AN EXACT MODEL FOR NON-EQUILIBRIUM PHENOMENA

Ioannis Bakas

National Technical University

Athens, Greece

Joint work with Kostas Skenderis [arXiv:1404.4824 [hep-th]] JHEP

MOTIVATION AND OUTLINE

After the plastic collision of two massive bodies a configuration may form describing a black hole far away from equilibrium. Subsequently, emission of gravitational radiation drives the system to equilibrium state – that of a static black hole – after sufficiently long time.

An analogous situation seems to arise in heavy ion collisions, leading to the formation of a new state of matter (quark–gluon plasma) that thermalizes, reaching equilibrium after sufficiently long time.

The problems have been studied extensively numerically/experimentally. There have also been theoretical attempts to connect them using ideas of holography in the context of AdS_5/CFT_4 correspondence. Black hole hydrodynamics arose as bi-product and it also serves as valuable (semi-realistic) tool to study the transport properties of quark–gluon plasma.

Non-equilibrium phenomena are interesting but notoriously difficult to study analytically far away from equilibrium.

We'll consider an exact model for such phenomena that shares many similarities with the mathematical theory of geometric flows, in the context of geometric analysis, which is a mathematician's framework for non-equilibrium phenomena.

- **It derives from general relativity as model of gravitational radiation emitted from bounded sources in four space-time dimensions. [As such, it can also been used to illustrate the validity of Penrose (and other related) inequalities].**
- **It admits a holographic description in the context of AdS_4/CFT_3 correspondence, allowing for comparison with hydrodynamics.**

The model is special in many respects, it does not admit any higher dimensional generalizations, but it is interesting to consider anyway.

SPHERICAL GRAVITATIONAL WAVES

Long time ago Robinson–Trautman considered special class of metrics in four space-time dimensions describing outgoing radiation emitted from bounded sources in the form of spherical gravitational waves. Using retarded time u they assumed metrics of the form

$$ds^2 = 2r^2 e^{\Phi(z,\bar{z};u)} dz d\bar{z} - 2du dr - F(r,u,z,\bar{z}) du^2 \ .$$

and found that Einstein equations with cosmological constant Λ, i.e., $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$, can be partially integrated to yield the front factor F,

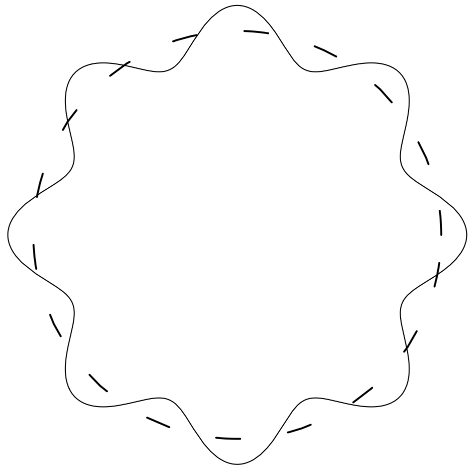
$$F = r\partial_u \Phi - \Delta\Phi - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \ ,$$

where $\Delta = e^{-\Phi}\partial_z \partial_{\bar{z}}$ is the Laplace–Beltrami operator on S^2. $\Delta\Phi$ is the curvature of S^2 at fixed r, which, as it turns out, varies with u.

Φ satisfies a parabolic fourth-order non-linear differential equation, called Robinson–Trautman equation,

$$3m\partial_u\Phi + \Delta\Delta\Phi = 0 \ .$$

Given sufficiently smooth data, the metric on S^2 evolves by dissipating curvature perturbations trying to reach the constant curvature metric, as in heat flow equations



Proper study of the evolution with respect to u relies on the theory of geometric flow equation noting the connection with Calabi flow on S^2. Recall the general definition of Calabi flow on a Kähler manifold M, say compact without boundaries,

$$\partial_u g_{a\bar{b}} = \frac{\partial^2 R}{\partial z^a \partial \bar{z}^b} \ ,$$

deforming the metric $g_{a\bar{b}}$ by derivatives of the Ricci scalar curvature. On S^2 with metric

$$ds_2^2 = 2e^{\Phi(z,\bar{z};u)} dz d\bar{z}$$

Calabi flow is identical to Robinson–Trautman equation (with $3m = 2$) for $R = -2\Delta\Phi$.

Calabi flow on S^2 cannot be solved in closed form, but it exhibits some properties that are sufficient for our purposes:

- **For any given initial data at u_0, a solution exists for all $u \geq u_0$.**
- **All trajectories flow to a fixed point, as $u \to \infty$, associated to the constant curvature metric on S^2,**

$$e^{\Phi_0} = \frac{1}{(1 + z\bar{z}/2)^2} \ .$$

- **The area of S^2 and the average curvature $< R > \sim \chi(S^2)$ remain fixed throughout the evolution, but not higher moments of the curvature, like Calabi's functional $< R^2 > = \int_{S^2} \sqrt{g}\, R^2$ that acts as an entropy functional, decreasing monotonically along the flow lines.**

Close to the fixed point the equation linearizes and one can easily compute the damping rate of different harmonics. Thus, assuming axial symmetry (for simplicity) and parametrizing all perturbations of the round S^2 in terms of Legendre polynomials ($l \geq 2$), as

$$ds_2^2 = [1 + \epsilon_l(u) P_l(\cos\theta)] \left(d\theta^2 + \sin^2\theta d\phi^2 \right) ,$$

we find that the perturbations are damped exponentially as

$$\epsilon_l(u) = \epsilon_l(0) \exp\left(-\frac{u}{12m}(l-1)l(l+1)(l+2) \right) \equiv \epsilon_l(0) e^{-i\omega_{\rm s} u}$$

with fall-off rate given by the characteristic imaginary frequencies

$$\omega_{\rm s} = -i \, \frac{(l-1)l(l+1)(l+2)}{12m} \; .$$

Returning back to the Robinson-Trautman metric in four space-time dimensions, we can assign a definite meaning to all previous results:

- **Starting from sufficiently smooth data at u_0, the space-time metric exists for all $u \geq u_0$ and after infinitely long time it settles to a static solution that is nothing else but the exterior of a Schwarzschild black-hole with mass m and cosmological constant Λ:**

$$ds^2 = \frac{2r^2}{(1 + z\bar{z}/2)^2} dz d\bar{z} - 2du dr - \left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right) du^2$$

written in Eddington-Filkenstein frame. To pass to ordinary frame we set $u = t - r_\star$, using the tortoise coordinate $r_\star$ defined as $dr_\star = dr/f(r)$ with profile function $f(r) = 1 - 2m/r - \Lambda r^3/3$.

- **At late times, the four-dimensional solution is a small perturbation of the black-hole metric associated to the algebraically special modes which are purely out-going total transmission modes that vanish on the horizon at $r = r_{\rm h}$. Comparison with the theory of quasi-normal modes reveals that such modes are zero energy states of an effective Schrödinger problem for the polar (vs axial) perturbations of the black-hole,**

$$[\frac{d^2}{dr_\star^2} + W^2(r) + \frac{dW(r)}{dr_\star}]\Psi(r) = E\,\Psi(r)\,,$$

where $E = \omega^2 - \omega_{\rm s}^2$ **and**

$$W(r) = \frac{6mf(r)}{r[(l-1)(l+2)r + 6m]} + i\omega_{\rm S}\,.$$

The algebraically special modes are purely dissipative corresponding to "ringing" frequencies

$$\omega = \omega_{\mathrm{S}} = -i\,\frac{(l-1)l(l+1)(l+2)}{12m} = -\frac{2i}{m},\ \ -\frac{10i}{m},\ \ -\frac{30i}{m},\cdots.$$

As in supersymmetric quantum mechanics, they satisfy a first order equation

$$Q\Psi(r_\star) = \left(-\frac{d}{dr_\star} + W(r_\star)\right)\Psi(r_\star) = 0\,.$$

Thus, the Robinson-Trautman metrics are formed by superposition of the algebraically special modes as compared to all other classes of radiative metrics that require making use of the entire set of quasi-normal modes.

Linear as well as not linear effects can be captured by the late time expansion of the solutions. Parametrizing deviations from equilibrium state as

$$e^{\Phi(z,\bar{z};u)} = \frac{1}{\sigma^2(z,\bar{z};u)\,(1+z\bar{z}/2)^2}$$

we may expand systematically as

$$\sigma(z,\bar{z};u) \;=\; 1+\sigma_1(z,\bar{z})e^{-2u/m}+\sigma_2(z,\bar{z})e^{-4u/m}+\cdots$$

showing only the quadrupole and the first non-linear correction to it. For axially symmetric solutions we obtain recursively, setting $x = \cos\theta$,

$$\sigma_1(x) = a\left(x^2-\frac{1}{3}\right), \quad \sigma_2(x) = -a^2\left(\frac{23}{78}x^4-\frac{47}{39}x^2+\frac{49}{234}\right), \quad \cdots .$$

In view of the holographic applications of AdS_4 Robinson-Trautman, we note the mixed boundary conditions satisfied by the algebraically special modes,

$$\frac{d}{dr_\star}\Psi_+^{(0)}(r_\star)\,|_{r_\star=0} \quad = \left(i\omega_{\rm S} - \frac{2m\Lambda}{(l-1)(l+2)}\right)\Psi_+^{(0)}(r_\star = 0)\,.$$

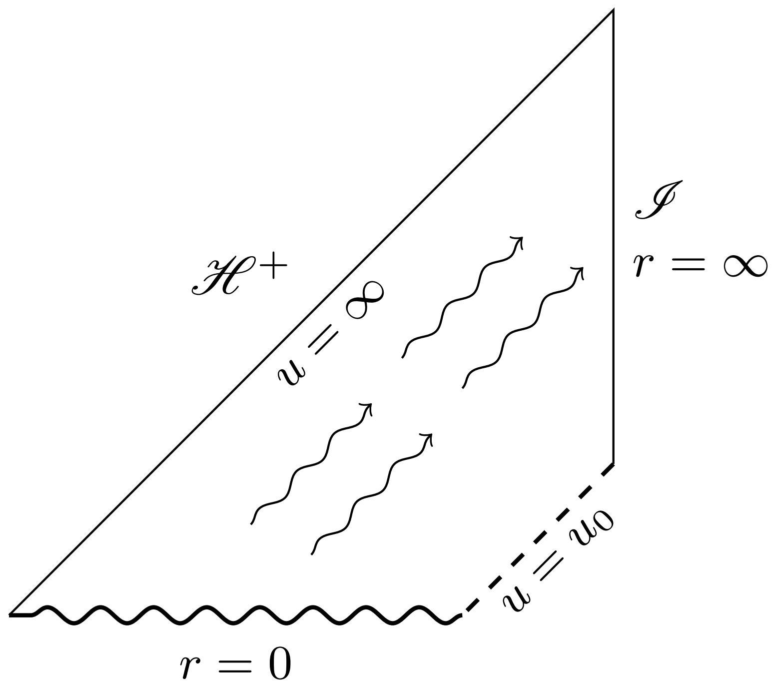
Here, $r_\star$ ranges from $-\infty$ to 0 as r ranges from $r_{\rm h}$ to $+\infty$.

The corresponding wave-functions are normalizable,

$$\int_{-\infty}^{0} dr_\star \mid \Psi_+^{(0)}(r_\star) \mid^2 < \infty \ .$$

They amount to a metric at conformal infinity $\mathcal{I} = \mathbb{R} \times S^2$ that is not conformally flat; its evolution is also governed by Calabi flow.

The Penrose diagram of AdS_4 Robinson–Trautman space-times is



but it turns out that Kruskal extension across the future horizon $\mathcal{H}^+$ breaks down completely for sufficiently large AdS_4 black-holes, which are thermodynamically favorable.

PENROSE (AND OTHER) INEQUALITIES

Robinson–Trautman space-times provide a nice example to illustrate the validity of Penrose (and other related) inequalities.

• **Bondi mass: Motivated by $\Lambda = 0$, we consider the mass formula**

$$\mathcal{M}_{\mathrm{Bondi}} = \frac{m}{4\pi} \int_{S^2} d\mu_0 \, \frac{1}{\sigma^3}$$

which can be shown to decrease monotonically with time u, reaching m as $u \to \infty$.

• **Past apparent horizon: It is a marginally trapped surface Σ defined by the embedding equation $r = U(z, \bar{z})$ for constant u with U satisfying a variant of Tod–Penrose equation in the presence of Λ**

$$2\Delta(\log U) + \Delta\Phi + \frac{2m}{U} + \frac{\Lambda}{3}U^2 = 0 \ .$$

A solution $U(z,\bar{z},u)$ always exist and it is unique for $\Lambda \leq 0$. Then, the area of the past apparent horizon is

$$\mathrm{Area}(\Sigma) = \int_{S^2} d\mu_0 \left(\frac{U}{\sigma}\right)^2 .$$

[It is not known if the horizon area varies monotonically wrt u].

It can be shown, using Holder and Sobolev inequalities, that although $\mathcal{M}_{\mathrm{Bondi}}$ and $\mathrm{Area}(\Sigma)$ both vary with u, the following version of Penrose inequality holds for all $\Lambda \leq 0$,

$$16\pi\mathcal{M}^2_{\mathrm{Bondi}} \geq \mathrm{Area}(\Sigma)\left(1 - \frac{\Lambda}{3}\frac{\mathrm{Area}(\Sigma)}{4\pi}\right)^2 .$$

To appreciate the non-triviality of the inequality, one may use the late time expansion of Calabi flow to compute $\mathcal{M}_{\rm Bondi}$ and $\mathrm{Area}(\Sigma)$ to any given order. For axially symmetric solutions, setting $x = \cos\theta$, we have

$$\sigma(x;u) \;=\; 1+a\left(x^2-\frac{1}{3}\right)e^{-2u/m}-a^2\left(\frac{23}{78}x^4-\frac{47}{39}x^2+\frac{49}{234}\right)e^{-4u/m}$$

up to $\mathcal{O}(e^{-6u/m})$ terms, leading to the following late time expansions

$$\mathcal{M}_{\rm Bondi}=m[1+\frac{2a^2}{15}e^{-4u/m}+\mathcal{O}\left(e^{-6u/m}\right)]$$

and

$$\mathrm{Area}(\Sigma)=4\pi r_{\rm h}^2[1+\frac{16a^2mr_{\rm h}^2}{15(3m-r_{\rm h})(2r_{\rm h}+3m)^2}e^{-4u/m}+\mathcal{O}\left(e^{-6u/m}\right)]\,.$$

Another inequality is provided by Thorne's hoop conjecture, stating (when generalized in the presence of cosmological constant)

$$4\pi\mathcal{M} \geq C(\Sigma)\left(1 - \frac{\Lambda}{3}\left(\frac{C(\Sigma)}{2\pi}\right)^2\right)$$

for appropriately defined mass $\mathcal{M}$ and circumference C lassoing the black-hole. The Robinson–Trautman space-times provide a realization of it, letting $\mathcal{M}_{\mathrm{Bondi}}$ be the mass and C be the length of the shortest closed geodesic on the past apparent horizon Σ. However, there might be a more stringent bound, choosing C to be the Birkhoff length of Σ.

- **All inequalities turn into equalities at the equilibrium state, $u = \infty$.**

HOLOGRAPHIC RENORMALIZATION

The AdS_4 Robinson–Trautman metric is an example of asymptotically locally AdS space-time. The boundary metric (after rescaling) is

$$ds^2_{\mathcal{I}} = -dt^2 - \frac{6}{\Lambda} e^{\hat{\Phi}} dz d\bar{z}$$

and the corresponding energy-momentum tensor turns out to be

$$\kappa^2 T_{tt} = -\frac{2m\Lambda}{3} \,, \qquad \kappa^2 T_{tz} = -\frac{1}{2} \partial_z (\hat{\Delta} \hat{\Phi}) \,,$$

$$\kappa^2 T_{z\bar{z}} = m \, e^{\hat{\Phi}}, \qquad \kappa^2 T_{zz} = -\frac{3}{4\Lambda} \partial_t \left((\partial_z \hat{\Phi})^2 - 2\partial_z^2 \hat{\Phi} \right) ,$$

whereas $T_{t\bar{z}} = \bar{T}_{tz} \,, \quad T_{\bar{z}\bar{z}} = \bar{T}_{zz}$ **.**

The energy-momentum tensor is traceless and conserved, as it should

$$T^a{}_a = 0\,, \qquad \nabla^a T_{ab} = 0$$

by the classical equations of motion provided by the boundary version of Calabi flow,

$$3m\partial_t \hat{\Phi} + \hat{\Delta}\hat{\Delta}\hat{\Phi} = 0\ ,$$

where $\hat{\Phi}$ **is the boundary value of** Φ

$$\hat{\Phi}(z, \bar{z}; t) = \lim_{r_\star \to 0} \Phi(z, \bar{z}; u)$$

and $\hat{\Delta} = e^{-\hat{\Phi}} \partial_z \partial_{\bar{z}}$ **is the corresponding Laplace–Beltrami operator on the spatial slices** S^2 **of** $\mathcal{I} = \mathbb{R} \times S^2$**.**

The boundary is not conformally flat, hereby accounting for part of the gravitational radiation that is being transmitted through space. The Cotton tensor of the boundary metric γ, which is a traceless and symmetric tensor that is covariantly conserved without employing the equations of motion,

$$C^{ab} = \frac{\epsilon^{acd}}{\sqrt{-\det\gamma}} \nabla_c \left(R^b{}_d - \frac{1}{4} \delta^b{}_d R \right) ,$$

turns out to be

$$C_{zz} = i \frac{\Lambda}{3} \kappa^2 T_{zz} \ , \qquad C_{\bar{z}\bar{z}} = -i \frac{\Lambda}{3} \kappa^2 T_{\bar{z}\bar{z}} \, ,$$

$$C_{tz} = i \frac{\Lambda}{3} \kappa^2 T_{tz} \ , \qquad C_{t\bar{z}} = -i \frac{\Lambda}{3} \kappa^2 T_{t\bar{z}} \, .$$

HYDRODYNAMIC CONSIDERATIONS

On the boundary we have a $2+1$ dimensional relativistic fluid that can be very far away from equilibrium. As $t \to \infty$, the system thermalizes.

To compare with first order hydrodynamics we have to linearize the energy-momentum tensor around the black-hole equilibrium state and determine the corresponding energy density ρ and the time-like unit vector u^a so that

$$T_{ab}\, u^b = -\rho\, u_a \,.$$

Then, T_{ab} takes the perfect fluid form plus a viscous term,

$$T^{ab} = \rho u^a u^b + p\Delta^{ab} + \Pi^{ab} \,,$$

where $\Delta^{ab} = u^a u^b + g^{ab}$. The viscous term is

$$\Pi^{ab} = -\eta\sigma^{ab} - \zeta\Delta^{ab}(\nabla_c u^c) \ ,$$

where

$$\sigma^{ab} = 2\nabla^{<a}u^{b>}$$

using the short-hand notation

$$A^{<ab>} = \frac{1}{2}\Big(\Delta^{ac}\Delta^{bd}(A_{cd} + A_{dc}) - \Delta^{ab}\Delta^{cd}A_{cd}\Big) \,.$$

The bulk viscosity coefficient $\zeta = 0$, since the fluid is conformal. For that reason we also have $\rho = 2p$.

The effective shear viscosity coefficient η can be determined for each algebraically special mode. Explicit computation shows that

$$\kappa^2 \eta = \frac{1}{4} l(l+1)$$

and so the ratio of shear viscosity to the entropy density for large AdS_4 black holes turns out to be

$$\frac{\eta}{s} = \frac{4}{r_{\rm h}^2}\left(-\frac{3}{\Lambda}\right)\eta = \frac{1}{4\pi} \cdot \frac{l(l+1) r_{\rm h}}{4m} \,.$$

As such, it violates the celebrated KSS bound for sufficiently low l. [Recall that large black holes have $m > r_{\rm h} > L = \sqrt{-3/\Lambda}$].

The modes giving rise to KSS bound are also purely dissipative with

$$\Omega_{\rm s} = -i\,\frac{(l-1)(l+2)}{3r_{\rm h}}$$

as compared the algebraically special modes, which have

$$\omega_{\rm s} = -i\,\frac{(l-1)l(l+1)(l+2)}{12m}\ .$$

Thus, the end result is neatly summarized as follows,

$$\frac{\eta}{s} = \frac{1}{4\pi}\frac{\omega_{\rm s}}{\Omega_{\rm s}}\ .$$

Curiously, $\Omega_{\rm s}$ dictate the late time expansion of the normalized Ricci flow on S^2, but they do not extend to non-linear gravitational regime.

Entropy production is established by considering the entropy current $s^a = su^a$, where s is the local entropy density that is related to the energy density ρ via the thermodynamic relations of $2+1$ dimensional conformal fluids,

$$s = \gamma \mathcal{T}^2, \qquad \rho = 2p = \frac{2\gamma}{3} \mathcal{T}^3$$

setting $\gamma = -4\pi^2/3\Lambda$ for large AdS_4 black-holes.

Expanding all fields to order $e^{-4t/m}$ (quadrupole plus first non-linear corrections) one finds that there is entropy production such that

$$\nabla_a s^a = \frac{\eta}{2\mathcal{T}_0} \sigma_{ab} \sigma^{ab} ,$$

as it should be on general grounds [Landau–Lifshitz].

SUMMARY AND OUTLOOK

We have considered an exact model for non-equilibrium dynamics encompassing linear as well as non-linear phenomena of gravity in four space-time dimensions (in the presence of cosmological constant) and studied its global and holographic aspects in connection with the mathematical theory of geometric flows.

This model can be used further to expose more general aspects of the theory of gravitational radiation in asymptotically locally AdS spaces as well as provide a systematic interpretation of the energy and entropy densities at the boundary.

THANK YOU!