On the evaluation of one-loop amplitudes: *the gluon case*

Giulia Zanderighi
Oxford Theoretical Physics & STFC
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work done in collaboration with W. Giele ⇒ see his talk!

References:

- *Ellis, Giele, Kunszt 0708.2398 ⇒ D=4, cut-constructable part*
- *Giele, Kunszt, Melnichov 0801.2237 ⇒ arbitrary integer D, full amplitude*
- *Giele & GZ 0805.2152 ⇒ algorithm of polynomial complexity, many new results*
- *references therein (Ossola et al.; Britto et al.; Bern et al. ...)*

Starting point

I assume that

- we know what we expect form the LHC
- we believe that NLO calculations might be crucial
- we know the bottleneck at NLO are virtual corrections
- we agree that the common aim in NLO calculations is to be able to do N-leg one-loop calculations for a general process ("N" is the key) $\Rightarrow$ e.g. Alpgen@NLO

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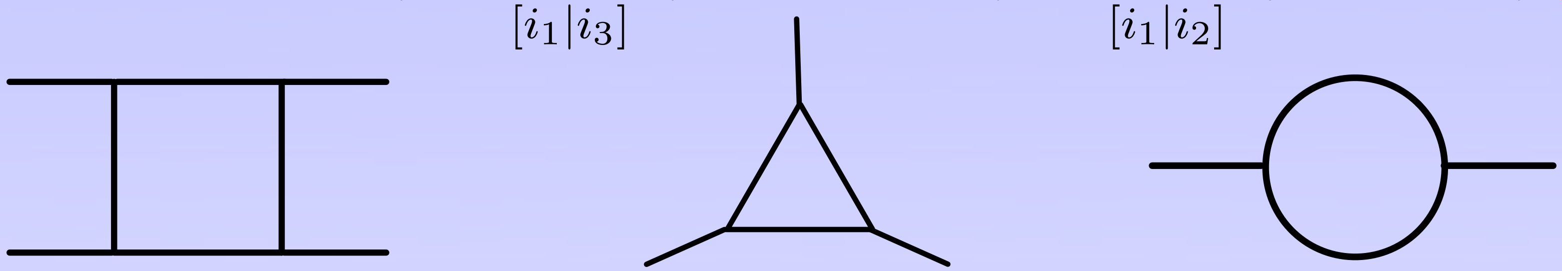
This work: merge analytical & numerical techniques, build a general, fully automated algorithm of polynomial complexity for the evaluation of one-loop amplitudes

One-loop virtual amplitudes

One-loop amplitudes can be decomposed into a cut-constructable part (coefficients times scalar master integrals) + rational terms

$$\mathcal{A}_N^{*} = \sum_{[i_1|i_4]} \left(d_{i_1i_2i_3i_4}\; I^{(D)}_{i_1i_2i_3i_4}\right) + \sum_{[i_1|i_3]} \left(c_{i_1i_2i_3}\; I^{(D)}_{i_1i_2i_3}\right) + \sum_{[i_1|i_2]} \left(b_{i_1i_2}\; I^{(D)}_{i_1i_2}\right) + \mathcal{R}$$



Get cut constructable part by taking residues: in D=4 up to 4 constraints on the loop momentum (4 onshell propagators) ⇒ get up to box integrals

Want rational part: need to think about $D \neq 4$

* if non-vanishing masses: tadpole term

Generic D dependence

Two sources of D dependence

dimensionality of loop momentum D

nr. of spin eigenstates/ polarization states D_s ($\geq D$)

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Keep D and D_s distinct

$$\mathcal{A}^D \Rightarrow \mathcal{A}^{(D,D_s)}$$

Two key observations

1. External particles in D=4 ⇒ no preferred direction in the extra space

$$\mathcal{N}(l) = \mathcal{N}(l_4, \tilde{l}^2) \qquad \tilde{l}^2 = -\sum_{i=5}^{D} l_i^2$$

☛ in arbitrary D up to 5 constraints ⇒ get up to pentagon integrals

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2. Dependence of $\mathcal{N}$ on D_s is linear
(appears from closed loops of contracted metrics)

$$\mathcal{N}^{D_s}(l) = \mathcal{N}_0(l) + (D_s - 4)\mathcal{N}_1(l)$$

☛ evaluate at any (integer!) D_{s1}, D_{s2} ⇒ get $\mathcal{N}_0$ and $\mathcal{N}_1$, i.e. full $\mathcal{N}$

[$D_s = 4 - 2\,\varepsilon$ ‘t-Hooft-Veltman scheme, $D_s = 4$ FDH scheme]

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D_{s1}, D_{s2} independent of ε ⇒ suitable for numerical implementation

Practically: pentagon cuts

$$\frac{\mathcal{N}^{(D_s)}(l)}{d_1 d_2 \cdots d_N} = \sum_{[i_1|i_5]} \frac{\overline{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\overline{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\overline{c}^{(D_s)}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\overline{b}^{(D_s)}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\overline{a}^{(D_s)}_{i_1}(l)}{d_{i_1}}$$

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Pentagon residue:

$$\overline{e}^{(D_s)}_{ijkmn}(l_{ijkmn}) = \mathrm{Res}_{ijkmn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1 \cdots d_N}\right) \quad \Leftrightarrow \quad d_i(l_{ijkmn}) = \cdots = d_n(l_{ijkmn}) = 0$$

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Solution: (trivial algebra)

$$l^\mu_{ijkmn} = V_5^\mu + \sqrt{\frac{-V_5^2 + m_n^2}{\alpha_5^2 + \cdots + \alpha_D^2}} \left(\sum_{h=5}^{D} \alpha_h n_h^\mu\right) \qquad \forall \alpha_i$$

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$$\times \mathcal{M}(l_k; p_{k+1}, \ldots, p_m; -l_m) \times \mathcal{M}(l_m; p_{m+1}, \ldots, p_n; -l_n) \times \mathcal{M}(l_n; p_{n+1} \ldots, p_i; -l_i)$$

V_5: function of the 4 inflow momenta

n_i: span trivial space, ⊥ to physical one

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Finally: $\bar{e}^{D_s}_{ijkmn}(l) = \bar{e}^{D_s}_{ijkmn}(l_{ijlmn}) \equiv \bar{e}^{Ds,(0)}_{ijkmn}$

(because $\bar{e}^{D_s}_{ijkmn}(l)$ depend only on even powers of $s_e \equiv -\sum_{i=5}^{D} (l \cdot n_i)^2$)

Practically: boxes cuts

Box residue:

$$\bar{d}_{ijkn}^{(D_s)}(l) = \mathrm{Res}_{ijkn}\left(\frac{\mathcal{N}^{(D_s)}(l)}{d_1\cdots d_N} - \sum_{[i_1|i_5]}\frac{e_{i_1i_2i_3i_4i_5}^{(D_s,(0))}}{d_{i_1}d_{i_2}d_{i_3}d_{i_4}d_{i_5}}\right) \quad \Leftrightarrow \quad d_i(l_{ijkm}) = \cdots = d_n(l_{ijkm}) = 0$$

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V_4: function of the 3 inflow momenta

n_i: span trivial space, orthogonal to physical one

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Most general parameterization of quadrupole cut:

$$\overline{d}_{ijkn}(l) = d^{(0)}_{ijkn} + d^{(1)}_{ijkn}s_1 + (d^{(2)}_{ijkn} + d^{(3)}_{ijkn}s_1)s_e^2 + d^{(4)}_{ijkn}s_e^4 \qquad s_1 = l\cdot n_1$$

➥ make 5 choices of α_i and solve for the 5 coefficients!

Triangles and bubbles

Triangle and bubble residue:

- ✓ follow exactly the same procedure with appropriate changes in the dimensions
- ✓ get infinite solutions of the unitarity constraints and solve in both cases for 10 coefficients
- ✓ box and pentagon coefficients feed back in the form of subtraction terms

Putting it all together

$$\frac{\mathcal{N}^{(D_s)}(l)}{d_1 d_2 \cdots d_N} = \sum_{[i_1|i_5]} \frac{\overline{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\overline{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\overline{c}^{(D_s)}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\overline{b}^{(D_s)}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\overline{a}^{(D_s)}_{i_1}(l)}{d_{i_1}}$$

Need to combine the two evaluations:

$$\mathcal{A}^{\mathrm{FDH}} = \left(\frac{D_2 - 4}{D_2 - D_1}\right) \mathcal{A}_{(D, D_s = D_1)} - \left(\frac{D_1 - 4}{D_2 - D_1}\right) \mathcal{A}_{(D, D_s = D_2)}$$

Need to evaluate loop integration, use:

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{D-4}{2} I^{D+2}_{i_1 i_2 i_3 i_4} \to 0$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I^{D+4}_{i_1 i_2 i_3 i_4} \to -\frac{1}{6}$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = \frac{(D-4)}{2} I^{D+2}_{i_1 i_2 i_3} \to \frac{1}{2}$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = \frac{(D-4)}{2} I^{D+2}_{i_1 i_2} \to \frac{m_{i1}^2 + m_{i2}^2}{2} - \frac{1}{6}\left(q_{i1}^2 - q_{i2}^2\right)^2$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_i}{d_{i_1} \cdots d_{i_N}} = 0$$

Final result

Full one-loop amplitude:

$$\mathcal{A}_{(D)} = \sum_{[i_1|i_5]} e^{(0)}_{i_1 i_2 i_3 i_4 i_5} \; I^{(D)}_{i_1 i_2 i_3 i_4 i_5}$$
$$+ \sum_{[i_1|i_4]} \left(d^{(0)}_{i_1 i_2 i_3 i_4} \; I^{(D)}_{i_1 i_2 i_3 i_4} - \frac{D-4}{2} d^{(2)}_{i_1 i_2 i_3 i_4} \; I^{(D+2)}_{i_1 i_2 i_3 i_4} + \frac{(D-4)(D-2)}{4} d^{(4)}_{i_1 i_2 i_3 i_4} \; I^{(D+4)}_{i_1 i_2 i_3 i_4} \right)$$
$$+ \sum_{[i_1|i_3]} \left(c^{(0)}_{i_1 i_2 i_3} \; I^{(D)}_{i_1 i_2 i_3} - \frac{D-4}{2} c^{(9)}_{i_1 i_2 i_3} \; I^{(D+2)}_{i_1 i_2 i_3} \right) + \sum_{[i_1|i_2]} \left(b^{(0)}_{i_1 i_2} \; I^{(D)}_{i_1 i_2} - \frac{D-4}{2} b^{(9)}_{i_1 i_2} \; I^{(D+2)}_{i_1 i_2} \right)$$

"Cut-constructable"

$$\mathcal{A}^{CC}_N = \sum_{[i_1|i_4]} d^{(0)}_{i_1 i_2 i_3 i_4} \; I^{(4-2\epsilon)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c^{(0)}_{i_1 i_2 i_3} \; I^{(4-2\epsilon)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b^{(0)}_{i_1 i_2} \; I^{(4-2\epsilon)}_{i_1 i_2}$$

Rational part:

$$R_N = - \sum_{[i_1|i_4]} \frac{d^{(4)}_{i_1 i_2 i_3 i_4}}{6} + \sum_{[i_1|i_3]} \frac{c^{(9)}_{i_1 i_2 i_3}}{2} - \sum_{[i_1|i_2]} \left(\frac{(q_{i_1} - q_{i_2})^2}{6} - \frac{m^2_{i_1} + m^2_{i_2}}{2} \right) \; b^{(9)}_{i_1 i_2}$$

Vanishing contributions: $\mathcal{A} = \mathcal{O}(\epsilon)$

Basis integrals: QCDloop

$\Rightarrow$ *see talk of K. Ellis*

Rocket

Rocket: an F90 package which fully automates the calculation of virtual amplitudes via tree level recursion + D-unitarity

Apollo 15 was the ninth manned mission in the Apollo program and the fourth mission to land on the Moon. It was the first of what were termed "J missions", long duration stays on the Moon with a greater focus on science than had been possible on previous missions. [Wikipedia]

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Output: (un)-renormalized virtual amplitude in FDH or t'HV scheme

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Computer automated one-loop

Issues:

- checks of the results
- numerical stabilities at special points (thresholds/coplanarities): is there a problem? how severe? how can one deal with it?
- numerical efficiency: how fast is the algorithm? how does time scale with N (for large N)?

Checks on the results

- pole structure

$$A_{\rm v} = c_\Gamma \left(\frac{N}{\epsilon^2} + \frac{1}{\epsilon} \left(\sum_{i=1}^{N} \ln \frac{-s_{i,i+1}}{\mu^2} - \frac{11}{3} \right) \right) A_{\rm tree}$$

NB: single pole checks coefficients of two-point functions, which because of subtraction terms are sensitive to higher-point coefficients as well

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- $\exists$ infinite solutions of the unitarity constraints, results independent of the specific choice

- results are independent of all auxiliary vectors used to construct both the orthonormal basis and the polarization vectors (gauge inv.)

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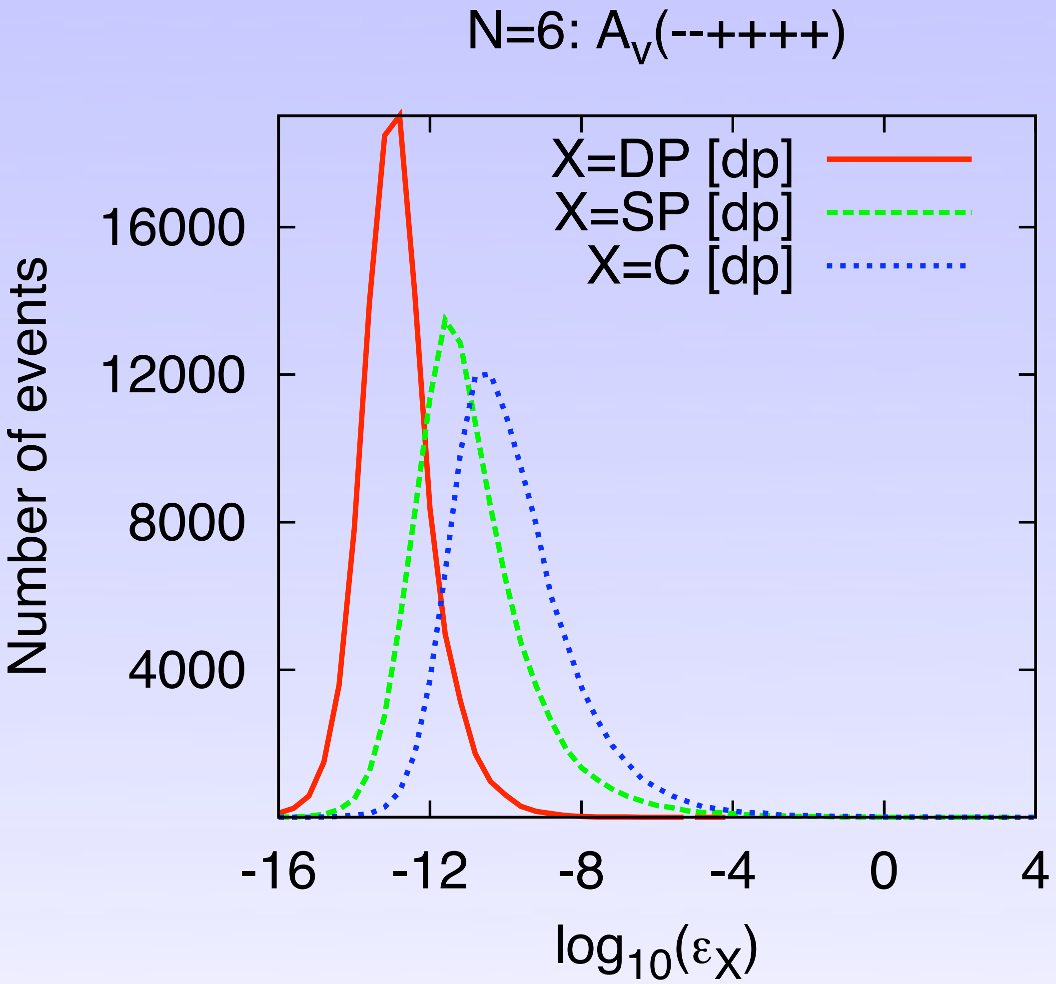
- results independent of the dimensionality run with any D_s=D_{s1},D_{s2}, where D_{s1}, D_{s2} are any integers larger than 4
- checks with some known analytical results (all N=6, finite and MHV amplitudes for larger N)

Study of the accuracy

Define:

$$\varepsilon_C = \log_{10} \frac{|A_N^{\mathrm{v,unit}} - A_N^{\mathrm{v,anly}}|}{|A_N^{\mathrm{v,anly}}|}$$

similar for ε_{DP} and ε_{SP}



- peak position of:
  - double pole: $10^{-12.8}$
  - single pole: $10^{-11.6}$
  - constant: $10^{-10.8}$

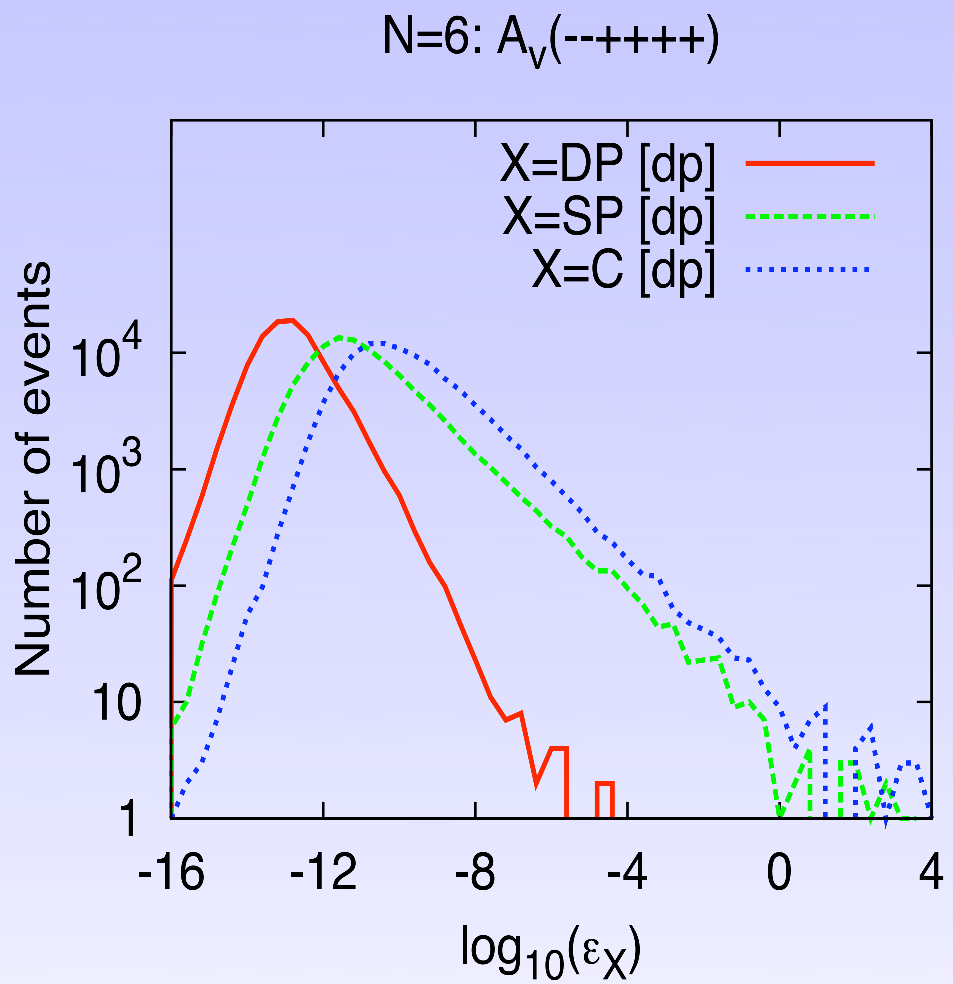
based on 10^5 flat phase space points with minimal cuts

Study of the accuracy

Define:

$$\varepsilon_C = \log_{10} \frac{|A_N^{\rm v,unit} - A_N^{\rm v,anly}|}{|A_N^{\rm v,anly}|}$$

similar for ε_{DP} and ε_{SP}



- peak position of:
  - double pole: $10^{-12.8}$
  - single pole: $10^{-11.6}$
  - constant: $10^{-10.8}$
- single pole and constant part, little tail at high ε $\Rightarrow$ well known exceptional configuration issue

based on 10^5 flat phase space points with minimal cuts

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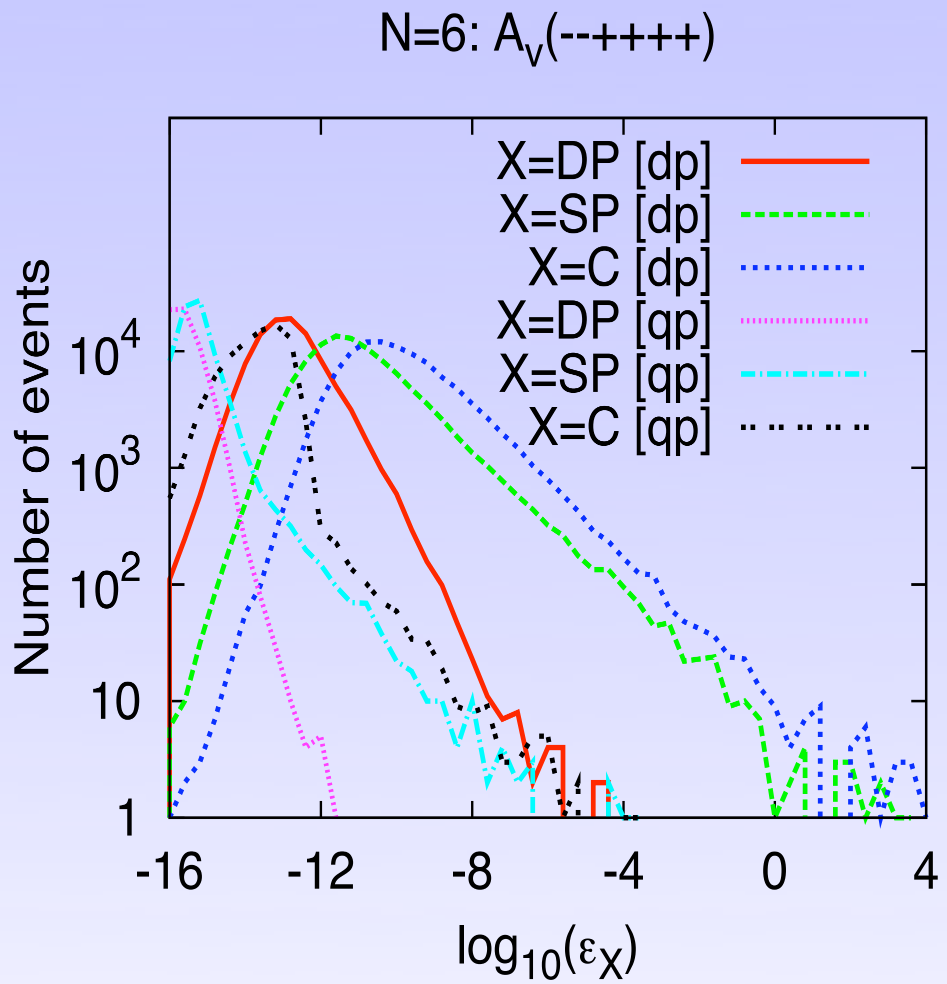

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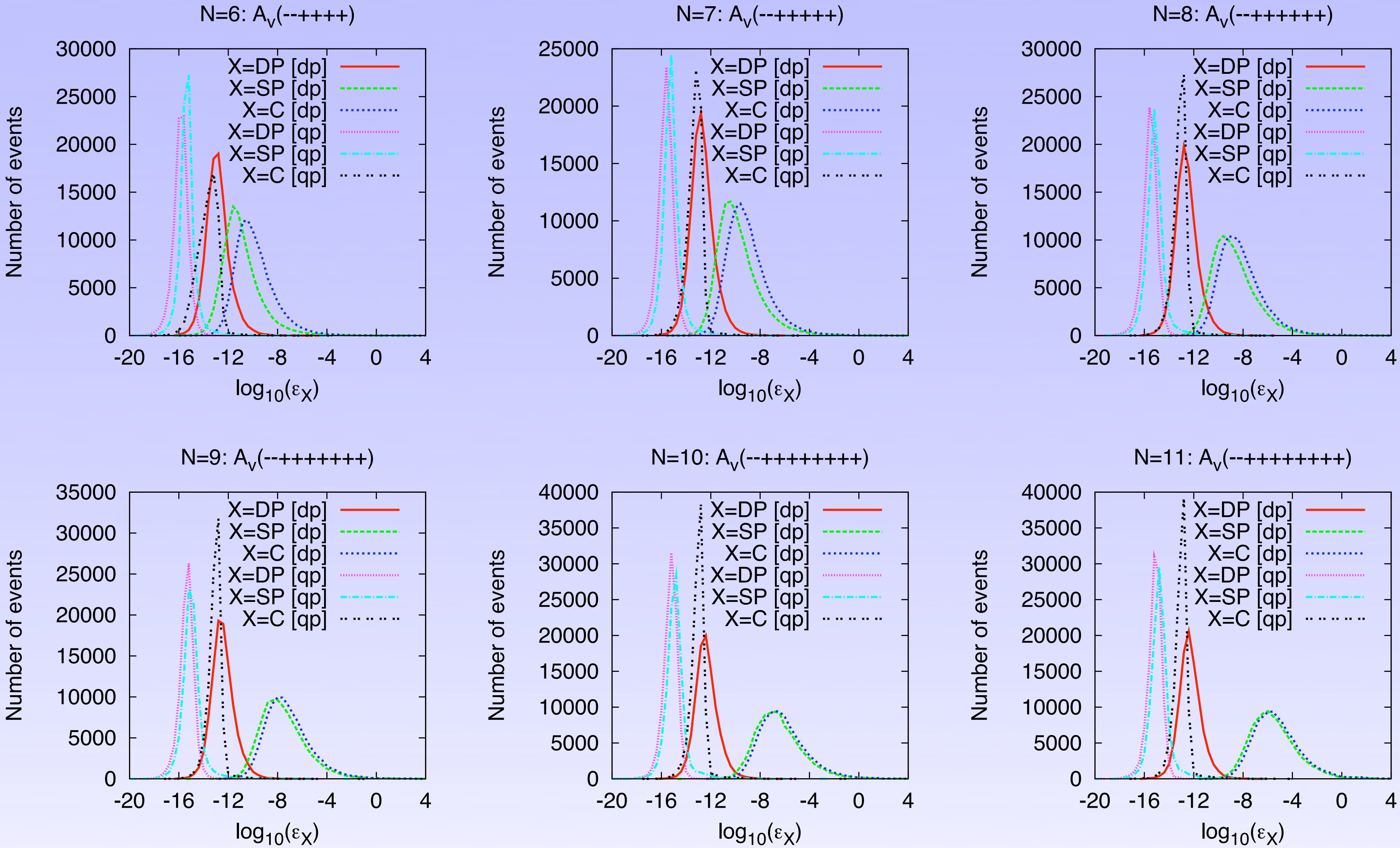
- single pole and constant part, little tail at high ε $\Rightarrow$ well known exceptional configuration issue
- switching to quadrupole precision kills the problem

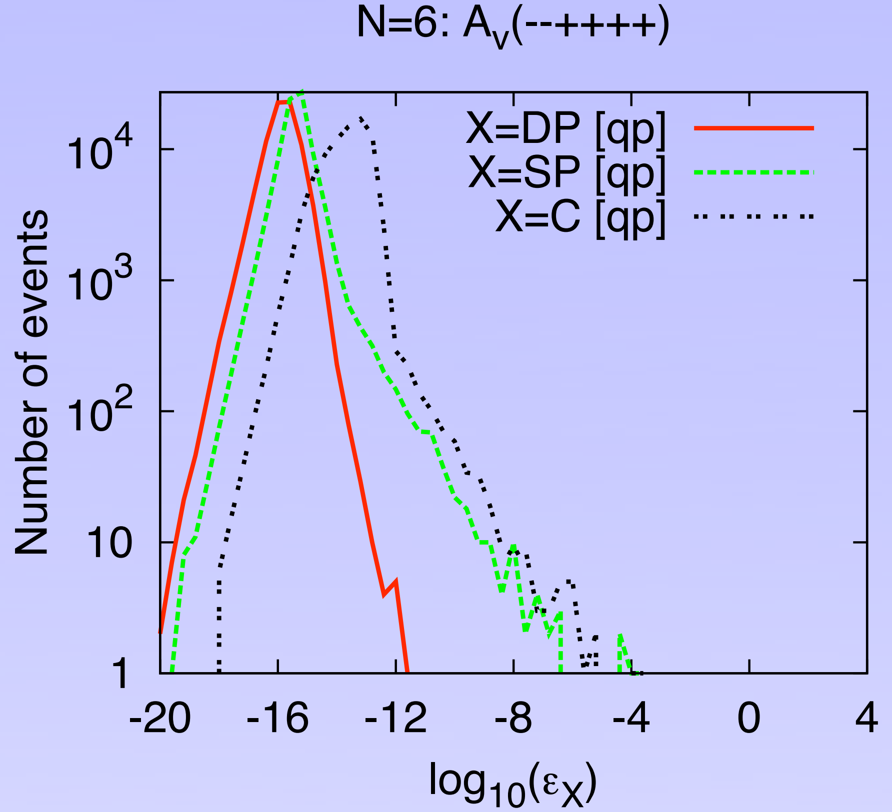
based on 10^5 flat phase space points with minimal cuts

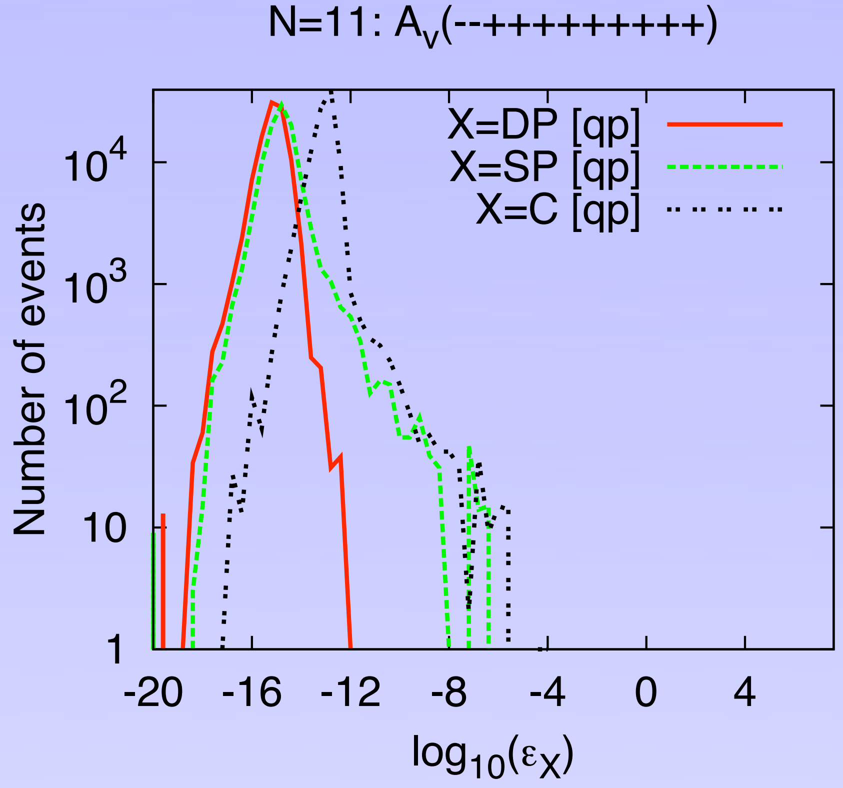
Study of the accuracy with increasing N



☛ accuracy gets worse with increasing N, but only very slowly

N=6 vs N=11





- peak position of:
  - double pole: $10^{-15.6}$
  - single pole: $10^{-15.2}$
  - constant: $10^{-13.2}$

- peak position of:
  - double pole: $10^{-15.2}$
  - single pole: $10^{-14.8}$
  - constant: $10^{-12.8}$

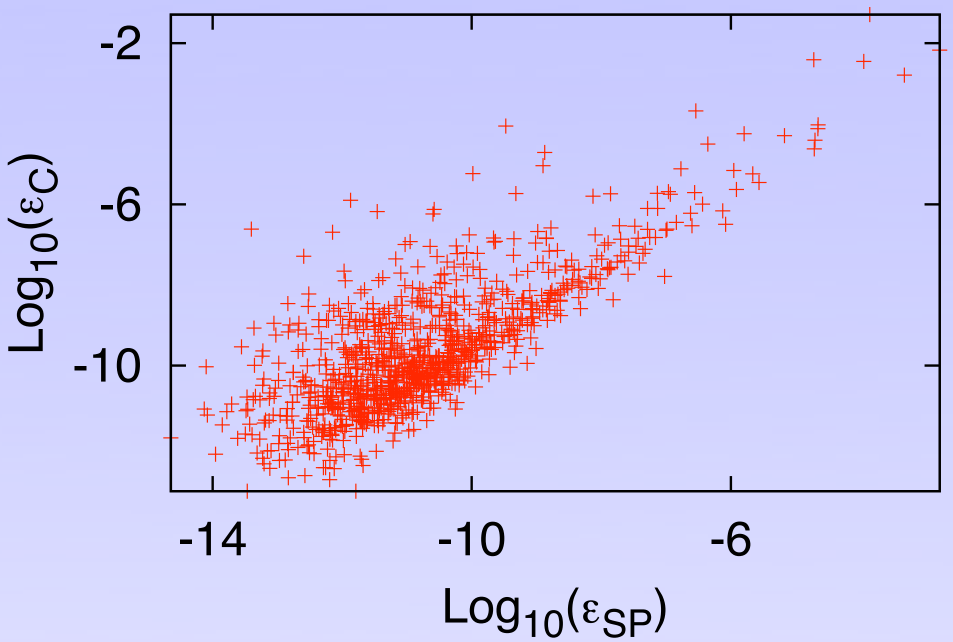
☛ out of 10^5 not a single event has accuracy worse than 10^{-4}

☛ up to N=11 (probably more) QP more than enough

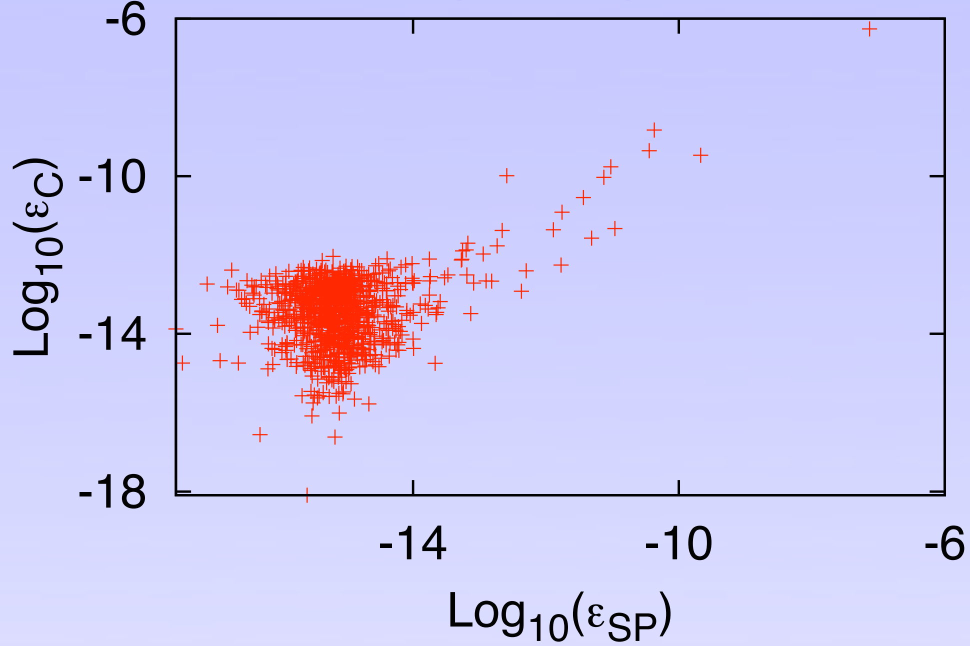
Hunting potential instabilities

Case study: N=6 MHV amplitudes

Double precision



Quadrupole precision



High correlation between accuracy of single pole and constant part
⇒ exploit it do decide which points need to be run in QP

Alternative criterion (or additional one?): run in QP whenever there is a small denominator

Time dependence of the algorithm

Constructive implementation of tree-level amplitudes (or recursive with memory)

$$\tau_{\text{tree}} = \binom{N}{3} E_3 + \binom{N}{4} E_4 \propto N^4$$

E_3 (E_4) → time for the evaluation of a 3 (4) gluon vertex

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Number of tree level amplitudes needed at one-loop

$$n_{\rm tree} = \left\{(D_{s1}-2)^2 + (D_{s2}-2)^2\right\} \times \left(5\, c_{5,\rm max}\binom{N}{5} + 4\, c_{4,\rm max}\binom{N}{4} + 3\, c_{3,\rm max}\binom{N}{3} + 2\, c_{2,\rm max}\left[\binom{N}{2} - N\right]\right)$$

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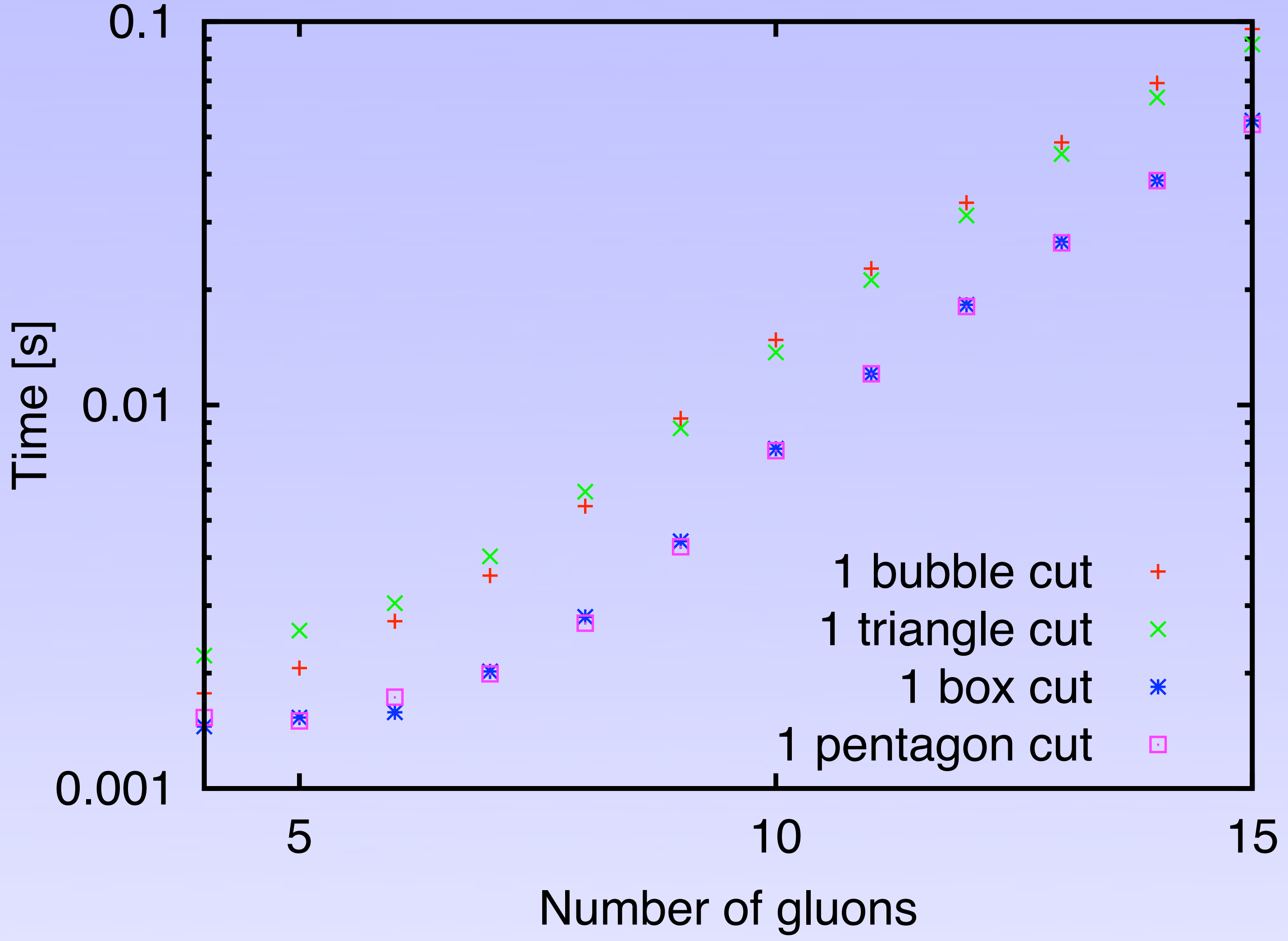
☛ $\tau_{\text{one-loop,N}} \sim n_{\text{tree}} \cdot \tau_{\text{tree,N}} \propto N^9$

[to be compared with factorial growth!]

Time dependence of the algorithm



☛ time for each cut ∝ N^4 (with different coefficients)

Time dependence: recall tree level status

[Duhr et al.'06]

Final State	BG		BCF		CSW	
	CO	CD	CO	CD	CO	CD
$2g$	0.24	0.28	0.28	0.33	0.31	0.26
$3g$	0.45	0.48	0.42	0.51	0.57	0.55
$4g$	1.20	1.04	0.84	1.32	1.63	1.75
$5g$	3.78	2.69	2.59	7.26	5.95	5.96
$6g$	14.2	7.19	11.9	59.1	27.8	30.6
$7g$	58.5	23.7	73.6	646	146	195
$8g$	276	82.1	597	8690	919	1890
$9g$	1450	270	5900	127000	6310	29700
$10g$	7960	864	64000	-	48900	-

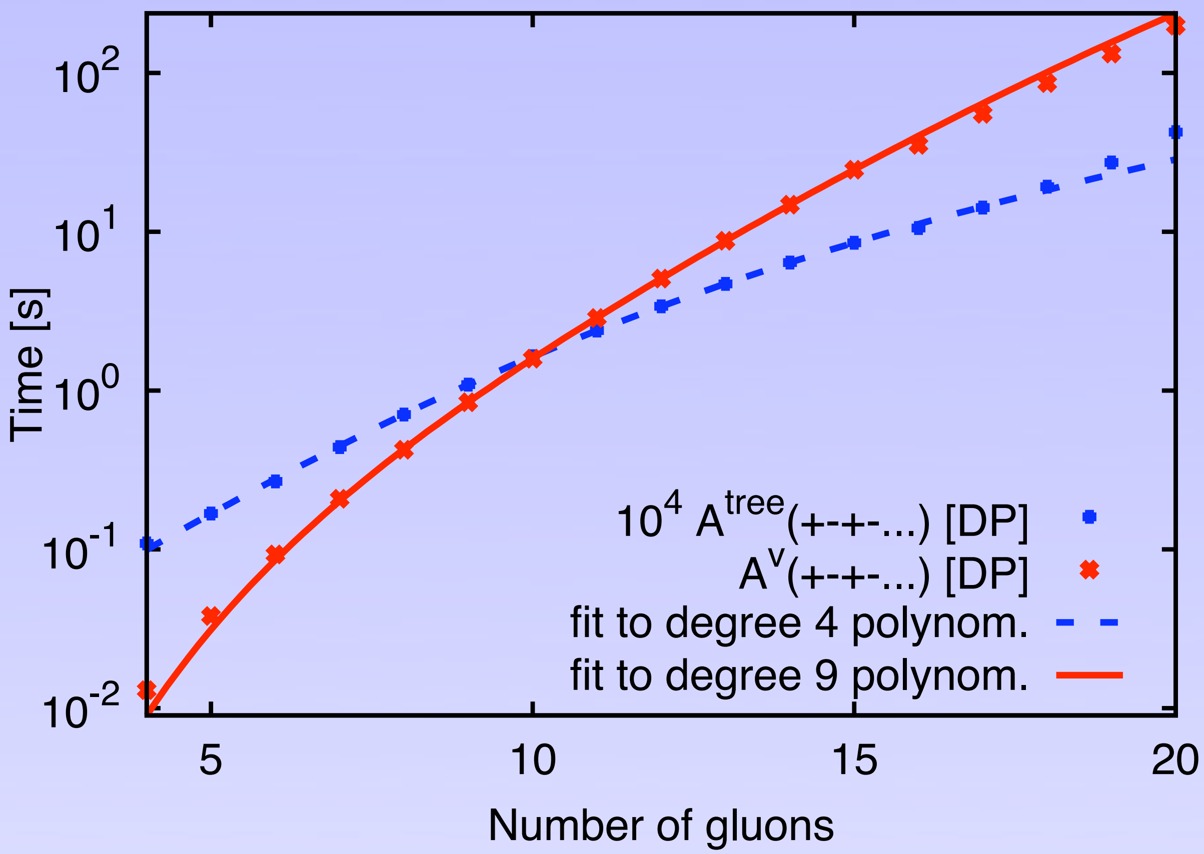
Tab. 3: Computation time (s) of the $2 \to n$ gluon amplitudes for 10^4 phase space points, sampled over helicity and color. Results are given for the color-ordered (CO) and the color-dressed (CD) Berends-Giele (BG), Britto-Cachazo-Feng (BCF) and Cachazo-Svrček-Witten (CSW) relations. Numbers were generated on a 2.66 GHz Xeon™ CPU.

n	4	5	6	7	8	9	10	11	12
Berends-Giele	0.00005	0.00023	0.0009	0.003	0.011	0.030	0.09	0.27	0.7
Scalar	0.00008	0.00046	0.0018	0.006	0.019	0.057	0.16	0.4	1
MHV	0.00001	0.00040	0.0042	0.033	0.24	1.77	13	81	—
BCF	0.00001	0.00007	0.0003	0.001	0.006	0.037	0.19	0.97	5.5

[Dinsdale et al.'06]

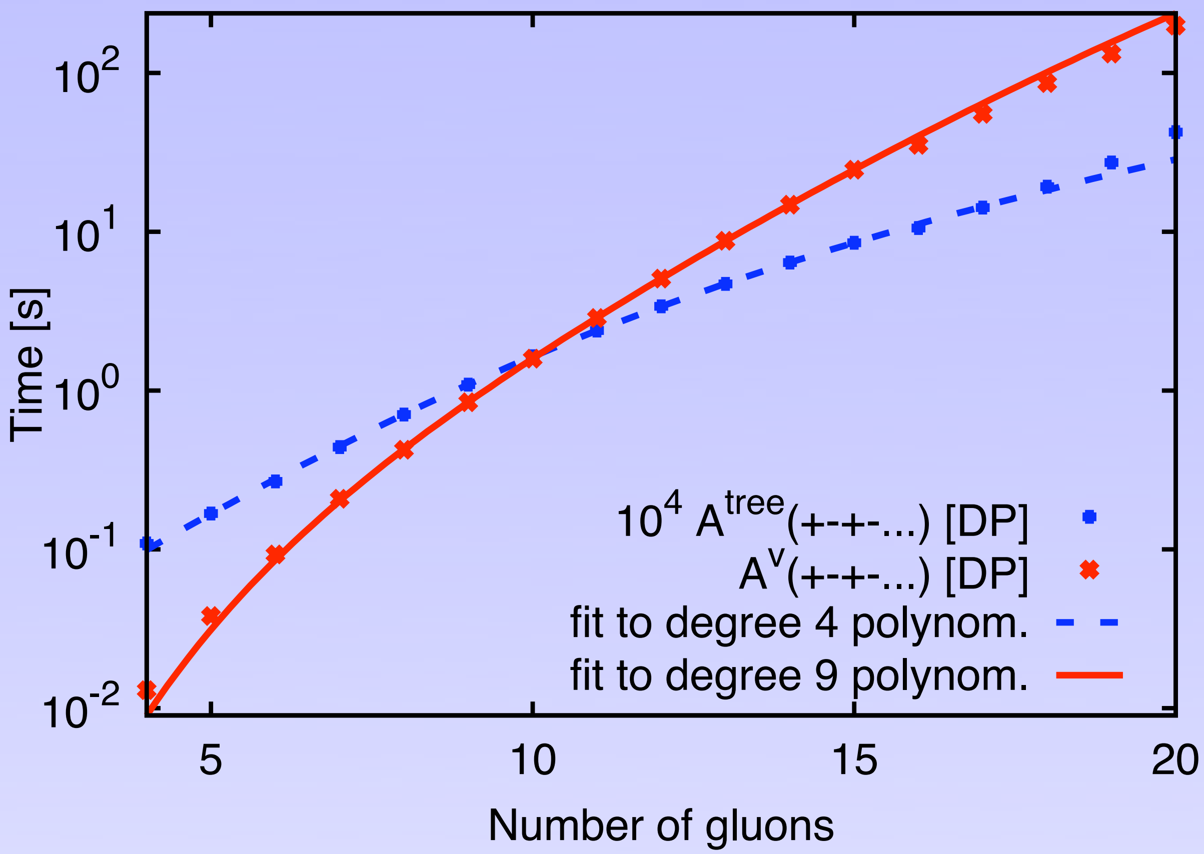
Table 1: CPU time in seconds for the computation of the n gluon amplitude on a standard PC (2 GHz Pentium IV), summed over all helicities.

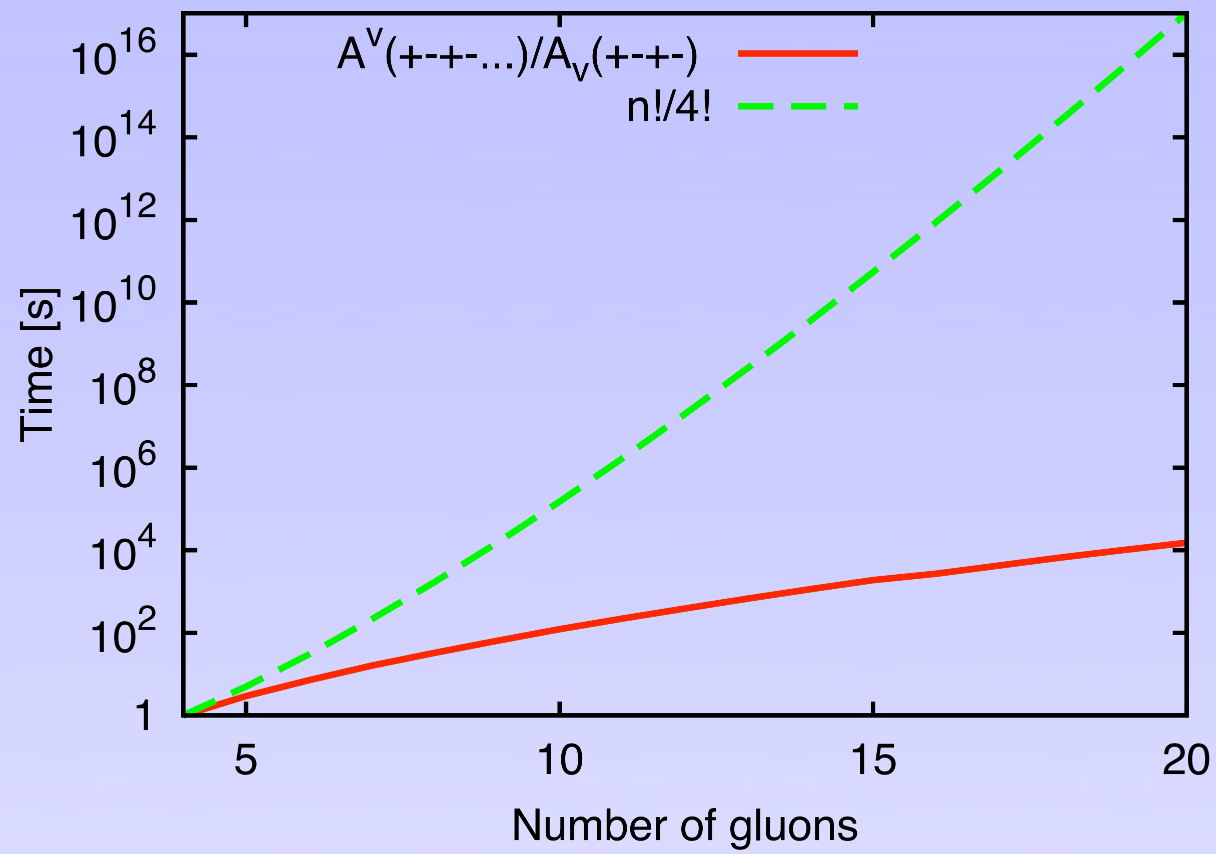
Time dependence of the algorithm up to N=20



- ☛ time $\propto N^9$ as expected
- ☛ independent of the helicity configuration

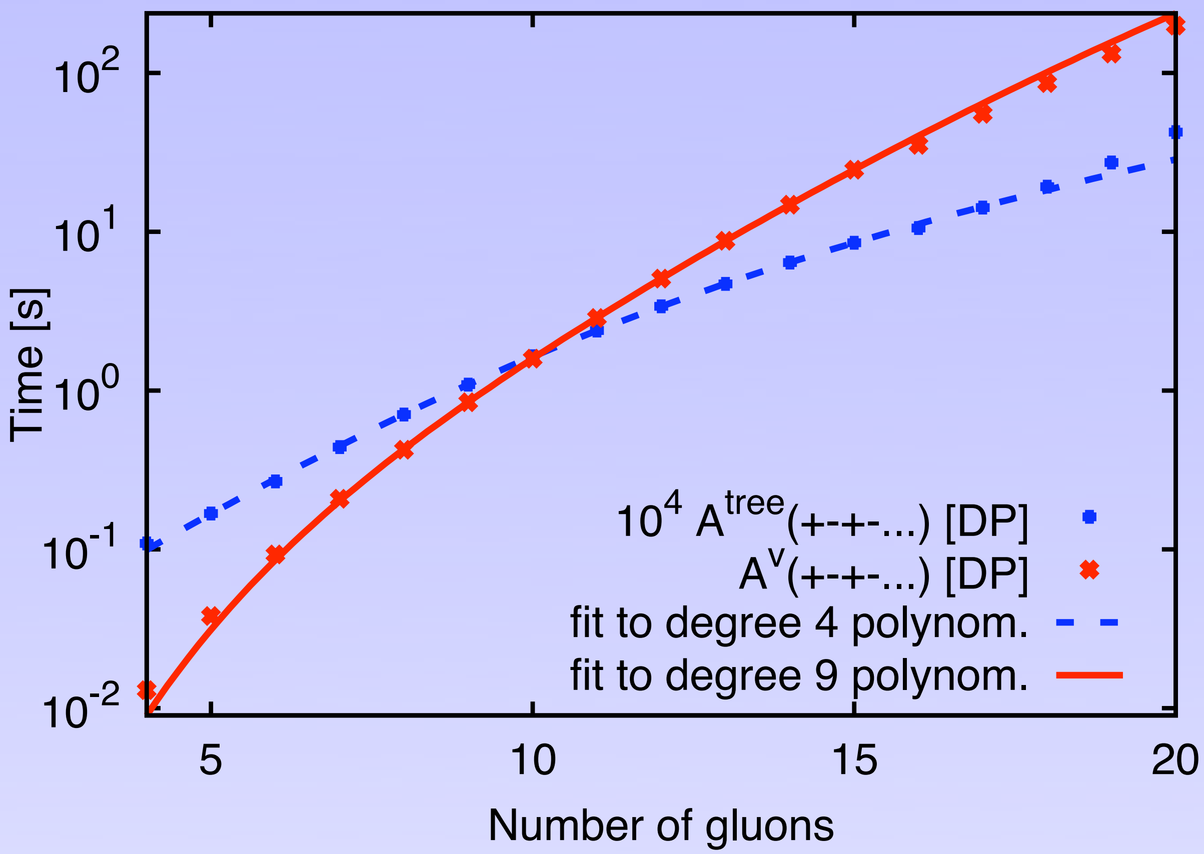
Time dependence of the algorithm up to N=20

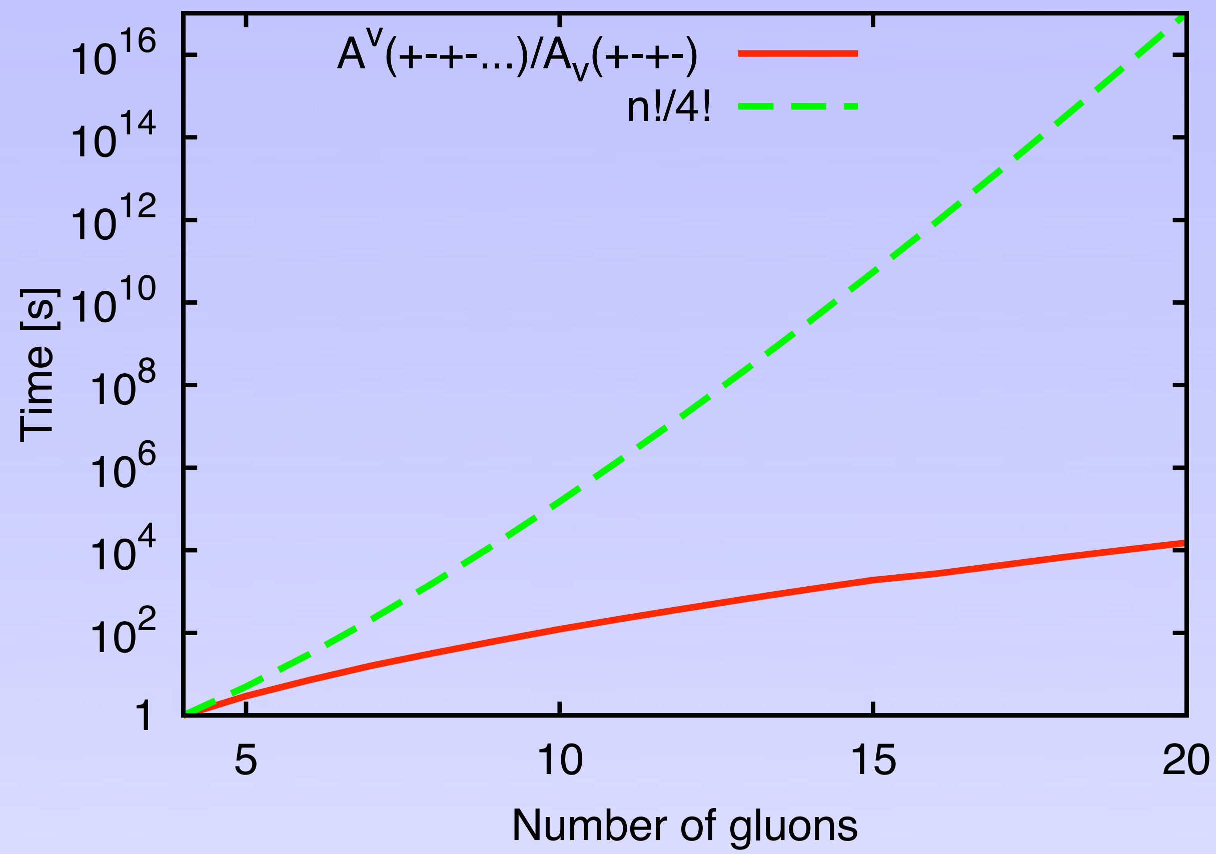




- ☛ time $\propto N^9$ as expected
- ☛ independent of the helicity configuration
- ☛ compare with factorial growth...

Time dependence of the algorithm up to N=20





- time $\propto N^9$ as expected
- independent of the helicity configuration
- compare with factorial growth...

Comparison with BlackHat: N=6 and N=7,8 MHV: slightly longer times (e.g for N=6 72ms vs 90ms), related to us using recursive tree amplitudes rather than analytic ones

Sample results at fixed points

Rocket can compute *any* N-gluon amplitude with *arbitrary helicities*, consider e.g. 15 gluon momenta random generated:*

$$
\begin{aligned}
p_1 &= (-7.500000000000000, 7.500000000000000, 0.000000000000000, 0.000000000000000) \\
p_2 &= (-7.500000000000000, -7.500000000000000, 0.000000000000000, 0.000000000000000) \\
p_3 &= (0.368648489648050, 0.161818085189973, 0.125609635286264, -0.306494430207942) \\
p_4 &= (0.985841964092509, -0.052394238926518, -0.664093578996812, 0.726717923425790) \\
p_5 &= (1.470453194926588, -0.203016239158633, 0.901766792550452, -1.143605551298596) \\
p_6 &= (2.467058579094687, -1.840106401193462, 0.715811527707121, 1.479189075734789) \\
p_7 &= (0.566021478235079, -0.406406330753485, -0.393435666409983, -0.020556861225509) \\
p_8 &= (0.419832726637289, -0.214182754609525, 0.074852807863799, -0.353245414886707) \\
p_9 &= (2.691168687878469, 1.868400546247601, 1.850615607221259, -0.571568175905795) \\
p_{10} &= (1.028090983779864, -0.986442664896249, -0.193408556327968, 0.215627155388572) \\
p_{11} &= (1.377779821947130, -0.155359745837053, -1.074009172530291, -0.848908054184264) \\
p_{12} &= (1.432526153404585, 0.621168997409793, -0.290964068761809, 1.257624811911176) \\
p_{13} &= (0.335532948820133, 0.244811479043329, 0.138986808214636, 0.182571538348285) \\
p_{14} &= (1.085581415795683, 0.330868645896313, -0.756382142822373, -0.704910635118478) \\
p_{15} &= (0.771463555739934, 0.630840621587917, -0.435349992994295, 0.087558618018677)
\end{aligned}
$$

* up to N=20 given in 0805.2152

Sample results at fixed points

Rocket can compute *any* N-gluon amplitude with *arbitrary helicities*, consider e.g. 15 gluon momenta random generated:*

Helicity amplitude	c_Γ/ϵ^2	c_Γ/ϵ	1
$\lvert A_{15}^{\rm tree}(++++\ldots)\rvert$	-	-	0
$\lvert A_{15}^{\rm v,unit}(++++\ldots)\rvert$	0	0	1.07572071884782
$\lvert A_{15}^{\rm v,anly}(++++\ldots)\rvert$	0	0	1.07572071880769 *
$\lvert A_{15}^{\rm tree}(-+++\ldots++)\rvert$	-	-	0
$\lvert A_{15}^{\rm v,unit}(-+++\ldots++)\rvert$	0	0	0.181194659968483
$\lvert A_{15}^{\rm v,anly}(-+++\ldots++)\rvert$	0	0	0.181194659846677 *
$\lvert A_{15}^{\rm tree}(--+++\ldots++)\rvert$	-	-	7.45782101450887
$\lvert A_{15}^{\rm v,unit}(--++\ldots++)\rvert$	111.867315217633	586.858955605213	1810.13038312828
$\lvert A_{15}^{\rm v,anly}(--++\ldots++)\rvert$	111.867315217633	586.858955605213	1810.13038312852 **
$\lvert A_{15}^{\rm tree}(-+-\ldots+-)\rvert$	-	-	$5.851039428822597 \cdot 10^{-3}$
$\lvert A_{15}^{\rm v,unit}(-+-\ldots+-)\rvert$	$8.776559143021942 \cdot 10^{-2}$	0.460420629357800	1.52033417713680
$\lvert A_{15}^{\rm v,anly}(-+-\ldots+-)\rvert$	$8.776559143233895 \cdot 10^{-2}$	0.460420661976678	N.A.
$\lvert A_{15}^{\rm tree}(+-+\ldots-+)\rvert$	-	-	$5.851039428822597 \cdot 10^{-3}$
$\lvert A_{15}^{\rm v,unit}(+-+\ldots-+)\rvert$	$8.776559143021942 \cdot 10^{-2}$	0.460420565320471	1.52960647292231
$\lvert A_{15}^{\rm v,anly}(+-+\ldots-+)\rvert$	$8.776559143233895 \cdot 10^{-2}$	0.460420661976678	N.A.

** Mahlon '93; Bern et al '05; ** Forde, Kosower '05*

* up to N=20 given in 0805.2152

Conclusions

We developed an algorithm of polynomial complexity for the evaluation of one-loop amplitudes and implemented it in Rocket. First step presented here: the gluon case.

Results presented demonstrate that:

- the time dependence of the algorithm is polynomial (as expected)
- results of excellent accuracy can be obtained
- N-gluon case fully solved: *all helicity amplitudes computed easily, efficiently and precisely with Rocket* (only limitation computer power)
- Next: include other interaction vertices and internal masses
  $\Rightarrow$ application to SM & BSM LHC processes


END
PLAY
GROUND
ZONE

Extra slides

Colour decomposition

- tree level decomposition

$$\mathcal{A}_n^{\rm tree}(\{p_i,\lambda_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\ A_n^{\rm tree}(p_{\sigma(1)}^{\lambda_{\sigma(1)}},\ldots,p_{\sigma(n)}^{\lambda_{\sigma(n)}})$$

Colour decomposition

- tree level decomposition

$$\mathcal{A}_n^{\text{tree}}(\{p_i,\lambda_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\; A_n^{\text{tree}}(p_{\sigma(1)}^{\lambda_{\sigma(1)}},\ldots,p_{\sigma(n)}^{\lambda_{\sigma(n)}})$$

- one-loop decomposition

$$\mathcal{A}_n^{[J]}(\{p_i,h_i,a_i\}) = g^{n} \sum_{c=1}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;c}} \text{Gr}_{n;c}(\sigma)\, A_{n;c}^{[J]}(\sigma)$$

  - leading in color: $\text{Gr}_{n;1}(1) = N_c \text{Tr}(T^{a_1}\ldots T^{a_n})$
  - subleading in color: $\text{Gr}_{n;c}(1) = \text{Tr}(T^{a_1}\ldots T^{a_c})\text{Tr}(T^{a_c}\ldots T^{a_n})$

Colour decomposition

- tree level decomposition

$$\mathcal{A}_n^{\rm tree}(\{p_i,\lambda_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\; A_n^{\rm tree}(p_{\sigma(1)}^{\lambda_{\sigma(1)}},\ldots,p_{\sigma(n)}^{\lambda_{\sigma(n)}})$$

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  - leading in color: ${\rm Gr}_{n;1}(1) = N_c {\rm Tr}(T^{a_1}\ldots T^{a_n})$
  - subleading in color: ${\rm Gr}_{n;c}(1) = {\rm Tr}(T^{a_1}\ldots T^{a_c}){\rm Tr}(T^{a_c}\ldots T^{a_n})$

- subleading amplitudes in color $A_{n;c}^{[1]}$ fully determined by the leading color ones

$$A_{n;c>1}^{[1]}(1,2,\ldots,c-1;c,c+1,\ldots,n) \;=\; (-1)^{c-1} \sum_{\sigma\in {\rm OP}\{\alpha\}\{\beta\}} A_{n;1}^{[1]}(\sigma_1,\ldots,\sigma_n)$$

$\Rightarrow$ *need only leading color amplitudes* $A_{n;1}^{[1]}$ *[Kleiss Kuijf ’89, Bern at al. ’93]*

History of pure QCD amplitudes at one-loop

Four parton processes

- ☑ $q\bar{q}q'\bar{q}'$ — *[Ellis, Furman, Haber, Hinchliffe 1980]*
- ☑ $q\bar{q}gg, gggg$ — *[Ellis, Sexton 1985]*

Five parton processes

- ☑ $ggggg$ — *[Bern et.al 1993]*
- ☑ $q\bar{q}ggg$ — *[Bern et. al 1994]*
- ☑ $q\bar{q}q'\bar{q}'g$ — *[Kunszt 1994]*

Six parton processes

- ☑ $gggggg$ *numerical* — *[Ellis, Giele, GZ '06]*
- ☑ $gggggg$ *analytical* — *[Bern et al. '06; Britto et al. '06; Xiao et al. '06]*