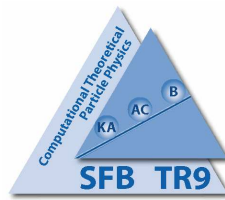


# Two and Three Loop Heavy Flavor Corrections in DIS

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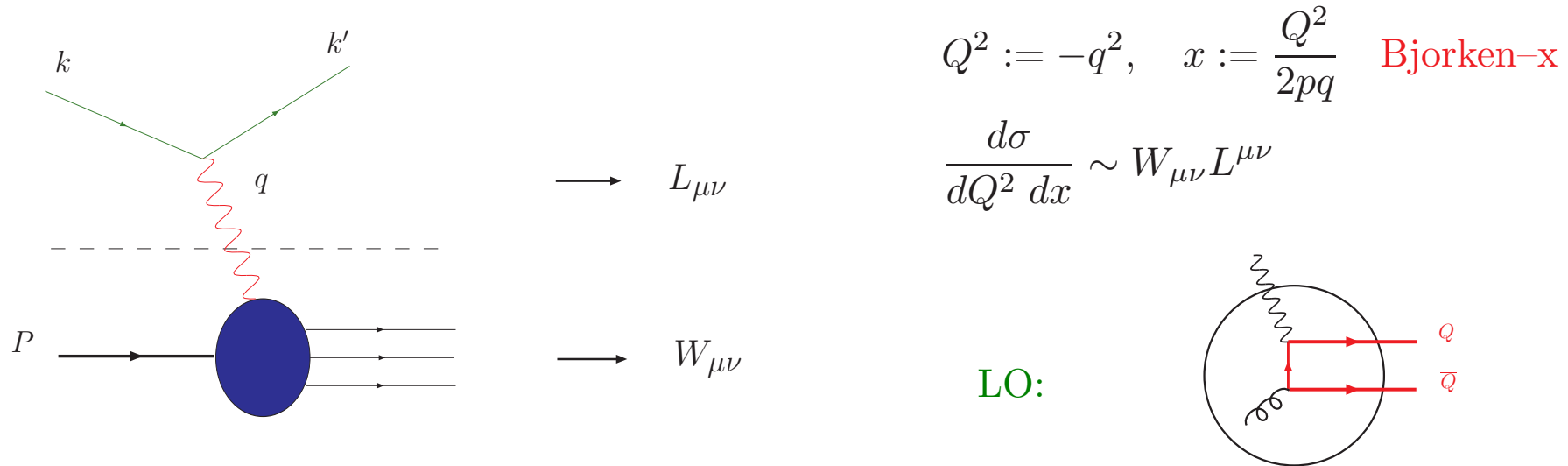
based on:

- Introduction
- Renormalization of the OME's to 3 Loops
- $O(\epsilon)$  terms at 2 Loops
- Towards Fixed Moments of  $A_{Qg}^{(3)}$
- Conclusions

- I. Bierenbaum, J. Blümlein, S. K., and C. Schneider  
arXiv:0707.4759 [math-ph];  
arXiv:0803.0273 [hep-ph].
- I. Bierenbaum, J. Blümlein, and S. K.,  
Phys. Lett. **B648** (2007) 195;  
Nucl. Phys. **B780** (2007) 40;  
arXiv:0706.2738 [hep-ph];  
Acta Phys. Polon. B **38** (2007) 3543;
- J. Blümlein, A. De Freitas, W.L. van Neerven, and S. K.,  
Nucl. Phys. **B755** (2006) 272.

# 1. Introduction

Deep-Inelastic Scattering (DIS):



Hadronic tensor for heavy quark production via single photon exchange:

$$W_{\mu\nu}^{Q\bar{Q}}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle_{Q\bar{Q}}$$

$$\text{unpol.} \left\{ \begin{aligned} &= \frac{1}{2x} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L^{Q\bar{Q}}(x, Q^2) + \frac{2x}{Q^2} \left( P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2^{Q\bar{Q}}(x, Q^2) \end{aligned} \right.$$

$$\text{pol.} \left\{ \begin{aligned} &-\frac{M}{2Pq} \varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[ s^\beta g_1^{Q\bar{Q}}(x, Q^2) + \left( s^\beta - \frac{sq}{Pq} p^\beta \right) g_2^{Q\bar{Q}}(x, Q^2) \right]. \end{aligned} \right.$$

- Light quarks: In Bjorken limit,  $\{Q^2, \nu\} \rightarrow \infty$ ,  $x$  fixed, at twist  $\tau = 2$ -level:

$$\underbrace{F_i(x, Q^2)}_{\text{structure functions}} = \sum_j \underbrace{C_i^j\left(x, \frac{Q^2}{\mu^2}\right)}_{\text{Wilson coefficients, perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{parton densities, non-perturbative}},$$

Different flavor contributions to  $F_2(x, Q^2)$ :

	light flavor	heavy flavor
LO:	$x[f_q + f_{\bar{q}}] _{q=u,d,s}$	$a_s \left[ C_g^{(1),H} \otimes xg \right]$
NLO:	$a_s \left[ C_q^{(1)} \otimes x[f_q + f_{\bar{q}}] + C_g^{(1)} \otimes xg \right]$	$a_s^2 \left[ C_{q,NS}^{(2),H} \otimes x[f_q + f_{\bar{q}}] + C_{q,PS}^{(2),H} \otimes x\Sigma + C_g^{(2),H} \otimes xg \right]$
N <sup>2</sup> LO:	⋮	⋮

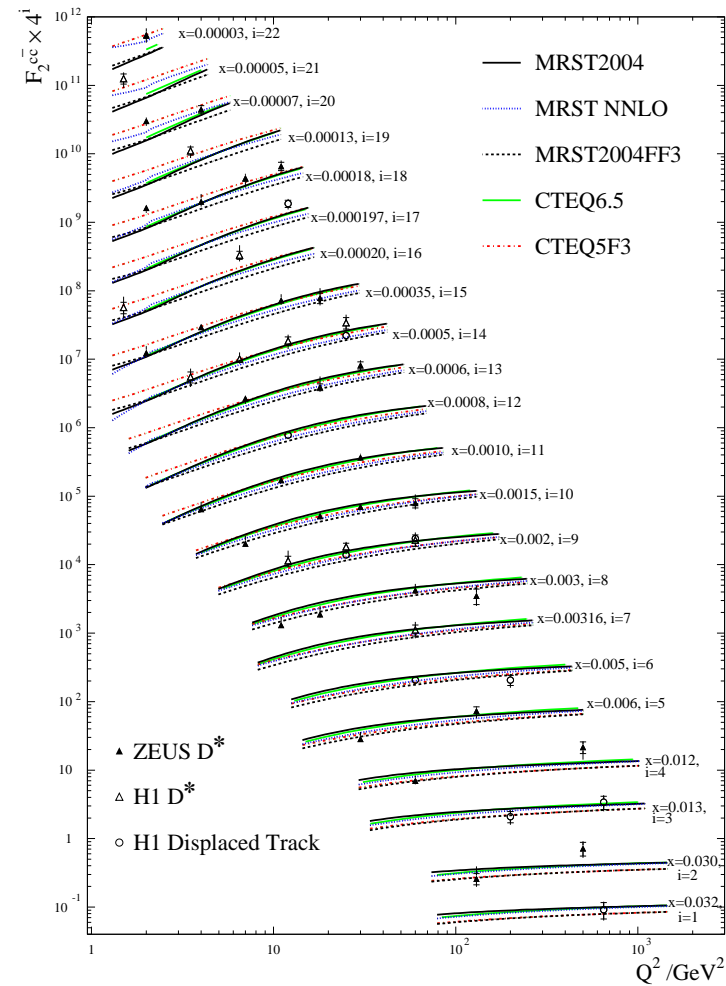
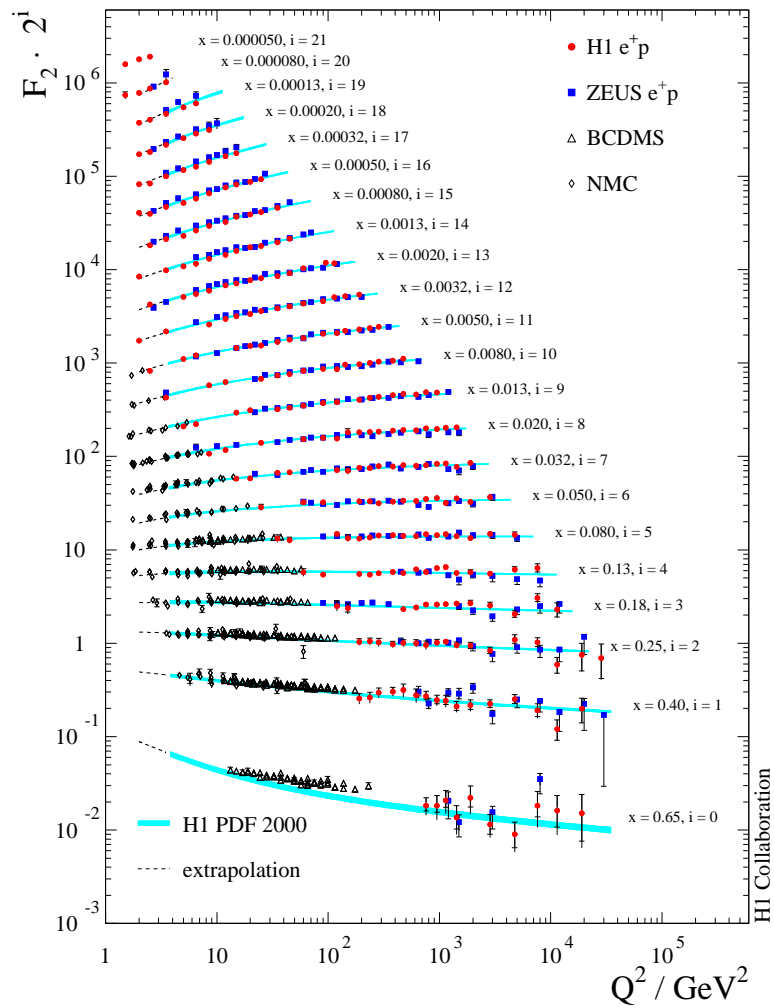
3-loop analysis:  $O(a_s^3)$  heavy flavor Wilson coefficients needed.

## Need for the calculation:

- **Heavy flavor** (charm) contributions to DIS **structure functions** are rather large [20–40 % at lower values of  $x$ ].
- Increase in accuracy of the perturbative description of DIS **structure functions**.
- $\iff$  QCD analysis and determination of  $\Lambda_{\text{QCD}}$ , resp.  $\alpha_s(M_Z^2)$ , from DIS data:  
 $\delta\alpha_s/\alpha_s < 1\%$ .
- $\iff$  Precise determination of the **gluon** and **sea quark** distributions.
- $\iff$  Derivation of **variable flavor number scheme** for **heavy quark** production to  $O(\alpha_s^3)$ .

### Goal:

Calculation of the **heavy flavor Wilson coefficients** to higher orders for  $Q^2 \geq 25 \text{ GeV}^2$  [sufficient in many applications].



[Thompson, 2007]

High statistics in both cases

$F_2^{c\bar{c}}(x, Q^2) \sim 20 - 40\% F_2(x, Q^2)$  for small values of  $x$ , but **different** scaling violations

## Previous calculations:

### Unpolarized DIS :

- **LO** : [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov 1978; Leveille, Weiler, 1979]
- **NLO** : [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]  
asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996; Bierenbaum, Blümlein, Klein, 2007]
- **Observation:**  $F_2^{c\bar{c}}(x, Q^2)$  is very well described by  $F_2^{c\bar{c}}(x, Q^2)|_{Q^2 \gg m^2}$  for  $Q^2 \gtrsim 10 m_c^2$ .

### Polarized DIS :

- **LO** : [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]
- **NLO** :  
asymptotic: [Buza, Matiounine, Smith, van Neerven, 1997; Bierenbaum, Blümlein, Klein, 2008, to appear]

Mellin–space expressions: [Alekhin, Blümlein, 2003].

Variable flavor number scheme at  $O(a_s^2)$ : [Buza, Matiounine, Smith, van Neerven, 1998]

In the following, we report on results for unpolarized and polarized Heavy Flavor Wilson coefficients beyond NLO.

## 2. The Method

- massless RGE and light-cone expansion in Bjorken-limit  $\{Q^2, \nu\} \rightarrow \infty, x$  fixed:

$$\lim_{\xi^2 \rightarrow 0} [J(\xi), J(0)] \propto \sum_{i, N, \tau} c_{i, \tau}^N(\xi^2, \mu^2) \xi_{\mu_1} \dots \xi_{\mu_N} O_{i, \tau}^{\mu_1 \dots \mu_m}(0, \mu^2) .$$

- Operators: flavor non-singlet ( $\leq 3$ ), pure-singlet and gluon; consider leading twist.
- RGE for collinear singularities  
 $\implies$  mass factorization of the structure functions into Wilson coefficients and parton densities:

$$F_i(x, Q^2) = \sum_j \underbrace{C_i^j \left( x, \frac{Q^2}{\mu^2} \right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{non-perturbative}}$$

- Light-flavor Wilson coefficients: process dependent

$$C_{(2,L);i}^{\text{fl}} \left( \frac{Q^2}{\mu^2} \right) = \delta_{i,q} + \sum_{l=1}^{\infty} a_s^l C_{(2,L),i}^{\text{fl},(l)}, \quad i = q, g$$

$\implies$  Known up to  $O(a_s^3)$  [Moch, Vermaseren, Vogt, 2005.]

- Heavy quark contributions given by heavy quark Wilson coefficient,  $H_{(2,L),i}^{\text{S,NS}} \left( \frac{Q^2}{\mu^2}, \frac{m^2}{Q^2} \right)$ .
- In the limit  $Q^2 \gg m_Q^2$  [ $Q^2 \approx 10 m_Q^2$  for  $F_2$ ]:  
**massive RGE**, derivative  $m^2 \partial / \partial m^2$  acts on Wilson coefficients only: all terms but power corrections calculable through **partonic operator matrix elements**,  $\langle i | A_l | j \rangle$ , which are **process independent objects!**

$$H_{(2,L),i}^{\text{S,NS}} \left( \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \underbrace{A_{k,i}^{\text{S,NS}} \left( \frac{m^2}{\mu^2} \right)}_{\text{massive OMEs}} \otimes \underbrace{C_{(2,L),k}^{\text{S,NS}} \left( \frac{Q^2}{\mu^2} \right)}_{\text{light-parton-Wilson coefficients}}.$$

- holds for **polarized** and **unpolarized** case. OMEs obey expansion

$$A_{k,i}^{\text{S,NS}} \left( \frac{m^2}{\mu^2} \right) = \langle i | O_k^{\text{S,NS}} | i \rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{\text{S,NS},(l)} \left( \frac{m^2}{\mu^2} \right), \quad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]



## 3. Renormalization

Unrenormalized massive operator matrix elements:

$$\hat{A}_{ij} = \delta_{ij} + \sum_{k=0}^{\infty} \hat{a}_s^k \hat{A}_{ij}^{(k)}$$

need for:

- Mass renormalization
- Charge renormalization
- Renormalization of ultraviolet singularities
- Factorization of collinear singularities

→ use  $\overline{\text{MS}}$  scheme and decoupling formalism [Ovrut, Schnitzer 1981; Bernreuther, Wetzel 1982].

Since the light-cone expansion is used, external legs obtain self-energy insertions due to heavy quarks.

## Mass renormalization:

on-mass-shell scheme for quarks

[Tarrach 1981; Nachtmann, Wetzel 1981; Gray, Broadhurst, Graefe, Schilcher 1990]

$$\hat{m} = m + \hat{a}_s \delta m_1 + \hat{a}_s^2 \delta m_2 + O(\hat{a}_s^3)$$

$$\delta m_1 = C_F S_\varepsilon m \left( \frac{m^2}{\mu^2} \right)^{\varepsilon/2} \left[ \frac{6}{\varepsilon} - 4 + \left( 4 + \frac{3}{4} \zeta_2 \right) \varepsilon \right]$$

$$\delta m_2 = C_F S_\varepsilon^2 m \left( \frac{m^2}{\mu^2} \right)^\varepsilon \left[ \frac{1}{\varepsilon^2} (18C_F + 22C_A - 8T_F(N_l + N_h)) + \frac{1}{\varepsilon} \left( -\frac{45}{2}C_F + \frac{91}{2}C_A - 14T_F(N_l + N_h) \right) \right. \\ \left. + C_F \left( \frac{199}{8} - \frac{51}{2}\zeta_2 + 48 \ln(2)\zeta_2 - 12\zeta_3 \right) + C_A \left( -\frac{605}{8} + \frac{5}{2}\zeta_2 - 24 \ln(2)\zeta_2 + 6\zeta_3 \right) + T_F \left[ N_l \left( \frac{45}{2} + 10\zeta_2 \right) + N_h \left( \frac{69}{2} - 14\zeta_2 \right) \right] \right]$$

## Charge renormalization: $\overline{\text{MS}}$ -scheme

$$\hat{a}_s(\varepsilon) = Z_g^2(\varepsilon, \mu^2) a_s(\mu^2) = a_s(\mu^2) \left[ 1 + \delta a_{s,1} a_s(\mu^2) + \delta a_{s,2} a_s^2(\mu^2) \right] + O(a_s^4)$$

$$\delta a_{s,1} = S_\varepsilon \frac{2\beta_0}{\varepsilon}, \quad \delta a_{s,2} = S_\varepsilon^2 \left[ \frac{4\beta_0^2}{\varepsilon^2} + \frac{\beta_1}{\varepsilon} \right]$$

## Operator renormalization:

ultraviolet divergences, renormalized by  $Z$ -factors.

Generic formula in terms of anomalous dimensions  $\gamma_{ij,k}$ . Has to be adapted e.g. for the various cases – three-loop non-singlet, pure-singlet, etc. ( $i, j, m, n \in \{q, g\}$ ):

$$\begin{aligned}
 Z_{ij}(N, a_s, \varepsilon) = & \delta_{i,j} + a_s S_\varepsilon \frac{\gamma_{ij,0}}{\varepsilon} + a_s^2 S_\varepsilon^2 \left\{ \frac{1}{\varepsilon^2} \left[ \frac{1}{2} \gamma_{im,0} \gamma_{mj,0} + \beta_0 \gamma_{ij,0} \right] + \frac{1}{2\varepsilon} \gamma_{ij,1} \right\} \\
 & + a_s^3 S_\varepsilon^3 \left\{ \frac{1}{\varepsilon^3} \left[ \frac{1}{6} \gamma_{in,0} \gamma_{nm,0} \gamma_{mj,0} + \beta_0 \gamma_{im,0} \gamma_{mj,0} + \frac{4}{3} \beta_0^2 \gamma_{ij,0} \right] \right. \\
 & + \frac{1}{\varepsilon^2} \left[ \frac{1}{6} (\gamma_{im,1} \gamma_{mj,0} + 2\gamma_{im,0} \gamma_{mj,1}) + \frac{2}{3} (\beta_0 \gamma_{ij,1} + \beta_1 \gamma_{ij,0}) \right] \\
 & \left. + \frac{\gamma_{ij,2}}{3\varepsilon} \right\}
 \end{aligned}$$

$$Z_{qq}^{PS} = Z_{qq} - Z_{NS}.$$

The anomalous dimensions  $\gamma_{ij,k}(N)$  are related to the splitting functions by

$$\gamma_{ij,k}(N) = - \int_0^1 dz z^{N-1} P_{ij}^{(k)}(z) .$$

Mass factorization: Collinear singularities are factored into  $\Gamma_{NS}$ ,  $\Gamma_{ij,S}$  and  $\Gamma_{qq,PS}$ .

For massless quarks:  $\Gamma_{NS} = Z_{NS}^{-1}$ ,  $\Gamma_{ij,S} = Z_{ij,S}^{-1}$ ,  $\Gamma_{qq,PS} = Z_{qq,PS}^{-1}$

$$\begin{aligned}\Gamma_{NS}(N, a_s, \varepsilon) &= 1 - a_s S_\varepsilon \frac{\gamma_{NS,0}}{\varepsilon} + a_s^2 S_\varepsilon^2 \left[ \frac{1}{\varepsilon^2} \left( \frac{1}{2} \gamma_{NS,0}^2 - \beta_0 \gamma_{NS,0} \right) - \frac{1}{2\varepsilon} \gamma_{NS,1} \right] \\ \Gamma_{ij,S}(N, a_s, \varepsilon) &= \delta_{ij} - a_s S_\varepsilon \frac{\gamma_{ij,0}}{\varepsilon} + a_s^2 S_\varepsilon^2 \left[ \frac{1}{\varepsilon^2} \left( \frac{1}{2} \gamma_{ik,0} \gamma_{kj,0} - \beta_0 \gamma_{ij,0} \right) - \frac{1}{2\varepsilon} \gamma_{ij,1} \right] \\ \Gamma_{qq,PS}(N, a_s, \varepsilon) &= -a_s^2 S_\varepsilon^2 \left[ \frac{1}{2\varepsilon^2} \gamma_{qq,0} \gamma_{qq,0} + \frac{1}{2\varepsilon} \gamma_{qq,PS,1} \right].\end{aligned}$$

Here: in each diagram at least one quark line is massive

$\Rightarrow$   $\Gamma$ -matrices apply to parts of the diagrams with massless lines only

$\leftrightarrow$  at most 2-loop sub-graphs

$\leftrightarrow$  mass factorization is different in various sub-classes of contributing Feynman diagrams

$\rightarrow$  singularities contained in  $\Gamma_{NS}$ ,  $\Gamma_{ij,S}$ , and  $\Gamma_{qq,PS}$  are absorbed into the bare parton densities, which become scale-dependent

The renormalized operator matrix elements are obtained removing the ultraviolet singularities and collinear singularities of the operator matrix elements,

$$A_{ij} = Z_{ik}^{-1} \hat{A}_{kl} \Gamma_{lj}^{-1} = \delta_{ij} + a_s A_{ij}^{(1)} + a_s^2 A_{ij}^{(2)} + a_s^3 A_{ij}^{(3)} .$$

For example:

$$A_{Qg} = Z_{qq}^{-1} \hat{A}_{Qq}^{PS} \Gamma_{qq}^{-1} + Z_{qq}^{-1} \hat{A}_{Qg} \Gamma_{gg}^{-1} + Z_{qq}^{-1} \hat{A}_{gq,Q} \Gamma_{qq}^{-1} + Z_{qq}^{-1} \hat{A}_{gg,Q} \Gamma_{gg}^{-1} .$$

$$\implies A_{Qg}^{(2)} = \hat{A}_{Qg}^{(2)} + Z_{qq}^{-1,(1)} \hat{A}_{Qg}^{(1)} + Z_{qq}^{-1,(1)} \hat{A}_{gq,Q}^{(1)} + Z_{qq}^{-1,(2)} + \left( \hat{A}_{Qg}^{(1)} + Z_{qq}^{-1,(1)} \right) \Gamma_{gg}^{-1,(1)} .$$

→ mixing with  $\hat{A}_{Qq}^{PS}$  and  $\hat{A}_{gg,Q}$  at  $O(a_s^3)$ .

→  $\hat{A}_{gq,Q}$  starts contributing from  $O(a_s^2) \Rightarrow$  at  $O(a_s^4)$  to  $\hat{A}_{Qg}$

For example: **Renormalized gluonic massive operator matrix elements** up to  $O(a_s^2)$ :

$$\begin{aligned}
 \text{Unrenormalized } \hat{A}_{Qg} \text{ to } O(a_s^2): \\
 \hat{A}_{Qg}^{(1)} &= S_\varepsilon \left( \frac{m^2}{\mu^2} \right)^{\varepsilon/2} \left\{ -\frac{1}{\varepsilon} \hat{P}_{qg}^{(0)} + a_{Qg}^{(1)} + \varepsilon \bar{a}_{Qg}^{(1)} + \varepsilon^2 \bar{\bar{a}}_{Qg}^{(1)} \right\} \\
 \hat{A}_{Qg}^{(2)} &= S_\varepsilon^2 \left( \frac{m^2}{\mu^2} \right)^\varepsilon \left[ \frac{1}{\varepsilon^2} \left\{ \frac{1}{2} \hat{P}_{qg}^{(0)} \otimes (P_{qq}^{(0)} - P_{gg}^{(0)}) + \beta_0 \hat{P}_{qg}^{(0)} \right\} \right. \\
 &\quad \left. + \frac{1}{\varepsilon} \left\{ -\frac{1}{2} \hat{P}_{qg}^{(1)} - 2\beta_0 a_{Qg}^{(1)} - a_{Qg}^{(1)} \otimes (P_{qq}^{(0)} - P_{gg}^{(0)}) \right\} + a_{Qg}^{(2)} + \varepsilon \bar{a}_{Qg}^{(2)} \right] \\
 &\quad - \frac{2}{\varepsilon} S_\varepsilon \sum_{H=Q}^t \beta_{0,H} \left( \frac{m_H^2}{\mu^2} \right)^{\varepsilon/2} \left( 1 + \frac{1}{8} \varepsilon^2 \zeta_2 + \frac{\varepsilon^3}{24} \zeta_3 \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{Renormalized } A_{Qg} \text{ to } O(a_s^2): \\
 A_{Qg}^{(1)} &= -\frac{1}{2} \hat{P}_{qg}^{(0)} \ln \left( \frac{m^2}{\mu^2} \right) \\
 A_{Qg}^{(2)} &= \frac{1}{8} \left\{ \hat{P}_{qg}^{(0)} \otimes [P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0] \right\} \ln^2 \left( \frac{m^2}{\mu^2} \right) - \frac{1}{2} \hat{P}_{qg}^{(1)} \ln \left( \frac{m^2}{\mu^2} \right) \\
 &\quad + \bar{a}_{Qg}^{(1)} [P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0] + a_{Qg}^{(2)} \quad ; \quad a_{Qg}^{(1)} \equiv 0
 \end{aligned}$$

- $\bar{a}_{Qg}^{(1)}$ :  $O(\varepsilon)$  of the OME  $A_{Qg}^{(1)}$  in the  $\overline{\text{MS}}$ -scheme. Enters  $A_{Qg}^{(2)}$  through renormalization.
- For the renormalization of  $A_{Qg}^{(3)}$ :  $\bar{a}_{Qg}^{(2)}$ ,  $\bar{a}_{gg,Q}^{(2)}$ ,  $\bar{a}_{Qq}^{(2),\text{PS}}$  are needed.

# 4. Calculation Techniques

- Calculation in **Mellin-space** for **space-like**  $q^2$ ,  $Q^2 = -q^2$ :  $0 \leq x \leq 1$
- use of **generalized hypergeometric functions** for general analytic results
- use of **Mellin-Barnes integrals** for numerical checks (**MB**, [Czakon, 2006]) and some analytic results
- Summation of lots of **new** infinite **one-parameter sums** into **harmonic sums**. E.g.:

$$N \sum_{i,j=1}^{\infty} \frac{S_1(i)S_1(i+j+N)}{i(i+j)(j+N)} = 4S_{2,1,1} - 2S_{3,1} + S_1 \left( -3S_{2,1} + \frac{4S_3}{3} \right) - \frac{S_4}{2} \\ - S_2^2 + S_1^2 S_2 + \frac{S_1^4}{6} + 6S_1 \zeta_3 + \zeta_2 (2S_1^2 + S_2) .$$

use of **integral techniques** and the **Mathematica package SIGMA** [C. Schneider, 2007],  
[I. Bierenbaum, J. Blümlein, S. K., C. Schneider, arXiv:0707.4659 [math-ph];  
arXiv:0803.0273 [hep-ph]]

- Partial checks for fixed values of  $N$  using **SUMMER**,  
[Vermaseren, Int. J. Mod. Phys. A **14** (1999)].
- Algebraic and structural simplification of the harmonic sums [J. Blümlein, 2003, 2007].

# 5. Results

## Unpolarized case, Singlet, $O(\varepsilon)$

$$\begin{aligned}
 \bar{a}_{Qg}^{(2)} = & T_F C_F \left\{ \frac{2}{3} \frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 + \frac{N^4 - 5N^3 - 32N^2 - 18N - 4}{N^2(N+1)^2(N+2)} S_1^2 \right. \\
 & + \frac{N^2 + N + 2}{N(N+1)(N+2)} \left( 16S_{2,1,1} - 8S_{3,1} - 8S_{2,1}S_1 + 3S_4 - \frac{4}{3}S_3S_1 - \frac{1}{2}S_2^2 - S_2S_1^2 - \frac{1}{6}S_1^4 + 2\zeta_2S_2 - 2\zeta_2S_1^2 - \frac{8}{3}\zeta_3S_1 \right) \\
 & - 8 \frac{N^2 - 3N - 2}{N^2(N+1)(N+2)} S_{2,1} + \frac{2}{3} \frac{3N+2}{N^2(N+2)} S_1^3 + \frac{2}{3} \frac{3N^4 + 48N^3 + 43N^2 - 22N - 8}{N^2(N+1)^2(N+2)} S_3 + 2 \frac{3N+2}{N^2(N+2)} S_2S_1 + 4 \frac{S_1}{N^2} \zeta_2 \\
 & \left. + \frac{N^5 + N^4 - 8N^3 - 5N^2 - 3N - 2}{N^3(N+1)^3} \zeta_2 - 2 \frac{2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12}{N^2(N+1)^3(N+2)} S_1 + \frac{P_2}{N^5(N+1)^5(N+2)} \right\} \\
 & + T_F C_A \left\{ \frac{N^2 + N + 2}{N(N+1)(N+2)} \left( 16S_{-2,1,1} - 4S_{2,1,1} - 8S_{-3,1} - 8S_{-2,2} - 4S_{3,1} - \frac{2}{3}\beta''' + 9S_4 - 16S_{-2,1}S_1 \right. \right. \\
 & + \frac{40}{3}S_1S_3 + 4\beta''S_1 - 8\beta'S_2 + \frac{1}{2}S_2^2 - 8\beta'S_1^2 + 5S_1^2S_2 + \frac{1}{6}S_1^4 - \frac{10}{3}S_1\zeta_3 - 2S_2\zeta_2 - 2S_1^2\zeta_2 - 4\beta'\zeta_2 - \frac{17}{5}\zeta_2^2 \left. \right) \\
 & + \frac{4(N^2 - N - 4)}{(N+1)^2(N+2)^2} \left( -4S_{-2,1} + \beta'' - 4\beta'S_1 \right) - \frac{2}{3} \frac{N^3 + 8N^2 + 11N + 2}{N(N+1)^2(N+2)^2} S_1^3 + 8 \frac{N^4 + 2N^3 + 7N^2 + 22N + 20}{(N+1)^3(N+2)^3} \beta' \\
 & + 2 \frac{3N^3 - 12N^2 - 27N - 2}{N(N+1)^2(N+2)^2} S_2S_1 - \frac{16}{3} \frac{N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6}{(N-1)N^2(N+1)^2(N+2)^2} S_3 - 8 \frac{N^2 + N - 1}{(N+1)^2(N+2)^2} \zeta_2S_1 \\
 & - \frac{2}{3} \frac{9N^5 - 10N^4 - 11N^3 + 68N^2 + 24N + 16}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3} S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2} \zeta_2 \\
 & \left. - \frac{P_5}{N(N+1)^3(N+2)^3} S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4} S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5} \right\}.
 \end{aligned}$$



Unpolarized case, Pure-Singlet and Non-singlet,  $O(\varepsilon)$

$$\bar{a}_{Qq}^{\text{PS},(2)} = C_F T_F \left\{ -2 \frac{(5N^3 + 7N^2 + 4N + 4)(N^2 + 5N + 2)}{(N-1)N^3(N+1)^3(N+2)^2} (2S_2 + \zeta_2) \right. \\ \left. - \frac{4}{3} \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} (3S_3 + \zeta_3) + 2 \frac{P_9}{(N-1)N^5(N+1)^5(N+2)^4} \right\}.$$

$$\bar{a}_{qq,Q}^{\text{NS},(2)} = C_F T_F \left\{ \frac{4}{3} S_4 + \frac{4}{3} S_2 \zeta_2 - \frac{8}{9} S_1 \zeta_3 - \frac{20}{9} S_3 - \frac{20}{9} S_1 \zeta_2 + 2 \frac{3N^2 + 3N + 2}{9N(N+1)} \zeta_3 \right. \\ \left. + \frac{112}{27} S_2 + \frac{3N^4 + 6N^3 + 47N^2 + 20N - 12}{18N^2(N+1)^2} \zeta_2 - \frac{656}{81} S_1 + \frac{P_8}{648N^4(N+1)^4} \right\}.$$

## Polarized case, Singlet $O(\varepsilon)$

$$\begin{aligned}
\Delta \bar{a}_{Qg}^{(2)} = & T_F C_F \left\{ \frac{N-1}{N(N+1)} \left( 16S_{2,1,1} - 8S_{3,1} - 8S_{2,1}S_1 + 3S_4 - \frac{4}{3}S_3S_1 - \frac{1}{2}S_2^2 - \frac{1}{6}S_1^4 - \frac{8}{3}S_1\zeta_3 - S_2S_1^2 + 2S_2\zeta_2 - 2S_1^2\zeta_2 \right) \right. \\
& - 8\frac{S_{2,1}}{N^2} + \frac{3N^2 + 3N - 2}{N^2(N+1)(N+2)} \left( 2S_2S_1 + \frac{2}{3}S_1^3 \right) + \frac{3N^4 + 48N^3 + 123N^2 + 98N + 8}{N^2(N+1)^2(2+N)} S_3 + 4\frac{N-1}{N^2(N+1)} S_1\zeta_2 \\
& + \frac{2(N-1)(3N^2 + 3N + 2)}{3N^2(N+1)^2} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 + \frac{N^3 - 6N^2 - 22N - 36}{N^2(N+1)(N+2)} S_1^2 + \frac{P_2}{N^3(N+1)^3} \zeta_2 \\
& \left. - 2\frac{2N^4 - 4N^3 - 3N^2 + 20N + 12}{N^2(N+1)^2(N+2)} S_1 + \frac{P_3}{N^5(N+1)^5(N+2)} \right\} \\
& + T_F C_A \left\{ \frac{N-1}{N(N+1)} \left( 16S_{-2,1,1} - 4S_{2,1,1} - 8S_{-3,1} - 8S_{-2,2} - 4S_{3,1} + \frac{2}{3}\beta''' - 16S_{-2,1}S_1 - 4\beta''S_1 + 8\beta'S_2 + 8\beta'S_1^2 + 9S_4 \right. \right. \\
& + \frac{40}{3}S_3S_1 + \frac{1}{2}S_2^2 + 5S_2S_1^2 + \frac{1}{6}S_1^4 + 4\zeta_2\beta' - 2\zeta_2S_2 - 2\zeta_2S_1^2 - \frac{10}{3}S_1\zeta_3 - \frac{17}{5}\zeta_2^2 \left. \right) + \frac{N-1}{N(N+1)^2} \left( -16S_{-2,1} - 4\beta'' + 16\beta'S_1 \right) \\
& - \frac{16}{3} \frac{N^3 + 7N^2 + 8N - 6}{N^2(N+1)^2(N+2)} S_3 + 2\frac{3N^2 - 13}{N(N+1)^2(N+2)} S_2S_1 - \frac{2}{3} \frac{N^2 + 4N + 5}{N(N+1)^2(N+2)} S_1^3 - 8\frac{\zeta_2S_1}{(N+1)^2} - \frac{2}{3} \frac{(N-1)(9N+8)}{N^2(N+1)^2} \zeta_3 \\
& - 8\frac{N^2 + 3}{N(N+1)^3} \beta' - \frac{P_4}{N^3(N+1)^3(N+2)} S_2 - \frac{N^4 + 2N^3 - 5N^2 - 12N + 2}{N(N+1)^3(N+2)} S_1^2 - 2\frac{P_5}{N^3(N+1)^3} \zeta_2 \\
& \left. + 2\frac{P_6}{N(N+1)^4(N+2)} S_1 - 2\frac{P_7}{N^5(N+1)^5(N+2)} \right\}.
\end{aligned}$$

[I. Bierenbaum, J. Blümlein and S. K., 2008, to appear]

Polarized case, Pure–Singlet and Non–singlet,  $O(\varepsilon)$

$$\begin{aligned} \Delta \bar{a}_{Qq}^{\text{PS},(2)} &= -\frac{4(N-1)(N+2)}{3N^2(N+1)^2} (3S_3 + \zeta_3) + 2\frac{(N+2)(N^3+2N+1)}{N^3(N+1)^3} (2S_2 + \zeta_2) \\ &\quad + 2\frac{(2+N)(N^5-7N^4+6N^3+7N^2+4N+1)}{(N+1)^5 N^5}, \\ \Delta \bar{a}_{qq,Q}^{\text{NS},(2)} &= \bar{a}_{qq,Q}^{\text{NS},(2)}. \end{aligned}$$

- Use of 't Hooft–Veltman–scheme for  $\gamma_5$ . A **finite renormalization** has to be applied to maintain the **Ward–identities**.
- The first moment of  $\Delta \bar{a}_{Qg}^{(2)}$  vanishes.  $\implies$  sum rule of  $\Delta H_g(x)$ .

$\hat{A}_{gg,Q}^{(2)}$  unpolarized

$$\begin{aligned}
\hat{A}_{gg,Q}^{(2)} = & \\
& T_F C_A \left\{ \frac{1}{\varepsilon^2} \left( -\frac{32}{3} S_1 + \frac{64(N^2 + N + 1)}{3(N-1)N(N+1)(N+2)} \right) + \frac{1}{\varepsilon} \left( -\frac{80}{9} S_1 + \frac{16P_1}{9(N-1)N^2(N+1)^2(N+2)} \right) \right. \\
& + \left( -\frac{8}{3} \zeta_2 S_1 + \frac{16(N^2 + N + 1)\zeta_2}{3(N-1)N(N+1)(N+2)} - 4 \frac{56N + 47}{27(N+1)} S_1 + \frac{2P_3}{27(N-1)N^3(N+1)^3(N+2)} \right) \\
& + \varepsilon \left( -\frac{8}{9} \zeta_3 S_1 - \frac{20}{9} \zeta_2 S_1 + \frac{16(N^2 + N + 1)}{9(N-1)N(N+1)(N+2)} \zeta_3 + \frac{2N+1}{3(N+1)} S_2 - \frac{S_1^2}{3(N+1)} \right. \\
& \left. + \frac{4P_1 \zeta_2}{9(N-1)N^2(N+1)^2(N+2)} - 2 \frac{328N^4 + 256N^3 - 247N^2 - 175N + 54}{81(N-1)N(N+1)^2} S_1 + \frac{P_5}{81(N-1)N^4(N+1)^4(N+2)} \right) \left. \right\} \\
& + T_F C_F \left\{ \frac{1}{\varepsilon^2} \left( \frac{16(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right) + \frac{1}{\varepsilon} \left( \frac{4P_2}{(N-1)N^3(N+1)^3(N+2)} \right) \right. \\
& + \left( \frac{4(N^2 + N + 2)^2 \zeta_2}{(N-1)N^2(N+1)^2(N+2)} - \frac{P_4}{(N-1)N^4(N+1)^4(N+2)} \right) \\
& \left. + \varepsilon \left( \frac{4(N^2 + N + 2)^2 \zeta_3}{3(N-1)N^2(N+1)^2(N+2)} + \frac{P_2 \zeta_2}{(N-1)N^3(N+1)^3(N+2)} + \frac{P_6}{4(N-1)N^5(N+1)^5(N+2)} \right) \right\}
\end{aligned}$$

→ Result obtained in terms of  $\Gamma$  and  $\Psi$  functions to **all orders in  $\varepsilon$**

→ to  **$O(\varepsilon^0)$  agreement** with van Neerven et al.

→  **$O(\varepsilon)$  new:** term needed too to derive **variable flavor number scheme** to  $O(\alpha_s^3)$ .

A remark on the mathematical structure of the  $O(1)$  and  $O(\varepsilon)$  terms:

van Neerven et al. to  $O(1)$ : unpolarized: 48 basic functions; polarized: 24 basic functions.

$O(1)$ :  $\{S_1, S_2, S_3, S_{-2}, S_{-3}\}, S_{-2,1} \implies 2$  basic objects.

$O(\varepsilon)$ :  $\{S_1, S_2, S_3, S_4, S_{-2}, S_{-3}, S_{-4}\}, S_{2,1}, S_{-2,1}, S_{-3,1}, S_{2,1,1}, S_{-2,1,1}$   
 $\implies 6$  basic objects

These objects are in common to all single scale higher order processes.

Str. Functions, DIS HQ, Fragn. Functions, DY, Hadr. Higgs-Prod., s+v contr. to Bhabha scatt., ...

harmonic sums with index  $\{-1\}$  cancel (holds even for each diagram)

[J. Blümlein, 2004; J. Blümlein, V. Ravindran, 2005,2006; J. Blümlein, S. Klein, arXiv: 0706.2426 [hep-ph];

J. Blümlein and S. Moch in preparation.]

Expectation for 3-loops: weight 5 (6) harmonic sums

## 6. Fixed moments of $A_{ij,Q}^{(3)}$

Contributing OMEs:

$$\begin{array}{lcl}
 \text{Singlet} & A_{Qg} & A_{gg,Q} & A_{qq,Q} \\
 \text{Pure-Singlet} & & A_{Qq}^{\text{PS}} & \\
 \text{Non-Singlet} & A_{qq,Q}^{\text{NS,+}} & A_{qq,Q}^{\text{NS,-}} & A_{qq,Q}^{\text{NS,v}}
 \end{array} \left. \vphantom{\begin{array}{l} \\ \\ \\ \end{array}} \right\} \text{mixing}$$

- All 2-loop  $O(\varepsilon)$ -terms in the **unpolarized** case are known:

$$\bar{a}_{Qg}^{(2)}, \bar{a}_{Qq}^{(2),\text{PS}}, \bar{a}_{gg,Q}^{(2)}, \bar{a}_{qq,Q}^{(2)}, \bar{a}_{qq,Q}^{(2),\text{NS}}.$$

- Unpolarized anomalous dimensions** are known up to  $O(a_s^3)$  [Moch, Vermaseren, Vogt, 2004.]

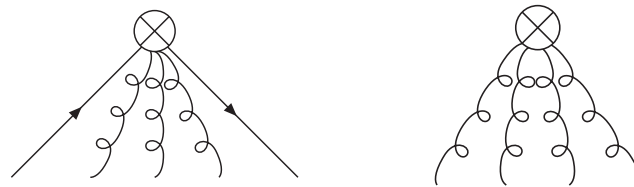
$\implies$  All terms needed for the renormalization of **unpolarized 3-Loop heavy OMEs** are present.

$\implies$  Calculation will provide first independent checks on  $\gamma_{qg}^{(3)}$ ,  $\gamma_{qq}^{(3),\text{PS}}$  and on respective color projections of  $\gamma_{qq}^{(3),\text{NS}\pm,\text{v}}$ ,  $\gamma_{gg}^{(3)}$  and  $\gamma_{gq}^{(3)}$ .

- Calculation proceeds in the same way in the **polarized** case. Known so far :

$$\Delta \bar{a}_{Qg}^{(2)}, \Delta \bar{a}_{Qq}^{(2),\text{PS}}, \Delta \bar{a}_{qq,Q}^{(2),\text{NS}} = \bar{a}_{qq,Q}^{(2),\text{NS}}.$$

- 3-loop OMEs are generated with QGRAF [Nogueira 1993]
- New operator insertions emerge:



We consider first the terms with 2 quark loops (at least one heavy): # of diagrams for  $A_{Qg}^{(3)}$

- 489 diagrams with two quark loops
- 1478 diagrams with one quark loop

$\implies$  number of diagrams can be reduced by using symmetry arguments.

**First step:** Calculation of **fixed moments** of  $A_{Qg}^{(3)}(N)$ ,  $N = 2, 4, 6, \dots$

- three-loop “self-energy” type diagrams with an operator insertion
- **Extension:** **additional scale** compared to massive propagators: **Mellin variable N**
- Genuine **tensor integrals** due to

$$\Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | O_{\mu_1 \dots \mu_n} | p \rangle = \Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | S \bar{\Psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_n} \Psi | p \rangle = A(N) \cdot (\Delta p)^N$$

$$D_\mu = \partial_\mu - i g t_a A_\mu^a \quad , \quad \Delta^2 = 0.$$

- Construction of a **projector** to obtain the desired moment in **N** [undo  $\Delta$ -contraction] ✓
- Color factors are calculated with [Ritbergen, Schellekens, Vermaseren 1998]
- Translation to suitable input for **MATAD** [Steinhauser, 2001] ✓

**Tests performed:** Various 2-loop calculations for  $N = 2, 4, 6, \dots$  were repeated  
 → agreement with our previous calculation; checked against result of  $A_{gg,Q}^{(2)}$

**Status:**

- automated chain to evaluate the OMEs ✓
- Terms  $\propto O(T_F^2 d_{abc} d^{abc})$  vanish
- **currently investigated:** terms  $\propto T_F^2 C_F, T_F^2 C_A$ -terms



# First Results

Non-singlet terms :  $O(T_F^2 C_F) A_{qq,Q}^{(3),NS^\pm}(N)$  : 2 heavy quark loops

$$\hat{A}_{qq,QQ}^{(3),NS} = \left( \frac{m^2}{\mu^2} \right)^{3\epsilon/2} \left( -\frac{8\beta_{0,Q}^2 \gamma_{qq,Q}^{(0),NS}}{3\epsilon^3} - \frac{4\beta_{0,Q} \gamma_{qq,Q}^{(1),NS}}{3\epsilon^2} + \frac{\gamma_{qq,QQ}^{(2),NS} - 12\beta_{0,Q} a_{qq,Q}^{(2),NS}}{3\epsilon} + a_{qq,QQ}^{(3),NS} \right).$$

$$A_{qq,QQ}^{(3),NS} = \frac{1}{6} \ln^3 \left( \frac{m^2}{\mu^2} \right) \beta_{0,Q}^2 \gamma_{qq,Q}^{(0),NS} + \frac{1}{2} \ln^2 \left( \frac{m^2}{\mu^2} \right) \beta_{0,Q} \gamma_{qq,Q}^{(1),NS} + \frac{1}{2} \ln \left( \frac{m^2}{\mu^2} \right) \gamma_{qq,QQ}^{(2),NS} + a_{qq,QQ}^{(3),NS} + 4\beta_{0,Q} \bar{a}_{qq,Q}^{(2),NS}.$$

$$\gamma_{qq,Q}^{(0),NS} = 4C_F \left[ 2S_1 - \frac{3N^2 + 3N + 2}{2N(N+1)} \right]$$

$$\gamma_{qq,Q}^{(1),NS} = 4C_F T_F \left\{ \frac{8}{3} S_2 - \frac{40}{9} S_1 + \frac{3N^4 + 6N^3 + 47N^2 + 20N - 12}{9N^2(N+1)^2} \right\}$$

$$\gamma_{qq,QQ}^{(2),NS} = C_F T_F^2 \left( \frac{128}{9} S_3 - \frac{640}{27} S_2 - \frac{128}{27} S_1 + 8 \frac{51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24}{27N^3(N+1)^3} \right).$$

$$\begin{aligned}
\hat{A}_{qq,QQ}^{(3),NS}(2) &= \frac{1}{\varepsilon^3} \left( -\frac{2048}{81} \right) + \frac{1}{\varepsilon^2} \left( -\frac{4096}{243} \right) + \frac{1}{\varepsilon} \left( -\frac{20224}{729} - \frac{256}{27} \zeta_2 \right) \\
&\quad - \frac{28736}{2187} - \frac{2048}{81} \zeta_3 - \frac{512}{81} \zeta_2 , \\
\hat{A}_{qq,QQ}^{(3),NS}(4) &= \frac{1}{\varepsilon^3} \left( -\frac{20096}{405} \right) + \frac{1}{\varepsilon^2} \left( -\frac{212336}{6075} \right) + \frac{1}{\varepsilon} \left( -\frac{5071846}{91125} - \frac{2512}{135} \zeta_2 \right) \\
&\quad - \frac{151928299}{5467500} - \frac{20096}{405} \zeta_3 - \frac{26542}{2025} \zeta_2 , \\
\hat{A}_{qq,QQ}^{(3),NS}(6) &= \frac{1}{\varepsilon^3} \left( -\frac{181504}{2835} \right) + \frac{1}{\varepsilon^2} \left( -\frac{13699808}{297675} \right) + \frac{1}{\varepsilon} \left( -\frac{2263340116}{31255875} - \frac{22688}{945} \zeta_2 \right) \\
&\quad - \frac{26884517771}{729303750} - \frac{181504}{2835} \zeta_3 - \frac{1712476}{99225} \zeta_2 , \\
\hat{A}_{qq,QQ}^{(3),NS}(8) &= \frac{1}{\varepsilon^3} \left( -\frac{632512}{8505} \right) + \frac{1}{\varepsilon^2} \left( -\frac{144967772}{2679075} \right) + \frac{1}{\varepsilon} \left( -\frac{285344205403}{3375634500} - \frac{79064}{2835} \zeta_2 \right) \\
&\quad - \frac{740566685766263}{17013197880000} - \frac{632512}{8505} \zeta_3 - \frac{36241943}{1786050} \zeta_2 .
\end{aligned}$$

$$A_{qq,QQ}^{(3),NS}(2) = T_F^2 C_F \left[ \frac{128}{81} \ln^3\left(\frac{m^2}{\mu^2}\right) + \frac{512}{81} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{896}{243} \ln\left(\frac{m^2}{\mu^2}\right) + \frac{25024}{2187} - \frac{1792}{81} \zeta_3 \right],$$

$$A_{qq,QQ}^{(3),NS}(4) = T_F^2 C_F \left[ \frac{1256}{405} \ln^3\left(\frac{m^2}{\mu^2}\right) + \frac{26542}{2025} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{384277}{60750} \ln\left(\frac{m^2}{\mu^2}\right) + \frac{243265699}{10935000} - \frac{17584}{405} \zeta_3 \right],$$

$$A_{qq,QQ}^{(3),NS}(6) = T_F^2 C_F \left[ \frac{11344}{2835} \ln^3\left(\frac{m^2}{\mu^2}\right) + \frac{1712476}{99225} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{80347571}{10418625} \ln\left(\frac{m^2}{\mu^2}\right) + \frac{125199267113}{4375822500} - \frac{22688}{405} \zeta_3 \right],$$

$$A_{qq,QQ}^{(3),NS}(8) = T_F^2 C_F \left[ \frac{39532}{8505} \ln^3\left(\frac{m^2}{\mu^2}\right) + \frac{36241943}{1786050} \ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{38920977797}{4500846000} \ln\left(\frac{m^2}{\mu^2}\right) + \frac{1128638049575063}{34026395760000} - \frac{79064}{1215} \zeta_3 \right].$$

All  $\zeta_2$  terms vanish after renormalization.

1 heavy - 1 massless quark

$$A_{qq,qQ}^{(3),NS} (2) = n_f T_F^2 C_F \left[ -\frac{128}{81} \ln^3\left(\frac{m^2}{\mu^2}\right) - \frac{4864}{243} \ln\left(\frac{m^2}{\mu^2}\right) - \frac{46336}{2187} + \frac{1024}{81} \zeta_3 \right],$$

$$A_{qq,qQ}^{(3),NS} (4) = n_f T_F^2 C_F \left[ -\frac{1256}{405} \ln^3\left(\frac{m^2}{\mu^2}\right) - \frac{2331077}{60750} \ln\left(\frac{m^2}{\mu^2}\right) - \frac{229618301}{5467500} + \frac{10048}{405} \zeta_3 \right],$$

$$A_{qq,qQ}^{(3),NS} (6) = n_f T_F^2 C_F \left[ -\frac{11344}{2835} \ln^3\left(\frac{m^2}{\mu^2}\right) - \frac{170378657}{3472875} \ln\left(\frac{m^2}{\mu^2}\right) - \frac{39653493629}{729303750} + \frac{90752}{2835} \zeta_3 \right],$$

$$A_{qq,qQ}^{(3),NS} (8) = n_f T_F^2 C_F \left[ -\frac{39532}{8505} \ln^3\left(\frac{m^2}{\mu^2}\right) - \frac{255097766597}{4500846000} \ln\left(\frac{m^2}{\mu^2}\right) - \frac{1076783602872937}{17013197880000} + \frac{316256}{8505} \zeta_3 \right],$$

## 7. Conclusions

- The **heavy flavor contributions** to  $F_2$  are **rather large** in the region of lower values of  $x$ .
- **QCD precision analyses** therefore require the description of the heavy quark contributions to **3-loop order**.
- We calculate the **heavy flavor DIS Wilson Coefficients** in the asymptotic regime [ $Q^2 \geq 10m^2$ ] using massive operator matrix elements.
- We **newly** presented **first contributions** to these corrections:
  - $\bar{a}_{Qg}^{(2)}, \bar{a}_{Qq}^{PS,(2)}, \bar{a}_{gg,Q}^{(2)}, \bar{a}_{qq}^{NS,(2)} = \Delta\bar{a}_{qq,Q}^{NS,(2)}$
  - $\Delta\bar{a}_{Qg}^{(2)}, \Delta\bar{a}_{Qg}^{PS,(2)}$

in the **unpolarized and polarized case** for **general values of the Mellin variable**. These terms contribute to  $H_{ij}^{(3)}, \Delta H_{ij}^{(3)}$  respectively, through renormalization.

- The calculation is performed in **Mellin space**, which allows to obtain compact results.
  - The **analytic results** were obtained using representations in terms of **generalized hypergeometric functions**.
  - Numerical checks were performed applying **Mellin–Barnes integrals**.
- **Integral techniques** and the summation package **SIGMA** have been used for summation. The results are given in the form of **nested harmonic sums**.
- We developed a programme–chain to calculate the massive operator matrix elements  $A_{ij}^{(3)}$  for **fixed Mellin moments** based on **QGRAF** and **MATAD**
- **First non-logarithmic 3–loop contributions** were calculated for a series of Mellin Moments. There are more to come soon for other color structures and higher values of Mellin N.
- The **3–loop anomalous dimensions** are obtained as a **by-product** from the single pole terms.