Scalar one-loop integrals for QCD

Keith Ellis

Fermilab

Work with Giulia Zanderighi

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Full calculation of one-loop tensor integrals

The calculation of one loop amplitudes

- The classical paradigm for the calculation of one-loop diagrams was established in 1979.
- Complete calculation of one-loop scalar integrals
- Reduction of tensors one-loop integrals to scalars.

Nuclear Physics B153 (1979) 365–401

SCALAR ONE-LOOP INTEGRALS

G. 't HOOFT and M. VELTMAN
Institute for Theoretical Physics, University of Utrecht, Netherlands*

Received 16 January 1979

Nuclear Physics B160 (1979) 151–207

ONE-LOOP CORRECTIONS FOR e^+e^- ANNIHILATION INTO $\mu^+\mu^-$ IN THE WEINBERG MODEL

G. PASSARINO* and M. VELTMAN
Institute for Theoretical Physics, University of Utrecht, Utrecht, The Netherlands

Received 22 March 1979

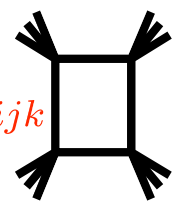
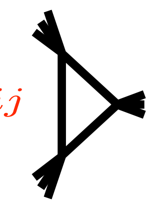
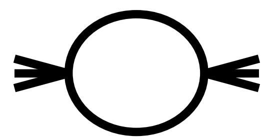
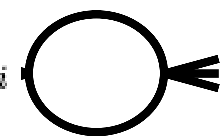
Neither are adequate for present-day purposes.

QCDloop: One-loop integrals

- Finite one-loop diagrams (in general with internal masses) solved in 1979. (‘t Hooft and Veltman).
- In QCD one has massless (or effectively massless lines) leading to one loop diagrams with divergences. In QCD these divergences are normally regulated using dimensional regularization ($D=4-2\,\varepsilon$).
- We have identified the 6 triangles and 16 boxes with soft and/or collinear divergences as basis sets.
- We have completed the calculation of the divergent diagrams (5 new boxes) and sorted the literature.
- Results are given at the webpage http://qcdloop.fnal.gov.

Basis set of scalar integrals

Any one-loop amplitude can be written as a linear sum of boxes, triangles, bubbles and tadpoles

$\mathcal{A}_N(\{p_i\}) = \sum d_{ijk}$  $+ \sum c_{ij}$  $+ \sum b_i$  $+ \sum_i a_i$ 

In addition, in the context of NLO calculations, scalar higher point functions, can always be expressed as sums of box integrals.

Passarino, Veltman - Melrose ('65)

Basis set for one-loop scalar integrals

- The tadpoles, bubbles, triangles and boxes form a complete basis set.
- What about higher point functions? Scalar pentagon can be expressed as a sum of boxes.

$$I_5^{\{D\}} = \sum_{i=1}^{5} c_i I_4^{\{D\}\,(i)} + \mathcal{O}(\epsilon)$$

- A general N point scalar integral in D dimensions can be recursively expressed as a linear combination of box integrals, if the external vectors can be taken to be in 4-dimensions

$$I_N^{\{D\}} = \sum_{i=1}^{N} d_i I_{N-1}^{\{D\}\,(i)}$$

Melrose, 1965, Vermaseren and van Neerven (1983), Bern, Dixon and Kosower (1992)

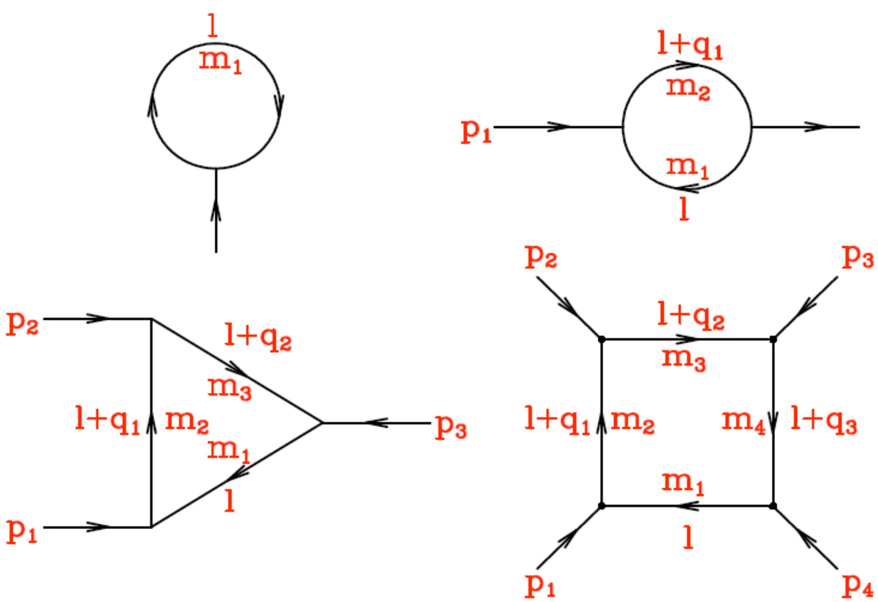
Definition of scalar integrals

$$I_1^D(m_1^2) = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma} \int d^D l \, \frac{1}{(l^2 - m_1^2 + i\varepsilon)},$$

$$I_2^D(p_1^2; m_1^2, m_2^2) = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma} \int d^D l \, \frac{1}{(l^2 - m_1^2 + i\varepsilon)((l+q_1)^2 - m_2^2 + i\varepsilon)},$$

$$I_3^D(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma}$$
$$\times \int d^D l \, \frac{1}{(l^2 - m_1^2 + i\varepsilon)((l+q_1)^2 - m_2^2 + i\varepsilon)((l+q_2)^2 - m_3^2 + i\varepsilon)},$$

$$I_4^D(p_1^2, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; m_1^2, m_2^2, m_3^2, m_4^2) = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}} r_\Gamma}$$
$$\times \int d^D l \, \frac{1}{(l^2 - m_1^2 + i\varepsilon)((l+q_1)^2 - m_2^2 + i\varepsilon)((l+q_2)^2 - m_3^2 + i\varepsilon)((l+q_3)^2 - m_4^2 + i\varepsilon)}$$



Integrals in D dimensions.

Calculation performed in space-like region, below all thresholds. Analytic continuation performed afterwards.

Distinguish between external momenta p and offsets in propagators q.

One-Loop Integral and Feynman parameters

$$I_N^{\{D\}}(\{s_{j_1\ldots j_n}\};\{m_i^2\}) = \mu^{4-D}(-1)^N\Gamma(N-\frac{D}{2})\int_0^1 \left(\prod_{i=1}^N da_i\right)\frac{\delta(1-\sum_i^N a_i)}{\left[a_i a_j Y_{ij} - i\varepsilon\right]^{N-\frac{D}{2}}}$$

$$Y_{ij} = \frac{1}{2}\left[m_i^2 + m_j^2 - (q_{i-1} - q_{j-1})^2\right]$$

Y is called the modified Cayley matrix. The Cayley matrix has a special structure for diagrams with a soft or collinear divergence. (Kinoshita)

$$Y_{\rm soft} = \begin{pmatrix} \ldots & 0 & \ldots & \ldots \\ 0 & 0 & 0 & \ldots \\ \ldots & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \quad Y_{\rm collinear} = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & 0 & \ldots \\ \ldots & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Basis set of divergent triangle integrals

By classsifying the integral in terms of the number of zero internal masses, and the number of distinct Cayley matrices we can create a basis set of divergent integrals

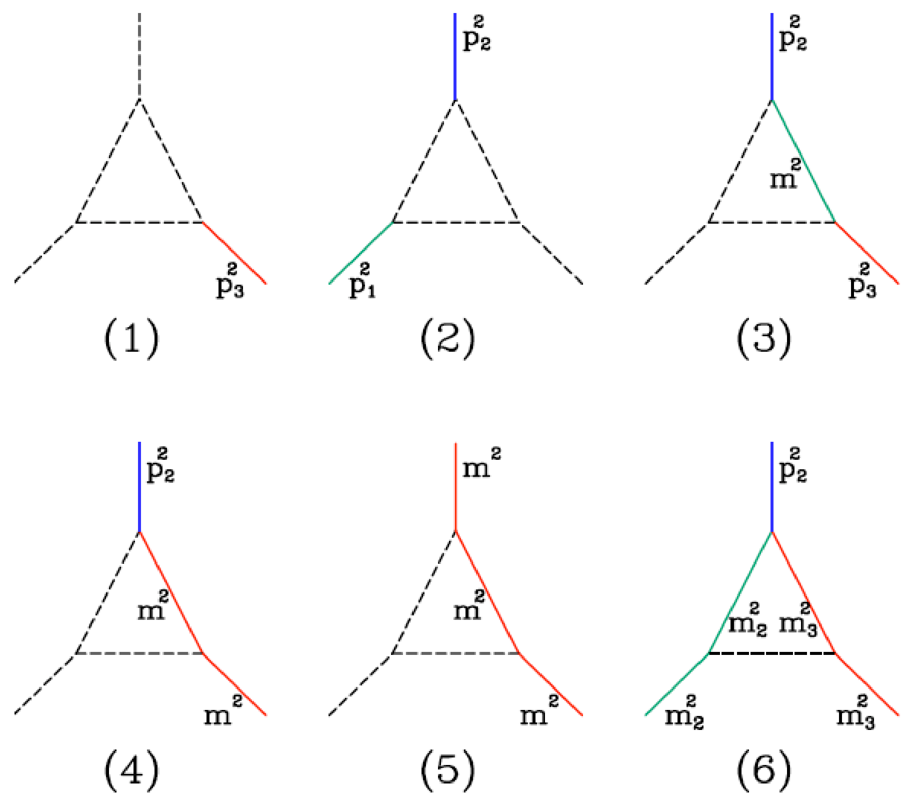
The basis set of divergent triangles contains 6 integrals

$$\begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}$$

(1) (2) (3)

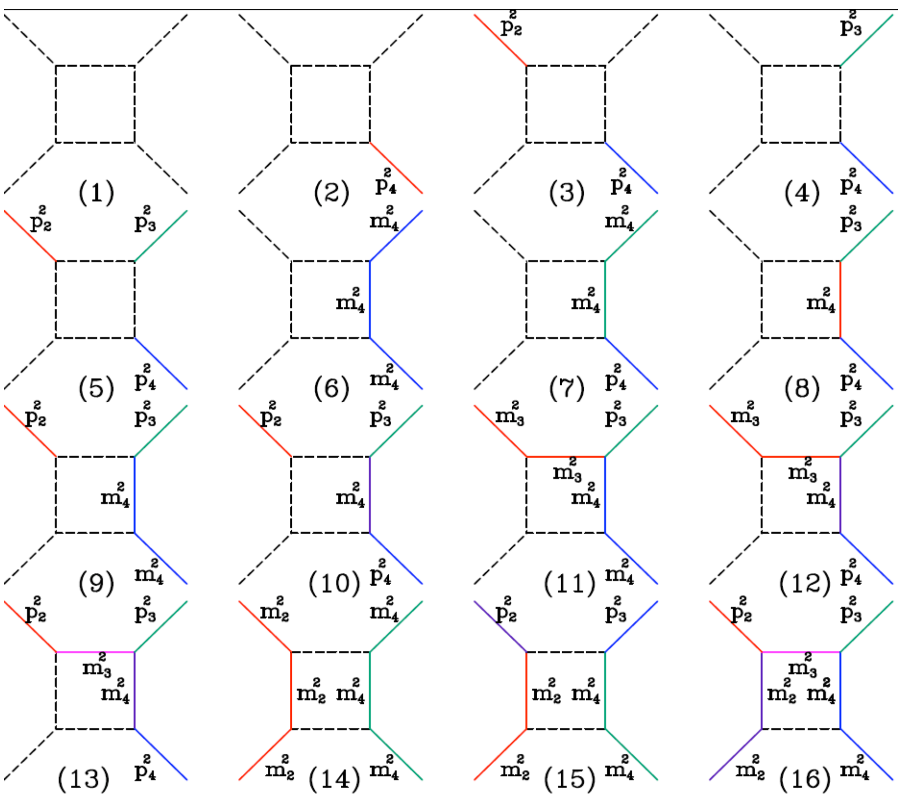
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$$

(4) (5) (6)



Modified Cayley Matrices for Box Integrals

There are 16 independent divergent box integrals.
11 exist in the literature.



$$Y_{ij} = \frac{1}{2}\left[m_i^2 + m_j^2 - (q_{i-1} - q_{j-1})^2\right]$$

$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \end{pmatrix}$ (1)
$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}$ (2)
$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}$ (3)
$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix}$ (4)

$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix}$ (5)
$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix}$ (6)
$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & \times \end{pmatrix}$ (7)
$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & \times \end{pmatrix}$ (8)

$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix}$ (9)
$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & \times \end{pmatrix}$ (10)
$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}$ (11)
$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$ (12)

$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$ (13)
$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix}$ (14)
$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix}$ (15)
$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}$ (16)

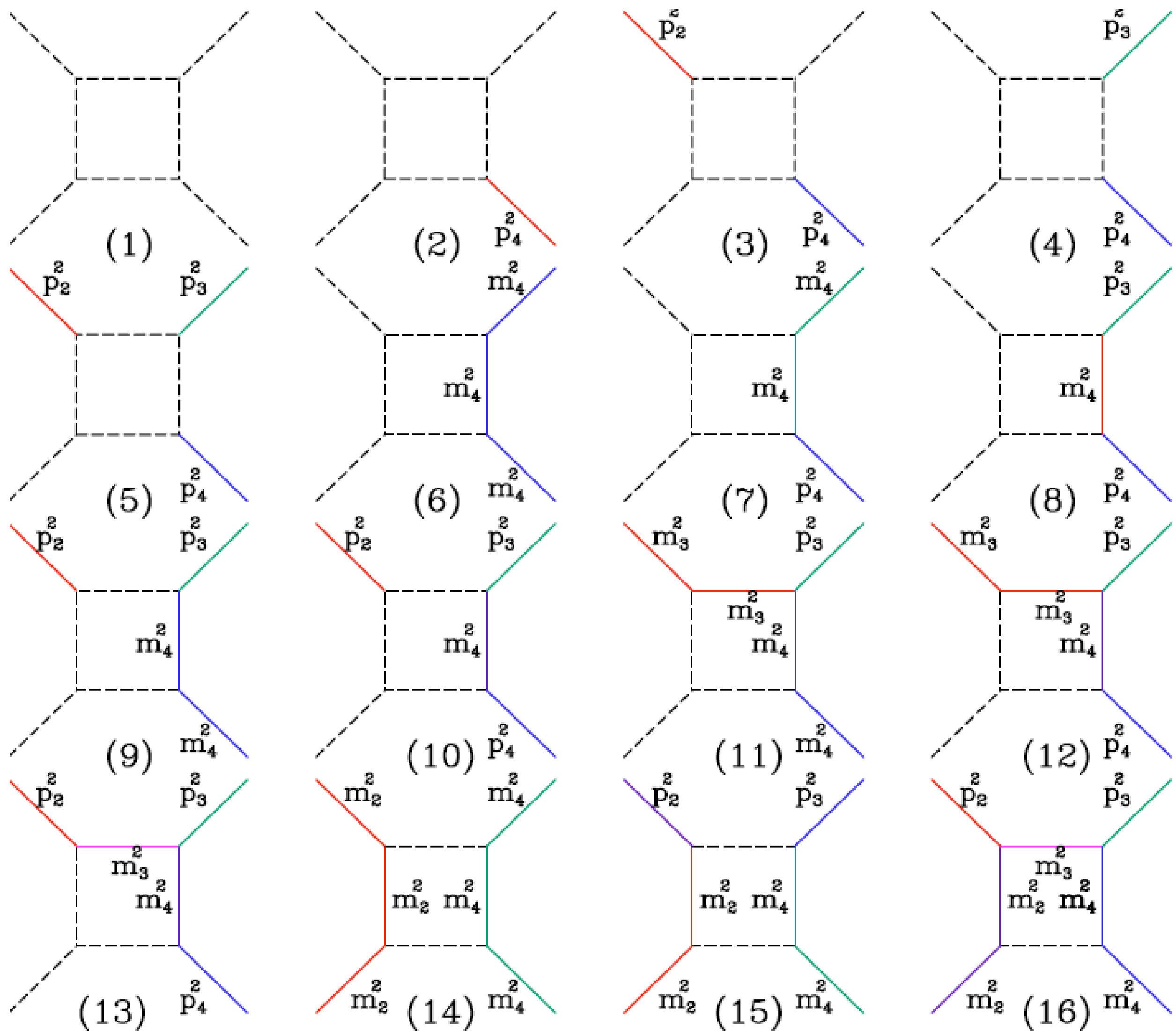
Integrals with no internal masses

There are five integrals with no internal masses,

1. $I_4^D(0,0,0,0;s_{12},s_{23};0,0,0,0)$
2. $I_4^D(0,0,0,p_4^2;s_{12},s_{23};0,0,0,0)$
3. $I_4^D(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0)$
4. $I_4^D(0,0,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0)$
5. $I_4^D(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0)$.

Ellis, Ross and Terrano (2), Bern. Dixon and Kosower (1?,3,4,5)
cf Duplancic and Nizic

The basis set of box integrals contains 16 integrals.



Integrals with one internal mass

$$Y = \begin{pmatrix} 0 & -\frac{1}{2}p_1^2 & -\frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ -\frac{1}{2}p_1^2 & 0 & -\frac{1}{2}p_2^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ -\frac{1}{2}s_{12} & -\frac{1}{2}p_2^2 & 0 & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix} . \tag{3.11}$$

With s_{12}, s_{23} fixed we can apply four conditions to potentially create a soft or collinear divergence, namely

$$p_1^2 = 0,\ p_2^2 = 0,\ p_3^2 = m_4^2,\ p_4^2 = m_4^2. \tag{3.12}$$

However performing the interchange $p_1^2 \leftrightarrow p_2^2, p_3^2 \leftrightarrow p_4^2$, (with m_4 fixed) corresponds to a relabelling of the diagram. In addition setting either $p_3^2 = m_4^2$ without setting $p_2^2 = 0$, or $p_4^2 = m_4^2$ without setting $p_1^2 = 0$ does not lead to a divergence. If we denote the application of the four conditions, eq. (3.12) on $p_1^2, p_2^2, p_3^2, p_4^2$ by (i, j, k, l) we have following 15 cases:

$$\begin{aligned}
&6.) && (1,2,3,4) \\
&7.) && (1,2,3) \equiv (1,2,4) \\
&8.) && (1,2) \\
&9.) && (1,4) \equiv (2,3) \equiv (1,3,4) \equiv (2,3,4) \\
&10.) && (1) \equiv (2) \equiv (1,3) \equiv (2,4) \\
& && (3) \equiv (4) \equiv (3,4) \equiv \text{finite}\,.
\end{aligned} \tag{3.13}$$

Credits

6), NDE, W. Beenakker, H. Kuijf, W. L. van Neerven and J. Smith,
7) W. Beenakker, S. Dittmaier, M. Kramer, B. Plumper, M. Spira and P. M. Zerwas
8) E. L. Berger, M. Klasen and T. M. P. Tait,
9,10) Ellis, Zanderighi

Integrals with 2 adjacent internal masses

Without loss of generality we can take the two non-zero adjacent internal masses to be m_3 and m_4. In this case the modified Cayley matrix is

$$Y = \begin{pmatrix} 0 & -\frac{1}{2}p_1^2 & \frac{1}{2}m_3^2 - \frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ -\frac{1}{2}p_1^2 & 0 & \frac{1}{2}m_3^2 - \frac{1}{2}p_2^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ \frac{1}{2}m_3^2 - \frac{1}{2}s_{12} & \frac{1}{2}m_3^2 - \frac{1}{2}p_2^2 & m_3^2 & \frac{1}{2}m_3^2 + \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_3^2 + \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix} . \qquad (3.14)$$

A necessary condition to have any divergence is $p_1^2 = 0$. This gives the integral 13. Applying either $p_2^2 = m_3^2$ or $p_4^2 = m_4^2$ gives a pair of integrals related by relabelling, integral 12. Applying both $p_2^2 = m_3^2$ and $p_4^2 = m_4^2$ gives integral 11,

11. $I_4^{\{D=4-2\epsilon\}}(0, m_3^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$

12. $I_4^{\{D=4-2\epsilon\}}(0, m_3^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$

13. $I_4^{\{D=4-2\epsilon\}}(0, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, m_3^2, m_4^2)$.

Ellis, Zanderighi 11,12,13

Integrals with 2 opposite masses

Without loss of generality we can take the two non-zero opposite internal masses to be m_2 and m_4. In this case the modified Cayley matrix is

$$Y = \begin{pmatrix} 0 & \frac{1}{2}m_2^2 - \frac{1}{2}p_1^2 & -\frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ \frac{1}{2}m_2^2 - \frac{1}{2}p_1^2 & m_2^2 & \frac{1}{2}m_2^2 - \frac{1}{2}p_2^2 & \frac{1}{2}m_2^2 + \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ -\frac{1}{2}s_{12} & \frac{1}{2}m_2^2 - \frac{1}{2}p_2^2 & 0 & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_2^2 + \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix} . \quad (3.15)$$

Here we can only have a soft divergence since there is no pair of adjacent zero internal masses. Setting $p_1^2 = m_2^2, p_4^2 = m_4^2$ or $p_2^2 = m_2^2, p_3^2 = m_4^2$ gives two integrals related by relabelling (15). Setting both conditions gives integral 14,

14. $I_4^{\{D=4-2\epsilon\}}(m_2^2, m_2^2, m_4^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)$

15. $I_4^{\{D=4-2\epsilon\}}(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)$.

Beenakker and Denner using $\ln \lambda^2 \to \frac{r_\Gamma}{\epsilon} + \ln \mu^2 + \mathcal{O}(\epsilon)$.

Integral with 3 internal masses

16. $I_4^D(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, m_3^2, m_4^2)$.

Beenakker and Denner using $$\ln\lambda^2 \to \frac{r_\Gamma}{\epsilon} + \ln\mu^2 + \mathcal{O}(\epsilon)\,.$$

QCDLoop web page

QCDLoop.fnal.gov

QCDLoop web page giving access to hyper-linked PDF web-pages which give the results for the basis integrals, together with references, special cases etc.

11 of the 16 divergent box integrals were known in the literature. The rest are new.

QCDloop: A repository for one-loop scalar integrals

This is a repository of one-loop scalar Feynman integrals, evaluated close to four dimensions. For integrals with all massive internal lines the integrals are all known, both analytically and numerically. This website therefore concentrates on integrals with some internal masses vanishing; in general, these integrals contain infra-red and collinear singularities which are here regulated dimensionally. The integrals are described in a PDF file for every known integral. The browser must be set to use hypertex-aware tool, such as Acrobat reader, and for best viewing, should open the pdf files in the browser. For general notation for the loop integrals click here

- Box integrals definitions and generalities
  - Basis set of divergent box integrals
  - Index of all box integrals currently in the repository
- Triangle integrals
  - Divergent triangle integrals
  - Finite triangle integrals
- Bubble integrals
- Tadpole integral

The results in this web-site are also available in the paper arXiv:0712.1851v1 by R.K. Ellis and G. Zanderighi
The corresponding fortran 77 code which calculates an arbitrary one-loop scalar integral, finite or divergent can be downloaded,
QCDLoop-1.4.tar.gz (version 1.4, date 2008-Apr-27). If you encounter any problems with the code, please notify the authors.

Other associated tools for one-loop diagrams:-
Looptools
the FF package by G.J. van Oldenborgh

R. Keith Ellis
Last modified: Sun Apr 27 14:43:28 CDT 2008

Divergent Box Integral 10: $I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2)$

Page contributed by R.K. Ellis

The result for this box (see figure) is

$$I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2) = \frac{1}{(s_{12}s_{23}-m^2 s_{12}-p_2^2 p_4^2+m^2 p_2^2)}$$

$$\times \left[\frac{1}{\epsilon}\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right)+\mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-s_{23})}{p_2^2 m^2}\right)-\mathrm{Li}_2\left(1+\frac{(m^2-p_3^2)(m^2-p_4^2)}{s_{12}m^2}\right)\right.$$

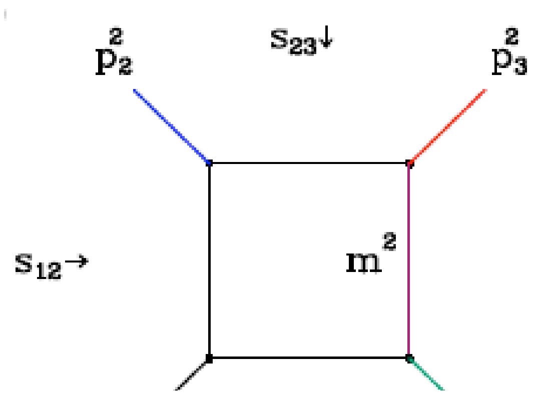
$$+\ 2\,\mathrm{Li}_2\left(1-\frac{m^2-s_{23}}{m^2-p_4^2}\right)-2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{12}}\right)+2\,\mathrm{Li}_2\left(1-\frac{p_2^2(m^2-p_4^2)}{s_{12}(m^2-s_{23})}\right)$$

$$\left.+\ 2\ln\left(\frac{\mu m}{m^2-s_{23}}\right)\ln\left(\frac{(m^2-p_4^2)p_2^2}{(m^2-s_{23})s_{12}}\right)\right]+\mathcal{O}(\epsilon)$$

See the file on notation.

References

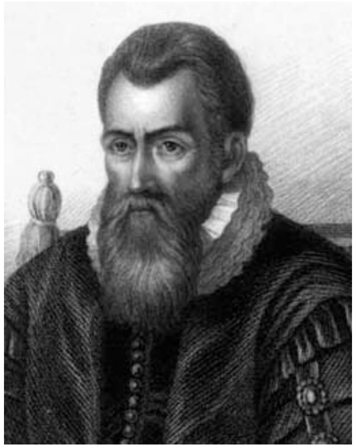
[1] R. K. Ellis and G. Zanderighi, "Scalar one-loop integrals for QCD," arXiv:0712.1851 [hep-ph]



$$Y=\frac{1}{2}\begin{pmatrix} 0 & 0 & -s_{12} & m^2-p_4^2 \\ 0 & 0 & -p_2^2 & m^2-s_{23} \\ -s_{12} & -p_2^2 & 0 & m^2-p_3^2 \\ m^2-p_4^2 & m^2-s_{23} & m^2-p_3^2 & 2m^2 \end{pmatrix}$$

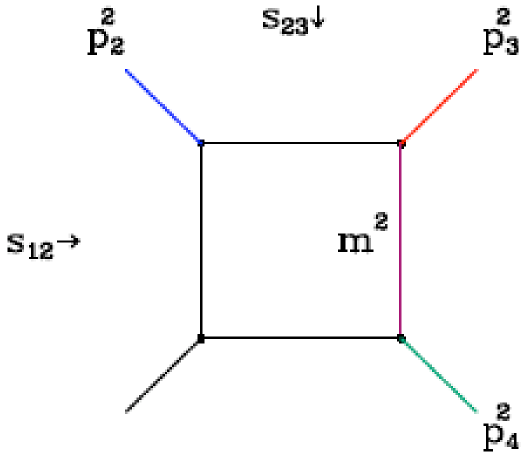
Credits

$\ln(x)$ (John Napier, 1619)



$Li_2(x)$ (WilliamSpence,1808)

Non-singular limits of this integral



$$Y = \frac{1}{2}\begin{pmatrix} 0 & 0 & -s_{12} & m^2 - p_4^2 \\ 0 & 0 & -p_2^2 & m^2 - s_{23} \\ -s_{12} & -p_2^2 & 0 & m^2 - p_3^2 \\ m^2 - p_4^2 & m^2 - s_{23} & m^2 - p_3^2 & 2m^2 \end{pmatrix}$$

Note that this basis integral encompasses also the case $p_3{}^2 = m^2$ because this limit does not lead to further singularities. The cases $p_4{}^2 = m^2$ or $p_2{}^2 = 0$ which do increase the degree of singularity are covered by different basis integrals.

QCDloop library

- We have created a numerical code which calculates an arbitrary scalar integral, (box, triangle, bubble, or tadpole).
- The program performs triage to determine which of the basis-set integrals can be used to calculate a given divergent integral.
- For the divergent integrals we use the basis set of integrals and program returns coefficients of $1/\varepsilon^2$,$1/\varepsilon$ and constant.
- For the finite integrals we use the implementation of the finite integrals due to van Oldenborgh (FF library).
- Further developments are possible, such as the extension to complex masses, and the inclusion pentagon, hexagon and other higher point functions.

Dilogarithm (Spence's function)

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dz}{z}\ln(1-z) = \frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots \text{ when } |x|\le 1$$

- Results of the integrals are given in terms of logarithms ln(x) and dilogarithms Li_2(1-x).
- For positive x both these functions are real.
- Simplification of these integrals using dilogarithm identities, eg

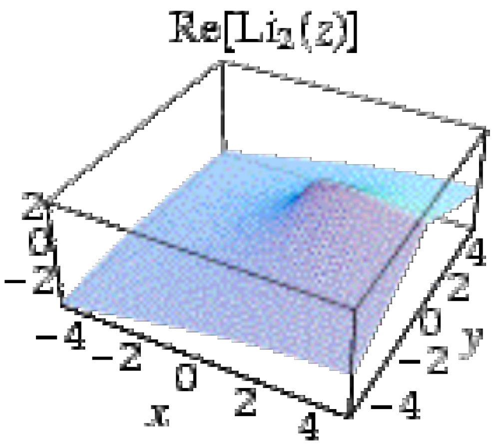
$$\mathrm{Li}_2(1-x)+\mathrm{Li}_2(1-\frac{1}{x}) = -\frac{1}{2}\ln^2 x, \quad \text{for } x>0$$

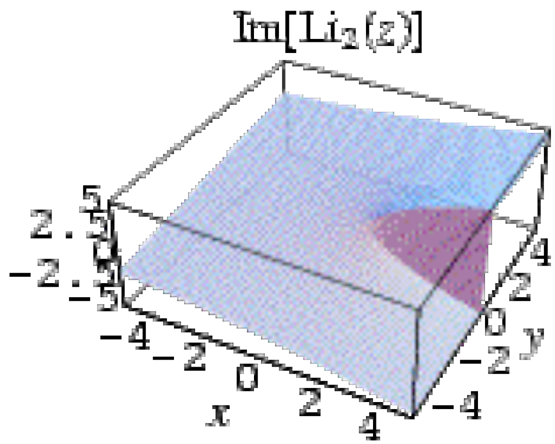
- A glorious example is shown below. Brandhuber, Spence and Travaglini

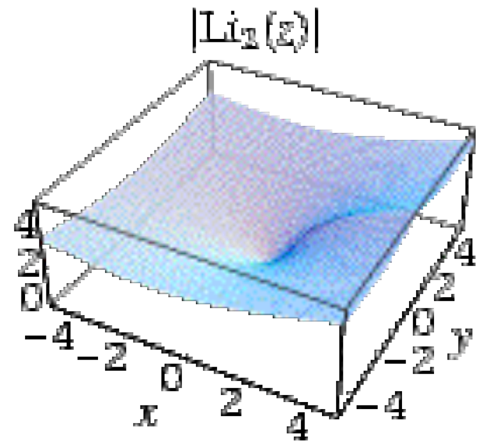
$$\mathrm{Li}_2(1-aP^2)+\mathrm{Li}_2(1-aQ^2)-\mathrm{Li}_2(1-as)-\mathrm{Li}_2(1-at) =$$
$$\mathrm{Li}_2\Big(1-\frac{P^2}{s}\Big)+\mathrm{Li}_2\Big(1-\frac{P^2}{t}\Big)+\mathrm{Li}_2\Big(1-\frac{Q^2}{s}\Big)+\mathrm{Li}_2\Big(1-\frac{Q^2}{t}\Big)$$
$$-\mathrm{Li}_2\Big(1-\frac{P^2Q^2}{st}\Big)+\frac{1}{2}\ln^2\Big(\frac{s}{t}\Big).$$

which holds for $a=(P^2+Q^2-s-t)/(P^2Q^2-st)$.

Spence's function







Analytic continuation for $-\pi<\arg(x_i)<\pi$

$$\ln\Big(\prod_{i=1}^{n} x_i\Big) \rightarrow \sum_{i=1}^{n} \ln(x_i)$$

$$\mathrm{Li}_2\Big(1-\prod_{i=1}^{n} x_i\Big) \rightarrow \mathrm{Li}_2\Big(1-\prod_{i=1}^{n} x_i\Big) + \Big[\ln\Big(\prod_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} \ln(x_i)\Big] \ln\Big(1-\prod_{i=1}^{n} x_i\Big)$$

for $\quad |\prod_{i=1}^{n} x_i| < 1$

Beenakker, Denner

Fortran code

Fortran code is available. It calculates finite integrals using the ff library for the finite integrals, and calculates divergent integrals using the QCDLoop library.

```
====================================================
 This is QCDLoop - version 1.3
 Authors: Keith Ellis and Giulia Zanderighi
 (ellis@fnal.gov, g.zanderighi1@physics.ox.ac.uk)
 For details see FERMILAB-PUB-07-633-T,OUTP-07/16P
 arXiv:0712.1851 [hep-ph]
====================================================
====================================================
 FF 2.0, a package to evaluate one-loop integrals
written by G. J. van Oldenborgh, NIKHEF-H, Amsterdam
====================================================
for the algorithms used see preprint NIKHEF-H 89/17,
'New Algorithms for One-loop Integrals', by G.J. van
Oldenborgh and J.A.M. Vermaseren, published in
Zeitschrift fuer Physik C46(1990)425.
====================================================
ffini: precx =   4.4408921E-16
ffini: precc =   4.4408921E-16
ffini: xalogm =   4.94065646E-324
ffini: xclogm =   4.94065646E-324

p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq  1.5 -1.7  2.3  2.9  37.
-15.7  3.  5.  9.  1.3  1.1

ep,qlI4(p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq,ep) -2 (0.,0.)
ep,qlI4(p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq,ep) -1 (0.,0.)
ep,qlI4(p1sq,p2sq,p3sq,p4sq,s12,s23,m1sq,m2sq,m3sq,m4sq,musq,ep) 0
(0.00404258529,0.0176915481)

test of divergent boxes
ep,qlI4(0d0,0d0,p3sq,p4sq,s12,s23,0d0,0d0,0d0,m4sq,musq,ep) -2
(-0.00158982512,0.)
ep,qlI4(0d0,0d0,p3sq,p4sq,s12,s23,0d0,0d0,0d0,m4sq,musq,ep) -1
(0.0138506054,0.00499458292)
ep,qlI4(0d0,0d0,p3sq,p4sq,s12,s23,0d0,0d0,0d0,m4sq,musq,ep) 0
(-0.043279838,0.0270258376)
```

Numerical calculation of scalar box function

```
      double complex function qlI4(p1,p2,p3,p4,s12,s23,
      . m1,m2,m3,m4,mu2,ep)
       implicit none
       double precision p1,p2,p3,p4,s12,s23,m1,m2,m3,m4,mu2
       integer ep
C      pi=p(i)^2, i=1,2,3,4 are momentum squared of the external lines
C      mi=m(i)^2  i=1,2,3,4 are masses squared of the internal lines
C      sij=(pi+pj)^2 are external invariants
C      mu2 is the square of the scale mu
C      ep=-2,-1,0 chooses the coefficient in the Laurent series.
```

Numerical checks

We can perform a numerical check of the code, by using the Relation between boxes, triangles and the six-dimensional box.

$$I_4^D = \frac{1}{2}\left(\sum_{i=1}^{4} c_i I_3^D[i] + (3-D)c_0 I_4^{D+2}\right) \qquad c_i = \sum_{j=1}^{4}(Y^{-1})_{ij}\,, \quad c_0 = \sum_{i=1}^{4} c_i\,.$$

In D=6, the box integral is finite - no UV,IR or collinear divergences
So we can check this relation numerically, (even in the physical region by setting the causal ε equal to a very small number).

$$I_4^D(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) = \frac{\mu^{2\epsilon}\Gamma(2+\epsilon)}{r_\Gamma}\prod_{i=1}^{4}\int_0^1 da_k\,\frac{\delta(1-\sum_k a_k)}{\left[\sum_{i,j} a_i a_j Y_{ij} - i\varepsilon\right]^{2+\epsilon}}$$

Using above relation to calculate box integrals does not seem to be the right strategy

Summary

- We have developed a program which returns a complex number for an arbitrary one loop integral, finite or divergent. The problem of divergent one-loop scalar integrals can now be considered completely solved.
- QCDloop program available for download.
- Importance of making software freely available

  (cf the Helas, Madgraph, Madevent, SUSYMadgraph experience).

Similarly, the modified Cayley matrices for 16 divergent box integrals

$$Y_{ij} = \frac{1}{2}\left[m_i^2 + m_j^2 - (q_{i-1} - q_{j-1})^2\right]$$

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix}$$

(1) (2) (3) (4)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & \times \end{pmatrix}$$

(5) (6) (7) (8)

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ 0 & \times & \times & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

(9) (10) (11) (12)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}$$

(13) (14) (15) (16)