# WHIZARD: Current and Future Developments

Tobias Kleinschmidt

LoopFest VII, SUNY Buffalo
May 2008

In collaboration with
W. Kilian (Siegen), T. Ohl (Würzburg), J. Reuter (Freiburg)

http://whizard.event-generator.org/

UNIVERSITÄT SIEGEN

# Outline

# Multi-Particle Simulations

**Physics processes:**

$$\underbrace{\text{pp} \to \underbrace{qq/qg/gg \to \text{partons (+leptons, photons)}}_{\text{Parton-level simulation}} \to \text{jets} \to \text{hadrons}}_{\text{Physics simulation}} \to \text{tracks} \to \dots$$

Many (most) interesting LHC processes have 4+ partons in final state:
SUSY, strongly interacting $W$s, Higgs processes, ...

Often, dominant pieces come from $2 \to 2$ processes and cascade decays
$\Rightarrow$ well done by on-shell process libraries (e.g. PYTHIA) ... but not always.

**And:** Need off-shell effects (why?)

1. ISR: PDF approach to initial state misses high-$p_T$ radiation that may spoil signal ID
2. Signal: few to few 10 percent: comparable to NLO corrections and PDF uncertainties
3. Background: much more, since kinematics is typically forced off-shell! On-shell or Breit-Wigner approximation underestimate background by 100% or more.

# Multi-Particle Simulations

**Physics processes:**

$$\underbrace{\text{pp} \to \underbrace{qq/qg/gg \to \text{partons (+leptons, photons)}}_{\text{Parton-level simulation}} \to \text{jets} \to \text{hadrons}}_{\text{Physics simulation}} \to \text{tracks} \to \ldots$$

Many (most) interesting LHC processes have 4+ partons in final state:
SUSY, strongly interacting $W$s, Higgs processes, ...

Often, dominant pieces come from $2 \to 2$ processes and cascade decays
$\Rightarrow$ well done by on-shell process libraries (e.g. PYTHIA) ... but not always.

**And:** Need off-shell effects (why?)

1. ISR: PDF approach to initial state misses high-$p_T$ radiation that may spoil signal ID
2. Signal: few to few 10 percent: comparable to NLO corrections and PDF uncertainties
3. Background: much more, since kinematics is typically forced off-shell! On-shell or Breit-Wigner approximation underestimate background by 100% or more.

# The Multi-Particle Generator WHIZARD

Kilian/Ohl/Reuter, 07

Very high level of Complexity:

- $e^+e^- \to t\bar{t}H \to b\bar{b}b\bar{b}jj\ell\nu$ (110,000 diagrams)
- $e^+e^- \to ZHH \to ZWWWW \to bb + 8j$ (12,000,000 diagrams)
- $pp \to \ell\ell + nj, n = 0, 1, 2, 3, 4, \ldots$ (2,100,000 diagrams with 4 jets + flavors)
- $pp \to \tilde{\chi}_1^0\tilde{\chi}_1^0 bbbb$ (32,000 diagrams, 22 color flows, $\sim 10,000$ PS channels)
- $pp \to VVjj \to jj\ell\ell\nu\nu$ incl. anomalous TGC/QGC
- Test case $gg \to 9g$ (224,000,000 diagrams)

Current versions:

WHiZard 1.51 / O'Mega 000.011beta $\Omega$ $\to$ joint version: WHIZARD 1.92

release date: April 2008

one grand unified package (incl. VAMP, Circe, Circe 2, WHiZard, O'Mega)

New web address: `http://whizard.event-generator.org`

Standard Reference for new versions: Kilian/Ohl/Reuter, 0708.4233

Major upgrade this summer: WHIZARD 2.0.0

# WHIZARD: Matrix Element Generation

**Full matrix-element calculation:**

- ✓ Complete (no missing background) for given final state
- ✓ Gauge invariance, Breit-Wigner distributions, polarization can be implemented
- ✗ Full matrix element is CPU costly (thousands of Feynman diagrams)
- ✗ Need good (adaptive) phase space parameterization: otherwise, no result at all
- ✗ Current implementations may not suit experimental needs (inclusive event generation?)

Take a process definition and a set of Feynman rules to produce a (Fortran/C) function:

- The (squared) amplitude as a function of given momenta and helicities.

**call O'Mega** T. Ohl

- ⇒ Complete helicity amplitudes computed numerically and recursively
- ⇒ All redundancies eliminated by organizing the calculation (DAG = Directed Acyclical Graph)
- ⇒ Computation cost $\propto n^k$ instead of $n!$!

# WHIZARD: Matrix Element Generation

**Full matrix-element calculation:**

- ✓ Complete (no missing background) for given final state
- ✓ Gauge invariance, Breit-Wigner distributions, polarization can be implemented
- ✗ Full matrix element is CPU costly (thousands of Feynman diagrams)
- ✗ Need good (adaptive) phase space parameterization: otherwise, no result at all
- ✗ Current implementations may not suit experimental needs (inclusive event generation?)

Take a process definition and a set of Feynman rules to produce a (Fortran/C) function:

- The (squared) amplitude as a function of given momenta and helicities.

**call O'Mega** T. Ohl

- ⇒ Complete helicity amplitudes computed numerically and recursively
- ⇒ All redundancies eliminated by organizing the calculation (DAG = Directed Acyclical Graph)
- ⇒ Computation cost $\propto n^k$ instead of $n!$!

# WHIZARD: Phase Space Integration

- Matrix elements are complicated and vary over orders of magnitude
  - $\Rightarrow$ Uniform phase space sampling yields no result
  - $\Rightarrow$ No single parameterization allows for mapping the function into a constant
- **Solution:** Multi-channel parameterization with mappings and parameterizations adapted to Feynman diagram structure
  $*$ WHIZARD: Improve by VEGAS adaptation within each channel
- What does this mean in practice?
  - WHIZARD has to find the *important* channels: The Feynman diagrams which have the strongest peaks $\Rightarrow$ correspond to good parameterizations
  - WHIZARD has many degrees of freedom to adapt:
    - The optimal binning of each integration dimension (10 – 50)
    - This is needed for each integration dimension (10 – 20)
    - The optimal relative weight of each channel (10 – 1000)

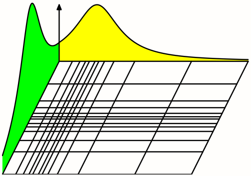
  $\Rightarrow 10^3 - 10^6$ degrees of freedom have to self-optimize

$*$ Apparently, this works – and at least as good as other methods

# WHIZARD: Phase Space Integration

- Matrix elements are complicated and vary over orders of magnitude
  - $\Rightarrow$ Uniform phase space sampling yields no result
  - $\Rightarrow$ No single parameterization allows for mapping the function into a constant
- **Solution:** Multi-channel parameterization with mappings and parameterizations adapted to Feynman diagram structure
  $\ast$ WHIZARD: Improve by VEGAS adaptation within each channel
- What does this mean in practice?
  - WHIZARD has to find the *important* channels: The Feynman diagrams which have the strongest peaks $\Rightarrow$ correspond to good parameterizations
  - WHIZARD has many degrees of freedom to adapt:
    - The optimal binning of each integration dimension (10 – 50)
    - This is needed for each integration dimension (10 – 20)
    - The optimal relative weight of each channel (10 – 1000)

  $\Rightarrow 10^3 - 10^6$ degrees of freedom have to self-optimize



$\ast$ Apparently, this works – and at least as good as other methods

# Implemented Physics Content

**Structured beams:**

For Tevatron/LHC: PDFs from LHAPDF (or PDFLIB)

For ILC physics:

- ISR (implemented: Skrzypek/Jadach, Kuraev/Fadin)
- arbitrarily polarized beams
- beamstrahlung, photon collider spectra (CIRCE/CIRCE 2)

external (user-defined) beam spectra can be read in

+ Parton Shower (final state)

**Supported Physics Models:**

- Test models: QED, QCD
- SM
- Littlest/Simplest Little Higgs, Little Higgs Models with $T$ parity
- Moose models: 3-site model
- MSSM, NMSSM, extended SUSY models, incl. gravitinos (SLHA/SLHA2)
- Graviton resonances, Universal extra dimensions, Randall-Sundrum
- Noncommutative Standard Model
- Higher-dimensional operators, SM effective field theory extensions
- Anomalous triple and quartic gauge couplings
- K-matrix/Padé unitarization, unitarized resonances Alboteanu/Kilian/Reuter

# MSSM

MSSM implementation: cross-check (over 500 processes) with Madgraph and Sherpa
hep-ph/0512260

| $e^+e^- \to X\ (I)$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Final state | status | Madgraph/Helas 0.5 TeV | Madgraph/Helas 2 TeV | Whizard/O'Mega 0.5 TeV | Whizard/O'Mega 2 TeV | Sherpa/A'Megic 0.5 TeV | Sherpa/A'Megic 2 TeV |
| $\tilde{e}_L \tilde{e}_L^*$ | ● | 54.687(2) | 78.864(6) | 54.687(3) | 78.866(4) | 54.6890(7) | 78.8670(8) |
| $\tilde{e}_R \tilde{e}_R^*$ | ● | 274.69(2) | 91.776(8) | 274.682(1) | 91.776(5) | 274.695(3) | 91.778(1) |
| $\tilde{e}_L \tilde{e}_R^*$ | ● | 75.168(5) | 7.237(1) | 75.167(3) | 7.2372(4) | 75.1693(7) | 7.23744(7) |
| $\tilde{\mu}_L \tilde{\mu}_L^*$ | ● | 22.5471(7) | 6.8263(2) | 22.5478(9) | 6.8265(3) | 22.5482(2) | 6.82638(7) |
| $\tilde{\mu}_R \tilde{\mu}_R^*$ | ● | 51.839(2) | 5.8107(2) | 51.837(2) | 5.8105(2) | 51.8401(5) | 5.81085(6) |
| $\tilde{\tau}_1 \tilde{\tau}_1^*$ | ● | 55.582(2) | 5.7139(2) | 55.580(2) | 5.7141(2) | 55.5835(6) | 5.71399(6) |
| $\tilde{\tau}_2 \tilde{\tau}_2^*$ | ● | 19.0161(6) | 6.5047(2) | 19.0174(7) | 6.5045(3) | 19.0163(2) | 6.50473(7) |
| $\tilde{\tau}_1 \tilde{\tau}_2^*$ | ● | 1.4118(4) | 0.21406(1) | 1.41191(5) | 0.214058(8) | 1.41187(1) | 0.214067(2) |
| $\tilde{\nu}_e \tilde{\nu}_e^*$ | ● | 493.35(2) | 272.15(2) | 493.38(2) | 272.15(1) | 493.358(5) | 272.155(3) |
| $\tilde{\nu}_\mu \tilde{\nu}_\mu^*$ | ● | 14.8632(4) | 2.9231(1) | 14.8638(6) | 2.9232(1) | 14.8633(1) | 2.92309(3) |
| $\tilde{\nu}_\tau \tilde{\nu}_\tau^*$ | ● | 15.1399(5) | 2.9246(1) | 15.1394(8) | 2.9245(1) | 15.1403(2) | 2.92465(3) |
| $\tilde{u}_L \tilde{u}_L^*$ | ● | — | 7.6185(2) | — | 7.6188(3) | — | 7.61859(8) |
| $\tilde{u}_R \tilde{u}_R^*$ | ● | — | 4.6933(1) | — | 4.6935(2) | — | 4.69342(5) |
| $\tilde{c}_L \tilde{c}_L^*$ | ● | — | 7.6185(2) | — | 7.6182(3) | — | 7.61859(8) |
| $\tilde{c}_R \tilde{c}_R^*$ | ● | — | 4.6933(1) | — | 4.6933(2) | — | 4.69342(5) |
| $\tilde{t}_1 \tilde{t}_1^*$ | ● | — | 5.9845(4) | — | 5.9847(2) | — | 5.98459(6) |
| $\tilde{t}_2 \tilde{t}_2^*$ | ● | — | 5.3794(3) | — | 5.3792(2) | — | 5.37951(6) |
| $\tilde{t}_1 \tilde{t}_2^*$ | ● | — | 1.2427(1) | — | 1.24264(5) | — | 1.24270(1) |
| $\tilde{d}_L \tilde{d}_L^*$ | ● | — | 5.2055(1) | — | 5.2059(2) | — | 5.20563(2) |
| $\tilde{d}_R \tilde{d}_R^*$ | ● | — | 1.17588(2) | — | 1.17595(5) | — | 1.17591(1) |
| $\tilde{s}_L \tilde{s}_L^*$ | ● | — | 5.2055(1) | — | 5.2058(2) | — | 5.20563(2) |
| $\tilde{s}_R \tilde{s}_R^*$ | ● | — | 1.17588(2) | — | 1.17585(5) | — | 1.17591(1) |
| $\tilde{b}_1 \tilde{b}_1^*$ | ● | — | 4.9388(3) | — | 4.9387(2) | — | 4.93883(5) |
| $\tilde{b}_2 \tilde{b}_2^*$ | ● | — | 1.1295(1) | — | 1.12946(4) | — | 1.12953(1) |
| $\tilde{b}_1 \tilde{b}_2^*$ | ● | — | 0.51644(3) | — | 0.516432(9) | — | 0.516447(6) |

# Upcoming Features

**WHIZARD version 2.0.0 coming out this summer**

- New syntax for defining cuts, scales and analyses: allows for arbitrary functions of kinematical variables
- fancier (and faster) color structures from O'Mega
- WHIZARD uses O'Mega info for better/faster phase space generation
- Cascade decays (apply with great care!!!)
  WHIZARD calls itself recursively, breaks double decay chains down into subprocesses
- Leading order (QCD) parton shower
  (so only fragmentation/hadronization and PDFs by external routines)
- Dark matter relic density calculator
- Support for `ROOT` data format
- TAUOLA interface

**All points close to finalization;**
**Major restructuring of the code**

- Interface to `FeynArts`: all MSSM $2 \to 2$ processes for ILC available T. Robens

# Outline

# Derivation of the Feynman Tree Theorem (FTT)

- Integrand $I(k)$ of a one-loop graph with loop momentum $k$: $I(k) = N(k) \prod_i F_i$

  with *Feynman* Green Functions $F_i$ (t'Hooft-Feynman gauge): $F_i \equiv \frac{i}{(k+p_i)^2 - m_i^2 + i\epsilon}$

- Partial fraction decomposition yields (also: *advanced* Green functions $A_i$)

$$F_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i - i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right),$$

$$A_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i + i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right), \quad E_i = \sqrt{(\vec{k} + \vec{p}_i)^2 + m_i^2}$$

$$\Delta_i^l \equiv F_i - A_i \overset{\epsilon \to 0}{=} \frac{2\pi}{2E_i}\delta(k^0 - (-p_i^0 + E_i)).$$

$\Rightarrow \Delta_i^l$ sets momentum $k + p_i$ on-shell with positive energy component $E_i$.

- Idea: Start with $0 = \int N(k) \prod_i^n A_i$, and replace $A_i \to F_i - \Delta_i^l$:

**Feynman Tree Theorem (FTT)**

$$0 = \int N(k)\left[F \cdots F - \sum \Delta^l F \cdots + \sum \Delta^l \Delta^l F \cdots - \ldots + (-1)^n \sum \Delta^l \cdots \Delta^l\right]$$

Acta. Phys. Polon. **24** (1963) 697

- Possible drawback: FTT still includes $i\epsilon$ terms. Not used in numerical calculations. Role of higher order terms in FTT?

# Derivation of the Feynman Tree Theorem (FTT)

- Integrand $I(k)$ of a one-loop graph with loop momentum $k$: $I(k) = N(k)\prod_i F_i$

  with *Feynman* Green Functions $F_i$ (t'Hooft-Feynman gauge): $F_i \equiv \frac{i}{(k+p_i)^2 - m_i^2 + i\epsilon}$
- Partial fraction decomposition yields (also: *advanced* Green functions $A_i$)

$$F_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i - i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right),$$

$$A_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i + i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right), \quad E_i = \sqrt{(\vec{k} + \vec{p}_i)^2 + m_i^2}$$

$$\Delta_i^l \equiv F_i - A_i \overset{\epsilon \to 0}{=} \frac{2\pi}{2E_i}\delta(k^0 - (-p_i^0 + E_i)).$$

$\Rightarrow$ $\Delta_i^l$ sets momentum $k + p_i$ on-shell with positive energy component $E_i$.

- Idea: Start with $0 = \int N(k)\prod_i^n A_i$, and replace $A_i \to F_i - \Delta_i^l$:

## Feynman Tree Theorem (FTT)

$$0 = \int N(k)\left[F\cdots F - \sum \Delta^l F \cdots + \sum \Delta^l \Delta^l F \cdots - \ldots + (-1)^n \sum \Delta^l \cdots \Delta^l\right]$$

Acta. Phys. Polon. **24** (1963) 697

- Possible drawback: FTT still includes $i\epsilon$ terms. Not used in numerical calculations. Role of higher order terms in FTT?

# Derivation of the Feynman Tree Theorem (FTT)

- Integrand $I(k)$ of a one-loop graph with loop momentum $k$: $I(k) = N(k) \prod_i F_i$
  with *Feynman* Green Functions $F_i$ (t'Hooft-Feynman gauge): $F_i \equiv \frac{i}{(k+p_i)^2 - m_i^2 + i\epsilon}$
- Partial fraction decomposition yields (also: *advanced* Green functions $A_i$)

$$F_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i - i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right),$$

$$A_i = \frac{i}{2E_i}\left(\frac{1}{k^0 - (-p_i^0 + E_i + i\epsilon)} - \frac{1}{k^0 - (-p_i^0 - E_i + i\epsilon)}\right), \quad E_i = \sqrt{(\vec{k} + \vec{p}_i)^2 + m_i^2}$$

$$\Delta_i^l \equiv F_i - A_i \overset{\epsilon \to 0}{=} \frac{2\pi}{2E_i}\delta(k^0 - (-p_i^0 + E_i)).$$

$\Rightarrow \Delta_i^l$ sets momentum $k + p_i$ on-shell with positive energy component $E_i$.

- Idea: Start with $0 = \int N(k) \prod_i^n A_i$, and replace $A_i \to F_i - \Delta_i^l$:

## Feynman Tree Theorem (FTT)

$$0 = \int N(k)\left[F \cdots F - \sum \Delta^l F \cdots + \sum \Delta^l \Delta^l F \cdots - \ldots + (-1)^n \sum \Delta^l \cdots \Delta^l\right]$$

Acta. Phys. Polon. **24** (1963) 697

- Possible drawback: FTT still includes $i\epsilon$ terms. Not used in numerical calculations. Role of higher order terms in FTT?

# FTT: Improved Version

- Make use of identity:

$$\frac{1}{x - a \pm i\epsilon} = \mathcal{P}\frac{1}{x-a} \mp i\pi\delta(x-a)$$

- Rewrite Feynman Green function $F_i$:

$$F_i = P_i + \frac{1}{2}\Delta_i^l + \frac{1}{2}\Delta_i^u$$

$$P_i = \mathcal{P}\frac{i}{(k+p_i)^2 - m_i^2}$$

$$\Delta_i^u = \frac{2\pi}{2E_i}\delta(k^0 - (-p_i^0 - E_i))$$

- Replace any $F_i$ in subleading terms of FTT:

## Feynman Tree Theorem - Improved Version

$$\int I(k) = \int N(k)[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1}\Delta_n^l]$$

$$+ \int N(k) \sum_{\substack{perm.\\ U+L \geq 2}} C_{LUP}\, \Delta^{l^L} \Delta^{u^U} P^P,$$

$$C_{LUP} = \frac{1}{2^{L+U}}\left(1 - (-1)^L\right)$$

# FTT: Improved Version

- Make use of identity:

$$\frac{1}{x-a\pm i\epsilon} = \mathcal{P}\frac{1}{x-a} \mp i\pi\delta(x-a)$$

- Rewrite Feynman Green function $F_i$:

$$F_i = P_i + \frac{1}{2}\Delta_i^l + \frac{1}{2}\Delta_i^u$$

$$P_i = \mathcal{P}\frac{i}{(k+p_i)^2-m_i^2}$$

$$\Delta_i^u = \frac{2\pi}{2E_i}\delta(k^0-(-p_i^0-E_i))$$

- Replace any $F_i$ in subleading terms of FTT:

## Feynman Tree Theorem - Improved Version

$$\begin{aligned}\int I(k) &= \int N(k)\left[\Delta_1^l P_2\cdots P_n + P_1\Delta_2^l P_3\cdots P_n + \ldots + P_1\cdots P_{n-1}\Delta_n^l\right] \\ &+ \int N(k) \sum_{\substack{perm.\\ U+L\geq 2}} C_{LUP}\,\Delta^{l^L}\Delta^{u^U}P^P, \\ C_{LUP} &= \frac{1}{2^{L+U}}\left(1-(-1)^L\right)\end{aligned}$$

Leading terms: $$\int N(k)\left[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1}\Delta_n^l\right]$$

- Performing $k_0$ integration, $\Delta_i^l$ act as *opening* or *cutting* the loop:

- Momentum $k + p_i$ is set on-shell
- Numerator of cut propagator is product of wave functions, summed over all internal states
- Loop integral is replaced by phase space integral

$$(\not k + \not p_i + m) = \sum_\lambda u_\lambda(k+p_i)\bar{u}_\lambda(k+p_i);$$

$$-g_{\mu\nu} \to \sum_\sigma \epsilon^*_\mu(k+p_i;\sigma)\epsilon_\nu(k+p_i;\sigma)$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^3k}{(2\pi)^3 2E_i}.$$



Loop corrections for a $2 \to n$ process can be computed by considering all possible $2+1 \to n+1$ tree graphs with an additional incoming and outgoing on-shell particle. A phase space integration over the additional particles' momenta has to be performed.

Leading terms: $$\int N(k)\left[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1}\Delta_n^l\right]$$

- Performing $k_0$ integration, $\Delta_i^l$ act as *opening* or *cutting* the loop:

  - Momentum $k + p_i$ is set on-shell
  - Numerator of cut propagator is product of wave functions, summed over all internal states
  - Loop integral is replaced by phase space integral

$$(\not{k} + \not{p}_i + m) = \sum_\lambda u_\lambda(k+p_i)\bar{u}_\lambda(k+p_i);$$

$$-g_{\mu\nu} \to \sum_\sigma \epsilon_\mu^*(k+p_i;\sigma)\epsilon_\nu(k+p_i;\sigma)$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^3k}{(2\pi)^3 2E_i}.$$



Loop corrections for a $2 \to n$ process can be computed by considering all possible $2+1 \to n+1$ tree graphs with an additional incoming and outgoing on-shell particle. A phase space integration over the additional particles' momenta has to be performed.

Leading terms: $$\int N(k)\left[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1}\Delta_n^l\right]$$

- Performing $k_0$ integration, $\Delta_i^l$ act as *opening* or *cutting* the loop:

- Momentum $k + p_i$ is set on-shell
- Numerator of cut propagator is product of wave functions, summed over all internal states
- Loop integral is replaced by phase space integral

$$(\not{k} + \not{p}_i + m) = \sum_\lambda u_\lambda(k+p_i)\bar{u}_\lambda(k+p_i);$$

$$-g_{\mu\nu} \to \sum_\sigma \epsilon_\mu^*(k+p_i;\sigma)\epsilon_\nu(k+p_i;\sigma)$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^3k}{(2\pi)^3 2E_i}.$$



Loop corrections for a $2 \to n$ process can be computed by considering all possible $2+1 \to n+1$ tree graphs with an additional incoming and outgoing on-shell particle. A phase space integration over the additional particles' momenta has to be performed.

Leading terms: $$\int N(k)\left[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1} \Delta_n^l\right]$$

- Performing $k_0$ integration, $\Delta_i^l$ act as *opening* or *cutting* the loop:

- Momentum $k + p_i$ is set on-shell
- Numerator of cut propagator is product of wave functions, summed over all internal states
- Loop integral is replaced by phase space integral

$$(\not{k} + \not{p}_i + m) = \sum_\lambda u_\lambda(k+p_i)\bar{u}_\lambda(k+p_i);$$

$$-g_{\mu\nu} \to \sum_\sigma \epsilon^*_\mu(k+p_i;\sigma)\epsilon_\nu(k+p_i;\sigma)$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^3k}{(2\pi)^3 2E_i}.$$



Loop corrections for a $2 \to n$ process can be computed by considering all possible $2+1 \to n+1$ tree graphs with an additional incoming and outgoing on-shell particle. A phase space integration over the additional particles' momenta has to be performed.

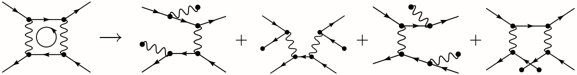
Leading terms: $$\int N(k)\left[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1}\Delta_n^l\right]$$

- Performing $k_0$ integration, $\Delta_i^l$ act as *opening* or *cutting* the loop:
  - Momentum $k + p_i$ is set on-shell
  - Numerator of cut propagator is product of wave functions, summed over all internal states
  - Loop integral is replaced by phase space integral

$$(\not{k} + \not{p}_i + m) = \sum_\lambda u_\lambda(k+p_i)\bar{u}_\lambda(k+p_i);$$

$$-g_{\mu\nu} \to \sum_\sigma \epsilon_\mu^*(k+p_i;\sigma)\epsilon_\nu(k+p_i;\sigma)$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^3k}{(2\pi)^3 2E_i}.$$



Loop corrections for a $2 \to n$ process can be computed by considering all possible $2+1 \to n+1$ tree graphs with an additional incoming and outgoing on-shell particle. A phase space integration over the additional particles' momenta has to be performed.

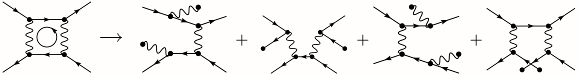
Leading terms: $$\int N(k)\left[\Delta_1^l P_2 \cdots P_n + P_1 \Delta_2^l P_3 \cdots P_n + \ldots + P_1 \cdots P_{n-1} \Delta_n^l\right]$$

- Performing $k_0$ integration, $\Delta_i^l$ act as *opening* or *cutting* the loop:
  - Momentum $k + p_i$ is set on-shell
  - Numerator of cut propagator is product of wave functions, summed over all internal states
  - Loop integral is replaced by phase space integral

$$(\not{k} + \not{p}_i + m) = \sum_\lambda u_\lambda(k+p_i)\bar{u}_\lambda(k+p_i);$$

$$-g_{\mu\nu} \rightarrow \sum_\sigma \epsilon^*_\mu(k+p_i;\sigma)\epsilon_\nu(k+p_i;\sigma)$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{d^3k}{(2\pi)^3 2E_i}.$$



Loop corrections for a $2 \rightarrow n$ process can be computed by considering all possible $2 + 1 \rightarrow n + 1$ tree graphs with an additional incoming and outgoing on-shell particle. A phase space integration over the additional particles' momenta has to be performed.

## Advantages

- Tree graphs simple to generate automatically (➨ O'Mega [Ohl et.al.,'01])
- Phase space integrations *under control* for up to 8 final state particles.
- Phase space integration over additional particles can be performed simultaneously with integrations over external particle momenta.

Make method ideally suited for implementation in existing matrix element and event generator frameworks.

In the following:

1. Renormalization and regularization scheme
2. Treatment of infrared divergences
3. Treatment of threshold singularities

### Advantages

- Tree graphs simple to generate automatically (➨ O'Mega [Ohl et.al.,'01])
- Phase space integrations *under control* for up to 8 final state particles.
- Phase space integration over additional particles can be performed simultaneously with integrations over external particle momenta.

Make method ideally suited for implementation in existing matrix element and event generator frameworks.

In the following:

1. Renormalization and regularization scheme
2. Treatment of infrared divergences
3. Treatment of threshold singularities

# 1. Renormalization and Regularization

- Use *on-shell* renormalization scheme [Ross and Taylor, '73]:

$$
\begin{aligned}
\text{Re } i\Gamma^{(2)}_{\alpha\beta}(-p,p)\Phi^{\beta}(p)\Big|_{p^2=m^2} &= 0 & \Gamma^{(3)}(p_i,\lambda)\big|_{p_i^2=m^2} &= \lambda_0^3 \\
\text{Res } \left(-\Gamma^{(2)}(p)\right)^{-1}_{\not{p}=m,p^2=m^2} &= 1 & \Gamma^{(4)}(p_i,\lambda)\big|_{p_i^2=m^2} &= \lambda_0^4
\end{aligned}
$$

- For fully numerical computations: do not introduce artificial regulators like $\epsilon$ in dimensional regularization.
  - Separate calculation of loop graphs and counterterms: Assign finite value to regulator
  - Subtract large values from each other: Numerical instabilities
- Idea: Define subtraction graphs which can be evaluated under same integral as loop integral/phase space integral and renormalization conditions are fulfilled.
- Use variation of BPHZ regularization prescription: [Bogoliubov, Parasiuk, Hepp, Zimmermann, '57,'70]

$$\hat{\Gamma}^n(p_1,\ldots,p_n) = \Gamma^n(p_1,\ldots,p_n) - T \circ \Gamma^n(p_1,\ldots,p_n)$$

# 1. Renormalization and Regularization

- Use *on-shell* renormalization scheme [Ross and Taylor, '73]:

$$\begin{array}{ll} \text{Re } i\Gamma^{(2)}_{\alpha\beta}(-p,p)\Phi^{\beta}(p)\Big|_{p^2=m^2} = 0 & \Gamma^{(3)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^3 \\ \text{Res } \left(-\Gamma^{(2)}(p)\right)^{-1}_{\not p=m,p^2=m^2} = 1 & \Gamma^{(4)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^4 \end{array}$$

- For fully numerical computations: do not introduce artificial regulators like $\epsilon$ in dimensional regularization.
  - Separate calculation of loop graphs and counterterms: Assign finite value to regulator
  - Subtract large values from each other: Numerical instabilities
- Idea: Define subtraction graphs which can be evaluated under same integral as loop integral/phase space integral and renormalization conditions are fulfilled.
- Use variation of BPHZ regularization prescription: [Bogoliubov, Parasiuk, Hepp, Zimmermann, '57,'70]

$$\hat{\Gamma}^n(p_1,\ldots,p_n) = \Gamma^n(p_1,\ldots,p_n) - T\circ\Gamma^n(p_1,\ldots,p_n)$$

# 1. Renormalization and Regularization

- Use *on-shell* renormalization scheme [Ross and Taylor, '73]:

$$
\begin{aligned}
\text{Re } i\Gamma^{(2)}_{\alpha\beta}(-p,p)\Phi^{\beta}(p)\Big|_{p^2=m^2} &= 0 \qquad \Gamma^{(3)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^3 \\
\text{Res } \left(-\Gamma^{(2)}(p)\right)^{-1}_{\not p=m,p^2=m^2} &= 1 \qquad \Gamma^{(4)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^4
\end{aligned}
$$

- For fully numerical computations: do not introduce artificial regulators like $\epsilon$ in dimensional regularization.
  - Separate calculation of loop graphs and counterterms: Assign finite value to regulator
  - Subtract large values from each other: Numerical instabilities
- Idea: Define subtraction graphs which can be evaluated under same integral as loop integral/phase space integral and renormalization conditions are fulfilled.
- Use variation of BPHZ regularization prescription: [Bogoliubov, Parasiuk, Hepp, Zimmermann, '57,'70]

$$\hat{\Gamma}^n(p_1,\dots,p_n) = \Gamma^n(p_1,\dots,p_n) - T\circ\Gamma^n(p_1,\dots,p_n)$$

$$
\begin{aligned}
T\circ\Gamma^n(p_1,\dots,p_n) &= \Gamma^n(\bar{p}_1,\dots,\bar{p}_n) + \sum_i^{n-1}(p_i-\bar{p}_i)^{\mu}\,\frac{\partial\Gamma^n}{\partial p_i^{\mu}}\Big|_{p_j=\bar{p}_j} + \dots \\
&\quad \dots + \frac{1}{d!}\sum_{i_1,\dots,i_d}^{n-1}(p_{i_1}-\bar{p}_{i_1})^{\mu_1}\cdots(p_{i_d}-\bar{p}_{i_d})^{\mu_d}\,\frac{\partial^d\Gamma^n}{\partial p_{i_1}^{\mu_1}\cdots\partial p_{i_d}^{\mu_d}}\Big|_{p_j=\bar{p}_j}
\end{aligned}
$$

# 1. Renormalization and Regularization

- Use *on-shell* renormalization scheme [Ross and Taylor, '73]:

$$\begin{aligned} \text{Re } i\Gamma^{(2)}_{\alpha\beta}(-p,p)\Phi^{\beta}(p)\Big|_{p^2=m^2} &= 0 \qquad \Gamma^{(3)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^3 \\ \text{Res } \left(-\Gamma^{(2)}(p)\right)^{-1}_{\not{p}=m,p^2=m^2} &= 1 \qquad \Gamma^{(4)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^4 \end{aligned}$$

- For fully numerical computations: do not introduce artificial regulators like $\epsilon$ in dimensional regularization.
  - Separate calculation of loop graphs and counterterms: Assign finite value to regulator
  - Subtract large values from each other: Numerical instabilities
- Idea: Define subtraction graphs which can be evaluated under same integral as loop integral/phase space integral and renormalization conditions are fulfilled.
- Use variation of BPHZ regularization prescription: [Bogoliubov, Parasiuk, Hepp, Zimmermann, '57,'70]

$$\hat{\Gamma}^n(p_1,\ldots,p_n) = \Gamma^n(p_1,\ldots,p_n) - T\circ\Gamma^n(p_1,\ldots,p_n)$$

$$\begin{aligned} T\circ\Gamma^n(p_1,\ldots,p_n) &= \Gamma^n(\bar{p}_1,\ldots,\bar{p}_n) + \sum_i^{n-1}(p_i-\bar{p}_i)^{\mu}\left.\frac{\partial\Gamma^n}{\partial p_i^{\mu}}\right|_{p_j=\bar{p}_j} + \ldots \\ &\ldots + \frac{1}{d!}\sum_{i_1,\ldots,i_d}^{n-1}(p_{i_1}-\bar{p}_{i_1})^{\mu_1}\ldots(p_{i_d}-\bar{p}_{i_d})^{\mu_d}\left.\frac{\partial^d\Gamma^n}{\partial p_{i_1}^{\mu_1}\ldots\partial p_{i_d}^{\mu_d}}\right|_{p_j=\bar{p}_j} \end{aligned}$$

# 1. Renormalization and Regularization

- Use *on-shell* renormalization scheme [Ross and Taylor, '73]:

$$\text{Re } i\Gamma^{(2)}_{\alpha\beta}(-p,p)\Phi^{\beta}(p)\Big|_{p^2=m^2} = 0 \qquad \Gamma^{(3)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^3$$

$$\text{Res } \left(-\Gamma^{(2)}(p)\right)^{-1}_{\not p=m,p^2=m^2} = 1 \qquad \Gamma^{(4)}(p_i,\lambda)\big|_{p_i^2=m^2} = \lambda_0^4$$

- For fully numerical computations: do not introduce artificial regulators like $\epsilon$ in dimensional regularization.
  - Separate calculation of loop graphs and counterterms: Assign finite value to regulator
  - Subtract large values from each other: Numerical instabilities
- Idea: Define subtraction graphs which can be evaluated under same integral as loop integral/phase space integral and renormalization conditions are fulfilled.
- Use variation of BPHZ regularization prescription: [Bogoliubov, Parasiuk, Hepp, Zimmermann, '57,'70]

$$\hat{\Gamma}^n(p_1,\ldots,p_n) = \Gamma^n(p_1,\ldots,p_n) - T\circ\Gamma^n(p_1,\ldots,p_n)$$

$$T\circ\Gamma^n(p_1,\ldots,p_n) = \Gamma^n(\bar{p}_1,\ldots,\bar{p}_n) + \sum_i^{n-1}(p_i-\bar{p}_i)^\mu \left.\frac{\partial\Gamma^n}{\partial p_i^\mu}\right|_{p_j=\bar{p}_j} + \ldots$$

$$\ldots + \frac{1}{d!}\sum_{i_1,\ldots,i_d}^{n-1}(p_{i_1}-\bar{p}_{i_1})^{\mu_1}\ldots(p_{i_d}-\bar{p}_{i_d})^{\mu_d}\left.\frac{\partial^d\Gamma^n}{\partial p_{i_1}^{\mu_1}\ldots\partial p_{i_d}^{\mu_d}}\right|_{p_j=\bar{p}_j}$$

- Example: Electron Self-Energy

- Virtual one loop cross section:

$$\sigma_v^{(1)} = \Phi \int d\Pi_n \, 2\,\mathrm{Re}(\mathcal{M}^{\mathrm{Born}}(\mathcal{M}_n^{\mathrm{loop}} + \mathcal{M}_{n,\mathrm{CT}}^{\mathrm{loop}})^*)$$

$$= \Phi \int d\Pi_n \int \frac{d^3k}{(2\pi)^3} 2\mathrm{Re}(\mathcal{M}_n^{\mathrm{Born}}(\mathcal{M}_{n+1}^{\mathrm{Tree}} + \mathcal{M}_{n+1,\mathrm{CT}}^{\mathrm{Tree}})^*)$$

3-dim integral UV convergent. ✓

2. Infrared divergent terms in $\mathcal{M}_{n+1}^{\mathrm{Tree}}$ *and* $\mathcal{M}_{n+1,\mathrm{CT}}^{\mathrm{Tree}}$. Compensated by addition of real emission graphs [Kinoshita, '63; Lee, Nauenberg, '64]

$$\sigma_{\mathrm{re}}^{(1)} = \Phi \int d\Pi_n \int \frac{d^3k}{(2\pi)^3 2E_k} |\mathcal{M}_{n+\gamma}^{\mathrm{Born}}|^2$$

Contains implicit $\delta$-function conserving overall momentum.
$\Rightarrow$ Need approximation for soft real emission diagrams.

- Example: Electron Self-Energy

- Virtual one loop cross section:

$$\begin{aligned}\sigma_v^{(1)} &= \Phi \int d\Pi_n \, 2\,\mathrm{Re}(\mathcal{M}^{\text{Born}}(\mathcal{M}_n^{\text{loop}} + \mathcal{M}_{n,\text{CT}}^{\text{loop}})^*) \\ &= \Phi \int d\Pi_n \int \frac{d^3k}{(2\pi)^3} 2\mathrm{Re}(\mathcal{M}_n^{\text{Born}}(\mathcal{M}_{n+1}^{\text{Tree}} + \mathcal{M}_{n+1,\text{CT}}^{\text{Tree}})^*)\end{aligned}$$

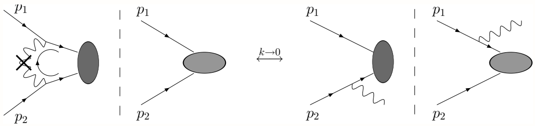
3-dim integral UV convergent. ✓

2. Infrared divergent terms in $\mathcal{M}_{n+1}^{\text{Tree}}$ *and* $\mathcal{M}_{n+1,\text{CT}}^{\text{Tree}}$. Compensated by addition of real emission graphs [Kinoshita, '63; Lee, Nauenberg, '64]

$$\sigma_{\text{re}}^{(1)} = \Phi \int d\Pi_n \int \frac{d^3k}{(2\pi)^3 2E_k} \, |\mathcal{M}_{n+\gamma}^{\text{Born}}|^2$$

Contains implicit $\delta$-function conserving overall momentum.
$\Rightarrow$ Need approximation for soft real emission diagrams.

# 2. Infrared Divergences



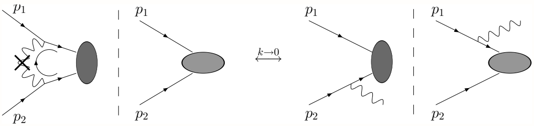

- Virtual IR-divergence arises solely from cut of massless particle.
- In limit $k \to 0$, expressions for cut loop and real emission compensate each other.
- Two equivalent approaches:
  - Project $n + \gamma$ amplitude on $n$-particle phase space.
  - Modify tree graph of cut massless propagator.

  $\to$ In the following (2nd approach):

$$\mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}} \to \mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}}\theta(|\vec{k}| - E_s), \qquad E_s\text{: soft cut}$$

- Infrared convergent

# 2. Infrared Divergences



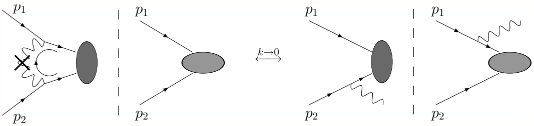

- Virtual IR-divergence arises solely from cut of massless particle.
- In limit $k \to 0$, expressions for cut loop and real emission compensate each other.
- Two equivalent approaches:
  - Project $n+\gamma$ amplitude on $n$-particle phase space.
  - Modify tree graph of cut massless propagator.

$\rightarrow$ In the following (2nd approach):

$$\mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}} \rightarrow \mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}}\theta(|\vec{k}| - E_s), \qquad E_s\text{: soft cut}$$

- Infrared convergent

# 2. Infrared Divergences




- Virtual IR-divergence arises solely from cut of massless particle.
- In limit $k \to 0$, expressions for cut loop and real emission compensate each other.
- Two equivalent approaches:
  - Project $n + \gamma$ amplitude on $n$-particle phase space.
  - Modify tree graph of cut massless propagator.

$\longrightarrow$ In the following (2nd approach):

$$\mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}} \to \mathcal{M}^{\text{Tree}}_{n,\gamma\text{-cut}} \theta(|\vec{k}| - E_s), \qquad E_s\text{: soft cut}$$

- Infrared convergent

# 3. Threshold Singularities

- Propagators of tree graphs can become singular in parts of integration region.

$$P_j = \frac{i}{(k+p_j)^2 - m_j^2} = \frac{i}{\left(k^0 - (-p_j^0 + E_j)\right)\left(k^0 - (-p_j^0 - E_j)\right)}$$

- After cutting propagator $P_i$, one of the two factors in $P_j$ can get zero:

$$(p_j^0 - p_i^0) + (E_i \mp E_j) = 0$$

  - Vanishing of first factor corresponds to coincidence of original poles in lower half plane.
    $\Rightarrow$ Singularities cancel in the sum of tree graphs
  - Vanishing of second factor corresponds to coincidence of poles in lower and upper half plane.
    $\Rightarrow$ Singularity not canceled in the sum of tree graphs.

- Terms in FTT with higher number of $\Delta$ function get support at these singularities.
  $\Rightarrow$ For each singular peak in sum of tree graphs, these terms give further imaginary or real contribution to final result.

# 3. Threshold Singularities

- Propagators of tree graphs can become singular in parts of integration region.

$$P_j = \frac{i}{(k+p_j)^2 - m_j^2} = \frac{i}{\left(k^0 - (-p_j^0 + E_j)\right)\left(k^0 - (-p_j^0 - E_j)\right)}$$

- After cutting propagator $P_i$, one of the two factors in $P_j$ can get zero:

$$(p_j^0 - p_i^0) + (E_i \mp E_j) = 0$$

  - Vanishing of first factor corresponds to coincidence of original poles in lower half plane.
    $\Rightarrow$ Singularities cancel in the sum of tree graphs
  - Vanishing of second factor corresponds to coincidence of poles in lower and upper half plane.
    $\Rightarrow$ Singularity not canceled in the sum of tree graphs.

- Terms in FTT with higher number of $\Delta$ function get support at these singularities.
  $\Rightarrow$ For each singular peak in sum of tree graphs, these terms give further imaginary or real contribution to final result.

# 3. Threshold Singularities

- Propagators of tree graphs can become singular in parts of integration region.

$$P_j = \frac{i}{(k+p_j)^2 - m_j^2} = \frac{i}{\left(k^0 - (-p_j^0 + E_j)\right)\left(k^0 - (-p_j^0 - E_j)\right)}$$

- After cutting propagator $P_i$, one of the two factors in $P_j$ can get zero:

$$(p_j^0 - p_i^0) + (E_i \mp E_j) = 0$$

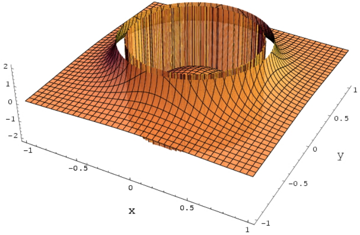
  - Vanishing of first factor corresponds to coincidence of original poles in lower half plane.
    $\Rightarrow$ Singularities cancel in the sum of tree graphs
  - Vanishing of second factor corresponds to coincidence of poles in lower and upper half plane.
    $\Rightarrow$ Singularity not canceled in the sum of tree graphs.

- Terms in FTT with higher number of $\Delta$ function get support at these singularities.
  $\Rightarrow$ For each singular peak in sum of tree graphs, these terms give further imaginary or real contribution to final result.

# Construction of Fix Functions

- In rest frame of $p_{ji}$, peak of threshold singularity is spherical:

$$I(\mathbf{k}') \propto \frac{1}{\mathbf{k}' - \mathbf{k}_s}, \qquad \mathbf{k}_s = \frac{\lambda^{\frac{1}{2}}({p^0_{ji}}^2, m_i^2, m_j^2)}{2|p^0_{ji}|}$$




- Problematic for integration algorithms.
- Idea: Subtract zero from integrand:

$$\frac{\mathrm{Res}(k'_s)}{\mathbf{k}' - \mathbf{k}_s}$$

- More precise, in rest frame:

$$\mathrm{Fix}(\mathbf{k}', k'_s) \equiv \frac{\mathbf{k}_s R(\Lambda^{-1} k'_s - p_i)}{4 p^0_{ji}} \left( \frac{1}{\mathbf{k}' - \mathbf{k}_s} - 2\frac{\mathbf{k}' - \mathbf{k}_s}{c^2} + \frac{(\mathbf{k}' - \mathbf{k}_s)^3}{c^4} \right) \theta(\mathbf{k}', k'_s, c)$$

- Add to cross section in integration system:

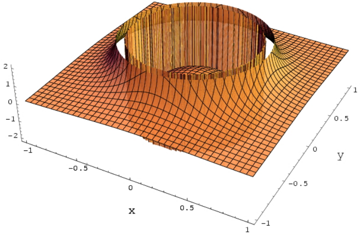
$$\sigma_{\mathrm{Fix}} = \Phi \int d\Pi_n \int \frac{\|\Lambda\| d^3 k}{\overrightarrow{\Lambda(k+p)}^2} \mathrm{Fix}(|\overrightarrow{\Lambda(k+p)}|, k'_s(\overrightarrow{\Lambda(k+p)}))$$

- Simple for numerical algorithms ✓

# Construction of Fix Functions

- In rest frame of $p_{ji}$, peak of threshold singularity is spherical:

$$I(\mathbf{k}') \propto \frac{1}{\mathbf{k}' - \mathbf{k}_s}, \qquad \mathbf{k}_s = \frac{\lambda^{\frac{1}{2}}({p^0_{ji}}^2, m_i^2, m_j^2)}{2|p^0_{ji}|}$$




- Problematic for integration algorithms.
- Idea: Subtract zero from integrand:

$$\frac{\mathrm{Res}(k'_s)}{\mathbf{k}' - \mathbf{k}_s}$$

- More precise, in rest frame:

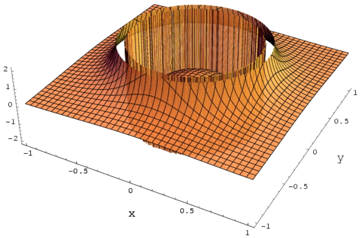
$$\mathrm{Fix}(\mathbf{k}', k'_s) \equiv \frac{\mathbf{k}_s R(\Lambda^{-1}k'_s - p_i)}{4p^0_{ji}} \left( \frac{1}{\mathbf{k}' - \mathbf{k}_s} - 2\frac{\mathbf{k}' - \mathbf{k}_s}{c^2} + \frac{(\mathbf{k}' - \mathbf{k}_s)^3}{c^4} \right) \theta(\mathbf{k}', k'_s, c)$$

- Add to cross section in integration system:

$$\sigma_{\mathrm{Fix}} = \Phi \int d\Pi_n \int \frac{\|\Lambda\| d^3k}{\overrightarrow{\Lambda(k+p)}^2} \mathrm{Fix}(|\overrightarrow{\Lambda(k+p)}|, k'_s(\overrightarrow{\Lambda(k+p)}))$$

- Simple for numerical algorithms ✓

# Construction of Fix Functions




- In rest frame of $p_{ji}$, peak of threshold singularity is spherical:

$$I(\mathbf{k}') \propto \frac{1}{\mathbf{k}' - \mathbf{k}_s}, \qquad \mathbf{k}_s = \frac{\lambda^{\frac{1}{2}}({p^0_{ji}}^2, m_i^2, m_j^2)}{2|p^0_{ji}|}$$

- Problematic for integration algorithms.
- Idea: Subtract zero from integrand:

$$\frac{\mathrm{Res}(k'_s)}{\mathbf{k}' - \mathbf{k}_s}$$

- More precise, in rest frame:

$$\mathrm{Fix}(\mathbf{k}', k'_s) \equiv \frac{\mathbf{k}_s R(\Lambda^{-1} k'_s - p_i)}{4p^0_{ji}} \left( \frac{1}{\mathbf{k}' - \mathbf{k}_s} - 2\frac{\mathbf{k}' - \mathbf{k}_s}{c^2} + \frac{(\mathbf{k}' - \mathbf{k}_s)^3}{c^4} \right) \theta(\mathbf{k}', k'_s, c)$$

- Add to cross section in integration system:

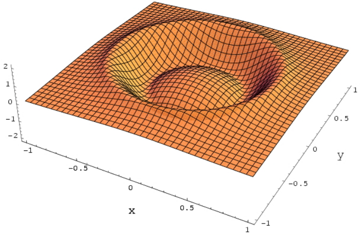
$$\sigma_{\mathrm{Fix}} = \Phi \int d\Pi_n \int \frac{\|\Lambda\| d^3k}{\overrightarrow{\Lambda(k+p)}^2} \mathrm{Fix}(|\overrightarrow{\Lambda(k+p)}|, k'_s(\overrightarrow{\Lambda(k+p)}))$$

- Simple for numerical algorithms ✓

# Construction of Fix Functions

- In rest frame of $p_{ji}$, peak of threshold singularity is spherical:

$$I(\mathbf{k}') \propto \frac{1}{\mathbf{k}' - \mathbf{k}_s}, \qquad \mathbf{k}_s = \frac{\lambda^{\frac{1}{2}}(p_{ji}^{0\,2}, m_i^2, m_j^2)}{2|p_{ji}^0|}$$




- Problematic for integration algorithms.
- Idea: Subtract zero from integrand:

$$\frac{\mathrm{Res}(k_s')}{\mathbf{k}' - \mathbf{k}_s}$$

- More precise, in rest frame:

$$\mathrm{Fix}(\mathbf{k}', k_s') \equiv \frac{\mathbf{k}_s R(\Lambda^{-1}k_s' - p_i)}{4p_{ji}^0}\left(\frac{1}{\mathbf{k}' - \mathbf{k}_s} - 2\frac{\mathbf{k}' - \mathbf{k}_s}{c^2} + \frac{(\mathbf{k}' - \mathbf{k}_s)^3}{c^4}\right)\theta(\mathbf{k}', k_s', c)$$

- Add to cross section in integration system:

$$\sigma_{\mathrm{Fix}} = \Phi \int d\Pi_n \int \frac{\|\Lambda\| d^3k}{\overrightarrow{\Lambda(k+p)}^2} \mathrm{Fix}(|\overrightarrow{\Lambda(k+p)}|, k_s'(\overrightarrow{\Lambda(k+p)}))$$

- Simple for numerical algorithms ✓

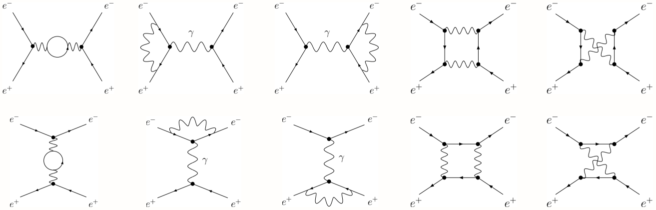
# Outline

# Bhabha Scattering: Cross Section Integration

## Application of FTT to QED Bhabha Scattering at NLO as Proof of Principle

- Includes 10 loop graphs, 2pt, 3pt and 4pt functions.
- Test subtration scheme for UV/IR divergences and internal singularities

$\Rightarrow$ Compare with automated packages; FeynArts/FormCalc [Hahn ea, '98]




## Recipe - 1st Approach

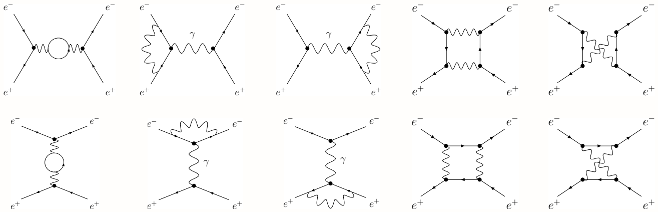
- Create loop graphs with FeynArts/FormCalc. Compute interference with Born graphs. Blocked tensor reduction. Obtain expressions in terms of scalar products.
- In Mathematica: Create subtraction graphs, cut loops, add fix functions, write out to Fortran

# Bhabha Scattering: Cross Section Integration

## Application of FTT to QED Bhabha Scattering at NLO as Proof of Principle

- Includes 10 loop graphs, 2pt, 3pt and 4pt functions.
- Test subtration scheme for UV/IR divergences and internal singularities

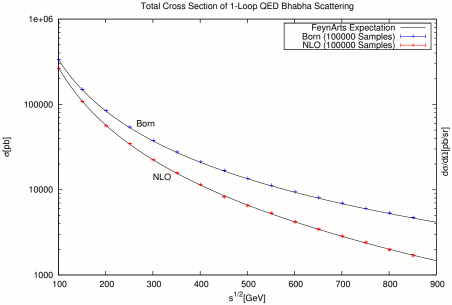
⇒ Compare with automated packages; FeynArts/FormCalc [Hahn ea, '98]

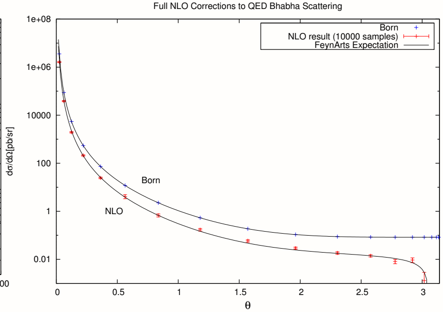


## Recipe - 1st Approach

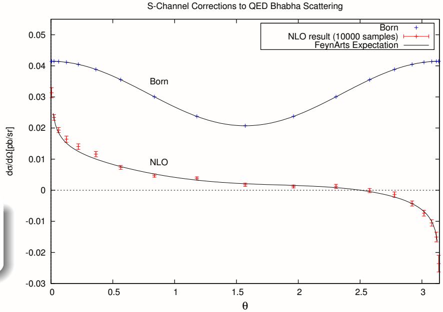
- Create loop graphs with FeynArts/FormCalc. Compute interference with Born graphs. Blocked tensor reduction. Obtain expressions in terms of scalar products.
- In Mathematica: Create subtraction graphs, cut loops, add fix functions, write out to Fortran

# Results




- $\Delta E_s = 5$ GeV
- $\Delta\theta = 1$deg




- $\sqrt{s} = 500$ GeV
- $\Delta E_s = 2.5$ GeV

# Future Approach

**Before:**

- Started with loops, calculated interference with born terms, cut loops
- Manipulation of integrand to enhance efficiency of single channel integration routine

**Now and Future:**

- Create tree amplitudes with O'Mega[Moretti, Ohl, Reuter, '01]
- Add fix functions and subtraction terms
- Remaining peak structure: Use multi channel routine VAMP[Ohl, '98]
- Use this setup to generate events, WHIZARD[Kilian ea, '01]

**First Result**

- One polarized box contribution to $e_L^- e_R^+ \to e_L^- e_R^+$ scattering

# Future Approach

Before:

- Started with loops, calculated interference with born terms, cut loops
- Manipulation of integrand to enhance efficiency of single channel integration routine

Now and Future:

- Create tree amplitudes with O'Mega[Moretti, Ohl, Reuter, '01]
- Add fix functions and subtraction terms
- Remaining peak structure: Use multi channel routine VAMP[Ohl, '98]
- Use this setup to generate events, WHIZARD[Kilian ea, '01]

First Result

- One polarized box contribution to $e^-_L e^+_R \to e^-_L e^+_R$ scattering

# Future Approach

## Before:

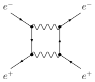
- Started with loops, calculated interference with born terms, cut loops
- Manipulation of integrand to enhance efficiency of single channel integration routine

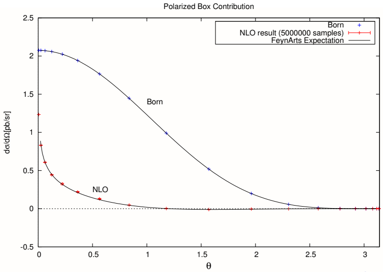
## Now and Future:

- Create tree amplitudes with O'Mega[Moretti, Ohl, Reuter, '01]
- Add fix functions and subtraction terms
- Remaining peak structure: Use multi channel routine VAMP[Ohl, '98]
- Use this setup to generate events, WHIZARD[Kilian ea, '01]

## First Result

- One polarized box contribution to $e_L^- e_R^+ \to e_L^- e_R^+$ scattering



# Summary and Outlook - Part I

- New version WHIZARD 1.92 $\longrightarrow$ 2.0.0

  http://whizard.event-generator.org

  Reference: arXiv:0708.4233

  Functional cut/analysis syntax, more models, recursive cascades, improved phase space, parton shower, . . .
- Extended WHIZARD: 1st NLO SUSY MC Event Generator for the ILC
  - Matching of resummed soft-coll. $\gamma$/virtual NLO avoids negative weights
  - Interface to `FeynArts`: all MSSM $2 \to 2$ processes for ILC available
- Important future developments:
  - ME + PS matching
  - Graphical and/or web interfaces

as usual: **we're open to users wish list!**

# Summary and Outlook - Part II

## Summary

- Presented Method for computation of loop diagrams from tree graphs.
  $\Rightarrow$ allows fully numerical evaluation in matrix element/event generator framework
- Simple prescription for cancellation of UV-, IR-, internal singularities
- Proof of principle: Application to Bhabha scattering
- No further manipulations necessary
  $\Rightarrow$ Level of complexity rises solely due to increasing number of terms
  $\Rightarrow$ Expect method to become efficient tool for mulit-leg processes

## Outlook

- Theoretical side: Extension to full Standard Model.
- Implementation in WHIZARD
- Far future: Extension to two loops

# Overlapping Peaks

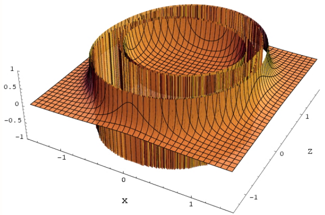
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!

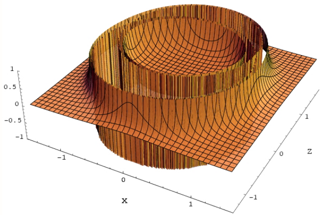


# Overlapping Peaks

- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.




- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)}$$
$$- \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!

# Overlapping Peaks

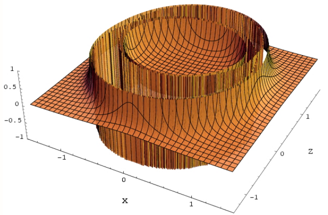
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)\overline{(r'(r,\theta,\phi)-b)}}$$
$$-\frac{f(a,\theta,\phi)}{\overline{(r-a)}(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{\overline{(r-a)}\,\overline{(r'(a,\theta,\phi)-b)}}$$

- However: Terms on right side are non-zero!  
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Overlapping Peaks

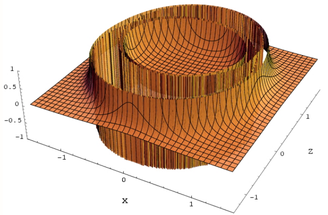
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)}$$
$$- \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Overlapping Peaks

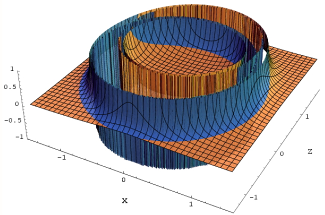
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)}$$
$$- \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Overlapping Peaks

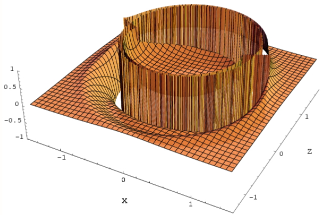
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi)-b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)}$$
$$- \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Overlapping Peaks

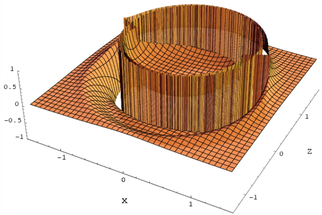
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)}$$
$$- \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Overlapping Peaks

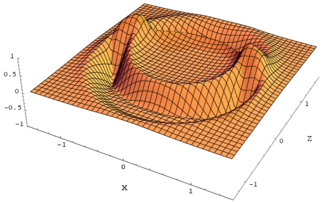
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)}$$
$$- \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Overlapping Peaks

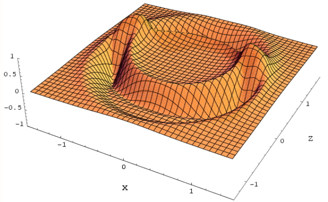
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)(r'(r,\theta,\phi)-b)}$$
$$- \frac{f(a,\theta,\phi)}{(r-a)(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{(r-a)(r'(a,\theta,\phi)-b)}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Overlapping Peaks

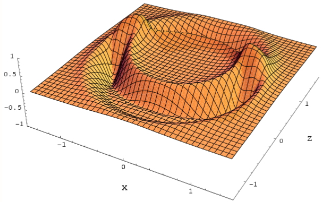
- Starting at 6-point functions with on-shell external particles or 3-point functions with unstable/off-shell external particles.
- Addition of fix function gives schematically:

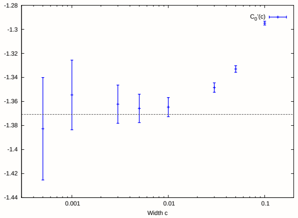
$$\frac{f(r,\theta,\phi) - f(r,\theta,\phi)|_{r'=b}}{(r'(r,\theta,\phi) - b)}$$

  - Equals derivative with respect to $r'$ in the limit $r' \to b$.
  - 1st term corresponds to original integrand; 2nd resembles fix function.
- Operate again on upper expression:

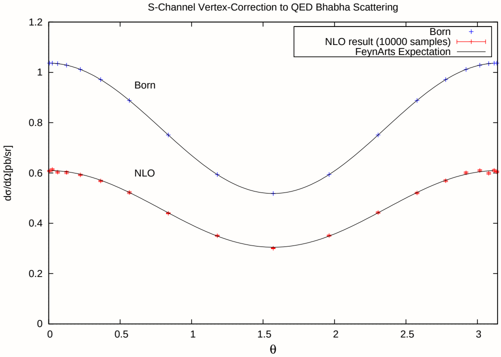
$$\frac{f(r,\theta,\phi)}{(r-a)(r'(r,\theta,\phi)-b)} - \frac{f(r,\theta,\phi)|_{r'=b}}{(r-a)\overline{(r'(r,\theta,\phi)-b)}}$$
$$- \frac{f(a,\theta,\phi)}{\overline{(r-a)}(r'(a,\theta,\phi)-b)} + \frac{f(a,\theta,\phi)|_{r'=b}}{\overline{(r-a)}\,\overline{(r'(a,\theta,\phi)-b)}}$$

- However: Terms on right side are non-zero!
  $\Rightarrow$ Trade-off between accuracy and efficiency!



# Results - Vertex Correction to S-Channel Bhabha Scattering Multi-Channel Sampling




- $\sqrt{s} = 100$ GeV
- $\Delta E_s = 0.5$ GeV

# Event Generation

## Include NLO by using FTT

- Additional 3 inclusive variables $k_i$ from phase space integral over additional particles in tree graphs
- Define event by $x_i$ *and* $k_i$. For each set of external momenta an internal momentum is chosen simultaneously.
  $\Rightarrow$ Expect gain in computation speed compared to (semi-)analytical methods.

## Negative Weights

- Integrand not positive definite
- Need to incorporate events with negative weights
  - Accept event if:

$$r \leq \frac{|w_i|}{w^{\pm}_{\text{max}}} \qquad w^{\pm}_{\text{max}} = \max(|w_{\text{max}}|, |w_{\text{min}}|)$$

  - Assign additional flag ($\pm 1$) to event, dependent on sign of $w_i$
- increase in relative error, drop in efficiency

# Event Generation

## Include NLO by using FTT

- Additional 3 inclusive variables $k_i$ from phase space integral over additional particles in tree graphs
- Define event by $x_i$ *and* $k_i$. For each set of external momenta an internal momentum is chosen simultaneously.
  $\Rightarrow$ Expect gain in computation speed compared to (semi-)analytical methods.

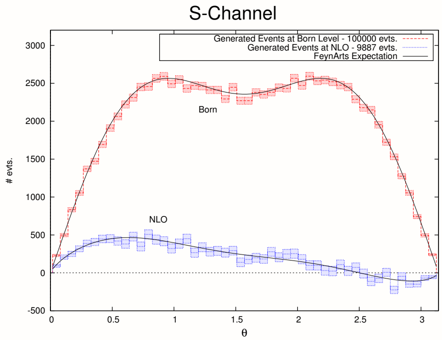
## Negative Weights

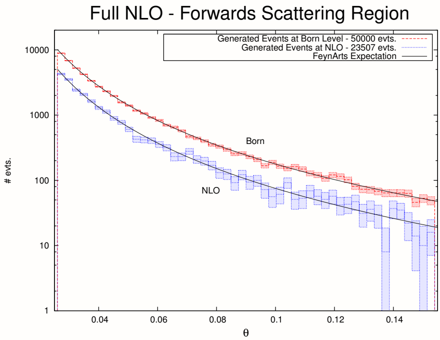
- Integrand not positive definite
- Need to incorporate events with negative weights
  - Accept event if:

$$r \leq \frac{|w_i|}{w^{\pm}_{\text{max}}} \qquad\qquad w^{\pm}_{\text{max}} = \max(|w_{\text{max}}|, |w_{\text{min}}|)$$

  - Assign additional flag $(\pm 1)$ to event, dependent on sign of $w_i$
- increase in relative error, drop in efficiency

# Results




- $\sigma^{\text{tot}}_{\text{Born}} = 0.347\text{pb}$
- $\text{eff}_{\text{Born}} = 66\%$
- $\mathcal{L} = 290\text{fb}^{-1}$
- $\sigma^{\text{tot}}_{\text{NLO}} = 0.0343\text{pb}$
- $\text{eff}^{\text{p+n}}_{\text{NLO}} = 1.8\%$
- $\text{eff}^{\text{hist}}_{\text{NLO}} = 0.14\%$

- $\sigma^{\text{tot}}_{\text{Born}} = 5980\text{pb}$
- $\text{eff}_{\text{Born}} = 65\%$
- $\sigma^{\text{tot}}_{\text{NLO}} = 2810\text{pb}$
- $\text{eff}^{\text{p+n}}_{\text{NLO}} = 3.0\%$
- $\text{eff}^{\text{hist}}_{\text{NLO}} = 0.8\%$