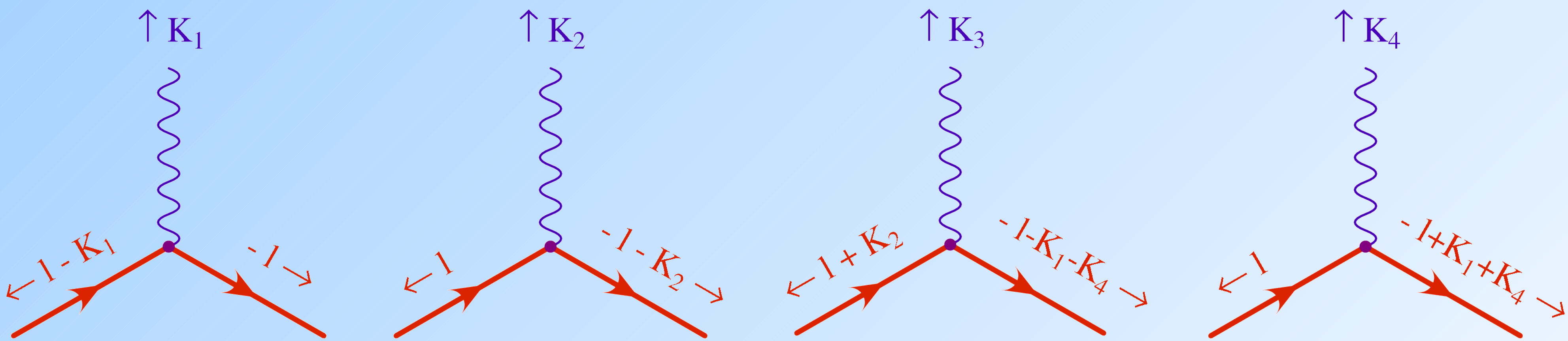
$$t = -\frac{\alpha \pm \sqrt{\alpha^2 - 2\beta \,\mathrm{Tr}\left(\not{K}_1^\flat \not{K}_4 \not{K}_2^\flat \not{K}_4\right)}}{\left\langle K_1^{\flat -} \left| \not{K}_4 \right| K_2^{\flat -} \right\rangle} \qquad \zeta = yw - \frac{m_0^2}{2K_1^\flat \cdot K_2^\flat}$$

$$\alpha = 2(y-1) K_1^\flat \cdot K_4 + 2\left(w - \frac{S_1}{\gamma_{12}}\right) K_2^\flat \cdot K_4 - S_4 + m_3^2 - m_1^2$$

$$\beta = yw - \frac{m_0^2}{\gamma_{12}}$$

Light-by-Light Box Terms (cont.)

With the loop momentum determined, we need only evaluate the tree-level amplitudes that appear at the corners of the box.

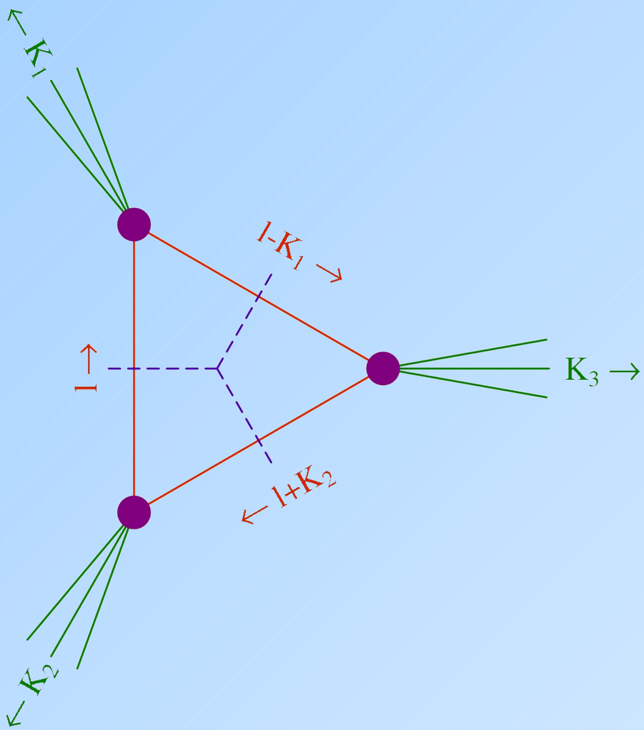


To evaluate, one averages over the solutions for the loop momentum and sums over internal polarization states using spin projectors and trace relations or spinors and helicity amplitudes.

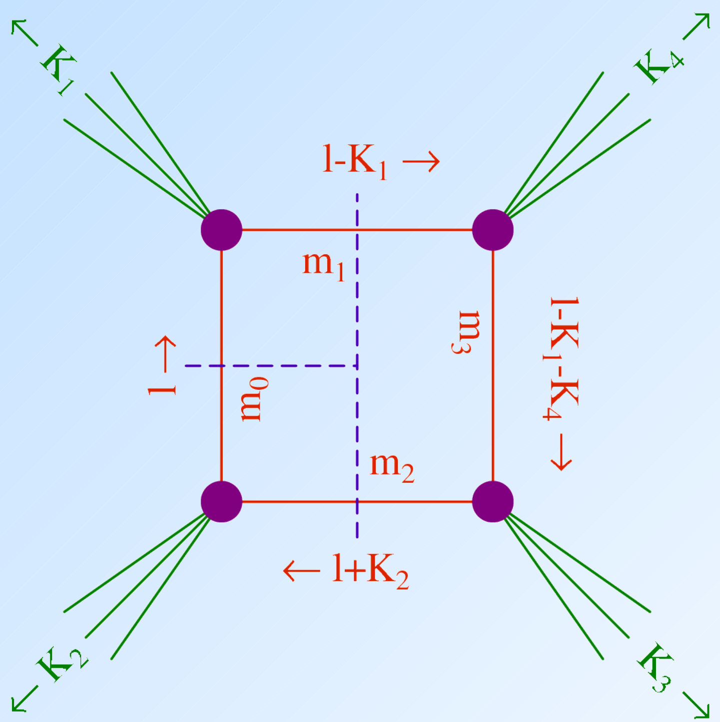
$$d = \frac{i}{2} \sum_{i=1}^{2} A(K_1, \ell_i) A(K_2, \ell_i) A(K_3, \ell_i) A(K_4, \ell_i)$$

Triple Cuts and Triangles

Quadruple cuts give the box, will triple cuts give the triangle?



$+\sum_k$



$$c_j C_0 + \sum_k d_k D_0^{(k)} = i \int \frac{d^4\ell}{(2\pi)^4} \frac{A_1(K_1;\ell)A_2(K_2;\ell)A_3(K_3;\ell)}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)((\ell + K_2)^2 - m_2^2)}$$

Triple Cuts (cont.)

Again parameterizing the loop momentum as

$$\ell^\mu = y K_1^{\flat\,\mu} + w K_2^{\flat\,\mu} + t\,\frac{\left\langle K_1^{\flat -}\,|\gamma^\mu|\,K_2^{\flat -}\right\rangle}{2} + \frac{\zeta}{t}\,\frac{\left\langle K_2^{\flat -}\,|\gamma^\mu|\,K_1^{\flat -}\right\rangle}{2}$$

y, w and ζ are the same as in the box, but t is a free parameter.

Imposing the δ-function constraints from the cuts, the loop integral is transformed into an integral over t,

$$c_j C_0^{\text{cut}} + \sum_k d_k D_0^{(k),\text{cut}} = i(-2i\pi)^3 \int \frac{dt}{(2\pi)^4} J_t\, A_1(K_1;t)\, A_2(K_2;t)\, A_3(K_3;t)$$

$$\to i(-2i\pi)^3 \int \frac{dt}{(2\pi)^4} J_t \left(\text{Inf}_t\,[A_1 A_2 A_3](t) + \sum_k \left[\frac{\text{Res}_{t\to t_k} A_1 A_2 A_3}{t - t_k} \right] \right)$$

Residues at finite t give boxes; terms at infinity give the triangle.

The Triangle Coefficient

The term at infinite t expands as

$$[\mathrm{Inf}_t\, A_1 A_2 A_3] = a_0 + a_1 t + a_2 t^2 + \ldots$$

Parameterizing the loop momentum in terms of $K_1^\flat$ and $K_2^\flat$ kills the contributions of tensor triangles, which would appear as non-vanishing powers of t.

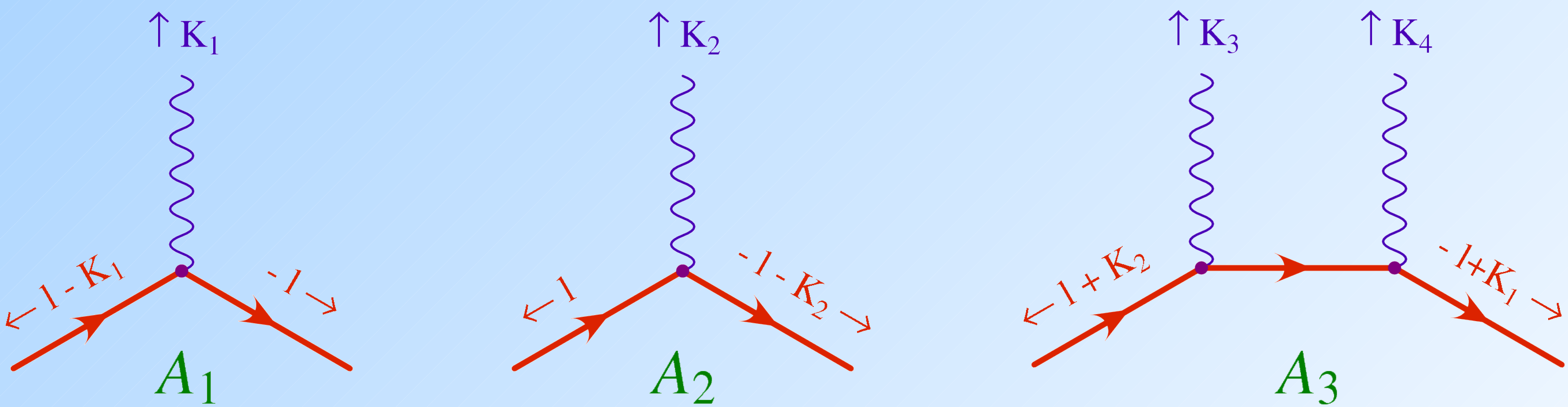
$$\int dt\, J_t\, t^{n\neq 0} = 0.$$

Thus, the scalar triangle coefficient takes the very simple form

$$c_j = -\mathrm{Inf}_t\, A_1 A_2 A_3(t)|_{t=0}$$

Light-by-Light Triangle Terms

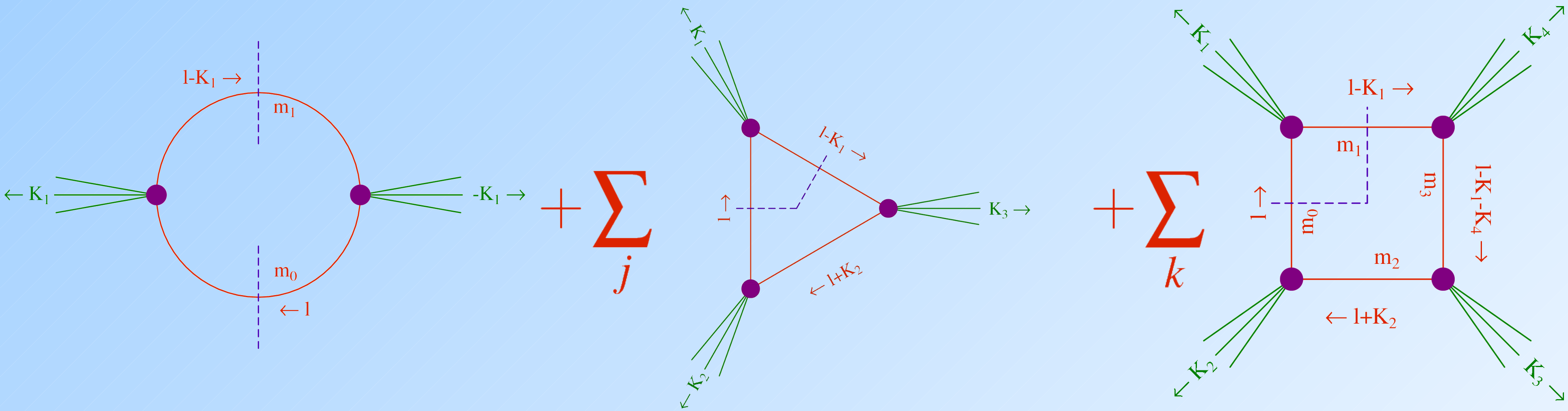
Using the parameterization defined before for the loop momentum, we evaluate the tree level amplitudes that appear at the corners of the triangle in terms of t.



The triangle coefficient is then given by the t^0 component of $\mathrm{Inf}[A_1 A_2 A_3](t)$.

Bubbles from Cuts

Extracting the bubbles from double cuts is more complicated.



$$b_i B_0 + \sum_j c_j C_0^{(j)} + \sum_k d_k D_0^{(k)} = i \int \frac{d^4\ell}{(2\pi)^4} \frac{A_1(K_1;\ell) A_2(K_2;\ell)}{(\ell^2 - m_0^2)((\ell - K_1)^2 - m_1^2)}$$

The loop momentum is parameterized in terms of K_1 and an arbitrary light-like vector χ.

$$\ell^\mu = y K_1^{\flat\,\mu} + w \chi^\mu + t \frac{\left\langle K_1^{\flat\,-} \middle| \gamma^\mu \middle| \chi^- \right\rangle}{2} + \frac{\zeta}{t} \frac{\left\langle \chi^- \middle| \gamma^\mu \middle| K_1^{\flat\,-} \right\rangle}{2}$$

Double Cuts

Imposing the δ-function constraints from the cuts fixes w and ζ, but leaves y and t undetermined.

$$w = \frac{S_1 + m_0^2 - m_1^2}{\gamma_{1\chi}} - \frac{S_1}{\gamma_{1\chi}} y \qquad \zeta = yw - \frac{m_0^2}{\gamma_{1\chi}}$$

The cut loop integral is then transformed into a contour integral over y and t.

$$b_i B_0^{\text{cut}} + \sum_j c_j C_0^{(j),\text{cut}} + \sum_k d_k D_0^{(k),\text{cut}} = i(-2i\pi)^2 \int \frac{dt}{(2\pi)^4} J_{t,y} A_1(K_1;t) A_2(-K_1;t)$$

$$\to i(-2i\pi)^2 \int \frac{dt}{(2\pi)^4} J_{t,y} \left(\text{Inf}_y \left[\text{Inf}_t [A_1 A_2]](t,y) + \sum_k \left[\frac{\text{Res}_{t\to t_k} A_1 A_2}{t - t_k} \right] (y) \right] \right.$$

$$\left. + \sum_j \frac{1}{y - y_j} \text{Res}_{y \to y_k} \left[\text{Inf}_t [A_1 A_2](t) + \sum_k \left[\frac{\text{Res}_{t\to t_k} A_1 A_2}{t - t_k} \right] (y) \right] \right)$$

Double Cuts (cont.)

The double residue terms correspond to boxes and can be eliminated. The Inf[Inf[A_1A_2]](t,y) terms are pure bubble contributions. The single residue terms correspond to triangles and must be kept since tensor components feed into the bubbles.

In the pure bubble terms, one needs an integral table for powers of y and t.

$$\int \frac{dt\,dy}{(2\pi)^4} J_{t,y} t^{n\neq 0} = 0 \qquad \int \frac{dt\,dy}{(2\pi)^4} J_{t,y} y^n = B(n)$$

$$B(n) = \frac{(-1)^n}{n+1} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \left(\frac{-m_0^2}{S_1}\right)^i \left(\frac{S_1+m_0^2-m_1^2}{S_1}\right)^{n-2i}$$

Double Cuts (cont.)

The single residue terms represent the triangle contributions to the bubble. The way to attack these is to split open the ends of the bubble to form all possible triangles and then solve for the value of y that puts the new propagator on shell. One then needs and integral table in the remaining variable t.

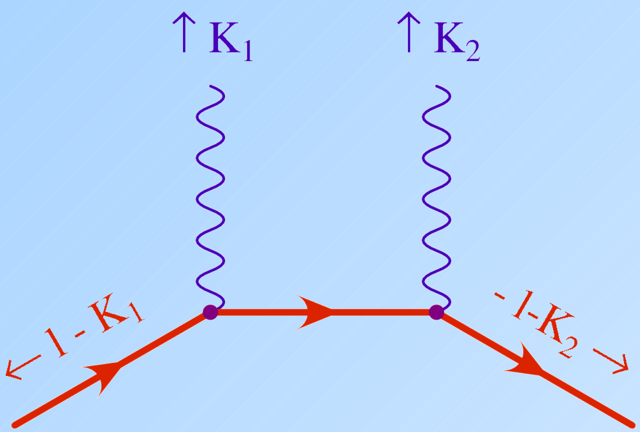
$$T(n) = \int \frac{dt\,dy}{(2\pi)^4} J'_t\, t^n = -\left(\frac{S_1}{2\gamma_{1\chi}}\right)^n \frac{\left\langle \chi^- \left| \not{K}_2 \right| K_1^{\flat -} \right\rangle^n (K_1 \cdot K_2)^{n-1}}{\Delta^n(K_1, K_2)} \mathscr{C}_n$$

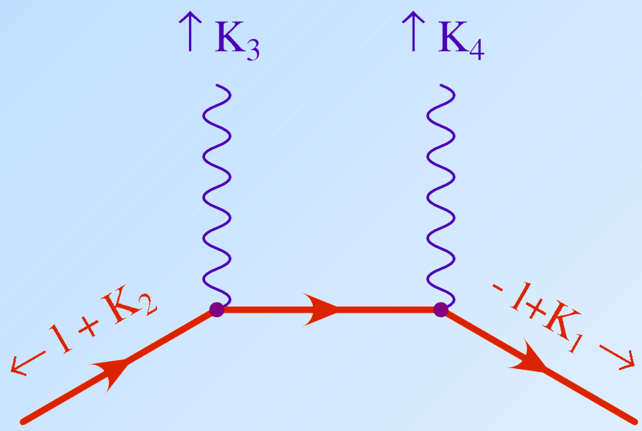
$$\mathscr{C}_0 = \mathscr{C}_1 = 0 \qquad \mathscr{C}_2 = \frac{3}{2}\left(\frac{S_1 + m_0^2 - m_1^2}{S_1} - \frac{S_2 + m_0^2 - m_2^2}{K_1 \cdot K_2}\right)$$

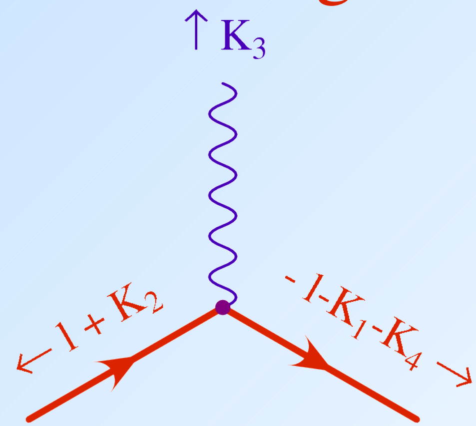
$$\mathscr{C}_3 = \frac{5}{2}\left(\frac{S_1 + m_0^2 - m_1^2}{S_1} - \frac{S_2 + m_0^2 - m_2^2}{K_1 \cdot K_2}\right)^2 - \frac{2}{3}\frac{\Delta(K_1, K_2)}{(K_1 \cdot K_2)^2}\left[\left(\frac{S_1 + m_0^2 - m_1^2}{S_1}\right)^2 - 4\frac{m_0^2}{S_1}\right]$$

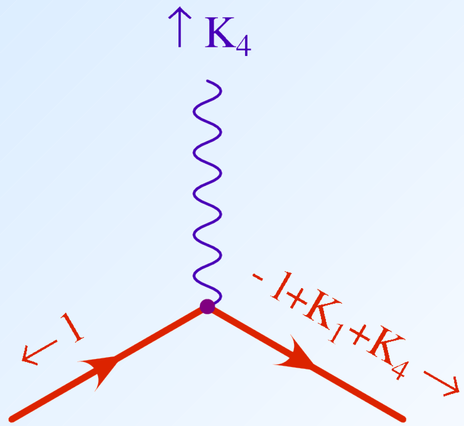
Light-by-Light Bubble Terms

The bubble terms are computed by evaluating the tree-level amplitudes that appear at the vertices of the bubble and triangles.







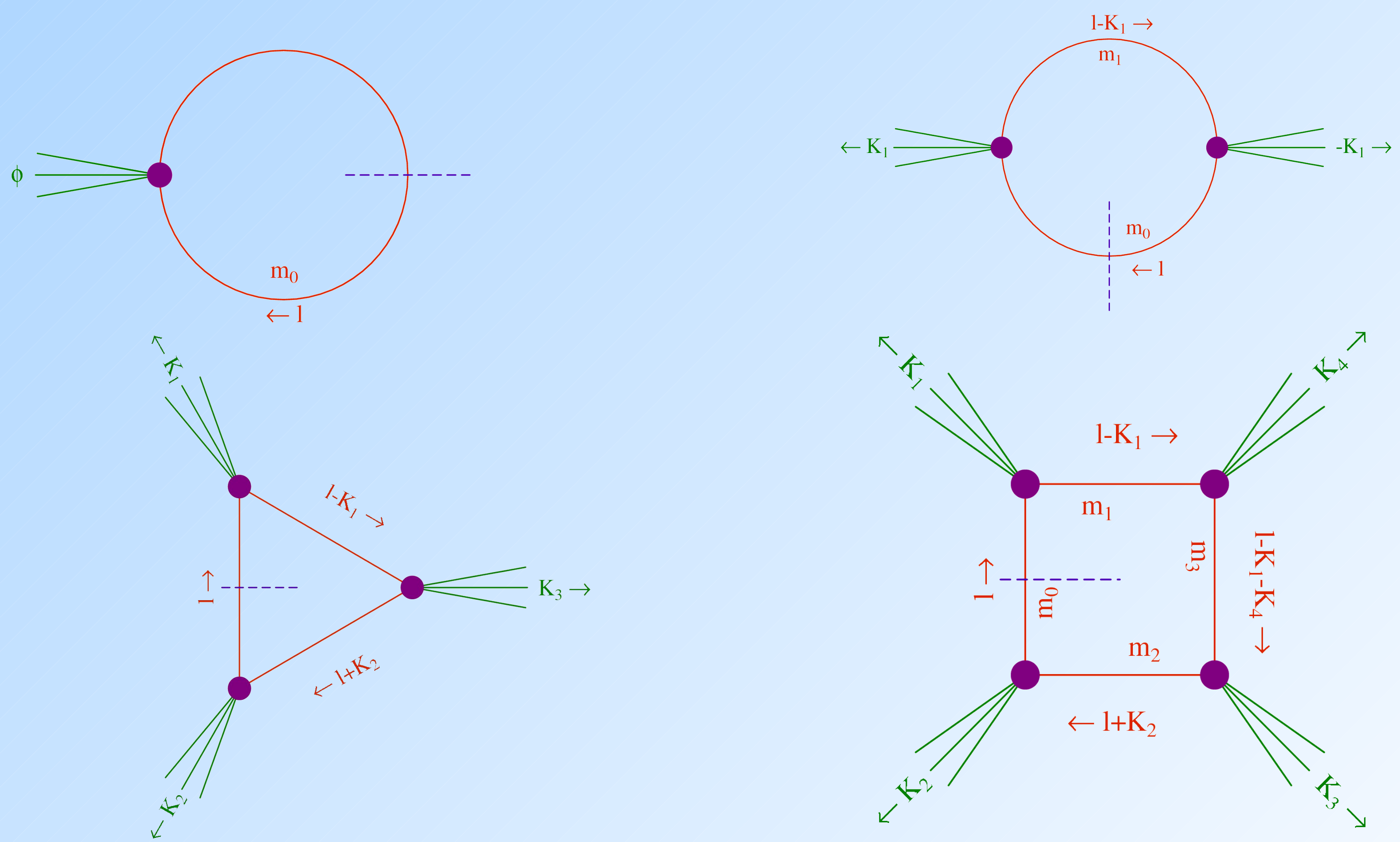


$$b_i = \; -i \; \mathrm{Inf}_y[\mathrm{Inf}_t[A_1 A_2]](t,y)\Big|_{t\to 0,\, y^n \to B(n)}$$

$$-\frac{1}{2}\sum_{\mathrm{tri}}\sum_{y=y_\pm} \; \mathrm{Inf}_t[\tilde{A}_1 \tilde{A}_2 \tilde{A}_3](t)\Big|_{t^n \to T(n)}$$

Tadpoles from Cuts

Extracting Tadpoles from cuts is similar to getting the bubbles, but with one more partial fractioning. Again, the box doesn't contribute, but tensor bubbles and triangles do.

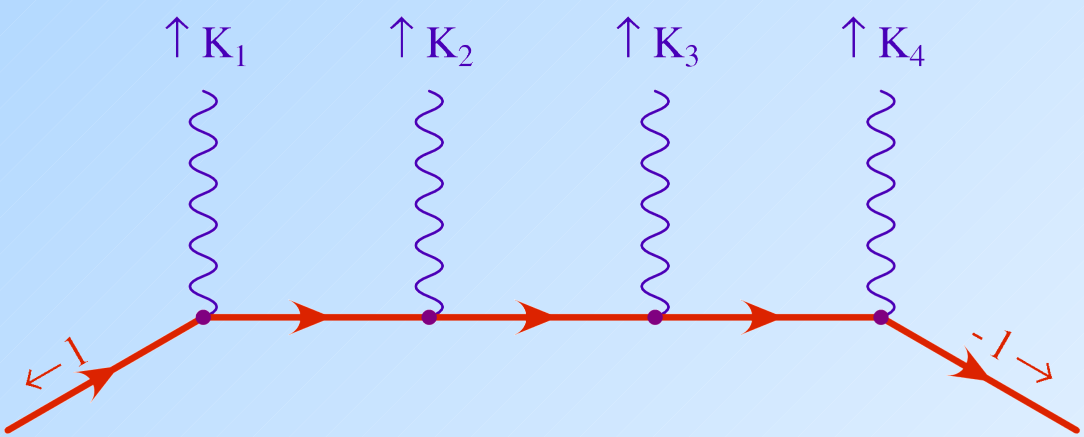


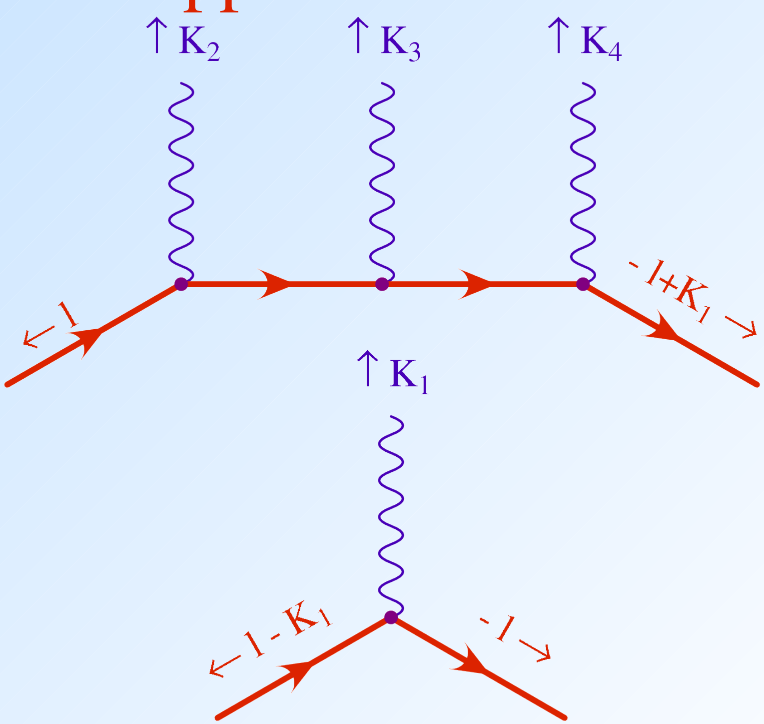
Light-by-Light Tadpole Terms

We parameterize the loop momentum in terms of two arbitrary massless momenta,

$$\ell^\mu = y\,\chi^\mu + w\,\psi^\mu + t\,\frac{\langle \chi^- | \gamma^\mu | \psi^- \rangle}{2} + \frac{y\,w - \frac{m^2}{2\chi\cdot\psi}}{t}\,\frac{\langle \psi^- | \gamma^\mu | \chi^- \rangle}{2}$$

and evaluate the tree-level amplitudes that appear at the vertices of the tadpoles, bubbles and triangles that appear.





Light-by-Light Tadpole Terms

Leaving out the details, I have constructed integral tables for pure tadpole terms, the single residue (bubble) and double residue (triangle) terms.

$$
\begin{aligned}
a_m = \; & \mathrm{Inf}_w[\mathrm{Inf}_y[\mathrm{Inf}_t[A]]](t,y,w)\Big|_{t\to 0,\,(wy)^n\to D(n)} \\
& -i\sum_{\mathrm{bub}}\Bigg[\;\mathrm{Inf}_y[\mathrm{Inf}_t[\tilde{A}_1\tilde{A}_2]]\Big|_{w=w_0}\Bigg|_{y^k t^m\to E_w(k,m)} \\
& \qquad + \mathrm{Inf}_w[\mathrm{Inf}_t[\tilde{A}_1\tilde{A}_2]]\Big|_{y=y_0}\Bigg|_{w^k t^m\to E_y(k,m)}\Bigg] \\
& -\frac{1}{2}\sum_{\mathrm{tri}}\sum_{(w,y)=(w_1,y_1)}^{(w_2,y_2)} \mathrm{Inf}_t[\hat{A}_1\hat{A}_2\hat{A}_3](t)\Big|_{t^n\to F(n)}
\end{aligned}
$$

Summary

- The Unitarity Bootstrap method of one-loop calculations uses generalized unitarity to obtain the "cut-constructible" pieces of a one-loop amplitude and on-shell recursion relations to obtain the "rational terms".
- I have applied D=4 dimensional unitarity to obtain the coefficients of loop integrals that appear in one-loop amplitudes with arbitrary internal masses.
- I think this method shows great potential for use in an automated system for NLO calculations for QCD and the electroweak theory.