Constructing Loop Amplitudes from Tree Amplitudes

"A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes"
R.K. Ellis, W.T.G. and Z. Kunszt
JHEP 0803:003,2008; arXiv:0708.2398 [hep-ph]

"Full one-loop amplitudes from tree amplitudes"
W.T.G., Z. Kunszt and K. Melnikov
JHEP 0804:049,2008 arXiv:0801.2237 [hep-ph]

"On the numerical evaluation of one-loop amplitudes"
W.T.G. and G. Zanderighi
arXiv:0805.2152 [hep-ph]

- Introduction
- The master integral basis and dimensionality
- Determining the master integral coefficients
- Outlook
- (See afternoon talk of Giulia Zanderighi)

W. Giele, Buffalo NY, 05/15/08

Introduction

- Lesson from the TEVATRON: shower matched fixed order calculations a must!
- This makes one-loop corrections with high number of external particles relevant
- The showers will provide the "phase space integrations", combine virtual and real, etc..
- One vertical integrated modular framework can be constructed:

Is an one-loop 2→15 gluon amplitude useful?
Within this framework: Yes!
(not necessarily associated with 15 jets at NLO, but with 15^{th} branching!

Fixed order ME → Vincia parton shower → Pythia 8 → experimental frame work

Tree amplitudes; One-loop amplitudes (Laurent series in (D-4))

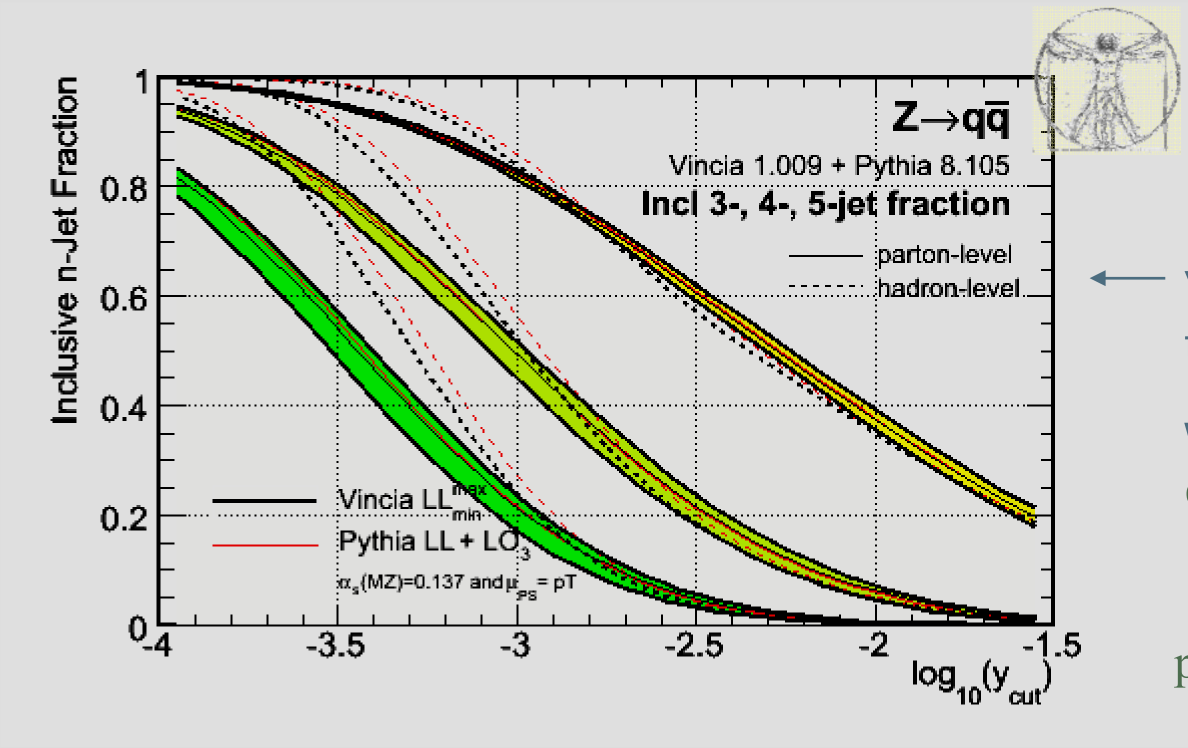
Dipole shower = re-summed dipole subtraction; expansion in a_S gives standard dipole subtracted NLO parton generator.

Deals with non-perturbative modeling & with interface to experimentalists

Experimentalists want this; the final interface is stable.

From Showers to Matrix Elements

- Matching is important:



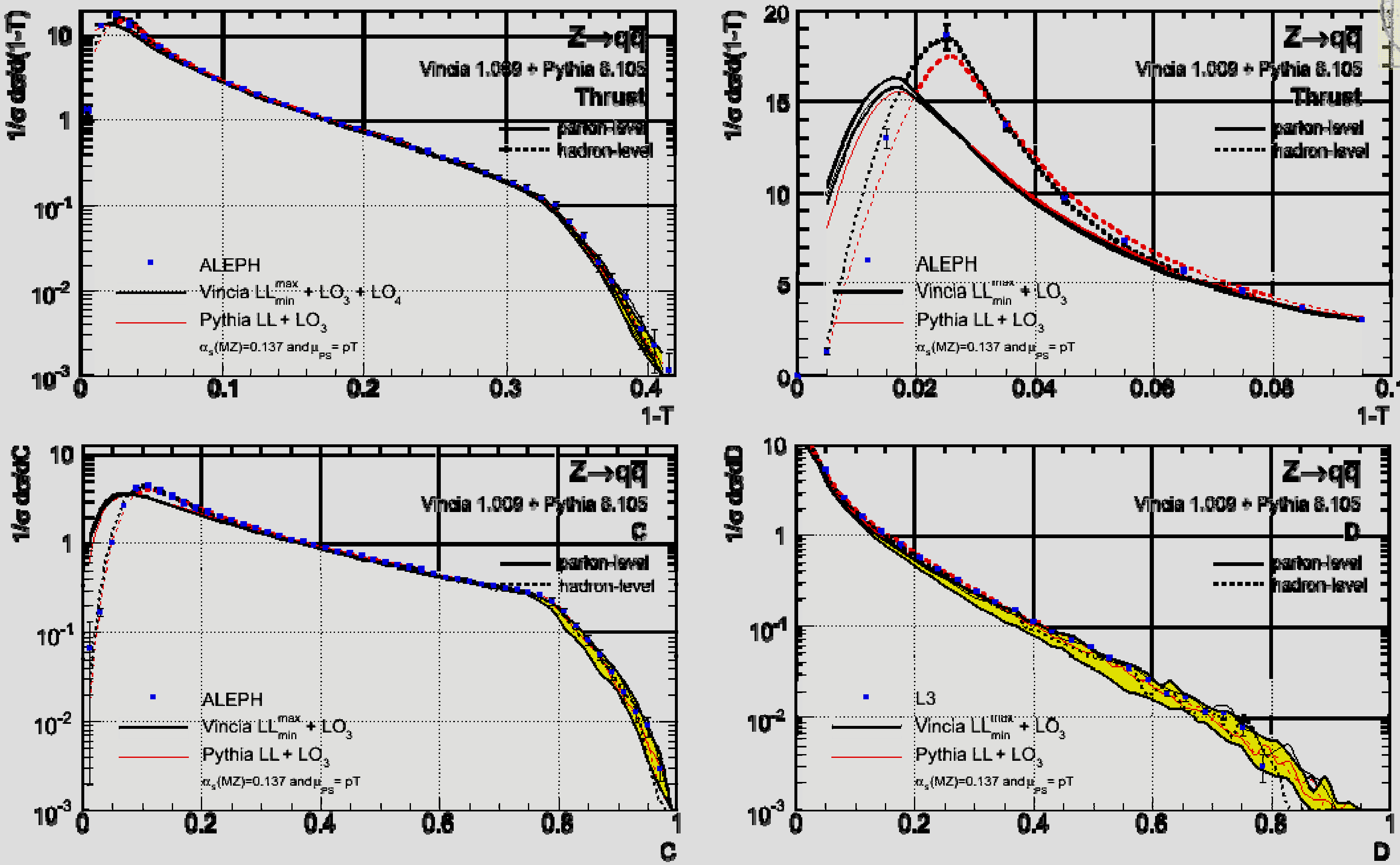
← Varying finite terms only

with $\alpha_s(M_Z)$=0.137,

μ_R=p_T,

p_{Thad} = 0.5 GeV

From Showers to Matrix Elements



Still with $\alpha_s(M_Z)=0.137$…

Dimensional dependence of master integral basis

- Calculating ordered one-loop amplitude is equivalent to calculating the coefficients of the master integrals

$$
\begin{aligned}
\mathcal{A}_{(D)} = & \sum_{[i_1|i_5]} e_{i_1i_2i_3i_4i_5} I^{(D)}_{i_1i_2i_3i_4i_5} + \sum_{[i_1|i_4]} d_{i_1i_2i_3i_4} I^{(D)}_{i_1i_2i_3i_4} \\
& + \sum_{[i_1|i_3]} c_{i_1i_2i_3} I^{(D)}_{i_1i_2i_3} + \sum_{[i_1|i_2]} b_{i_1i_2} I^{(D)}_{i_1i_2} + \sum_{[i_1|i_1]} a_{i_1} I^{(D)}_{i_1}, \qquad (3)
\end{aligned}
$$

where we introduced the short-hand notation $[i_1|i_n] = 1 \leq i_1 < i_2 < \cdots < i_n \leq N$. The master integrals on the r.h.s. of Eq. (3) are defined as

$$
I^{(D)}_{i_1 \cdots i_M} = \int \frac{d^D l}{i(\pi)^{D/2}} \frac{1}{d_{i_1} \cdots d_{i_M}} \,. \qquad (4)
$$

- Dimensional regularization makes the dimensional dependence of the coefficients a crucial consideration.

- Dimensional regularization tells us to calculate the one-loop amplitude in (integer) higher dimensions:
  - The Feynman diagram calculation is done in D_s dimensions
  - After the calculation we continue the parametric dimension D_s to a non-integer dimension $D_s \rightarrow 4-2e$
  - The external, "observable" particles have only components in the 4-dimensional embedded physical space

$$p_\mu = \left(p_0, p_1, p_2, p_3, 0, \ldots, 0\right)$$

  - We have the freedom to choose the loop momentum in $D \leq D_s$ dimensions

$$l_\mu = \left(l_0, l_1, \ldots, l_D, 0, \ldots, 0\right)$$

    This allows the implementation of different "schemes"
- Note, continuing the dimensionality to non-integer values makes no sense except at the end of the calculation

- By extending the master integral basis we can make the loop momentum dimensionality explicit

$$\mathcal{A}_{(D)} = \sum_{[i_1|i_5]} e^{(0)}_{i_1i_2i_3i_4i_5} \, I^{(D)}_{i_1i_2i_3i_4i_5}$$
$$+ \sum_{[i_1|i_4]} \left(d^{(0)}_{i_1i_2i_3i_4} \, I^{(D)}_{i_1i_2i_3i_4} - \frac{D-4}{2} d^{(2)}_{i_1i_2i_3i_4} \, I^{(D+2)}_{i_1i_2i_3i_4} + \frac{(D-4)(D-2)}{4} d^{(4)}_{i_1i_2i_3i_4} \, I^{(D+4)}_{i_1i_2i_3i_4} \right)$$
$$+ \sum_{[i_1|i_3]} \left(c^{(0)}_{i_1i_2i_3} \, I^{(D)}_{i_1i_2i_3} - \frac{D-4}{2} c^{(9)}_{i_1i_2i_3} \, I^{(D+2)}_{i_1i_2i_3} \right)$$
$$+ \sum_{[i_1|i_2]} \left(b^{(0)}_{i_1i_2} \, I^{(D)}_{i_1i_2} - \frac{D-4}{2} b^{(9)}_{i_1i_2} \, I^{(D+2)}_{i_1i_2} \right) + \sum_{i_1=1}^{N} a^{(0)}_{i_1} \, I^{(D)}_{i_1} \, . \qquad (27)$$

- The D_s dependence of the coefficients is straightforward

$$C^{(D_s)} = C_1 + (D_s - 4) \times C_2$$

- Once we determined the dimensionless coefficients we can make the dimensional continuation.

- The dimensional continuation can be performed is several schemes
  - The 't Hooft-Veltman scheme:

$$D_S \to 4-2\varepsilon;\; D \to 4-2\varepsilon;\; D \le D_S$$

  - The Four Dimensional Helicity (FDH) scheme: Bern, De Freitas, Dixon & Wong

$$D_S \to 4;\; D \to 4-2\varepsilon;\; D \le D_S$$

  - Schemes are related by a simple "renormalization"-shift (i.e. can be absorbed in the coupling constant renormalization.
  - After we have determined the dimensionless coefficients we can take the FDH dimensional continuation:

$$\mathcal{A}_N = \mathcal{A}_N^{CC} + R_N$$

$$\mathcal{A}_N^{CC} = \sum_{[i_1|i_4]} \tilde{d}^{(0)}_{i_1 i_2 i_3 i_4}\; I^{(4-2\epsilon)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c^{(0)}_{i_1 i_2 i_3}\; I^{(4-2\epsilon)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b^{(0)}_{i_1 i_2}\; I^{(4-2\epsilon)}_{i_1 i_2} + \sum_{i_1=1}^{N} a^{(0)}_{i_1}\; I^{(4-2\epsilon)}_{i_1}$$

$$R_N = -\sum_{[i_1|i_4]} \frac{d^{(4)}_{i_1 i_2 i_3 i_4}}{3} - \sum_{[i_1|i_3]} \frac{c^{(9)}_{i_1 i_2 i_3}}{2} - \sum_{[i_1|i_2]} \left(\frac{(q_{i_1} - q_{i_2})^2}{6} - \frac{m_{i_1}^2 + m_{i_2}^2}{2} \right) b^{(9)}_{i_1 i_2}$$

Determination of the master integral coefficients

Ossolla, Papadopoulos, Pittau

- We now have to determine the coefficients by applying (integer) D_s-dimensional cuts.
- To do this we adapt the OPP method to higher dimensions
- The OPP method is basically a parametric integration method: the integrand is parameterized
- Generalized unitarity/on-shell methods is transformed on the integrand level to an algebraic partional fractioning procedure of the rational integrand:

$$\frac{\mathcal{N}^{(D_s)}(l)}{d_1 d_2 \cdots d_N} = \sum_{[i_1|i_5]} \frac{\overline{e}^{(D_s)}_{i_1 i_2 i_3 i_4 i_5}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\overline{d}^{(D_s)}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\overline{c}^{(D_s)}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\overline{b}^{(D_s)}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\overline{a}^{(D_s)}_{i_1}(l)}{d_{i_1}}$$

- For example the triple denominator term.
- To determine the numerator using partional fractioning we simply demand $d_2=d_3=d_4=0$.
- Note the “form” of the propagator is irrelevant (can contain masses, imaginary masses or even more complicated forms).

- The solution to the partial fraction constraint is

$$l^\mu_{ijk} = V_3^\mu + \sqrt{\frac{-V_3^2 + m_k^2}{\alpha_1^2 + \alpha_2^2 + \alpha_5^2 + \cdots + \alpha_D^2}} \left(\alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \sum_{h=5}^{D} \alpha_h n_h^\mu \right)$$

- The amplitude now factorizes into tree amplitudes with 2 integer D_s-dimensional particles with (possible) complex momenta:

$$\sum_{i=1}^{D_s-2} e_\mu^{(i)}(l) e_\nu^{(i)}(l) = -g_{\mu\nu}^{(D_s)} + \frac{l_\mu b_\nu + b_\mu l_\nu}{l \cdot b}$$

$$\sum_{i=1}^{2^{(D_s-2)/2}} u^{(i)}(l)\overline{u}^{(i)}(l) = \not{l} + m = \sum_{\mu=1}^{D} l_\mu \gamma^\mu + m$$

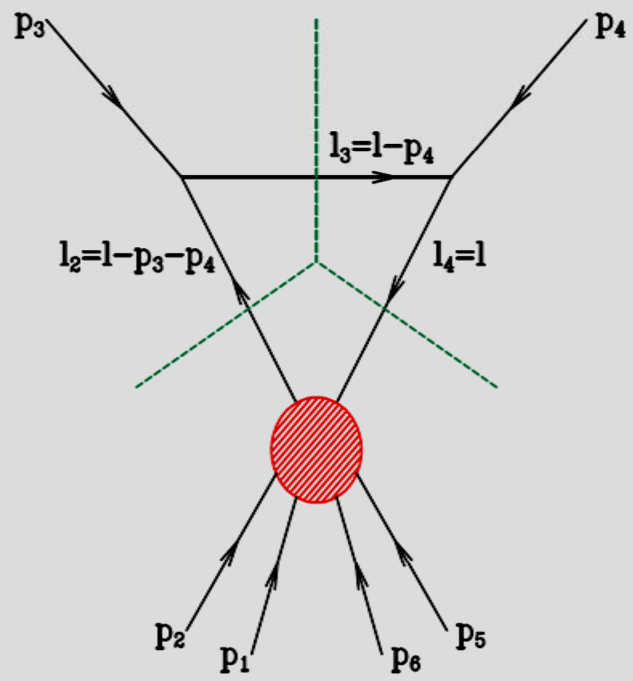


FIG. 3: The factorization of the 6-gluon ordered amplitude for the calculation of the $\overline{c}_{234}(l)$ residue

- The tree matrix amplitudes are still perfectly defined as far as Feynman rules goes
→ Use recursion relations to generate the tree amplitudes (with complex momenta and in any dimension) for arbitrary number of external particles.

$$\overline{c}_{ijk}(l) = \mathrm{Res}_{ijk}\left(\mathcal{A}_N(l) - \sum_{l\neq i,j,k} \frac{\overline{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$

$$\overline{c}^{\mathrm{FDH}}_{ijk}(l) = c^{(0)}_{ijk} + c^{(1)}_{ijk} s_1 + c^{(2)}_{ijk} s_2 + c^{(3)}_{ijk}(s_1^2 - s_2^2) + s_1 s_2 (c^{(4)}_{ijk} + c^{(5)}_{ijk} s_1 + c^{(6)}_{ijk} s_2)$$
$$+ c^{(7)}_{ijk}\, s_1\, s_e^2 + c^{(8)}_{ijk}\, s_2\, s_e^2 + c^{(9)}_{ijk} s_e^2,$$

$$s_1 = l\cdot n_1 \, s_2 = l \cdot n_2$$

$$s_e^2 = -\sum_{i=5}^{D} (l\cdot n_i)^2 = -\sum_{i=5}^{D} (\tilde{l}\cdot n_i)^2$$

- The parametric form of the triple pole residue is simply a decomposition in the projective basis of the unconstraint part of the loop momentum
- By choosing several values of (D_s,D) and choosing the appropriate set of solutions to the partial fractioning constraint we can determine all coefficients
- Note we have an infinite set of equations with a fixed number of unknowns; this easily solvable in a numerical stable manner.

- Once we determined the all master integral coefficients we can integrate the parametric form.
- The integration over the projective components of the loop momentum is straightforward

$$\int d^D l \, \frac{\overline{c}_{ijk}(l)}{d_i d_j d_k} = c_{ijk}^{(0)} \times \int d^D l \, \frac{1}{d_i d_j d_k} + c_{ijk}^{(9)} \times \int d^D l \, \frac{s_e^2}{d_i d_j d_k}$$

$$= c_{ijk}^{(0)} \times I_{ijk}^{(D)} - \left(\frac{D-4}{2}\right) c_{ijk}^{(9)} \times I_{ijk}^{(D+2)} = c_{ijk}^{(0)} \times I_{ijk}^{4-2e} - \frac{1}{9} c_{ijk}^{(9)} + O(e)$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = -\frac{D-4}{2} I_{i_1 i_2 i_3 i_4}^{D+2},$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^4}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} = \frac{(D-2)(D-4)}{4} I_{i_1 i_2 i_3 i_4}^{D+4}$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2} d_{i_3}} = -\frac{(D-4)}{2} I_{i_1 i_2 i_3}^{D+2},$$

$$\int \frac{\mathrm{d}^D l}{(i\pi)^{D/2}} \frac{s_e^2}{d_{i_1} d_{i_2}} = -\frac{(D-4)}{2} I_{i_1 i_2}^{D+2}.$$

The other extra-dimensional integrals

Note that for NLO calculations we only need to keep the UV divergent integrals and pick up the UV pole through the (D-4) term

- The dimensionality is now explicit, i.e. the value of the individual coefficients are independent of the dimensionality.
- We now know the one-loop amplitude for arbitrary integer dimension for the phase space point under consideration we can continue the dimension to 4-2e:

$$\mathcal{A}_N = \mathcal{A}_N^{CC} + R_N$$

$$\mathcal{A}_N^{CC} = \sum_{[i_1|i_4]} \tilde{d}^{(0)}_{i_1 i_2 i_3 i_4}\; I^{(4-2\epsilon)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c^{(0)}_{i_1 i_2 i_3}\; I^{(4-2\epsilon)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b^{(0)}_{i_1 i_2}\; I^{(4-2\epsilon)}_{i_1 i_2} + \sum_{i_1=1}^{N} a^{(0)}_{i_1}\; I^{(4-2\epsilon)}_{i_1}$$

$$R_N = -\sum_{[i_1|i_4]} \frac{d^{(4)}_{i_1 i_2 i_3 i_4}}{3} - \sum_{[i_1|i_3]} \frac{c^{(9)}_{i_1 i_2 i_3}}{2} - \sum_{[i_1|i_2]} \left(\frac{(q_{i_1} - q_{i_2})^2}{6} - \frac{m_{i_1}^2 + m_{i_2}^2}{2} \right) b^{(9)}_{i_1 i_2}$$

- We checked numerically against the analytic known 4,5 and 6 gluon ordered one-loop helicity amplitudes and found full agreement.

Outlook

- The *algebraic* method developed here reduces the calculation of one-loop amplitudes to calculating tree amplitudes (in higher dimensions)
- The calculation of the tree amplitudes can easily be handled by existing well-developed formalisms such as recursion relations
- This will lead very quickly to the development of one-loop generators for multi-parton one-loop amplitudes (including external vector bosons & higgs particles).
- Furthermore parton level shower monte carlo's which can integrate these one-loop amplitudes through matching are in full development
- This will lead to modular integrated tools for experimentalists:

Fixed order ME → Vincia parton shower → Pythia 8 → experimental frame work