An Overview of On-shell Methods and their Applications

LoopFest VII

Buffalo

May 15, 2008

Zvi Bern, UCLA

Outline

This talk will give an overview of on-shell methods for loops and describe three applications.

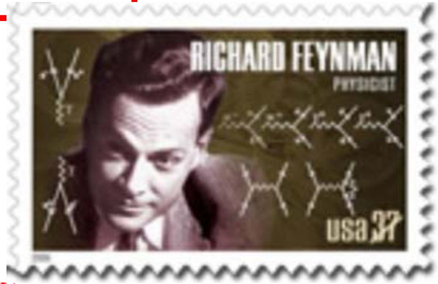
- **QCD: multi-parton scattering for the LHC**

  Talks from Britto, Bernicot, Forde, Giele, Kilgore, Zanderighi

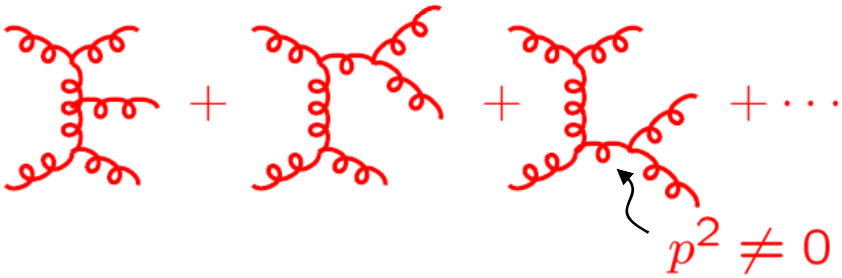
- **Supersymmetric gauge theory: resummation of planar maximally super-Yang-Mills scattering amplitudes to *all* loop orders.**

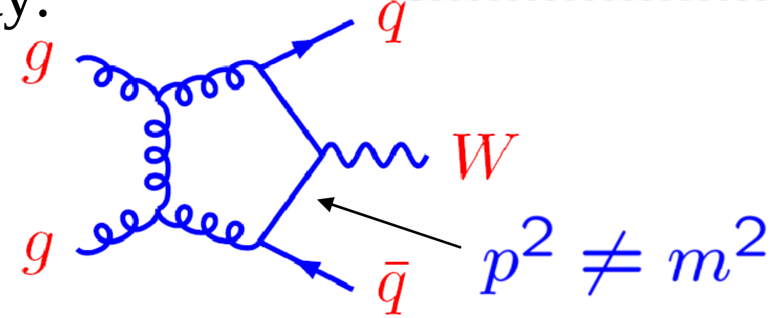
- **Quantum gravity: reexamination of standard wisdom on ultraviolet properties of quantum gravity.**

Why are Feynman diagrams clumsy for high-loop or high-multiplicity processes?



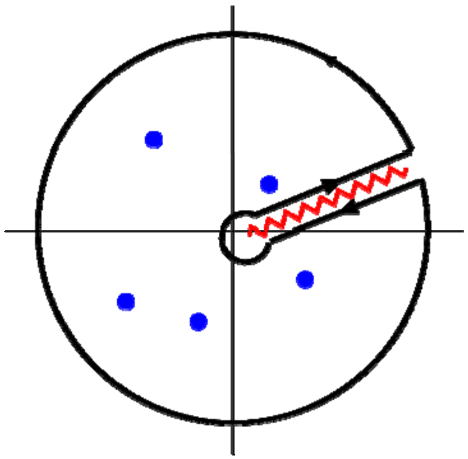
- Vertices and propagators involve gauge-dependent off-shell states. An important origin of the complexity.

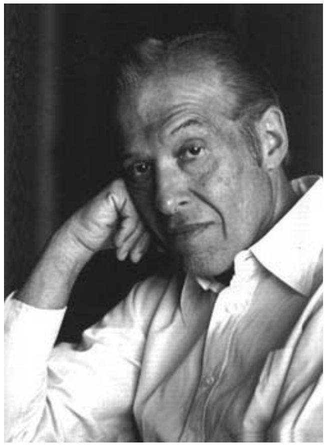




- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

- **All steps should be in terms of gauge invariant on-shell states. On-shell formalism.** $p^2 = m^2$





“One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane.”

J. Schwinger in “Particles, Sources and Fields” Vol 1

On-shell methods reconstruct amplitudes from their poles and cuts. Each of these corresponds to propagation of particles. Automatically gauge independent.

QCD: Experimenter's Wish List

Les Houches 2007

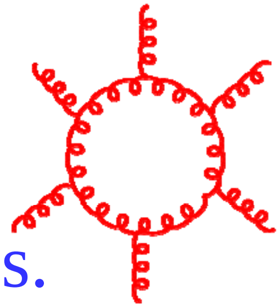
Process ($V \in \{Z, W, \gamma\}$)	Comments
4. $pp \to t\bar{t}\, b\bar{b}$	relevant for $t\bar{t}H$
5. $pp \to t\bar{t}$+2jets	relevant for $t\bar{t}H$
6. $pp \to VV\, b\bar{b}$,	relevant for VBF $\to H \to VV$, $t\bar{t}H$
7. $pp \to VV$+2jets	relevant for VBF $\to H \to VV$ VBF contributions calculated by (Bozzi/)Jäger/Oleari/Zeppenfeld.
8. $pp \to V$+3jets	various new physics signatures
NLO calculations added to list in 2007	
9. $pp \to b\bar{b}b\bar{b}$	Higgs and new physics signatures

Five-particle processes under good control with Feynman diagram based approaches.

Dittmaier,Kallweit,Uwer; Campbell, Ellis, Zanderighi; Binoth, Karg, Kauer,Sanguinetti
Ciccolini, Denner, Dittmaier; Lazapolous, Melnikov, Petriello; Hankele, Zeppenfeld
Binoth, Ossola, Papadopoulos; Figy, Hankele, Zeppenfeld; Jager, Oleari, Zeppenfeld;

Six-particle processes still difficult.

Approaches for higher points



Numerical or traditional Feynman approaches.

Anastasiou. Andersen, Binoth, Ciccolini; Czakon, Daleo, Denner, Dittmaier, Ellis; Heinrich, Karg, Kauer; Giele, Glover, Guffanti, Lazopoulos, Melnikov, Nagy, Pilon, Roth, Passarino, Petriello, Sanguinetti, Schubert; Smillie, Soper, Reiter, Wieders, Zanderighi, and many more.

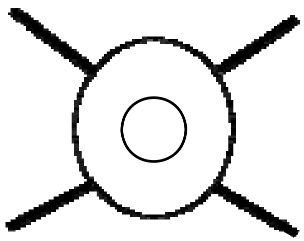
On-shell methods: unitarity method, on-shell recursion

Anastasiou, Badger, Bedford, Berger, Bern, Bernicot, Brandhuber, Britto, Buchbinder, Cachazo, Del Duca, Dixon, Dunbar, Ellis, Feng, Febres Cordero, Forde, Giele, Glover, Guillet, Ita, Kilgore, Kosower, Kunszt; Mastrolia; Maitre, Melnikov, Spence, Travaglini; Ossola, Papadopoulos, Pittau, Risager, Yang; Zanderighi, etc

Unitarity Method

- In the 1960's dispersion relations popular.
- Cutkosky rules for individual diagrams.
- Reconstructions of amplitudes from dispersion relations limited to 2 kinematic variables, massive theories, and contained subtraction ambiguities.

A(s,t)

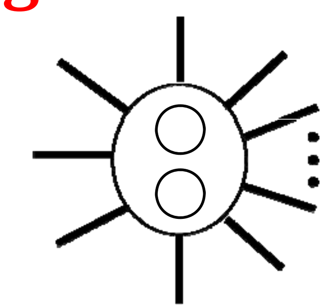


Mandelstam representation
double dispersion relation
Only 2 to 2 processes.

With modern unitarity method we obtain complete amplitudes with *any* number of external legs to *any* loop order starting with tree amplitude.

ZB, Dixon, Dunbar and Kosower

$A(s_1, s_2, s_3, \ldots)$

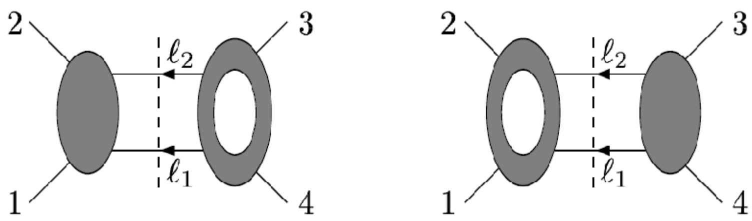


Unitarity method builds
loops from tree ampitudes.

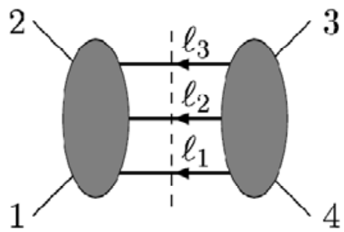
Bern, Dixon, Dunbar and Kosower

Unitarity Method

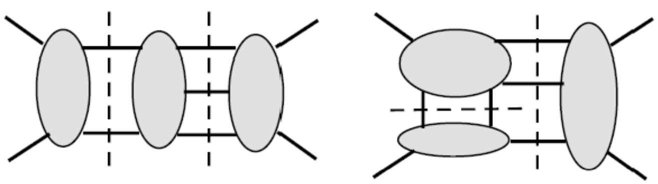
Two-particle cut:



Three- particle cut:



Generalized unitarity as a practical tool:



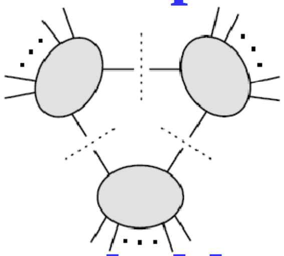
Different cuts merged to give an expression with correct cuts in all channels.

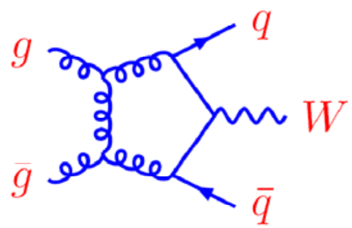
Bern, Dixon and Kosower

Generalized cut interpreted as cut propagators not canceling.

Unitarity Method: Some Developments

- **Generalized cuts – used to produce $pp \to W, Z + 2$ partons**
  Used in MCFM





ZB, Dixon, Kosower (1998).

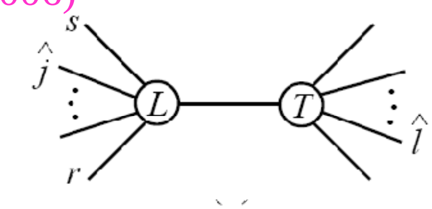
- **Realization of the remarkable power of complex momenta in generalized cuts. Inspiration fromWitten and twistors. Very important.** Britto, Cachazo, Feng (2004); Britto et al series of papers.

- ***D* dimensional unitarity to capture rational pieces of loops.**

  van Neerven(1986); ZB, Morgan (1995); ZB, Dixon, Dunbar, Kosower (1996), Anastasiou, Britto, Feng, Kunszt, Mastrolia (2006)

- **On-shell recursion for loops (based on BCFW)**

  Berger, ZB, Dixon, Forde, Kosower; + Febres Cordero, Ita, Maitre



- **Efficient on-shell reduction of integrals, in a way designed for numerical integration.**

  Ossola, Papadopoulos, Pittau (2006)

- **Efficient on-shell integration using analytic properties.**

  Forde (2007)

Unitarity Method: Programs and Results

- **General analytic formulas for coefficients of integrals.**

  Britto, Cachazo, Feng; Anastasiou, Britto, Feng, Kunszt, Mastrolia; Britto, Feng, Yang **See Britto's talk**

- **LoopTools for general purpose loop integration. Three vector boson production.**

  Ossola, Papadopoulos, Pittau (OPP); Binoth+OPP; Mastrolia + OPP

- **An OPP style evaluation. Gets rational terms from *D*-dimensional cuts.**

  Ellis, Giele and Kunszt; Giele, Kunszt and Melnikov **See Giele's & Zanderighi's talks**

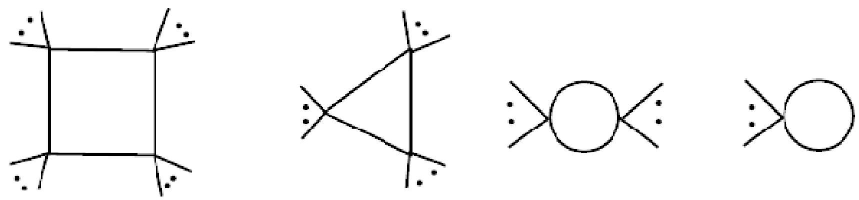
- **BlackHat: Automated evaluation. Demonstration of numerical stability and speed for six-gluon helicities.**

  Berger, ZB, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre **See Forde's talk**

Quadruple Cut Freezes Box Integral

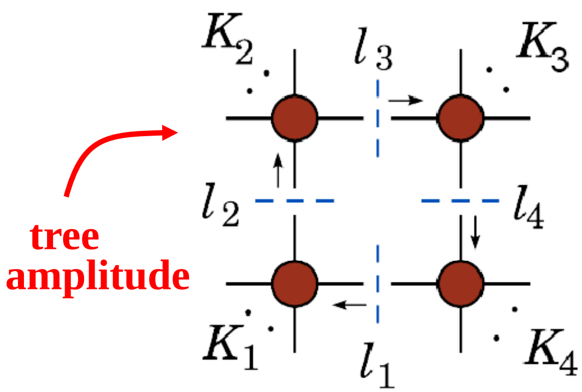
Basis of scalar integrals:

Britto, Cachazo, Feng



Box integral coefficient

$$d_i = \frac{1}{2}\sum_{\sigma=\pm} A^{\text{tree}}_{(1)} A^{\text{tree}}_{(2)} A^{\text{tree}}_{(3)} A^{\text{tree}}_{(4)}\Big|_{l_i=l_i^{(\sigma)}}$$



Solve on-shell conditions $l_i^2 = m_i^2$ and plug 2 solutions into product of tree amplitudes. Coefficient evaluated.

Very neat and very powerful!

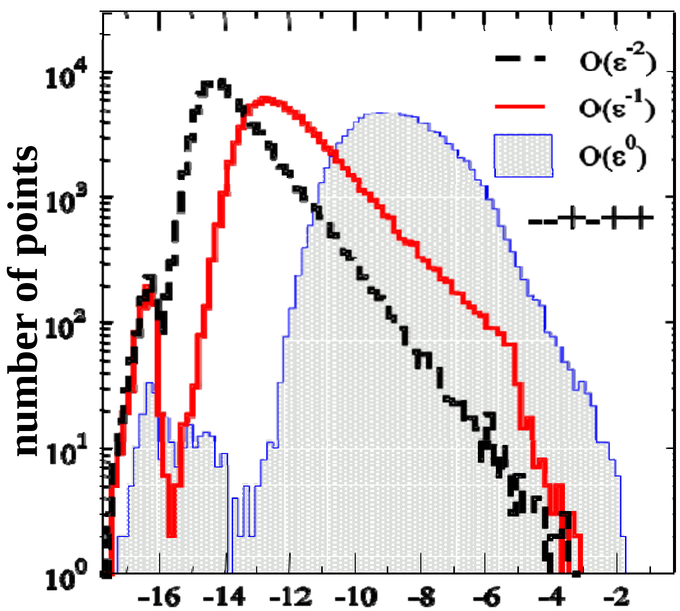
Bubble, triangle and tadpole coefficient can also be solved along these lines.

See talks from Britto, Bernicot, Forde, Giele, Kilgore, Zanderighi

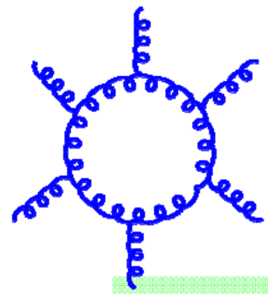
BlackHat: An automated implementation of on-shell methods for one-loop amplitudes

Berger, ZB, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre

six-gluon amplitude



$$\text{Precision} = \log_{10}\left(\frac{|A^{\text{num}} - A^{\text{ref}}|}{|A^{\text{ref}}|}\right)$$



See Forde's talk

Blackhat is stable and pretty fast.

Uses both unitarity method and on-shell recursion

Need to merge with automated phase-space integrators – e.g. Krauss and Gleisberg implementation of Catani-Seymour dipole subtraction.

See Gleisberg's talk

$\mathcal{N}$ = 4 Super-Yang-Mills to All Loops

Since 't Hooft's paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. $\mathcal{N} = 4$ sYM is much more promising.

- Special theory because of AdS/CFT correspondence:

  Strong coupling sYM $\longleftrightarrow$ weak coupling AdS gravity.
- Maximally supersymmetric.
- Simplicity both at strong and weak coupling.

Can we solve planar $\mathcal{N} = 4$ super-Yang-Mills theory?

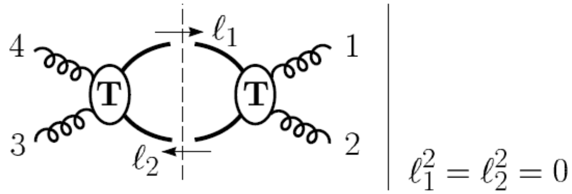
Initial Goal: Resum planar amplitudes to *all* loop orders for all values of the coupling.

Here we will present recent progress on this.

$\mathcal{N}$ = 4 Multi-loop Amplitude

ZB, Dunbar, Dixon, Kosower

Consider one-loop in $\mathcal{N}$ = 4:

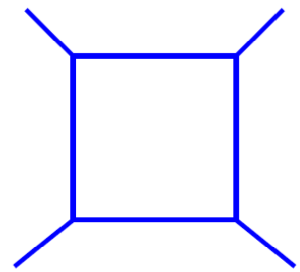


The basic *D*-dimensional two-particle sewing equation

$$\sum_{N=4 \text{ states}} A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \times A_4^{\text{tree}}(-\ell_2, 3, 4, \ell_1) = -\frac{st A_4^{\text{tree}}(1,2,3,4)}{(\ell_1 - k_1)^2 (\ell_2 - k_3)^2}$$

Applying this at one-loop gives

$$\mathcal{A}_4^{\text{1-loop}}(1,2,3,4) = -st A_4^{\text{tree}} \mathcal{I}_4^{\text{1-loop}}(s,t)$$



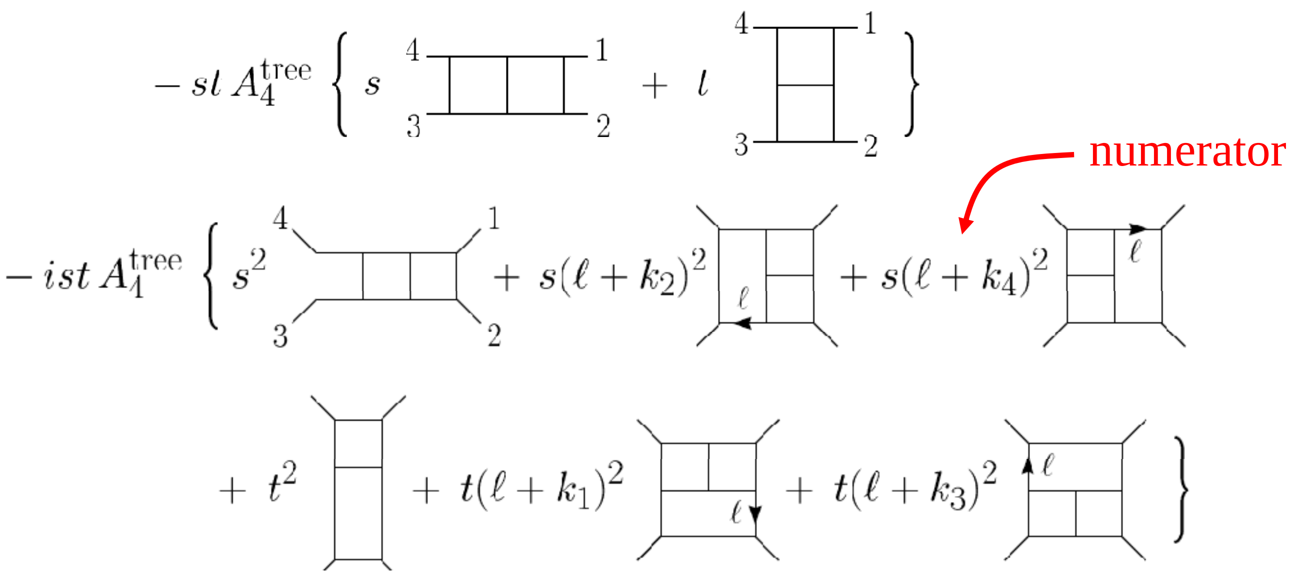
Agrees with known result of Green, Schwarz and Brink.

The two-particle cuts algebra recycles to all loop orders!

Higher-particle cuts of course more difficult.

Two- and Three-Loop Calculations

Combining *all* cuts we get two- and three-loop planar integrand:



ZB, Rozowsky, Yan

At three-loops:

- **Use Mellin-Barnes integration technology.**
- **Get harmonic polylogarithms.**

V. Smirnov

Vermaseren and Remiddi

Loop Iteration of the $\mathcal{N}$=4 Amplitude

The planar four-point two-loop amplitude undergoes fantastic simplification.

Anastasiou, ZB, Dixon, Kosower

$$M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\Big(M_4^{\text{1-loop}}(s,t)\Big)^2 + f(\epsilon) M_4^{\text{1-loop}}(s,t)\Big|_{\epsilon\to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

where

$$M_4^{\text{loop}} = A_4^{\text{loop}}/A_4^{\text{tree}}, \qquad f(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$$

$f(\epsilon)$ is universal function related to IR singularities $\quad D = 4 - 2\epsilon$

This gives two-loop four-point planar amplitude as iteration of one-loop amplitude.

Three loop satisfies similar iteration relation. Rather nontrivial.

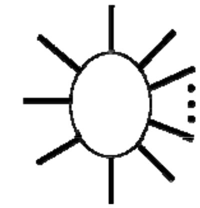
ZB, Dixon, Smirnov

All-Leg All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop and all legs, at least for MHV amplitudes?

constant independent of kinematics.

$$\mathcal{A}_n = A_n^{\rm tree} A_n^{\rm divergent} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$



all-loop resummed amplitude — **IR divergences** — **cusp anomalous dimension** — **finite part of one-loop amplitude**

"BDS conjecture"

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

- **IR singularities agree with Magnea and Sterman formula.**
- **Collinear limits gives us the key analytic information, at least for MHV amplitudes.**

Gives a definite prediction for *all* values of coupling given BES integral equation for the cusp anomalous dimension. Checked to 4 loops (and against string theory). Beisert, Eden, Staudacher

ZB, Czakon, Dixon, Kosower, Smirnov

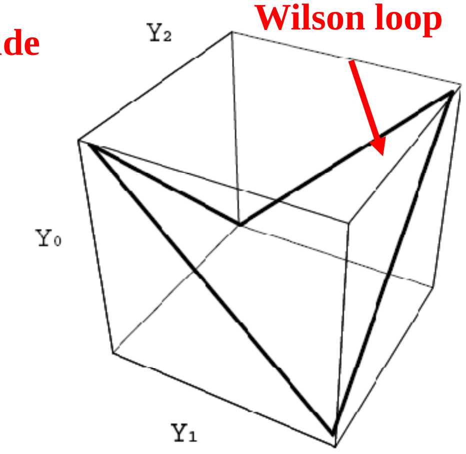
Alday and Maldacena Strong Coupling

For MHV amplitudes:

ZB, Dixon, Smirnov

$$\mathcal{A}_4 = A_4^{\text{tree}} A_4^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_4^{\text{1-loop}} + C\right]$$

- $\mathcal{A}_4$: all-loop resummed amplitude
- $A_4^{\text{divergent}}$: IR divergences
- γ_K: cusp anomalous dimension
- $F_4^{\text{1-loop}}$: finite part of one-loop amplitude
- C: constant independent of kinematics.



In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at strong coupling from an AdS string computation. Minimal surface calculation.

$$e^{S_{\text{cl}}}$$

Minimize Nambu-Goto string action

Corners separated by $2\pi k_i$

Minimal surface that ends on curve

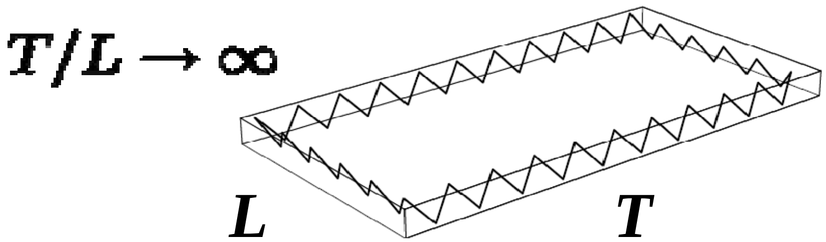
Very suggestive link to Wilson loops even at weak coupling.

Drummond, Korchemsky, Sokatchev ; Brandhuber, Heslop, and Travaglini

Trouble at Higher Points

For various technical reasons it is hard to solve for minimal surface for large number of gluons.

In a recent paper, Alday and Maldacena realized certain terms can be calculated at strong coupling for an infinite number of gluons

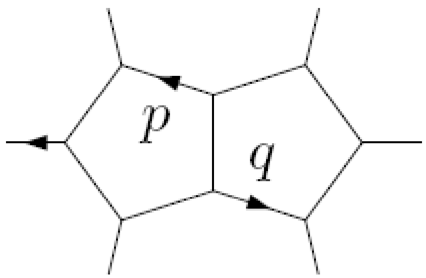


Disagrees with BDS conjecture

MB to evaluate integrals

Trouble also in the Regge limit.

Bartels, Lipatov, Sabio Vera



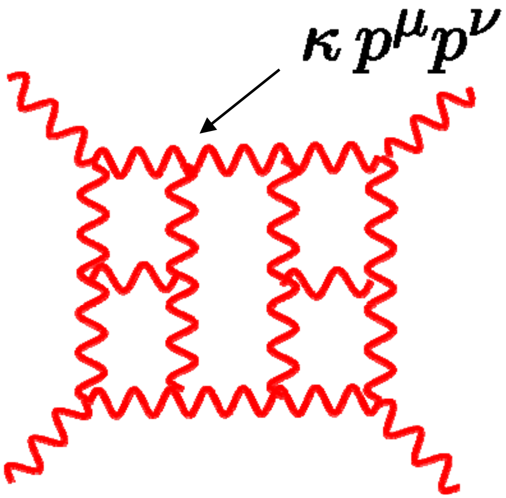
Explicit computation at 2-loop six points.
Need to modify conjecture!

ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich
Drummond, Henn, Korchemsky, Sokatchev

Can the BDS conjecture be repaired for six and higher points? We don't know, but stay tuned!

Is a UV finite theory of gravity possible?

$\kappa = \sqrt{32\pi G_N}$ ← Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

Reasons to focus on $N = 8$ supergravity:

- With more susy suspect better UV properties.
- High symmetry implies technical simplicity.

Cremmer and Julia

UV Finiteness of point-like gravity?

- **We are interested in UV finiteness because it would imply a new symmetry or non-trivial dynamical mechanism. The discovery of either would have a fundamental impact on our understanding of gravity.**

- **Non-perturbative issues and viable models of Nature are *not* the goal for now. Understanding the mechanism of cancellation is the only current goal.**

Opinions from the 80's

If certain patterns that emerge should persist in the higher orders of perturbation theory, then … $N = 8$ supergravity in four dimensions would have ultraviolet divergences starting at three loops.

Green, Schwarz, Brink, (1982)

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

The idea that *all* supergravity theories diverge at 3 loops has been widely accepted for over 20 years

There are a number of very good reasons to reanalyze this.

Non-trivial one-loop cancellations: no triangle & bubble integrals

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Dunbar , Ita, Perkins, Risager

Unitarity method implies higher loop cancellations.

ZB, Dixon, Roiban

Gravity Feynman Rules

Propagator in de Donder gauge:

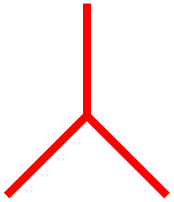
$$\mathcal{L}_{\text{gravity}} = \sqrt{g}\, R$$

$$P_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\left[\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\alpha}\eta_{\nu\beta}\right]\frac{i}{k^2 + i\epsilon}$$

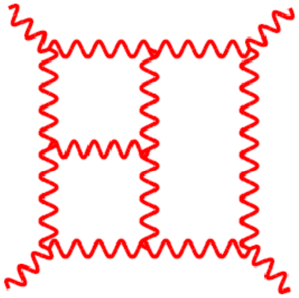
Three vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) =$$

$$\begin{aligned}
&\text{sym}[\,-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\quad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\quad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\quad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



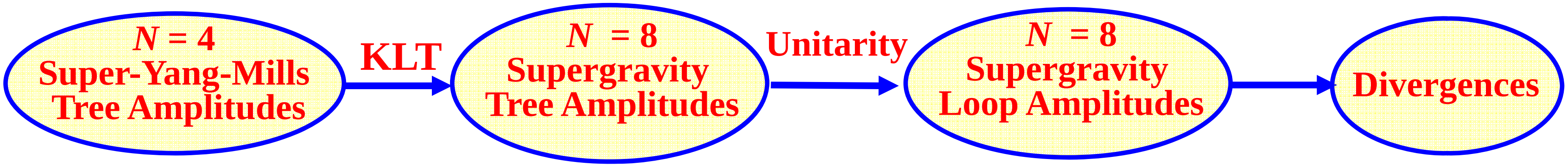
It's a mess!



If we attack this directly get 10^{20} terms in diagram. There is a reason why this hasn't been evaluated.

Basic Strategy

ZB, Dixon, Dunbar, Perelstein and Rozowsky (1998)



- Kawai-Lewellen-Tye relations: sum of products of gauge theory tree amplitudes gives gravity tree amplitudes.
- Unitarity method: efficient formalism for perturbatively quantizing gauge and gravity theories. Loop amplitudes from tree amplitudes. ZB, Dixon, Dunbar, Kosower (1994)

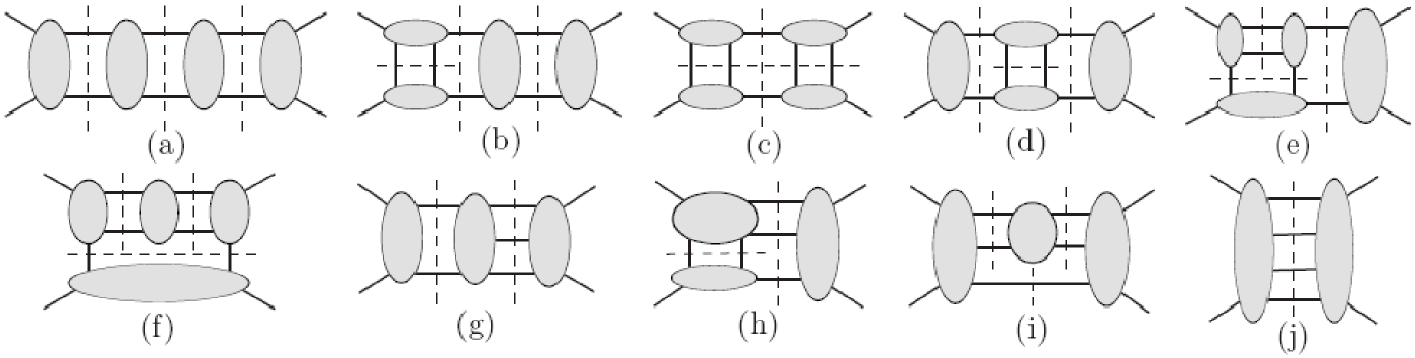
Key features of this approach:

- **Gravity calculations mapped into much simpler gauge theory calculations.**
- **Only on-shell states appear.**

Full Three-Loop Calculation

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Need following cuts:



reduces everything to product of tree amplitudes

For cut (g) have:

$$\sum_{N=8\text{ states}} M_4^{\text{tree}}(1,2,l_3,l_1)\times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use KLT

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12}A_4^{\text{tree}}(1,2,l_3,l_1)A_4^{\text{tree}}(2,1,l_3,l_1)$$

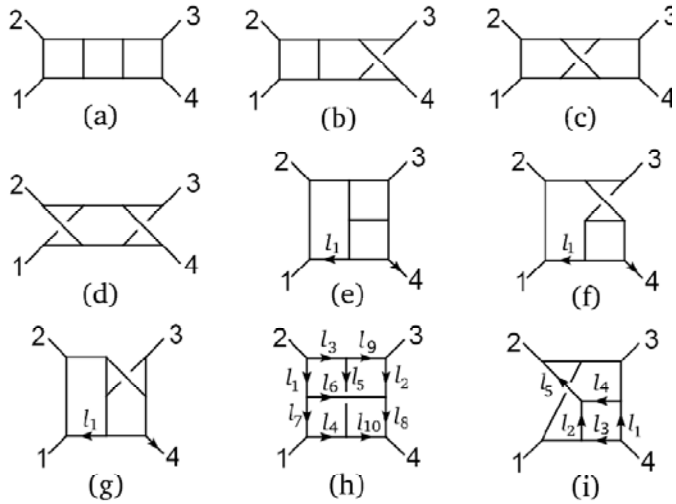
$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\, s_{l_1q_1}s_{l_3q_3}A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\,A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1\leftrightarrow l_3\}\,,$$

supergravity — super-Yang-Mills

N = 8 supergravity cuts are sums of products of N = 4 super-Yang-Mills cuts

Complete three loop result

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112



$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\rm tree} \sum_{S_3} \Big[I^{(a)} + I^{(b)} + \tfrac{1}{2} I^{(c)} + \tfrac{1}{4} I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \tfrac{1}{2} I^{(h)} + 2I^{(i)} \Big]$$

$$l_{i,j}^2 = (l_i + l_j)^2$$

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills	$\mathcal{N}=8$ Supergravity
(a)–(d)	s^2	$[s^2]^2$
(e)–(g)	$s(l_1 - k_4)^2$	$[s\,(l_1+k_4)^2]^2$
(h)	$sl_{1,2}^2 + tl_{3,4}^2 - sl_5^2 - tl_6^2 - st$	$(sl_{1,2}^2 + tl_{3,4}^2 - st)^2 - s^2(2(l_{1,2}^2 - t) + l_5^2)l_5^2 - t^2(2(l_{3,4}^2 - s) + l_6^2)l_6^2$ $- s^2(2l_1^2 l_8^2 + 2l_2^2 l_7^2 + l_1^2 l_7^2 + l_2^2 l_8^2) - t^2(2l_3^2 l_{10}^2 + 2l_4^2 l_9^2 + l_3^2 l_9^2 + l_4^2 l_{10}^2) + 2st l_5^2 l_6^2$
(i)	$sl_{1,2}^2 - tl_{3,4}^2 - \frac{1}{3}(s-t)l_5^2$	$(sl_{1,2}^2 - tl_{3,4}^2)^2 - (s^2 l_{1,2}^2 + t^2 l_{3,4}^2 + \frac{1}{3}stu)l_5^2$

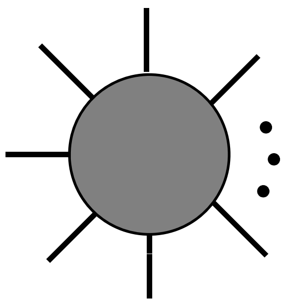
The leading UV behavior cancels!

“Superfinite” --- no divergence in $D < 6$

Origin of Cancellations?

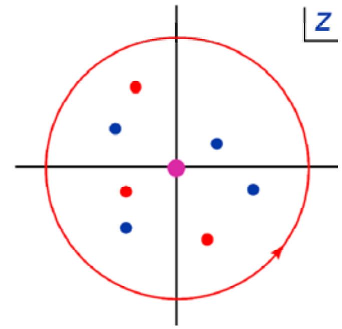
There does not appear to be a supersymmetry explanation for observed cancellations.

If it is *not* supersymmetry what might it be?



$$k_1^\mu \to k_1^\mu + \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle \qquad A^{\rm tree}(z) \to 0$$

$$k_2^\mu \to k_2^\mu - \frac{z}{2}\langle k_1^-|\gamma^\mu|k_2^-\rangle, \qquad z \to \infty$$



This property useful for constructing BCFW recursion relations for gravity .

Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Benincasa, Boucher-Veronneau , Cachazo; Arkani-Hamed, Kaplan; Hall

This same property appears to be directly related to the novel non-supersymmetric cancellations observed in the loops.

ZB, Carrasco, Forde, Ita, Johansson

Can we prove finiteness of *N* = 8 supergravity?

Time will tell…

Summary

On-shell methods offer a gauge invariant way to compute, simplifying complicated multi-particle or multi-loop calculations.

- **QCD: On shell methods are mature enough to attack 6-point problems of interest at the LHC**

  See talks of Britto, Bernicot, Forde, Giele, and Kilgore, Zanderighi

- ***N*=4 *S*upersymmetric gauge theory: New venue opened for studying Maldacena's AdS/CFT conjecture. Resummations to *all* loop orders. Can we solve planar theory? Looks good for 4, 5 point scattering. Obstacle at 6 points.**

- **Quantum gravity: Is a point-like perturbatively UV finite quantum gravity theory possible? New unitarity method calculations are challenging the conventional wisdom.**