# Two-loop amplitudes for gauge boson pair production in QCD

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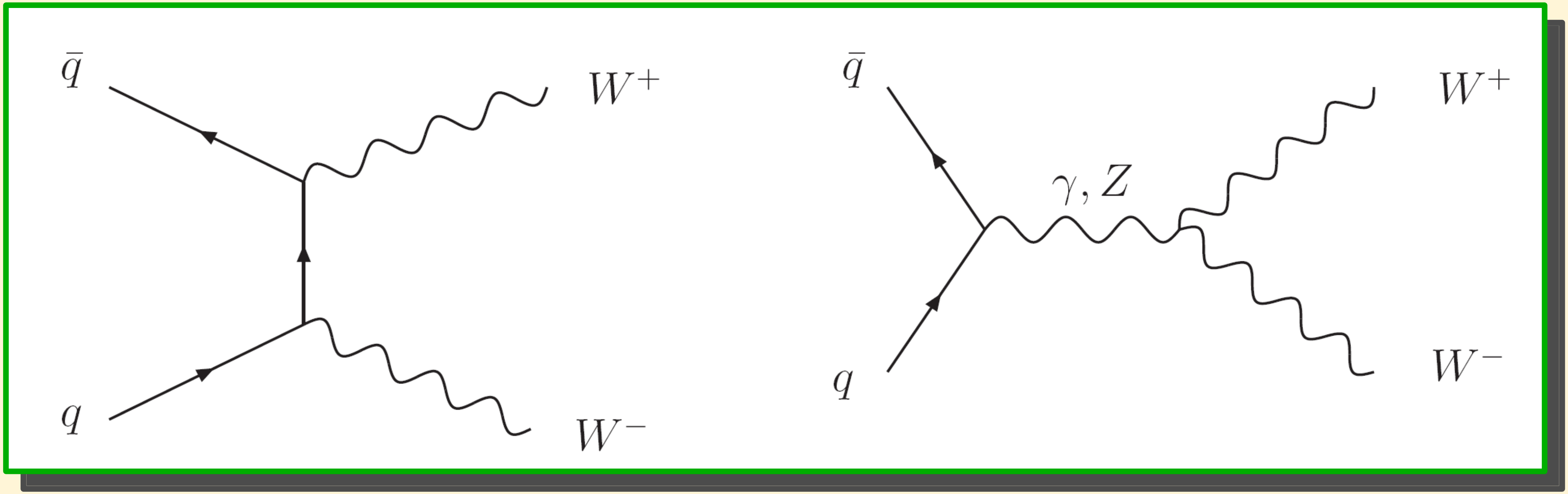
LoopFest VII 2008, Buffalo, 15 May 2008

# The process

W Pair Production in the quark-anti-quark annihilation channel




$2 \rightarrow 2$ process with massive particles

# One is bound to ask:

## Do we really need to go up to NNLO?

NNLO is needed when

- NLO corrections are large
- For benchmark measurements

*Theoretical uncertainty smaller or matching the experimental errors*

## State of the art (for LHC):

| Relative Order | $2 \to 1$ | $2 \to 2$ | $2 \to 3$ | $2 \to 4$ | $2 \to 5$ | $2 \to 6$ |
|---|---|---|---|---|---|---|
| 1 | LO | | | | | |
| $\alpha_s$ | NLO | LO | | | | |
| $\alpha_s^2$ | NNLO | NLO | LO | | | |
| $\alpha_s^3$ | | NNLO | NLO | LO | | |
| $\alpha_s^4$ | | | | NLO | LO | |
| $\alpha_s^5$ | | | | | NLO | LO |

Glover

# Outline

- Focus on hadronic W pair production:
  Virtual two-loop and one-loop squared amplitudes
- Motivation for studying $q\, q \rightarrow W\, W$ accurately
- Results: NNLO Virtual Corrections in the High Energy Limit
- Conclusions - Outlook

Power corrections

Full mass dependence

# Motivation I

W pair production important as a **signal** in searches for **New Physics.** Testing ground for non-abelian structure of SM, triple gauge couplings, γWW, ZWW

$$\sigma(p\bar{p} \rightarrow W^+W^-) =$$

$$14.6^{+5.8}_{-5.1}(\text{stat})\ ^{+1.8}_{-3.0}(\text{syst}) \pm 0.9(\text{lum})\ \text{pb}$$ CDF

$$13.8^{+4.3}_{-3.8}(\text{stat})\ ^{+1.2}_{-0.9}(\text{syst}) \pm 0.9(\text{lum})\ \text{pb}$$ DØ

| $\sqrt{s} = 2$ TeV ($p\bar{p}$) | $W^+W^-$ MRS98 | $W^+W^-$ CTEQ5 |
|---|---|---|
| Born [pb] | 10.0 | 10.3 |
| Full [pb] | 13.0 | 13.5 |

Tevatron

| $\sqrt{s} = 14$ TeV ($pp$) | $W^+W^-$ MRS98 | $W^+W^-$ CTEQ5 |
|---|---|---|
| Born [pb] | 81.8 | 86.7 |
| Full [pb] | 120.6 | 127.8 |

?

LHC

Campbell, Ellis ('99)

# Motivation II

## The 'elusive' Higgs boson

**Higgs:**

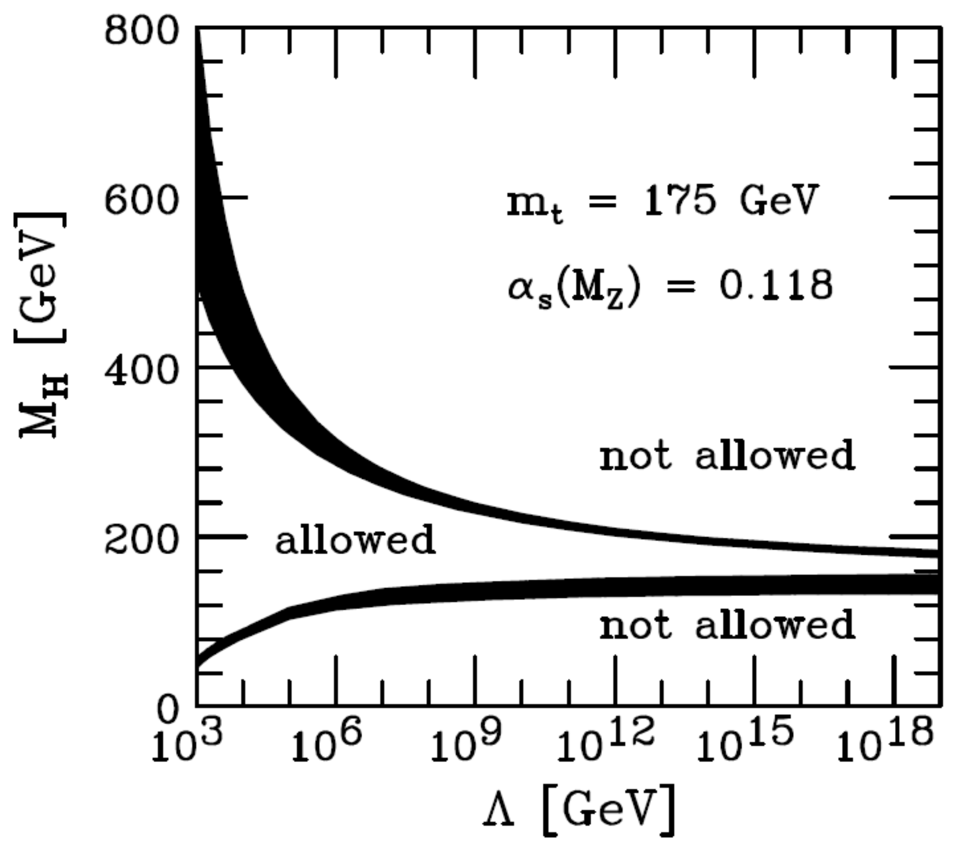
- Only constituent of the SM not experimentally observed yet.
- Electroweak symmetry breaking
- Description of particle masses

Discovery by itself is not enough!
Properties of the Higgs needed
to exclude or verify
alternative models

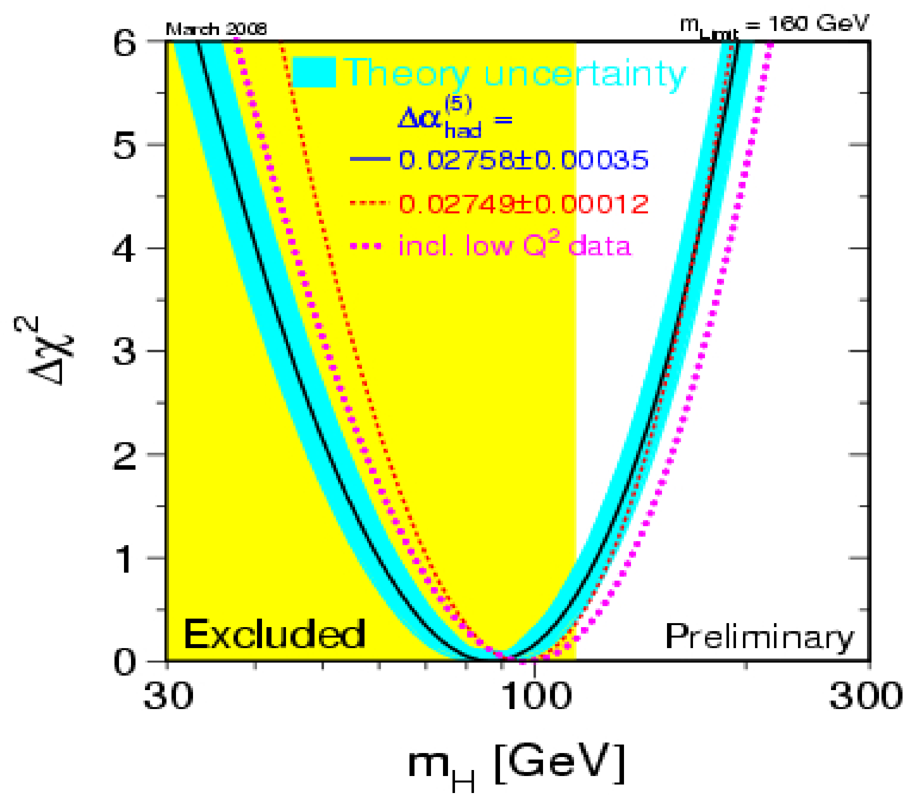
# Motivation II

## Soon in the LHC era

VIP: Higgs! LHC has the energy and luminosity required to discover the Higgs in all the allowed range, $114 < M_H < 600$ GeV

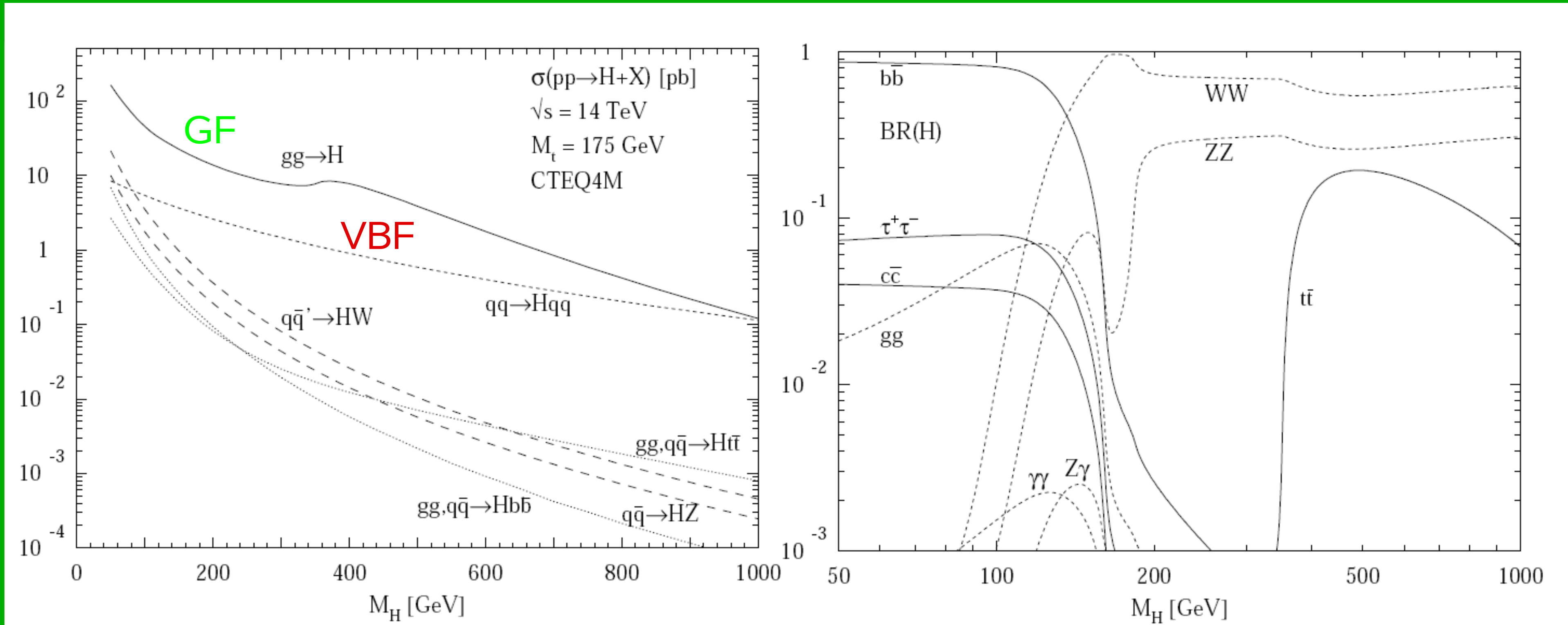



Hambye, Riesselmann ('97)




LEP EW working Group

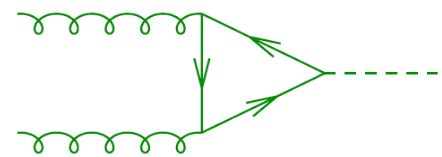
# Motivation II




Spira '97

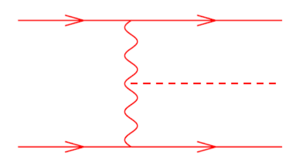
# LHC Higgs production ...

Gluon Fusion channel is the dominant production mechanism up to $M_H$ ~ 1 TeV : $g\, g \rightarrow H$



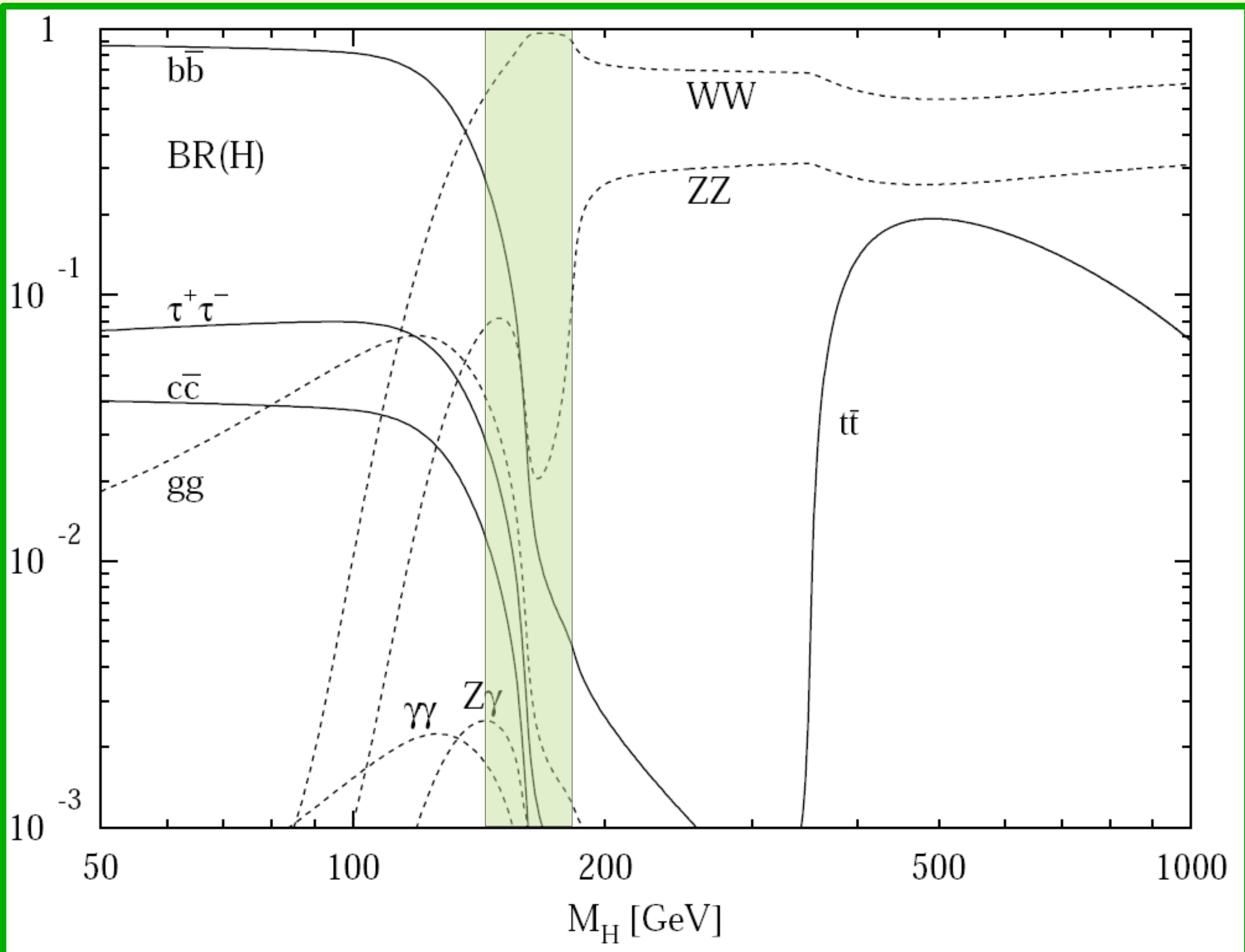
gluon fusion (GF)

Sub-dominant production process is
Vector Boson Fusion: $q\, q \rightarrow V\, V \rightarrow q\, q\, H$



vector boson fusion (VBF)

# Motivation II




… and decay

Once the Higgs is produced it will eventually decay into different particles depending on its mass. In the Higgs mass range $140 - 180$ GeV the main decay mode is into W pairs

# Motivation II

## Going after the Higgs: Main discovery Channels

$M_H$ : 114 – 140 GeV
$H \rightarrow \gamma\,\gamma$

$M_H$ : 180 – 600 GeV
$H \rightarrow Z\,Z \rightarrow 4\,l$

$M_H$: 140 – 180 GeV
$H \rightarrow W\,W \rightarrow 2\,l$ + missing Energy $E_T$

- Pick up the signal process
- Avoid or suppress the usually large **background**
- Accurate theoretical predictions for both signal and background

→ Main background (irreducible): W pair production

# Mini Summary

W Pair Production is important at the LHC:

- Searches for New Physics
- Irreducible background to Higgs production

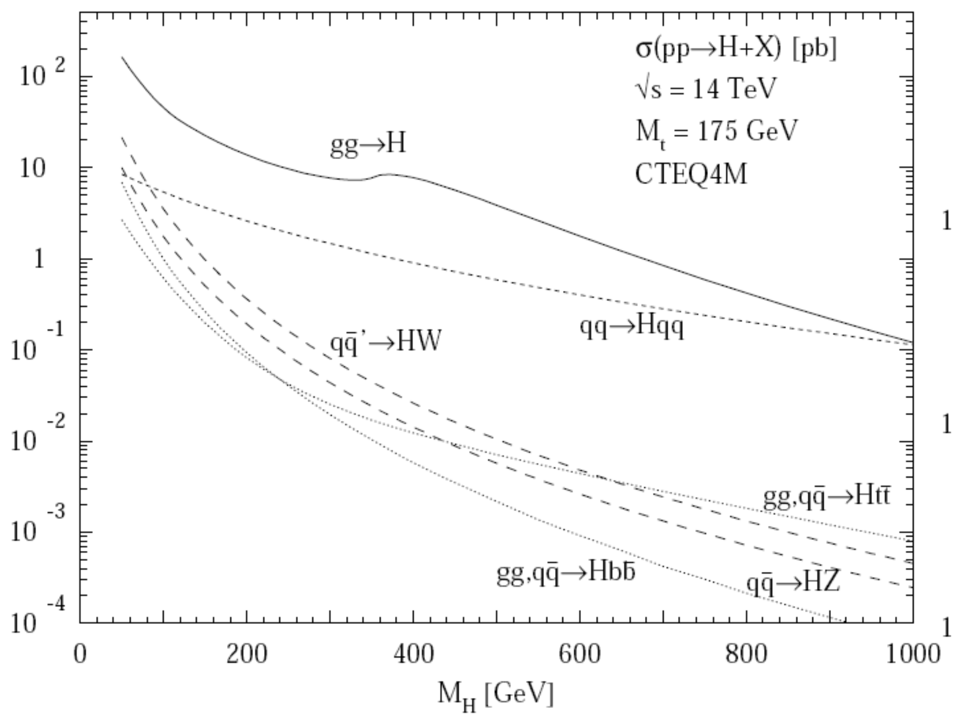
# All these are nice but still ...

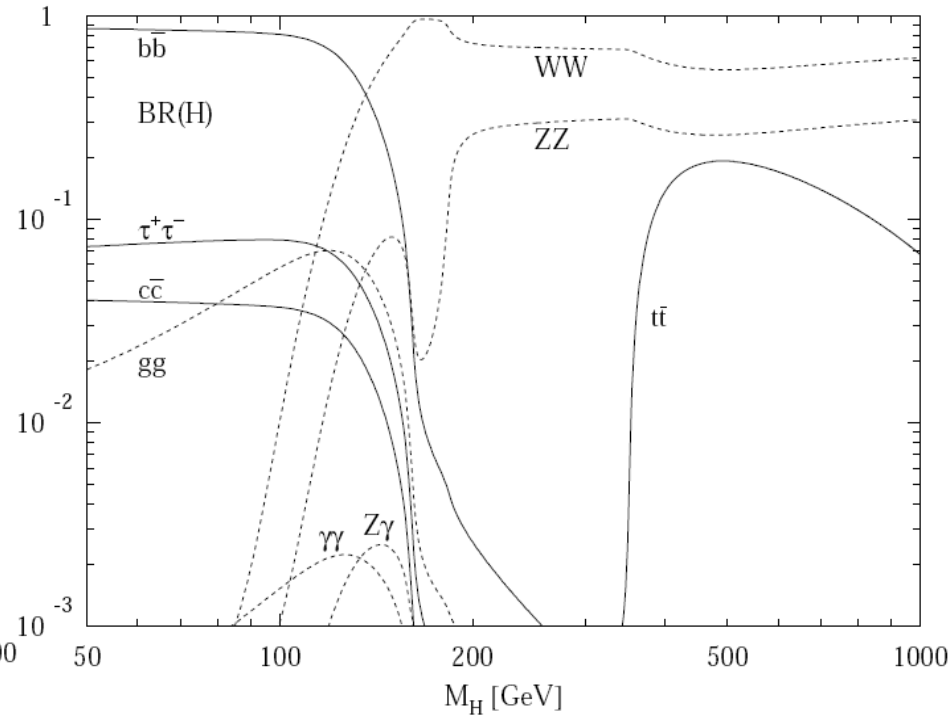
... do we really need to go up to NNLO?

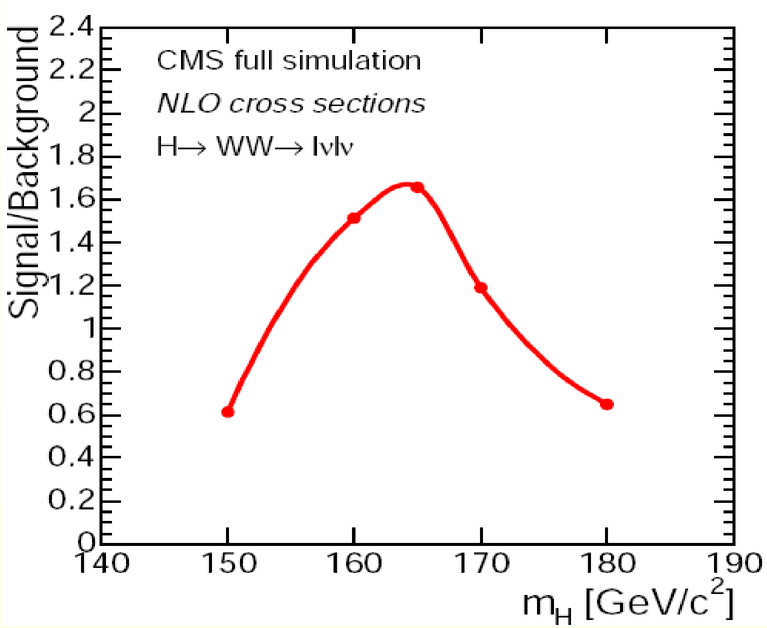
The answer is

YES!

# Signal/background




Signal – background ratio:
of order unity
in the mass range
150 – 180 GeV

# Signal known to NNLO

**QCD corrections to :**

$\underline{g\,g \rightarrow H}$

NLO: Contribute ~ 70%

Dawson ('91); Djouadi, Graudenz, Spira, Zerwas ('95)

NNLO: Contribute an additional 20% for LHC

Harlander, Kilgore ('02); Anastasiou, Melnikov ('02)

Ravindran, Smith, van Neerven ('03)

With a Jet veto at NNLO: corrections ~ 85%

Catani, de Florian, Grazzini ('02)

Davatz, Dissertori, Dittmar, Grazzini, Pauss ('04)

Anastasiou, Melnikov, Petrielo ('04)

NNLO $\underline{H \rightarrow W\,W \rightarrow l\,\nu\,l\,\nu}$

Anastasiou, Dissertori, Stöckli, Webber ('08)

Grazinni ('08)

# Background

- qq→WW

# Background

- qq→WW

  70% enhancement at NLO. With a jet veto the enhancements fall to 20-30%

  Dixon, Kunszt, Signer ('98, '99)

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  Dixon, Kunszt, Signer ('98, '99)
- loop induced gg→WW

# Background

- qq→WW

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  Dixon, Kunszt, Signer ('98, '99)

- loop induced gg→WW

  Contributes to the quark annihilation channel at $\mathcal{O}(\alpha_s^2)$. Enhanced by the **large gluon flux.** After Higgs search cuts it increases the background by 30%, with no cuts by 5%

  Glover, van der Bij ('89); Kao, Dicus ('91)
  Binoth, Ciccolini, Kauer, Krämer ('05)
  Duhrssen, Jackobs, v. d. Bij, Marquard ('05)

- EW corrections — Accomando, Denner, Kaiser('05)

Necessity of NNLO calculation for a few % level accuracy

# W Pair Production

1980 1990 2000 Present

LO — Brown, Mikaelian ('79)

NLO — Ohnemus ('91); Frixione ('93);
Ohnemus('94);
Dixon, Kunszt, Signer ('98,'99);
Campbell, K. Ellis ('99)

NNLO massless — Anastasiou, Glover, Tajeda-Yeomans ('02)

resummation — Grazzini ('06)

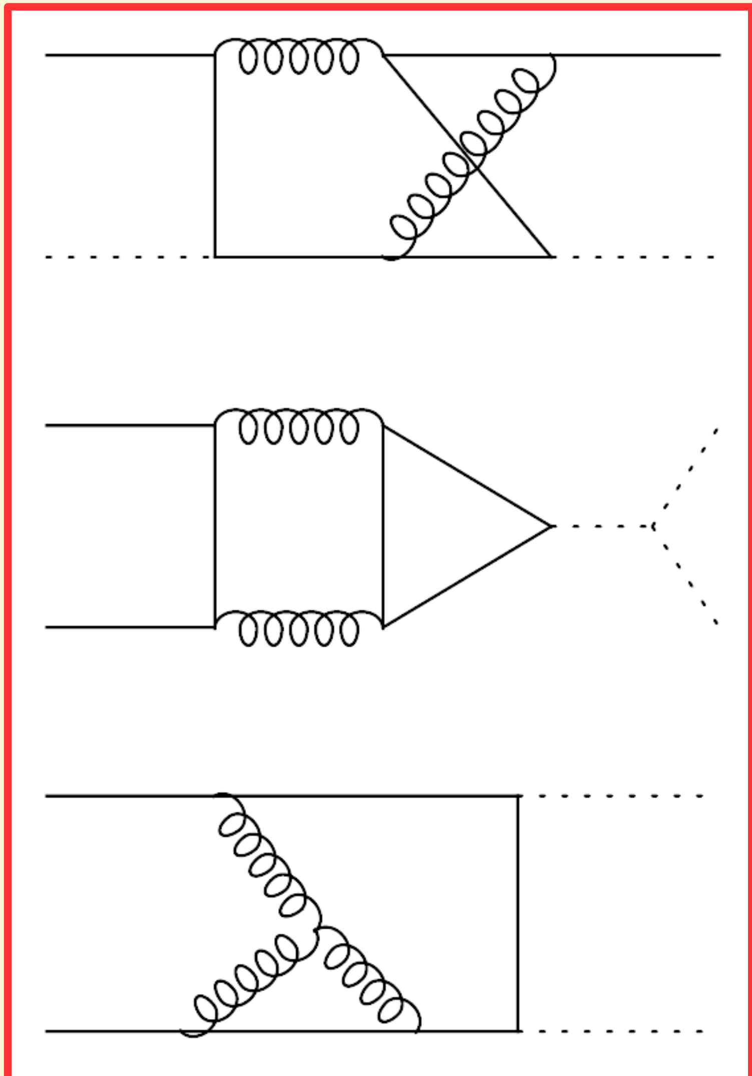
NNLO

The p p $\rightarrow$ W W story till recently...

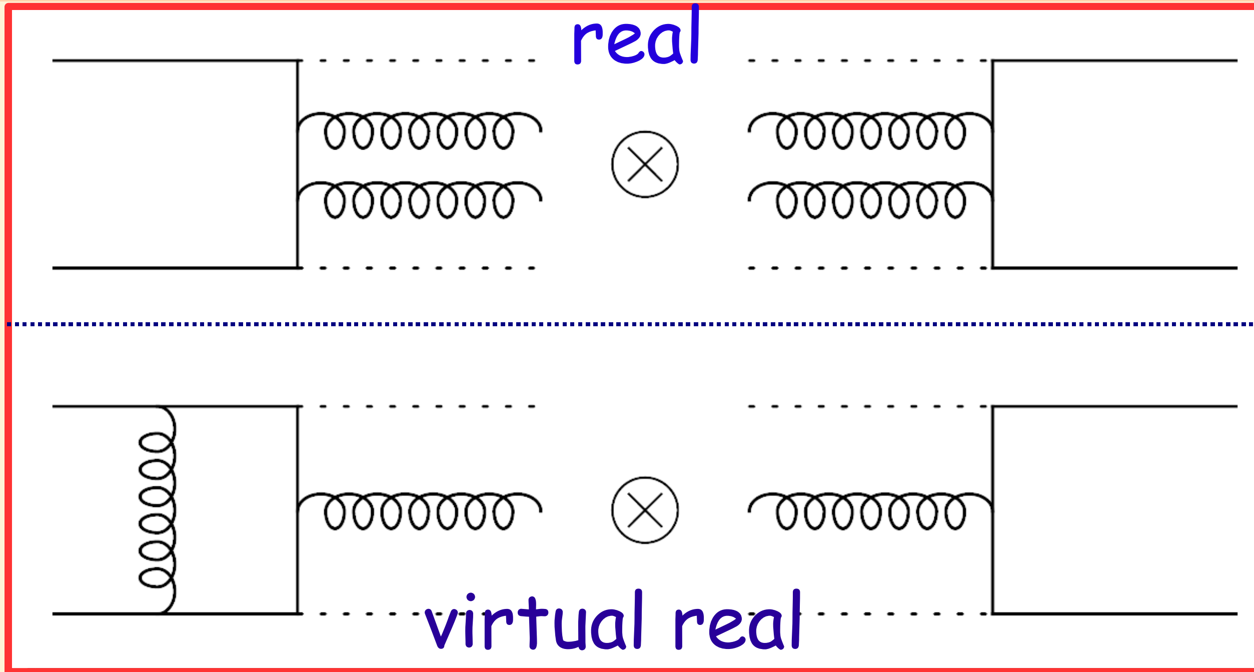
# We would now like to have...

- ... Cross sections for W Pair production at NNLO with full mass dependence...
- ... Then start with the amplitudes...
- ... The difficult part on the amplitude level is the virtual corrections, in particular the two-loop diagrams contracted with the Born

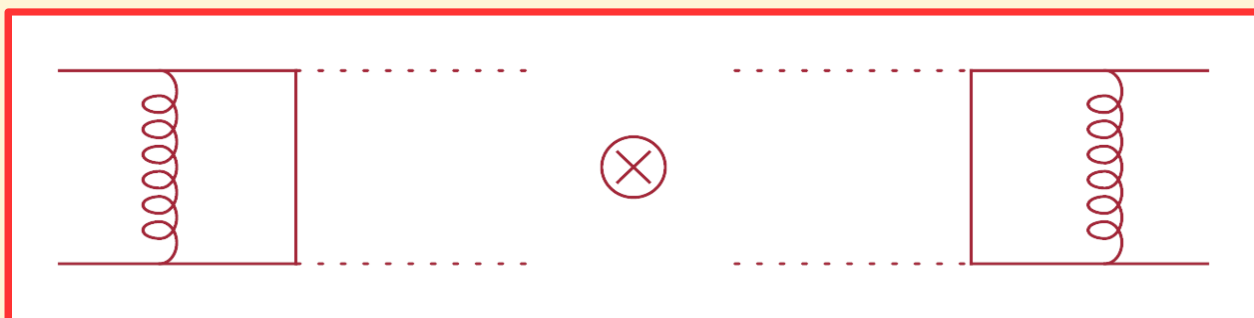
# Well, that means... many diagrams



2 loop virtual




(one loop) $\otimes$ (one loop)*



# Contributions to the cross section

$$d\sigma_n = d\Phi_n \left| M_n \right|^2$$

at NNLO:

$$d\sigma_n = d\sigma_n^{(\text{virtual diagrams})} + d\sigma_{n+1}^{(\text{virtual-real diagrams})}$$
$$+ \; d\sigma_{n+2}^{(\text{real diagrams})}$$

Difficulties lie at red (lesser ones at pink)

# So what is it at stake?

- A NNLO (4 legs, 2 loops) calculation of a process with massive particles (similar features to the recent "heavy quark production") Czakon, Mitov, Moch ('07) Czakon ('08)
- Color and spin averaged amplitudes
- Kinematical region: all kinematical invariants large compared to the mass of W:

$$M_W^2 \ll s, t, u$$

- We expand with respect to $m_s = M_W^2/s$
- Exact analytic result (up to terms suppressed by powers of $m_s$)

# Mellin-Barnes

Do a reduction a la Laporta into Masters, then starting from the Feynman parameters representation of a master, “walk” the following

## Steps:

- produce representations (**MBrepresentations.m**)
- analytically continue in ε to the vicinity of 0 (**MB.m**)
- expand in mass (**MBasymptotics.m, Czakon**)
- perform as many as possible integrations using Barnes lemmas (**BarnesRoutines.m**, Kosower)
- resum the remaining integrals by transforming into harmonic series (**Xsummer**)
- resum remaining constants by high-precision numerical evaluation (**quadprec.m**) and fit them to a transcendental basis (**PSLQ**)

# Software

MBrepresentations.m (GC, Czakon)

Produces representations for any **multi**-loop, **planar** or **non-planar**, **scalar** or **tensor** integral of any **rank**!

MB.m (Czakon)

Determination of contours, analytic continuation, expansion in a chosen parameter, numerical integration

XSummer (Moch, Uwer)

Evaluation of harmonic sums

PSLQ (Bailey)

Fitting to a transcendental basis

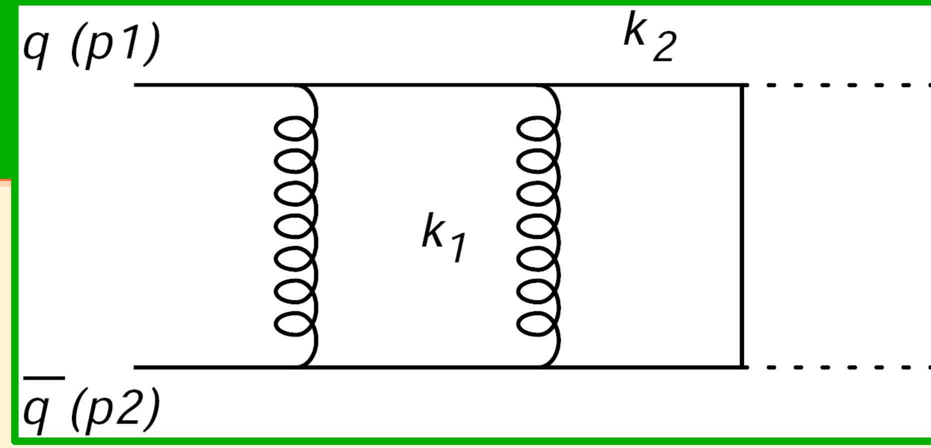
quadprec.m (Czakon)

High precision numerical evaluation with up to 64 digits

W Pair Production

# A tensor example:

(k1.p3) x



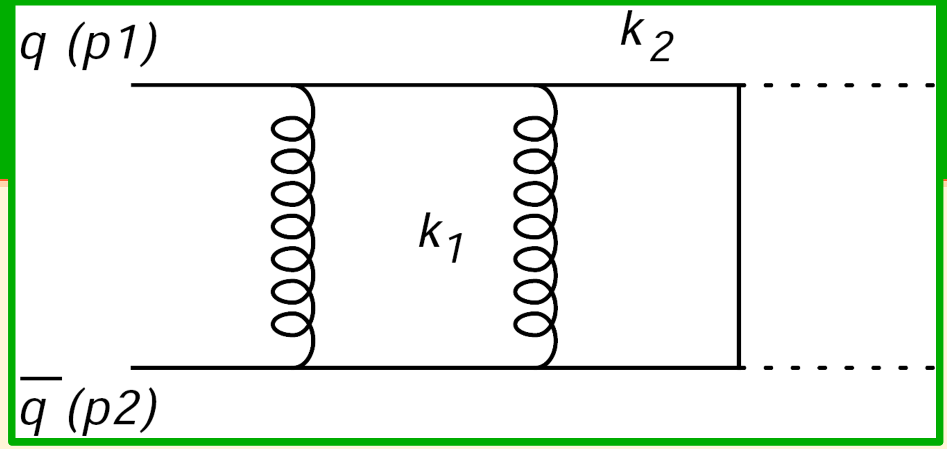

$$\int\int d^dk_1 d^dk_2 \frac{k_1.p_3}{k_1^2k_2^2(k_1+k_2)^2(k_1+k_2-p_1)^2(k_2-p_1-p_2)^2(k_1+k_2-p_1-p_2)^2(k_2-p_1-p_2+p_3)^2}$$

```
{(M2^(-1-ep-z5) S^(-2-ep-z1) U^(z1+z5) (-M2 + S + U) Gamma[-ep + z1] Gamma[2 + z2] Gamma[-1 - ep - z2 - z3]
   Gamma[-z3] Gamma[1 - z1 + z2 + z3] Gamma[-1 - ep - z1 + z3 - z4] Gamma[-z1 + z2 + z3 - z4]
   Gamma[-z4] Gamma[1 + z1 + z4] Gamma[2 + ep + z1 + z4] Gamma[1 - z3 + z4]
   Gamma[-z1 - z5] Gamma[1 - ep - z1 + z2 + z4 - z5] Gamma[1 + ep + z5] Gamma[z1 - z2 + z5]) /
  (2 Gamma[1 - 2 ep] Gamma[1 - z3] Gamma[1 - z1 + z2 + z3 - z4]
   Gamma[1 - 2 ep + z1 + z4] Gamma[2 - z1 + z2 + z4]),

 (M2^(-1-ep-z5) S^(-1-ep-z1) U^(z1+z5) Gamma[-ep + z1] Gamma[1 + z2] Gamma[-1 - ep - z2 - z3] Gamma[-z3]
   Gamma[1 - z1 + z2 + z3] Gamma[-ep - z1 + z3 - z4] Gamma[-z1 + z2 + z3 - z4] Gamma[-z4]
   Gamma[1 + z1 + z4] Gamma[2 + ep + z1 + z4] Gamma[1 - z3 + z4] Gamma[-z1 - z5]
   Gamma[1 - ep - z1 + z2 + z4 - z5] Gamma[1 + ep + z5] Gamma[z1 - z2 + z5]) /
  (2 Gamma[1 - 2 ep] Gamma[1 - z3] Gamma[1 - z1 + z2 + z3 - z4]
   Gamma[1 - 2 ep + z1 + z4] Gamma[2 - z1 + z2 + z4]),

 (M2^(-1-ep-z5) (2 M2 + S) S^(-2-ep-z1) U^(z1+z5) Gamma[1 - ep + z1] Gamma[2 + z2] Gamma[-1 - ep - z2 - z3]
   Gamma[-z3] Gamma[1 - z1 + z2 + z3] Gamma[-1 - ep - z1 + z3 - z4] Gamma[-z1 + z2 + z3 - z4]
   Gamma[-z4] Gamma[1 + z1 + z4] Gamma[2 + ep + z1 + z4] Gamma[1 - z3 + z4]
   Gamma[-z1 - z5] Gamma[1 - ep - z1 + z2 + z4 - z5] Gamma[1 + ep + z5] Gamma[z1 - z2 + z5]) /
  (2 Gamma[1 - 2 ep] Gamma[1 - z3] Gamma[1 - z1 + z2 + z3 - z4]
   Gamma[2 - 2 ep + z1 + z4] Gamma[2 - z1 + z2 + z4]),
```

Mellin Barnes representation

• • • 11 terms in total

# A tensor example:




(k1.p3) x

$$(ms = M_W^{\,2}/s,\ x = -t/s)$$

Analytic result

```
-      1
   ───────────
   480 (1 - x)

 (40 π^4 - 132 π^4 x + 360 π^2 H[0, 1, x] - 80 π^2 x H[0, 1, x] + 160 π^2 H[1, 0, x] - 160 π^2 x H[1, 0, x] +
  620 π^2 H[1, 1, x] - 400 π^2 x H[1, 1, x] + 240 H[0, 0, 1, 1, x] + 480 H[0, 1, 1, 0, x] -
  480 x H[0, 1, 1, 0, x] + 720 H[0, 1, 1, 1, x] + 480 H[1, 0, 1, 0, x] - 480 x H[1, 0, 1, 0, x] +
  960 H[1, 0, 1, 1, x] + 480 H[1, 1, 0, 0, x] - 480 x H[1, 1, 0, 0, x] + 2400 H[1, 1, 1, 0, x] -
  2400 x H[1, 1, 1, 0, x] + 1800 H[1, 1, 1, 1, x] + 100 π^2 H[1, x] Log[ms] -
  240 π^2 x H[1, x] Log[ms] - 240 H[0, 1, 1, x] Log[ms] + 1440 H[1, 1, 0, x] Log[ms] -
  1440 x H[1, 1, 0, x] Log[ms] - 600 H[1, 1, 1, x] Log[ms] + [illegible]0 π^2 Log[ms]^2 - 300 π^2 x Log[ms]^2 -
  180 H[1, 1, x] Log[ms]^2 + 260 H[1, x] Log[ms]^3 + 45 Log[ms]^4 + 130 x Log[ms]^4 +
  160 π^2 H[0, x] Log[1 - x] - 160 π^2 x H[0, x] Log[1 - x] + 80 π^2 H[1, x] Log[1 - x] -
  80 π^2 x H[1, x] Log[1 - x] + 480 H[0, 1, 0, x] Log[1 - x] - 480 x H[0, 1, 0, x] Log[1 - x] +
  480 H[1, 0, 0, x] Log[1 - x] - 480 x H[1, 0, 0, x] Log[1 - x] + 480 H[1, 1, 0, x] Log[1 - x] -
  480 x H[1, 1, 0, x] Log[1 - x] - 680 π^2 Log[ms] Log[1 - x] + 680 π^2 x Log[ms] Log[1 - x] +
  1440 H[1, 0, x] Log[ms] Log[1 - x] - 1440 x H[1, 0, x] Log[ms] Log[1 - x] +
  40 Log[ms]^3 Log[1 - x] - 40 x Log[ms]^3 Log[1 - x] + 300 π^2 Log[1 - x]^2 -
  300 π^2 x Log[1 - x]^2 + 240 H[0, 0, x] Log[1 - x]^2 - 240 x H[0, 0, x] Log[1 - x]^2 -
  720 H[1, 0, x] Log[1 - x]^2 + 720 x H[1, 0, x] Log[1 - x]^2 + 720 H[0, x] Log[ms] Log[1 - x]^2 -
  720 x H[0, x] Log[ms] Log[1 - x]^2 - 60 Log[ms]^2 Log[1 - x]^2 + 60 x Log[ms]^2 Log[1 - x]^2 -
  560 H[0, x] Log[1 - x]^3 + 560 x H[0, x] Log[1 - x]^3 - 200 Log[ms] Log[1 - x]^3 +
  200 x Log[ms] Log[1 - x]^3 + 190 Log[1 - x]^4 - 190 x Log[1 - x]^4 -
  6320 H[1, x] Zeta[3] + 1920 x H[1, x] Zeta[3] + 1280 Log[ms] Zeta[3] -
  1120 x Log[ms] Zeta[3] - 2080 Log[1 - x] Zeta[3] + 2080 x Log[1 - x] Zeta[3])
```

# Catani's recipe: An important test

One loop: The IR pole structure of the renormalized amplitude can be known by only knowing the tree level amplitude:

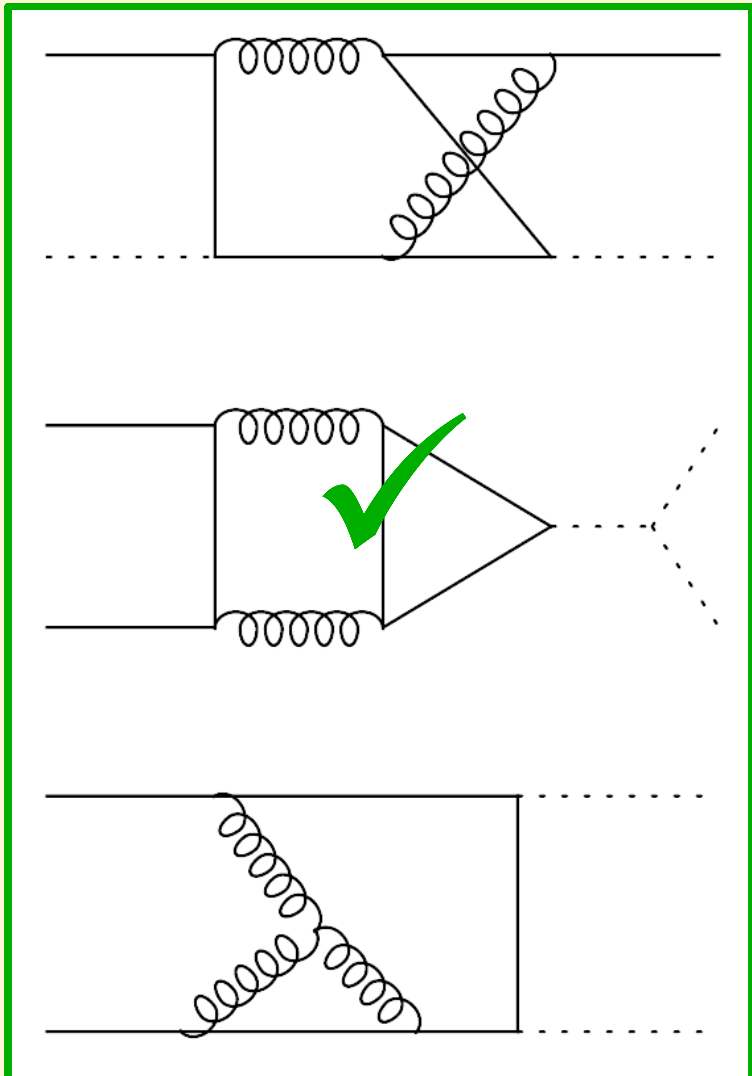
$$|\mathcal{M}_m^{(1)}(\mu^2;\{p\})\rangle_{\rm R.S.} = \boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\,|\mathcal{M}_m^{(0)}(\mu^2;\{p\})\rangle_{\rm R.S.} + |\mathcal{M}_m^{(1)\,{\rm fin}}(\mu^2;\{p\})\rangle_{\rm R.S.}$$

Two loop: Now you need tree and one loop level amplitude:

$$\begin{aligned}|\mathcal{M}_m^{(2)}(\mu^2;\{p\})\rangle_{\rm R.S.} &= \boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\})\,|\mathcal{M}_m^{(1)}(\mu^2;\{p\})\rangle_{\rm R.S.}\\ &+ \boldsymbol{I}_{\rm R.S.}^{(2)}(\epsilon,\mu^2;\{p\})\,|\mathcal{M}_m^{(0)}(\mu^2;\{p\})\rangle_{\rm R.S.} + |\mathcal{M}_m^{(2){\rm fin}}(\mu^2;\{p\})\rangle_{\rm R.S.}\end{aligned}$$

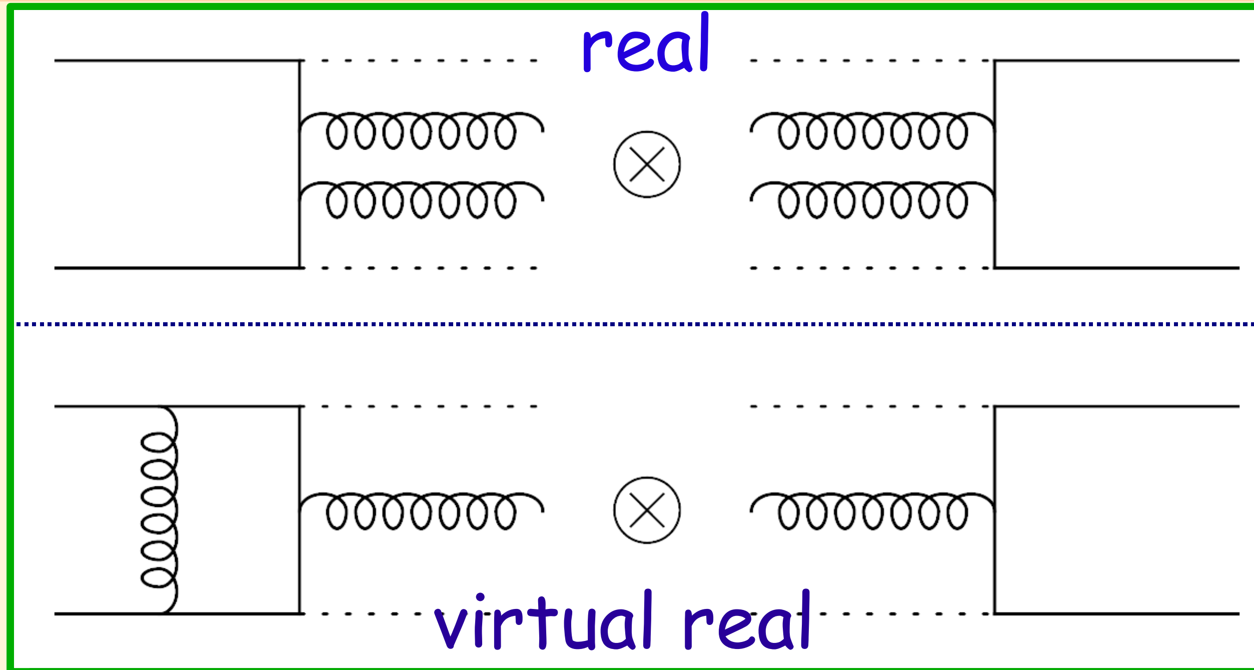
Singular dependence embodied in the operators $I^{(1)}$ and $I^{(2)}$

# Check list:



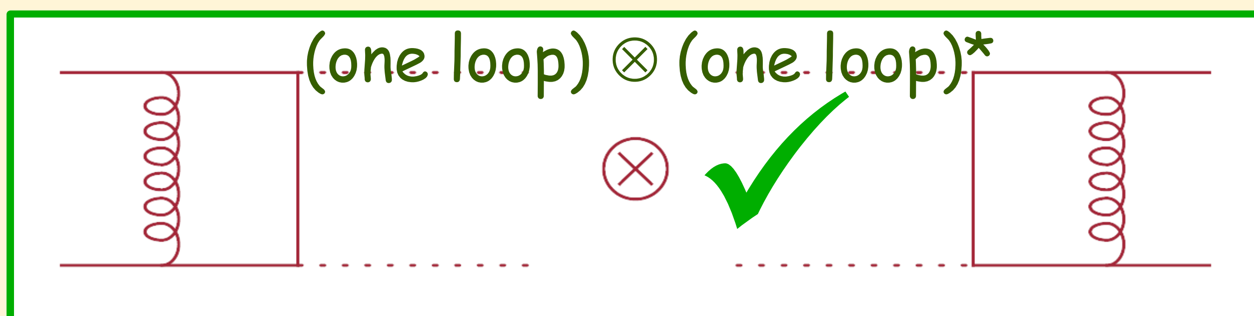
2 loop virtual

G.C., Czakon, Eiras ('08)




Campbell, Ellis, Zanderighi ('07)

Dittmaier, Kallweit, Uwer ('08)




G.C., Czakon, Eiras (to be published)

# Conclusions

- We have finally the full (virtual) result up to $O(m_s^0)$
- Mellin Barnes representations approach is a powerful technique
- Not an easy one though (especially for the non-planar graphs)
- Higher power corrections are being produced (to be published )

Next step: Full mass dependence

Similar to M. Czakon [arXiv:0803.1400]

•Deep expansion in mass around the high energy limit.

•Numerical Differential Equation method (more in the talks of M. Czakon and R. Boughezal)

Caffo, Czyz, Laporta, Remiddi ('98)

# Outlook

- The ultimate goal is to have a

  NNLO Monte Carlo generator

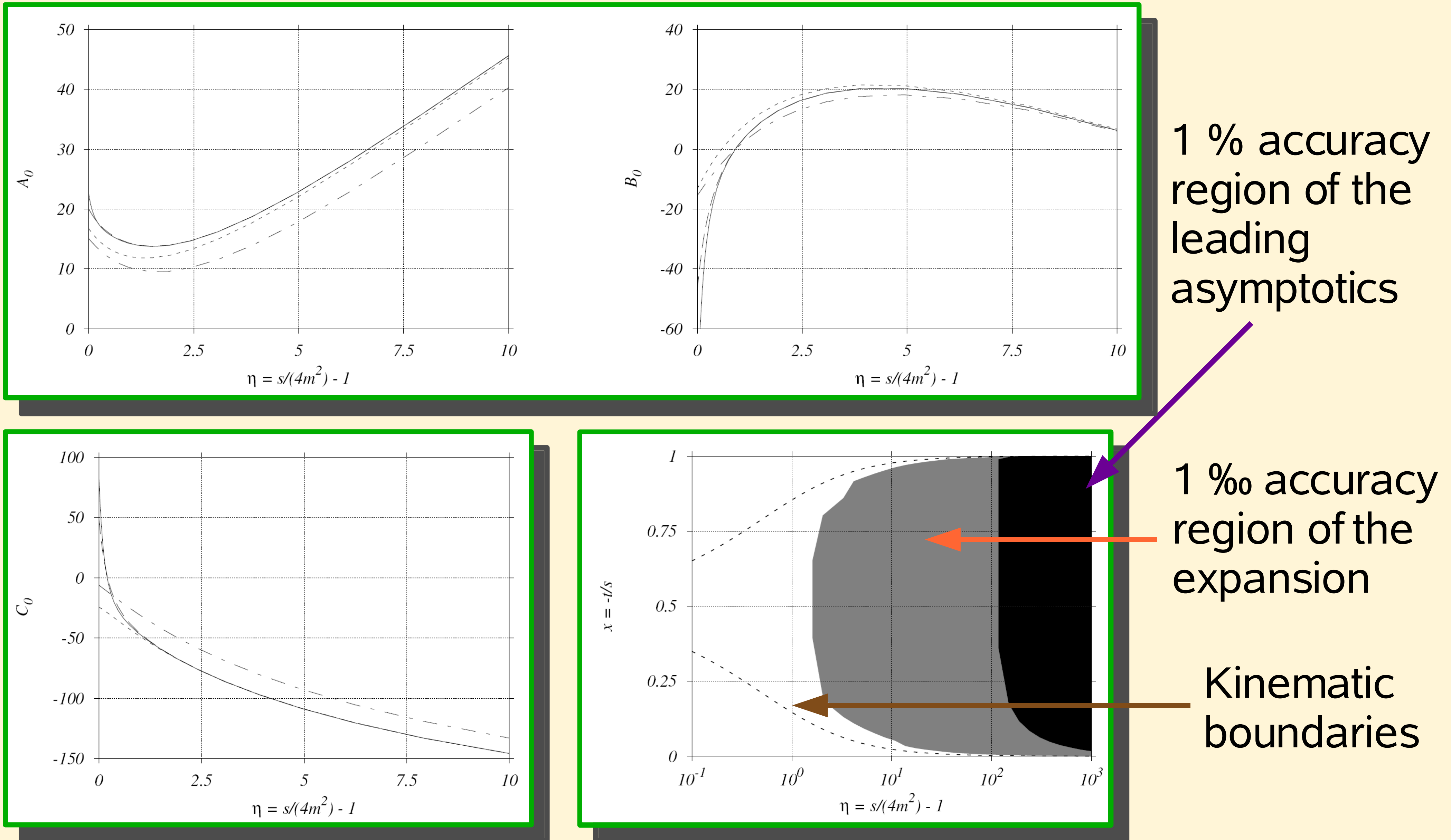
  for Gauge Boson Pair Production
- The first steps in this direction (and probably not the easier ones) have been done, many more are to follow

# Back up slides

# Power corrections

- expansion in $m_s$ (11 terms)

Czakon ('08 )

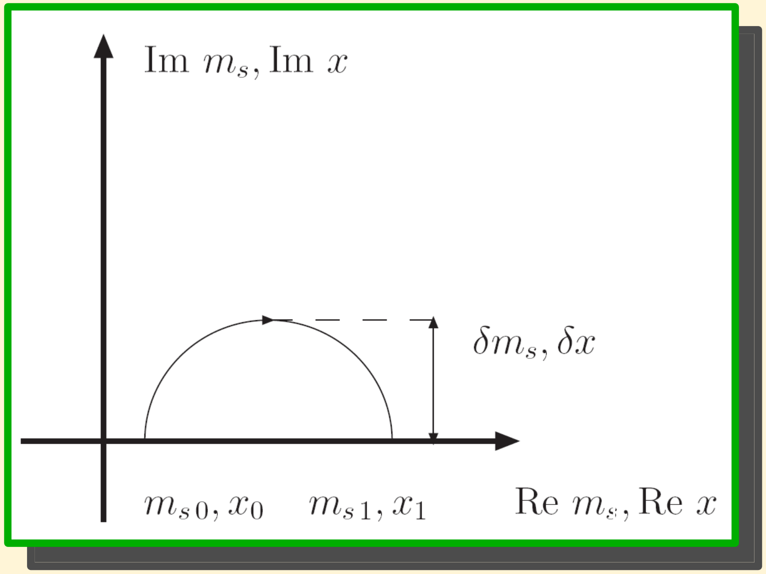


# Towards a numerical solution

- Compute the high energy asymptotics of the master integrals obtaining the leading behaviour of the amplitude
- Determine the coefficients of the mass expansions using differential equations in $m_s$ obtaining the power corrections

$$m_s \frac{d}{d m_s} M_i(m_s, x, \epsilon) = \sum_j C_{ij}(m_s, x, \epsilon) M_j(m_s, x, \epsilon)$$

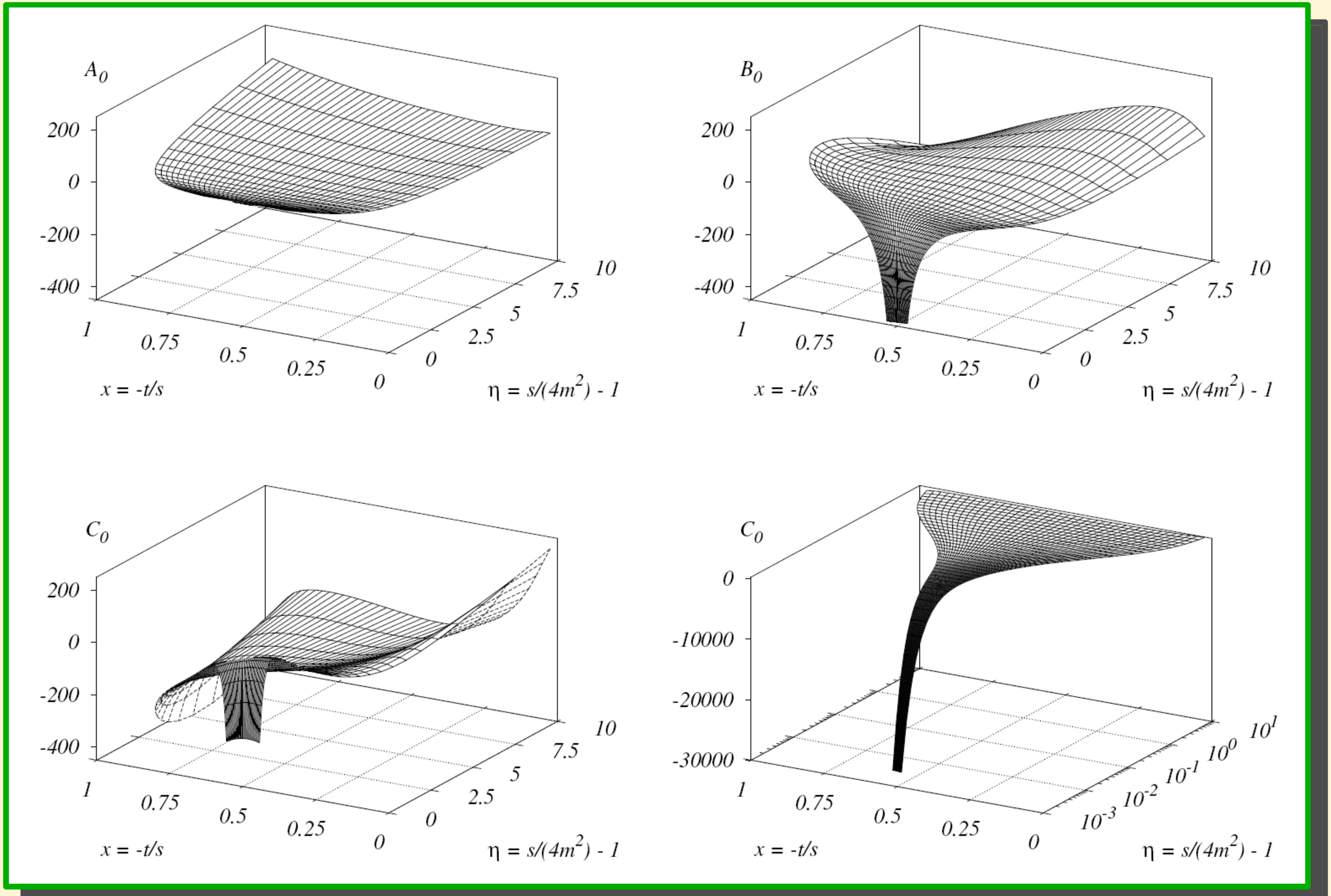
- Evaluate the expansions for $m_s \ll 1$ to obtain the desired numerical precision of the boundaries
- Evolve the functions from the boundary point with differential equations first in $m_s$ and then in $x$ (**ZVODE**)



# Full mass dependence

- Numerical solution of differential equations

Czakon ('08)



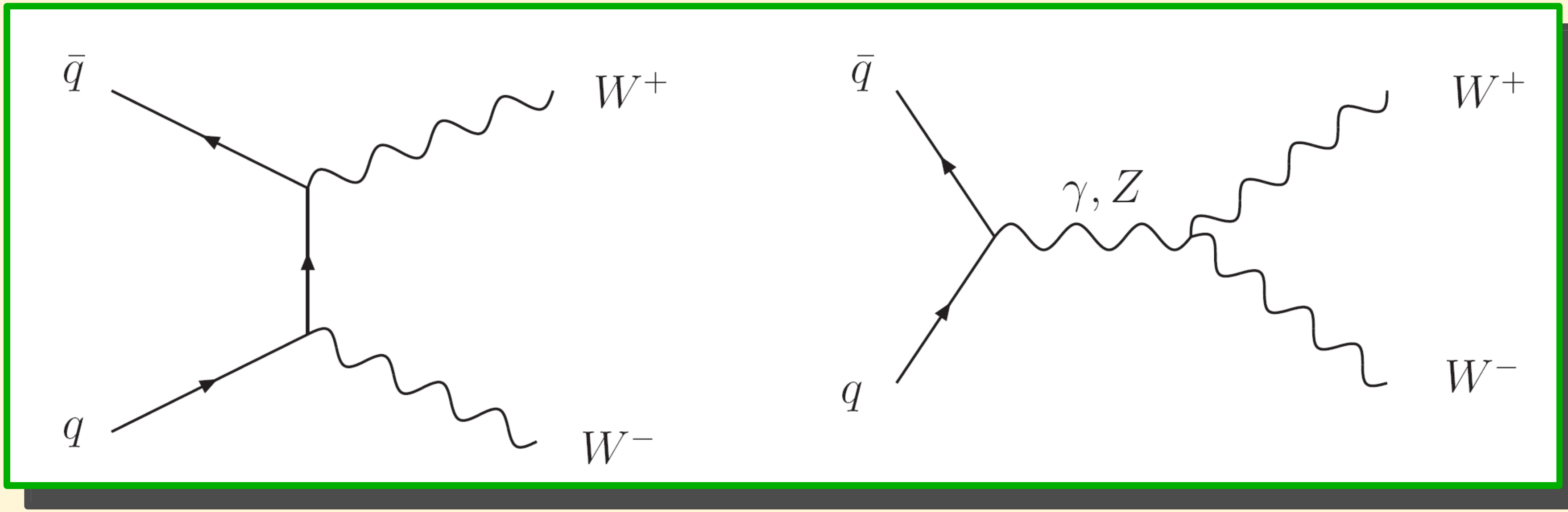
# Example point

$m^2 = .2\,s,\ x = 0.45$

| | leading color | | | full color | | |
|---|---|---|---|---|---|---|
| number of masters | 36 | | | 145 | | |
| number of functions | 155 | | | 595 | | |
| precision | quadruple | | double | quadruple | | double |
| evolution in $m^2/s$ | | | | | | |
| requested local error | $10^{-20}$ | $10^{-12}$ | $10^{-12}$ | $10^{-20}$ | $10^{-12}$ | $10^{-12}$ |
| contour deformation | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| number of steps taken | 2319 | 618 | 534 | 2932 | 777 | 1302 |
| jacobian evaluation time [ms] | 3.4 | 3.4 | 0.2 | 37 | 37 | 4.9 |
| evolution in $x$ | | | | | | |
| requested local error | $10^{-18}$ | $10^{-10}$ | $10^{-10}$ | $10^{-18}$ | $10^{-10}$ | $10^{-10}$ |
| contour deformation | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 |
| number of steps taken | 545 | 139 | 139 | 739 | 174 | 432 |
| jacobian evaluation time [ms] | 8.3 | 8.3 | 0.4 | 150 | 150 | 17 |
| total evaluation time [s] | 49 | 13 | $< 1$ | 957 | 243 | 26 |

| | $\epsilon^{-4}$ | $\epsilon^{-3}$ | $\epsilon^{-2}$ | $\epsilon^{-1}$ | $\epsilon^{0}$ |
|---|---|---|---|---|---|
| $A$ | 0.22625 | 1.391733154 | -2.298174307 | -4.145752449 | 17.37136599 |
| $B$ | -0.4525 | -1.323646320 | 8.507455541 | 6.035611156 | -35.12861106 |
| $C$ | 0.22625 | -0.06808683395 | -18.00716652 | 6.302454931 | 3.524044913 |
| $D_l$ | | -0.22625 | 0.2605057339 | -0.7250180282 | -1.935417247 |
| $D_h$ | | | 0.5623350684 | 0.1045606449 | -1.704747998 |
| $E_l$ | | 0.22625 | -0.3323207300 | 7.904121951 | 2.848697837 |
| $E_h$ | | | -0.5623350684 | 4.528240788 | 12.73232424 |
| $F_l$ | | | | | -1.984228442 |
| $F_{lh}$ | | | | | -2.442562819 |
| $F_h$ | | | | | -0.07924540546 |

# History I (... and some diagrams)



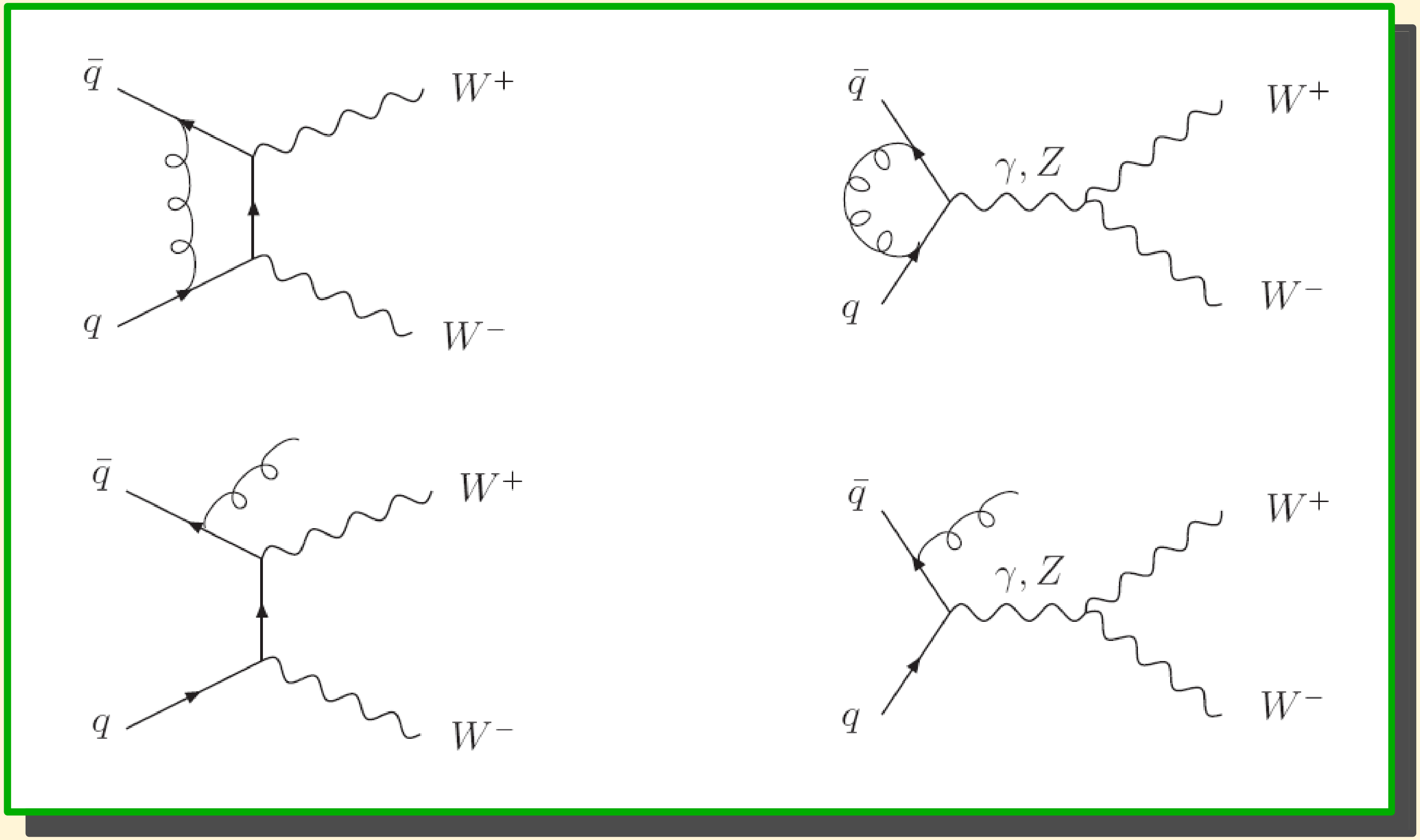

LO Calculation

Brown, Mikaelian ('79)

CERN Discovery of Z and W bosons

('83)

# History II (... and more diagrams)




NLO Calculation — Ohnemus ('91); Frixione ('93)

Also, — Ohnemus('94)

Dixon, Kunszt, Signer ('98,'99)

Campbell, K. Ellis ('99)

# The amplitude would look like:

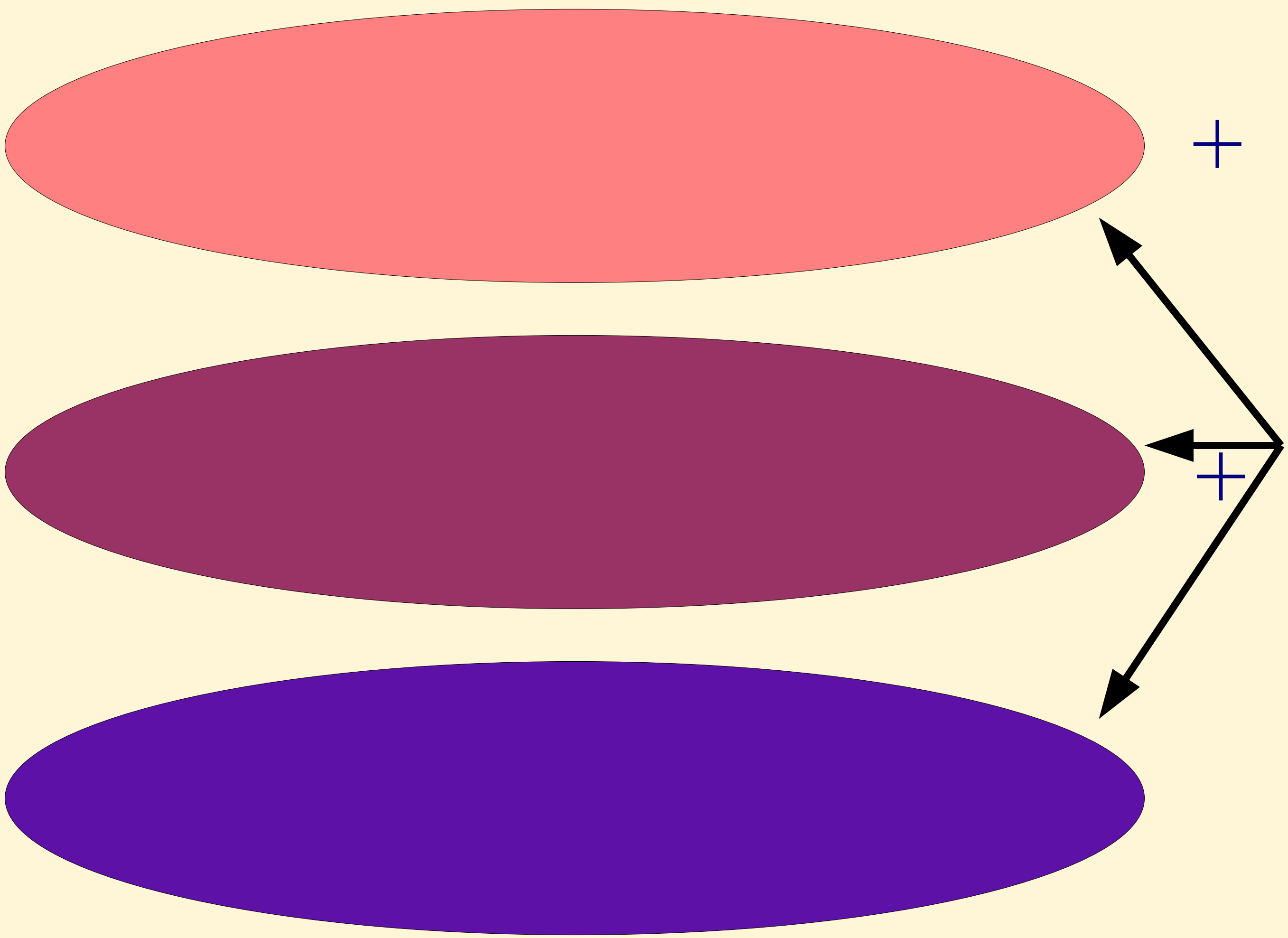
## Color Decomposition:

$$\sum \langle \mathcal{M}^2 | \mathcal{M}^0 \rangle =$$

$C_F^{\ 2}$

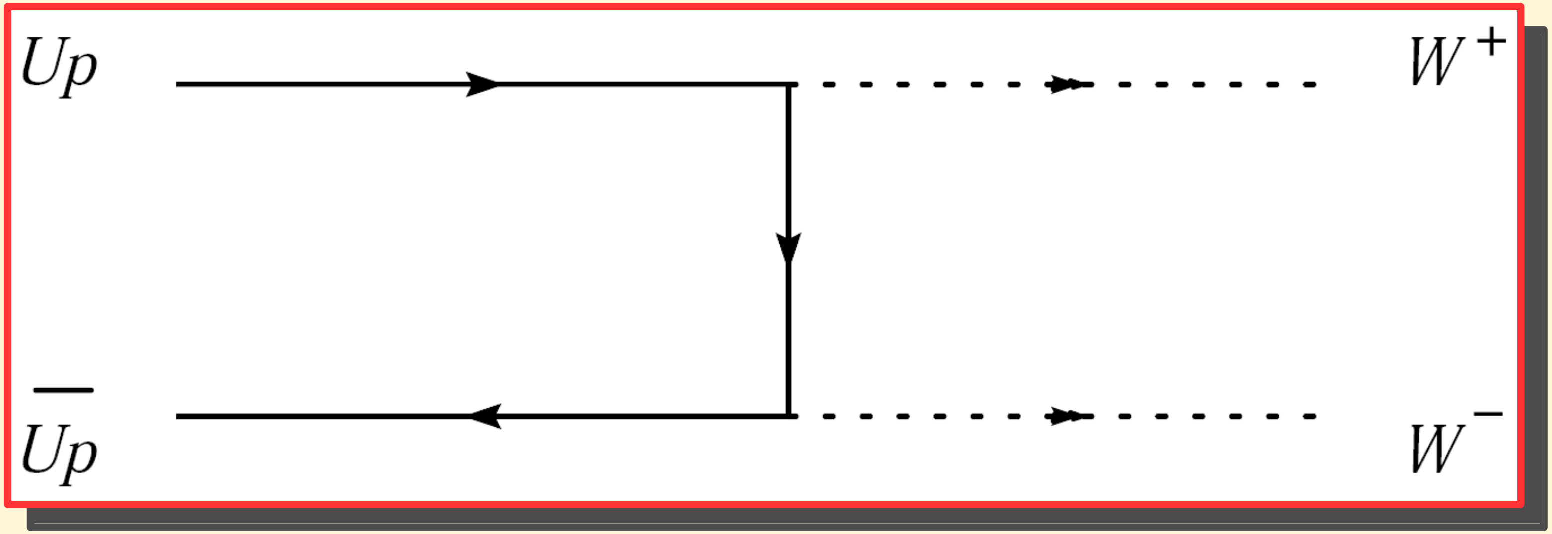
$C_F\, C_A$

$n_f\, T_F\, C_F$



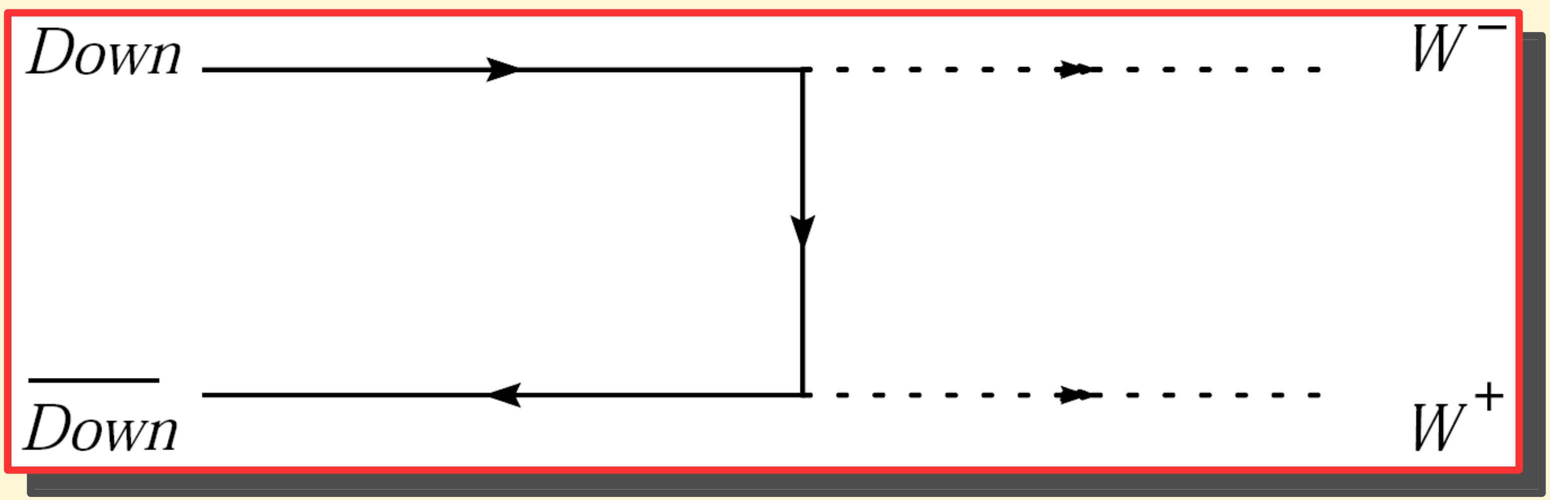

Color coefficients. Calculate these and you are done!

# “ Split the work in half… ”




From an up-type diagram to a down-type diagram use the formal substitution

$W^+ \leftrightarrow W^-$



# A convinient decomposition:

The invariant squared amplitude for the Born process can be decomposed as :

$$\sum_{\text{spin,color}} |\mathcal{M}|^2 = N_c \left[ c_i^{tt} F_i(s, t) - c_i^{ts} J_i(s, t) + c_i^{ss} K_i(s, t) \right]$$

where $c_i^{tt}$, $c_i^{ts}$, $c_i^{ss}$ are "coupling constants".

They generally depend on the mass $M_Z$, weak mixing angle, $\theta_w$, charge and isospin of the quark

# Done!

For the amplitude squared the change up-type $\leftrightarrow$ down-type is given by:

$$F_{down}(s, t) = F_{up}(s, u)$$

$$J_{down}(s, t) = -J_{up}(s, u)$$

$$K_{down}(s, t) = K_{up}(s, u)$$

and the corresponding changes to the couplings...

# The main difficulty remains though

Having a way to make the transition between up-type and down-type quarks is very nice!

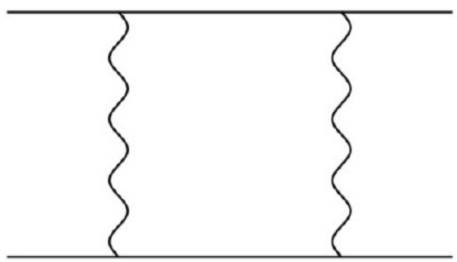
But still one needs to compute the color coefficients for either the up-type or down-type quarks...

The difficult part here is to compute the masters
We choose to do that by the
Mellin-Barnes representations technique

# Mellin-Barnes: a simple example

$$\frac{1}{(A+B)^n} = \frac{1}{2\pi i \Gamma(n)} \int_{-i\infty}^{i\infty} dz \frac{A^z}{B^{n+z}} \Gamma(-z)\Gamma(n+z)$$

- Example

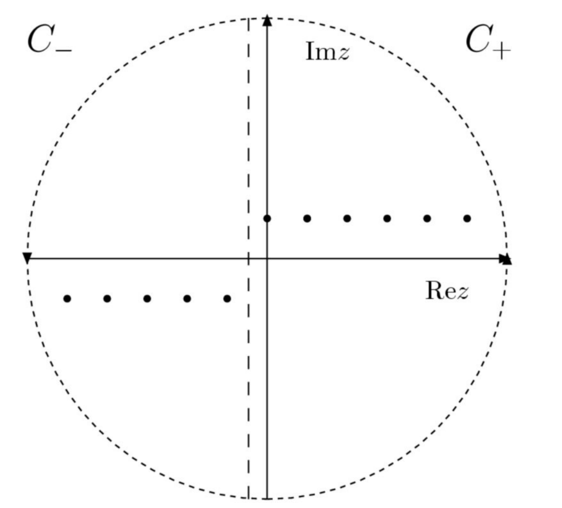


$$e^{\epsilon\gamma}\Gamma(2+\epsilon)\int dx_1 ... dx_4 \delta(1-x_1-...-x_4)\frac{1}{(-sx_2x_3-tx_1x_4)^{2+\epsilon}}$$

$$\frac{e^{\epsilon\gamma}}{2\pi i}\frac{1}{(-s)^{2+\epsilon}}\int_{-i\infty}^{i\infty} dz \ \left(\frac{t}{s}\right)^z \frac{\Gamma^2(-1-\epsilon-z)\Gamma(-z)\Gamma^2(1+z)\Gamma(2+\epsilon+z)}{\Gamma(-2\epsilon)}$$

$$\mathrm{Re}\ \epsilon = -\frac{1}{2}, \quad \mathrm{Re}\ z = -\frac{3}{4}$$

# Mellin-Barnes representations




Under the assumption that n>0 and that the contour separates the poles of the Gamma functions

$$\frac{1}{(A+B)^n} = \frac{1}{2\pi i \Gamma(n)} \int_{-i\infty}^{i\infty} dz \frac{A^z}{B^{n+z}} \Gamma(-z)\Gamma(n+z)$$

Behaviour of the Gamma functions around non-positive arguments

$$z\Gamma(z) = \Gamma(1+z) \quad \Rightarrow \quad \Gamma(-n+z) = \frac{\Gamma(1+z)}{(-n+z)...(z)} \sim \frac{(-)^n}{n!}\frac{1}{z}$$

Take residues depending on the values of A and B

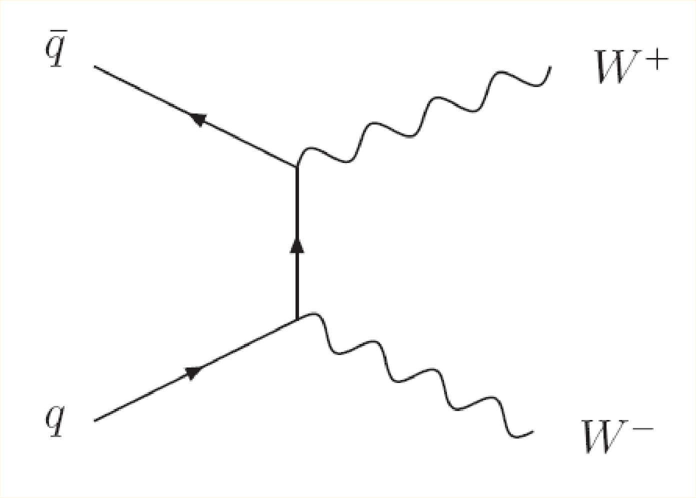
For $A > B \quad \Rightarrow \quad z < 0 \quad \Rightarrow \quad z = -N-n,\; N = 0, 1, ...$

$$\sum_{N=0}^{\infty} \frac{(-)^N}{N!} \frac{\Gamma(N+n)}{\Gamma(n)} \frac{B^N}{A^{N+n}} = \frac{1}{A^n} \sum_{N=0}^{\infty} \frac{(N+n-1)!}{(n-1)!N!} \left(-\frac{B}{A}\right)^N = \frac{1}{A^n} \frac{1}{\left(1+\frac{B}{A}\right)^n} = \frac{1}{(A+B)^n}$$

For $A < B \quad \Rightarrow \quad z > 0 \quad \Rightarrow \quad z = N,\; N = 0, 1, ...$

$$\sum_{N=0}^{\infty} \frac{(-)^N}{N!} \frac{\Gamma(N+n)}{\Gamma(n)} \frac{A^N}{B^{N+n}} = \frac{1}{B^n} \sum_{N=0}^{\infty} \frac{(N+n-1)!}{(n-1)!N!} \left(-\frac{A}{B}\right)^N = \frac{1}{B^n} \frac{1}{\left(1+\frac{A}{B}\right)^n} = \frac{1}{(A+B)^n}$$

# Born result for the coupling $c_i^{tt}$




$$-\frac{4\,(-1+x)\,x}{ms^2}-\frac{8\,(-2+2\,ep+x)}{ms}-\frac{4\,(-1+(5-8\,ep+4\,ep^2)\,x)}{-1+x}$$

The amplitude in ms starts at order $ms^{-2}$

That is due to:

$$\sum_{\lambda=0,\pm1}\varepsilon_\mu(p,\lambda)\varepsilon_\nu(p,\lambda)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}$$

1st Barnes' lemma

$$\int_{-i\infty}^{i\infty} dz\, \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z) = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

2nd Barnes' lemma

$$\int\limits_{-i\infty}^{i\infty} dz \; \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d-z)\Gamma(e-z)}{\Gamma(a+b+c+d+e+z)} =$$

$$\frac{\Gamma(a+d)\Gamma(a+e)\Gamma(b+d)\Gamma(b+e)\Gamma(c+d)\Gamma(c+e)}{\Gamma(a+b+d+e)\Gamma(a+c+d+e)\Gamma(b+c+d+e)}.$$

$$\mathrm{I}^{(1)}(\varepsilon) = -C_F \frac{e^{\varepsilon\gamma}}{\Gamma(1-\varepsilon)} \left(\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon}\right) \left(-\frac{\mu^2}{s}\right)^{\varepsilon}$$

$$\begin{aligned}
\mathrm{I}^{(2)}(\varepsilon) \;=\;& -\frac{1}{2}\mathrm{I}^{(1)}(\varepsilon)\left(\mathrm{I}^{(1)}(\varepsilon) + \frac{2\beta_0}{\varepsilon}\right) + \frac{e^{-\varepsilon\gamma}\Gamma(1-2\varepsilon)}{\Gamma(1-\varepsilon)}\left(\frac{\beta_0}{\varepsilon} + K\right)\mathrm{I}^{(1)}(2\varepsilon) \\
& + H^{(2)}(\varepsilon),
\end{aligned}$$

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_F n_f .$$

$$\begin{aligned}
H^{(2)}(\varepsilon) \;=\;& 2\frac{e^{\varepsilon\gamma}}{4\varepsilon\Gamma(1-\varepsilon)}\left(-\frac{\mu^2}{s}\right)^{2\varepsilon}\left\{\left(\frac{\pi^2}{2} - 6\,\zeta_3 - \frac{3}{8}\right) C_F^2\right. \\
& \left. + \left(\frac{13}{2}\zeta_3 + \frac{245}{216} - \frac{23}{48}\pi^2\right) C_A C_F + \left(-\frac{25}{54} + \frac{\pi^2}{12}\right) C_F T_F n_f\right\}
\end{aligned}$$

$$
\begin{aligned}
&g^{3}(m_s,x) = \\
&C_A C_F \left\{ \frac{1}{ms^2}\left[\frac{31}{240}(1-x)x\pi^4 - \frac{107}{72}(1-x)x\pi^2 - \frac{51157(1-x)x}{1296} + \frac{659}{36}(1-x)x\zeta_3 + \frac{44}{3}(1-x)x\mathrm{L}_s\right]\right. \\
&+\frac{1}{ms}\left[\frac{31}{240}\left(4x^2-4x+3\right)\pi^4 - \frac{107}{72}\left(4x^2-4x+3\right)\pi^2 - \frac{51157\left(4x^2-4x+3\right)}{1296} + \frac{659}{36}\left(4x^2-4x+3\right)\zeta_3\right. \\
&\left.+\frac{44}{3}\left(4x^2-4x+3\right)\mathrm{L}_s\right] + \left[-\frac{31}{20}\left(x^2-x+1\right)\pi^4 + \frac{107}{6}\left(x^2-x+1\right)\pi^2 + \frac{1}{108}\left(51157x^2 - 51157x\right.\right. \\
&\left.\left.\left. + \left(-23724x^2+23724x-23724\right)\zeta_3 + 51157\right) - 176\left(x^2-x+1\right)\mathrm{L}_s\right]\right\} \\
&+C_F{}^2 \left\{ \frac{1}{ms^2}\left[-\frac{11}{90}(1-x)x\pi^4 + \frac{29}{12}(1-x)x\pi^2 + \frac{255}{16}(1-x)x - 15(1-x)x\zeta_3\right]\right. \\
&+\frac{1}{ms}\left[-\frac{11}{90}\left(4x^2-4x+3\right)\pi^4 + \frac{29}{12}\left(4x^2-4x+3\right)\pi^2 + \frac{255}{16}\left(4x^2-4x+3\right) - 15\left(4x^2-4x+3\right)\zeta_3\right] \\
&\left.+\left[\frac{22}{15}\left(x^2-x+1\right)\pi^4 - 29\left(x^2-x+1\right)\pi^2 + \frac{45}{4}\left(-17x^2+17x+\left(16x^2-16x+16\right)\zeta_3 - 17\right)\right]\right\} \\
&+n_f T_F C_F \left\{ \frac{1}{ms^2}\left[\frac{7}{18}(1-x)x\pi^2 + \frac{4085}{324}(1-x)x - \frac{1}{9}(1-x)x\zeta_3 - \frac{16}{3}(1-x)x\mathrm{L}_s\right]\right. \\
&+\frac{1}{ms}\left[\frac{7}{18}\left(4x^2-4x+3\right)\pi^2 + \frac{4085}{324}\left(4x^2-4x+3\right) + \frac{1}{9}\left(-4x^2+4x-3\right)\zeta_3 - \frac{16}{3}\left(4x^2-4x+3\right)\mathrm{L}_s\right] \\
&\left.+\left[-\frac{14}{3}\left(x^2-x+1\right)\pi^2 + \frac{1}{27}\left(-4085x^2+4085x+\left(36x^2-36x+36\right)\zeta_3 - 4085\right) + 64\left(x^2-x+1\right)\mathrm{L}_s\right]\right\}
\end{aligned}
$$

$$
\mathrm{L}_m = \log\left(m_s\right), \quad \mathrm{L}_s = \log\left(\frac{s}{\mu^2}\right), \quad \mathrm{L}_x = \log\left(x\right), \quad \mathrm{L}_y = \log\left(1-x\right)
$$