Max Planck Institute for Mathematics
California Institute of Technology

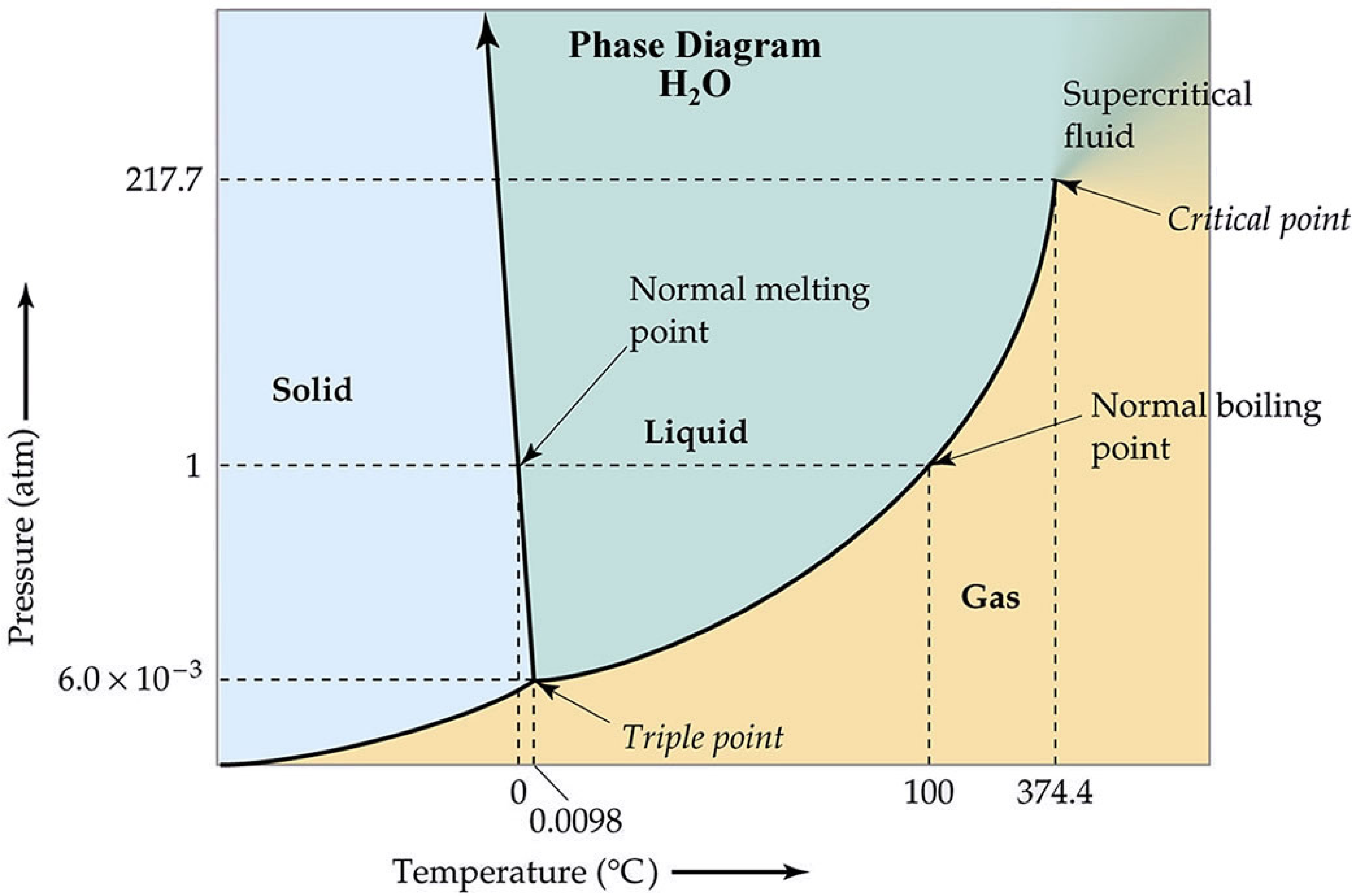
Confinement and 4-manifolds

based on:

arXiv:1306.4320 (2d $\mathcal{N} = (0,2)$ theories labeled by 4-manifolds)
arXiv:1404.2929 (duality defects and Lefschetz fibrations)
arXiv:1404.5314 (exact solutions of $\mathcal{N} = (0,2)$ gauge theories)

with A.Gadde and P.Putrov

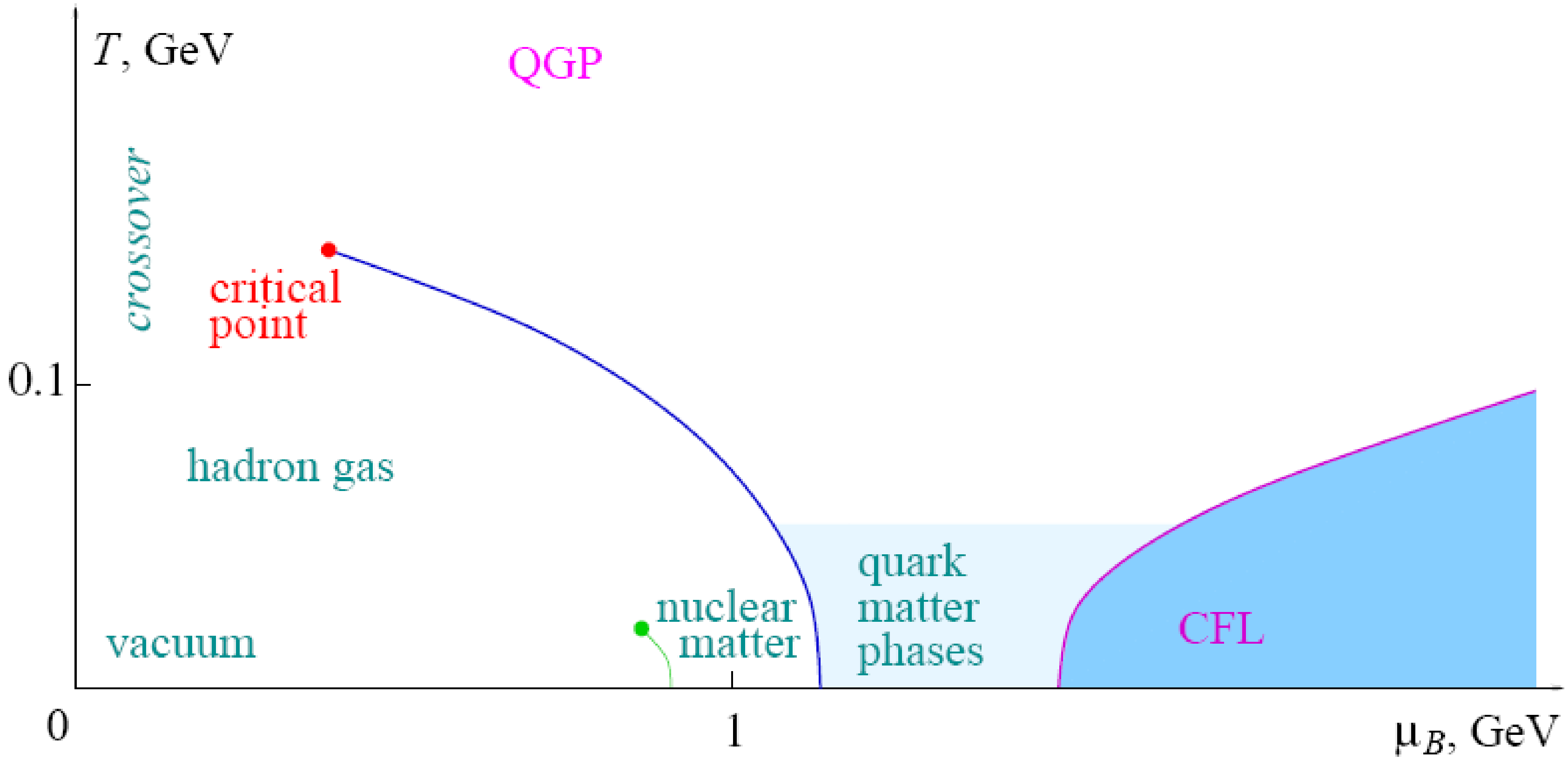
Phases



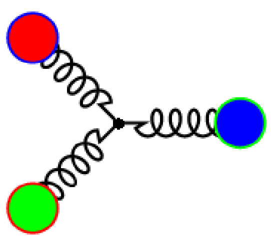
Phase Diagram
H_2O
Supercritical fluid
217.7
Critical point
Normal melting point
Solid
Liquid
Normal boiling point
1
Pressure (atm)
Gas
6.0×10^{-3}
Triple point
0
0.0098
100
374.4
Temperature (°C)

Phases

[M.Stephanov]



Can we quantitatively understand confinement and the mass gap?



- Extensively tested in computer simulations
- Paper-and-pencil computation?

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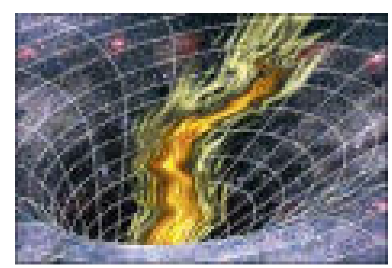
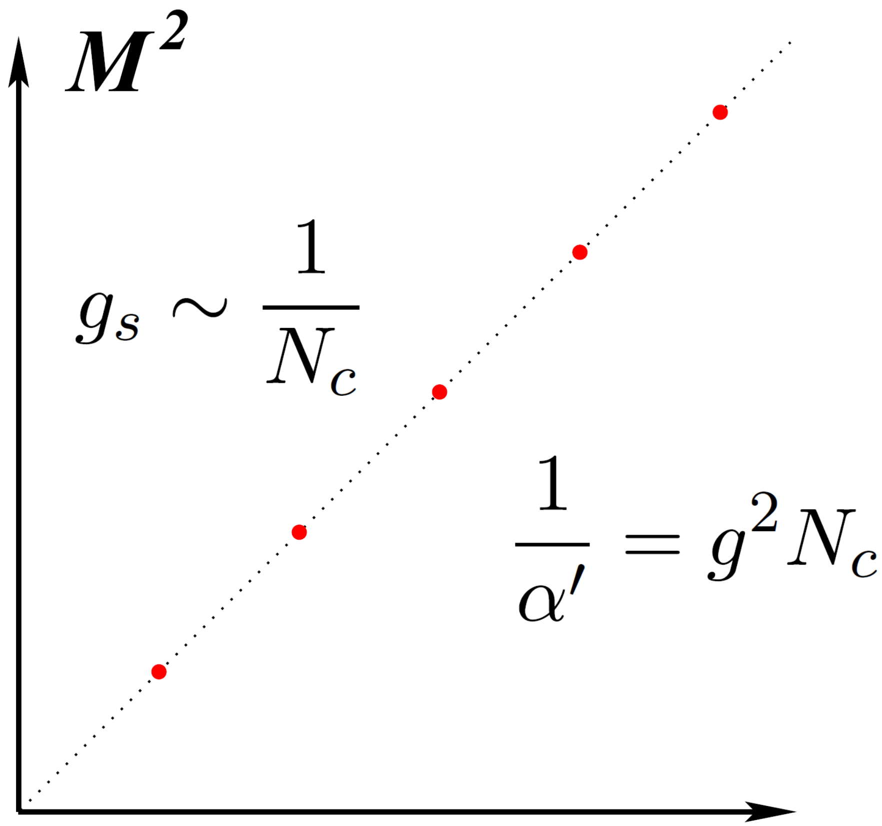
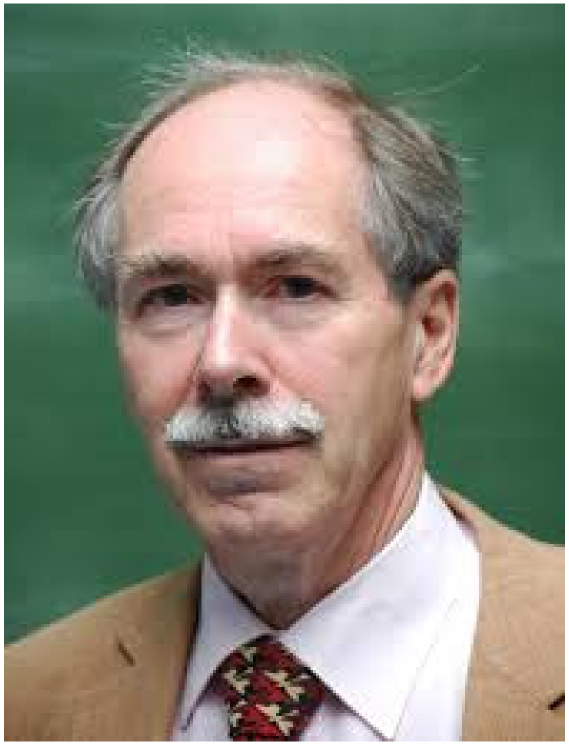
$1,000,000 Prize

The answer may involve gravity!



Solvable Gauge Theories





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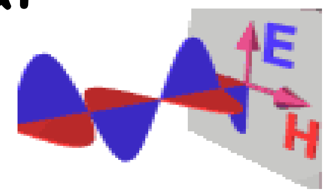
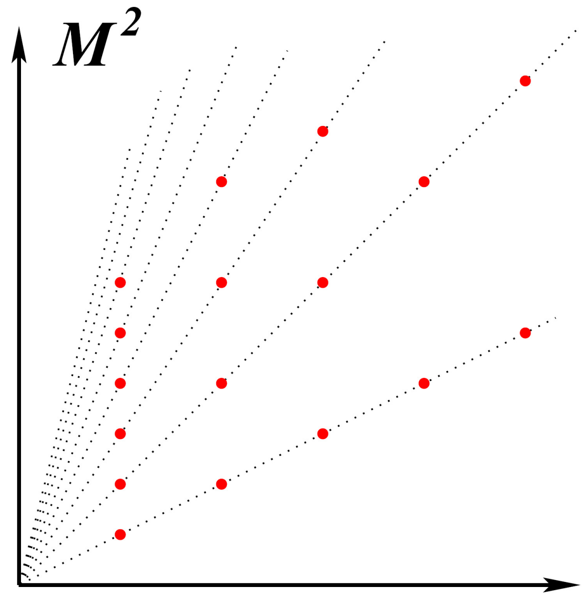
Solvable Gauge Theories

- 2d Yang-Mills: almost "topological"
- QED: confinement / screening

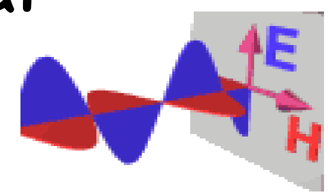


- 2d $\mathcal{N} = 0$ QCD: one Regge trajectory
- QCD with a massive adjoint: higher Regge



Solvable Gauge Theories

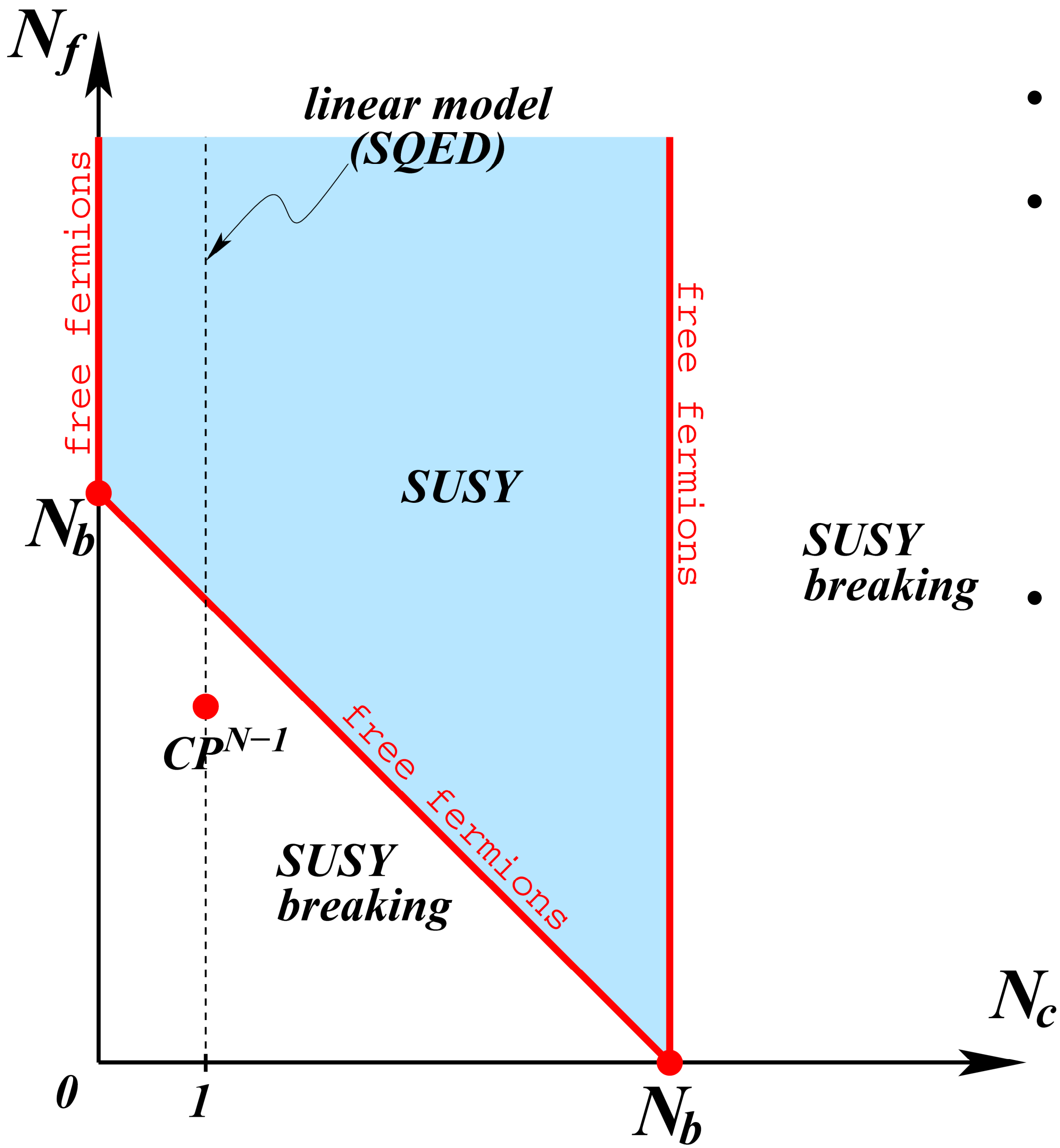
- 2d Yang-Mills: almost "topological"
- QED: confinement / screening



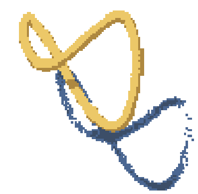
- 2d $\mathcal{N} = 0$ QCD: one Regge trajectory
- QCD with a massive adjoint: higher Regge
- 2d $\mathcal{N} = (1,1)$ SQCD: very similar
- 2d $\mathcal{N} = (0,2)$ SQCD: ?
- 2d $\mathcal{N} = (2,2)$ SQCD: ?

$\vdots$

2d $\mathcal{N} = (0,2)$ SQCD



- New (0,2) SCFTs
- Exactly solvable string phenomenology

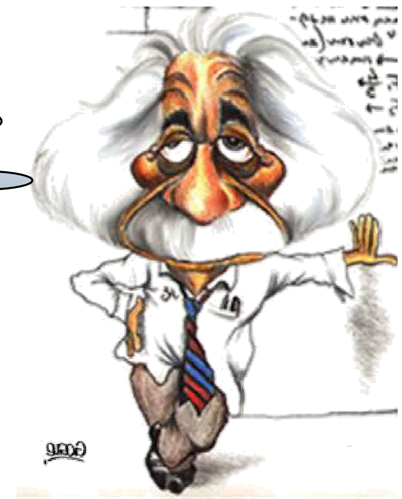


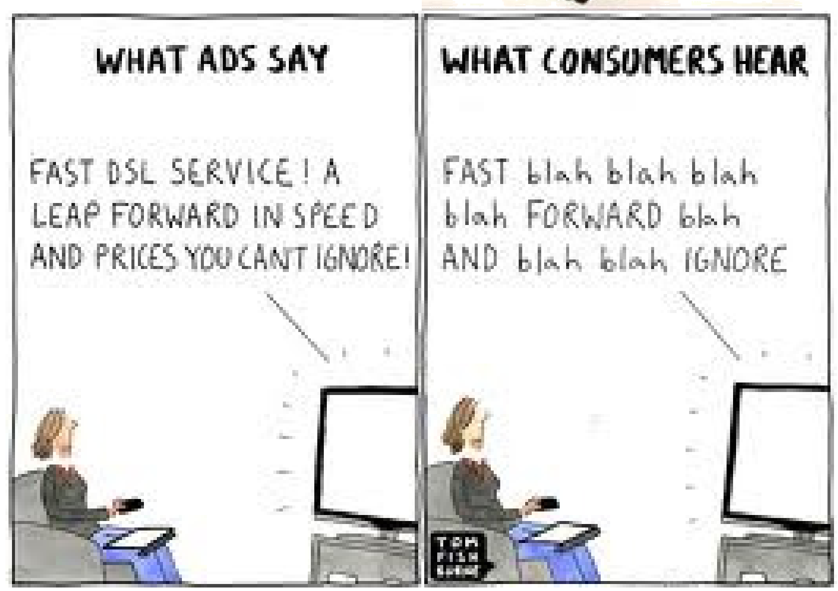
- Holographic dual of the Veneziano limit

⇓

higher spin theory

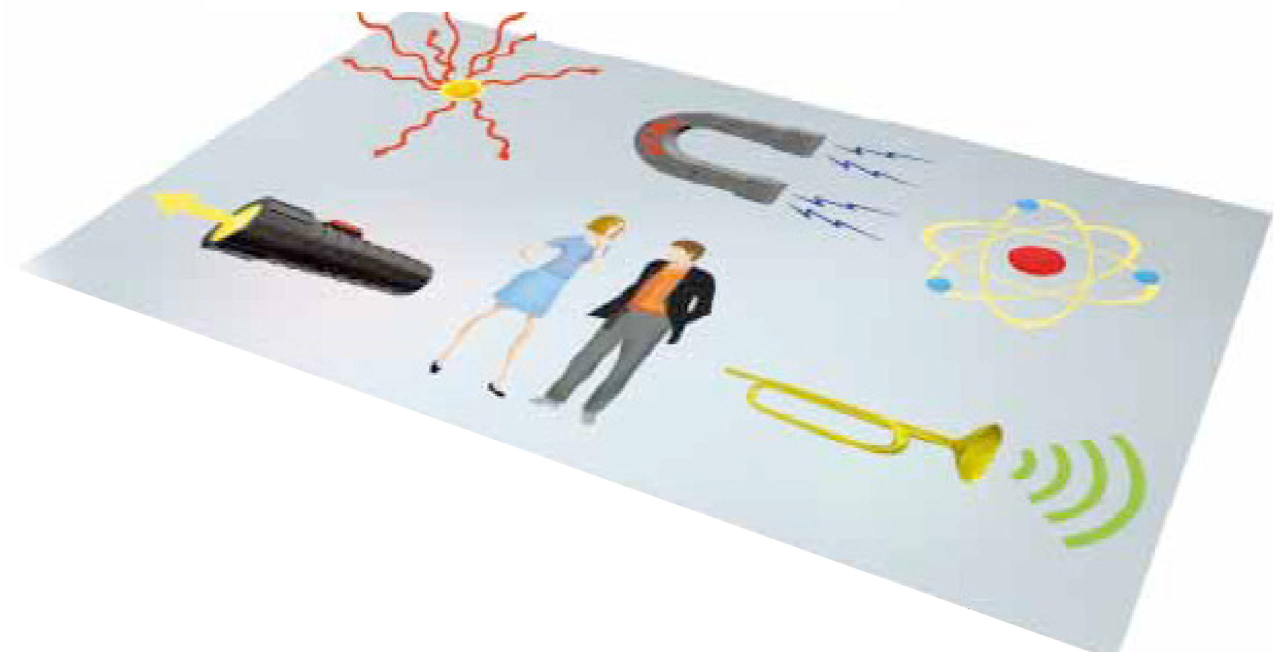
Okay, let us consider fivebranes on a coassociative 4-manifold in a G2 holonomy space …





What are Fivebranes?

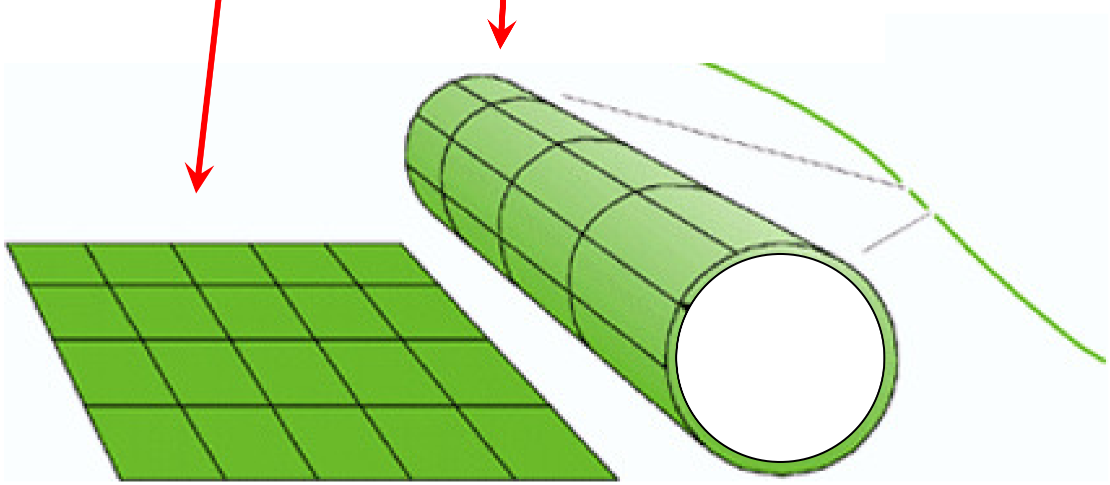
- 6-dimensional submanifolds in 11-dimensional space-time of M-theory



Kaluza-Klein compactification

6d fivebrane theory

on $\mathbb{R}^{6-n} \times M_n$



"effective" theory

$T[M_n]$

in 6-n dimensions

depends on topology and geometry of M_n

6 = 3 + 3

3-manifold M_3 ⇒ 3d $\mathcal{N}=2$ theory $T[M_3]$

complex flat connections ⇐ supersymmetric vacua

$$dA + A \wedge A = 0 \qquad\qquad \frac{\partial \mathcal{W}}{\partial \sigma_i} = 0$$

$$\mathcal{M}_{\text{flat}}(M_3; G_{\mathbb{C}}) = \mathcal{M}_{\text{SUSY}}(T[M_3])$$

6 = 3 + 3

3-manifold M_3 ⇒ 3d $\mathcal{N}=2$ theory $T[M_3]$

complex flat connections ⇐ supersymmetric vacua

$$\frac{\partial \mathcal{W}}{\partial \sigma_i} = 0$$

<u>Example:</u> $M_3 = L(k,1)$

ρ = highest weight integrable representation of the loop group $\mathcal{L}G$ at level k

$6 = 3 + 3$

3-manifold M_3 → 3d $\mathcal{N} = 2$ theory $T[M_3]$

complex flat connections ← supersymmetric vacua

- Knot complements: *A*-polynomial
- generalized / quantum / homological Volume Conjecture …

$$\hat{A}(\hat{x},\hat{y})\, Z(\mathbf{M}) = 0$$

$6 = 3 + 3$

3-manifold M_3 → 3d $\mathcal{N}=2$ theory **$T[M_3]$**

complex flat connections ← supersymmetric vacua

complex Chern-Simons partition function ← vortex partition function

$$Z_{CS}(M_3; G_{\mathbb{C}}) \; = \; Z_{\text{vortex}}(T[M_3])$$

$$\hat{A}(\hat{x},\hat{y})\, Z(\mathbf{M}) = 0$$

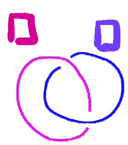
$6 = 3 + 3$

3-manifold M_3 ⇒ 3d $\mathcal{N} = 2$ theory $T[M_3]$

complex flat connections ⇐ supersymmetric vacua

complex Chern-Simons partition function ⇐ vortex partition function

Knot homology ⇐ Q-cohomology

$\mathcal{H}$ () $= \mathcal{H}_{\text{BPS}}$

Embedding in 11 dimensions

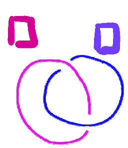
q-grading

homological grading

$$U(1)_P \times U(1)_F$$

space-time:	$\mathbb{R}$	$\times$	$\mathbb{R}^4_{q,t}$	$\times$	T^*M_3	
	$\|\|$		$\cup$		$\cup$	
N M5-branes:	$\mathbb{R}$	$\times$	$\mathbb{R}^2_q$	$\times$	M_3	

$$M_4 = \mathbb{R} \times M_3$$

$\mathcal{H}\Big($  $\Big) = \mathcal{H}^{\text{(open)}}_{\text{refined BPS}}$

Embedding in 11 dimensions

q-grading

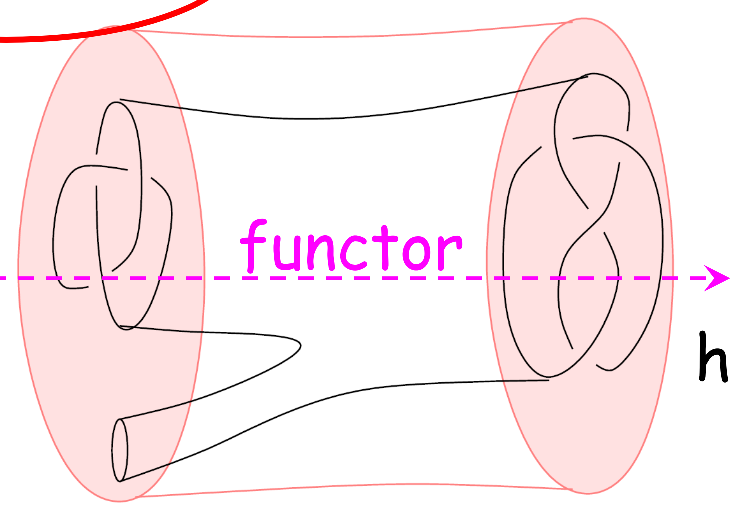
homological grading

$$U(1)_P \times U(1)_F$$

$$\begin{array}{lccccc} \text{space-time:} & \mathbb{R} & \times & \mathbb{R}^4_{q,t} & \times & T^* M_3 \\ & \| & & \cup & & \cup \\ N \ \text{M5-branes:} & \mathbb{R} & \times & \mathbb{R}^2_q & \times & M_3 \end{array}$$

$$M_4 = \mathbb{R} \times M_3$$

Link homology



Knot homology

Embedding in 11 dimensions

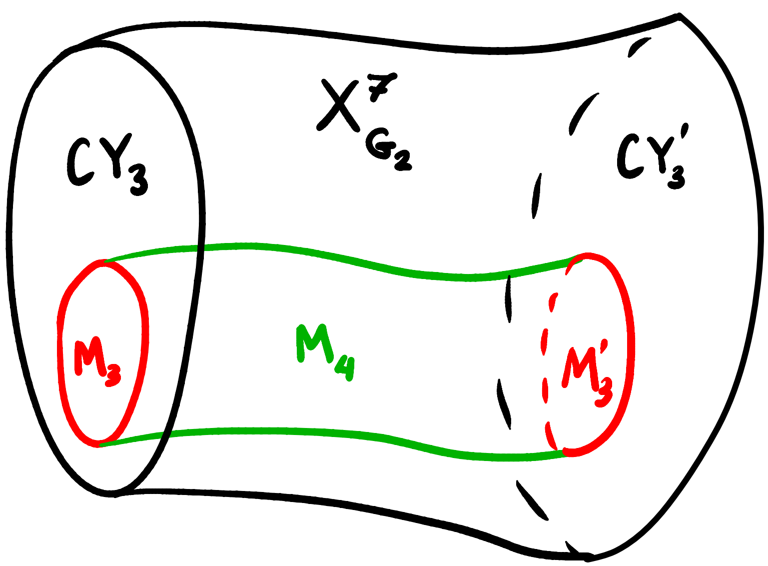
space-time: $\mathbb{R} \times \mathbb{R}^4_{q,t} \times T^* M_3$

$\quad\quad\quad\quad\quad \| \quad\quad \cup \quad\quad \cup$

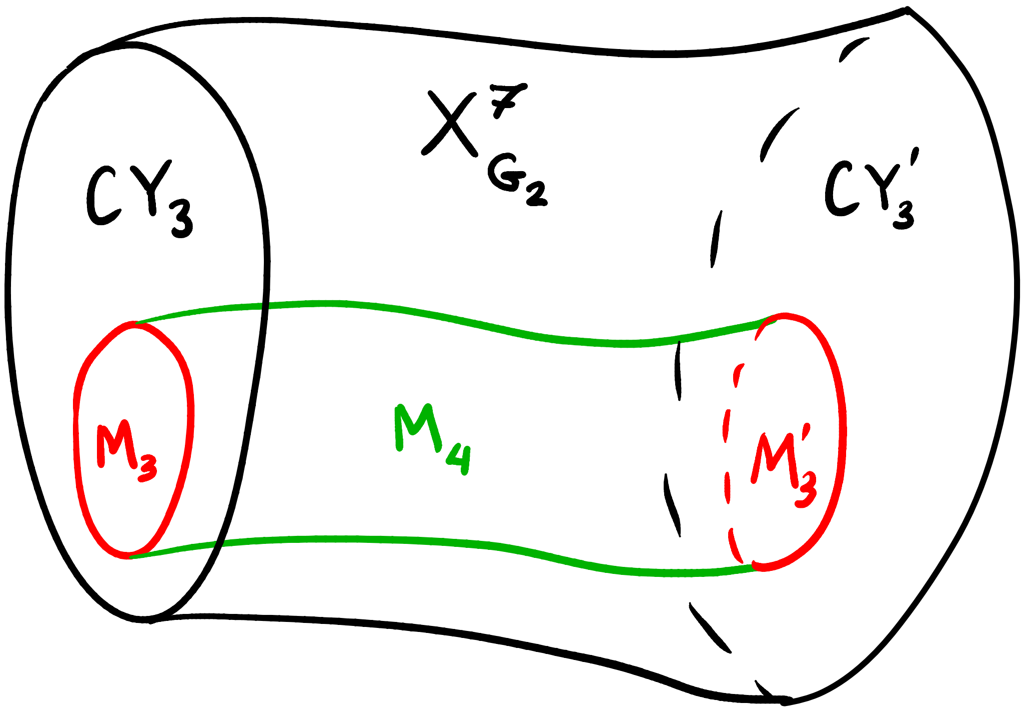
N M5-branes: $\mathbb{R} \times \mathbb{R}^2_q \times M_3$

$X^7 = \mathbb{R} \times T^* M_3$

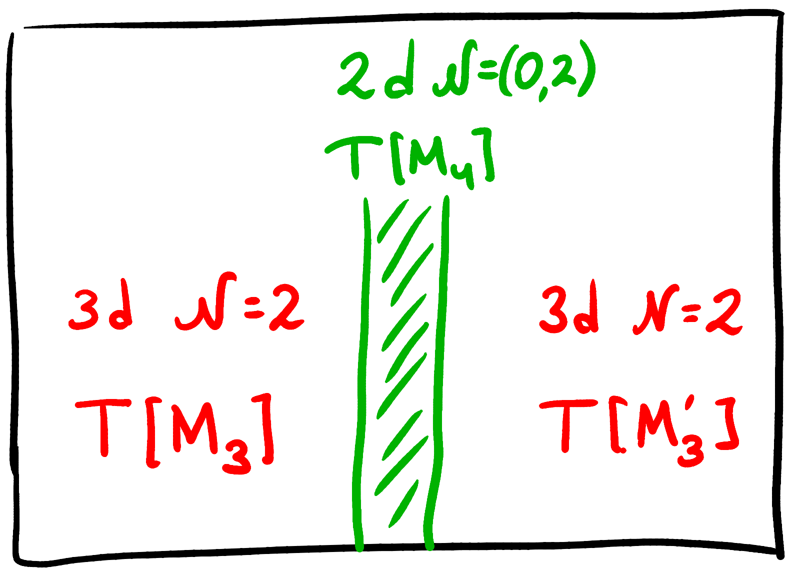
$M_4 = \mathbb{R} \times M_3$



cobordism



half-BPS domain wall

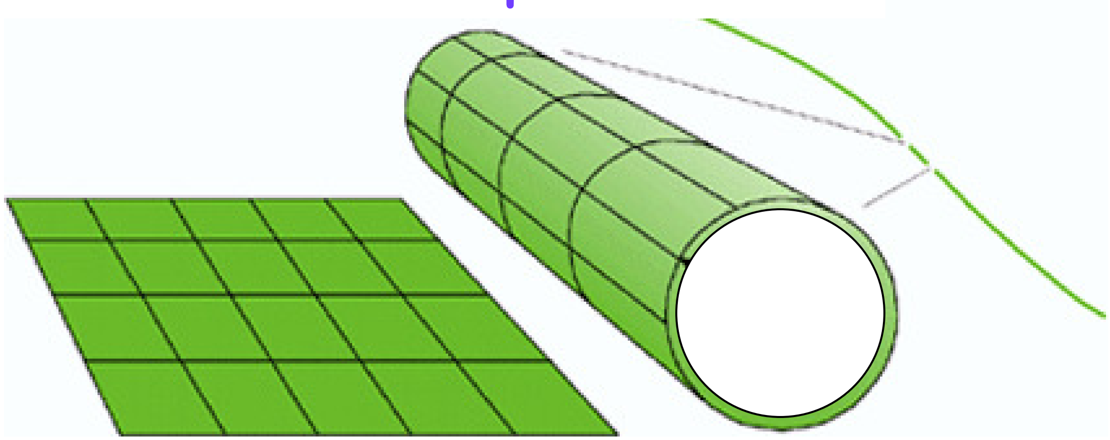


$6 = 2 + 4$

4-manifold M_4 ⇨ 2d $\mathcal{N} = (0,2)$ theory $T[M_4]$

depends on topology and geometry of M_4

6d fivebrane theory on $\mathbb{R}^2 \times M_4$

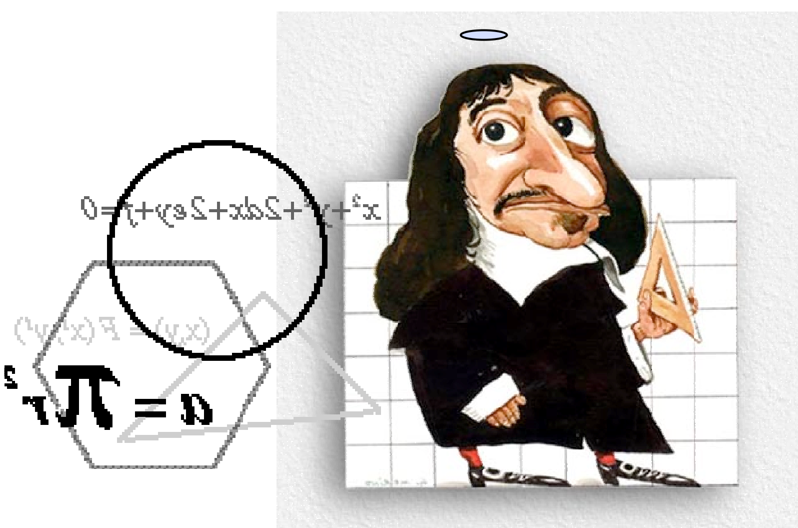


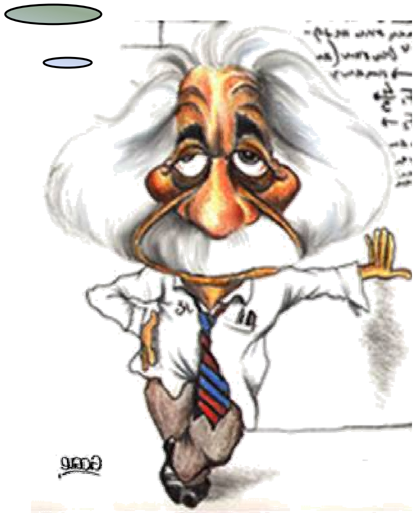
$6 = 2 + 4$

4-manifold M_4 ⇒ 2d $\mathcal{N} = (0,2)$ theory $T[M_4]$

very rich math

very rich physics





$6 = 2 + 4$

4-manifold M_4 ⇒ 2d $\mathcal{N} = (0,2)$ theory $T[M_4]$

$$Z[M_4] = \sum_n \mathbf{x}^c q^n \, \chi(\mathcal{M}_{n,c})$$ = equivariant elliptic genus of the 2d $\mathcal{N} = (0,2)$ theory

moduli space of instantons

$$G_{\text{flavor}}(T[M_4]) = U(1)^{b_2}$$

$$F_A^+ = 0$$

modular !

6 = 2 + 4

6d fivebrane
on $T^2 \times M_4$

[C.Vafa, E.Witten]

$\swarrow$ $\searrow$

$\mathcal{N} = 4$ super-Yang-Mills
on M_4

2d $(0,2)$ theory $T[M_4]$
on T^2

$Z[M_4] = \sum_n x^c q^n \chi(\mathcal{M}_{n,c})$ = equivariant elliptic genus of the 2d $\mathcal{N} = (0,2)$ theory

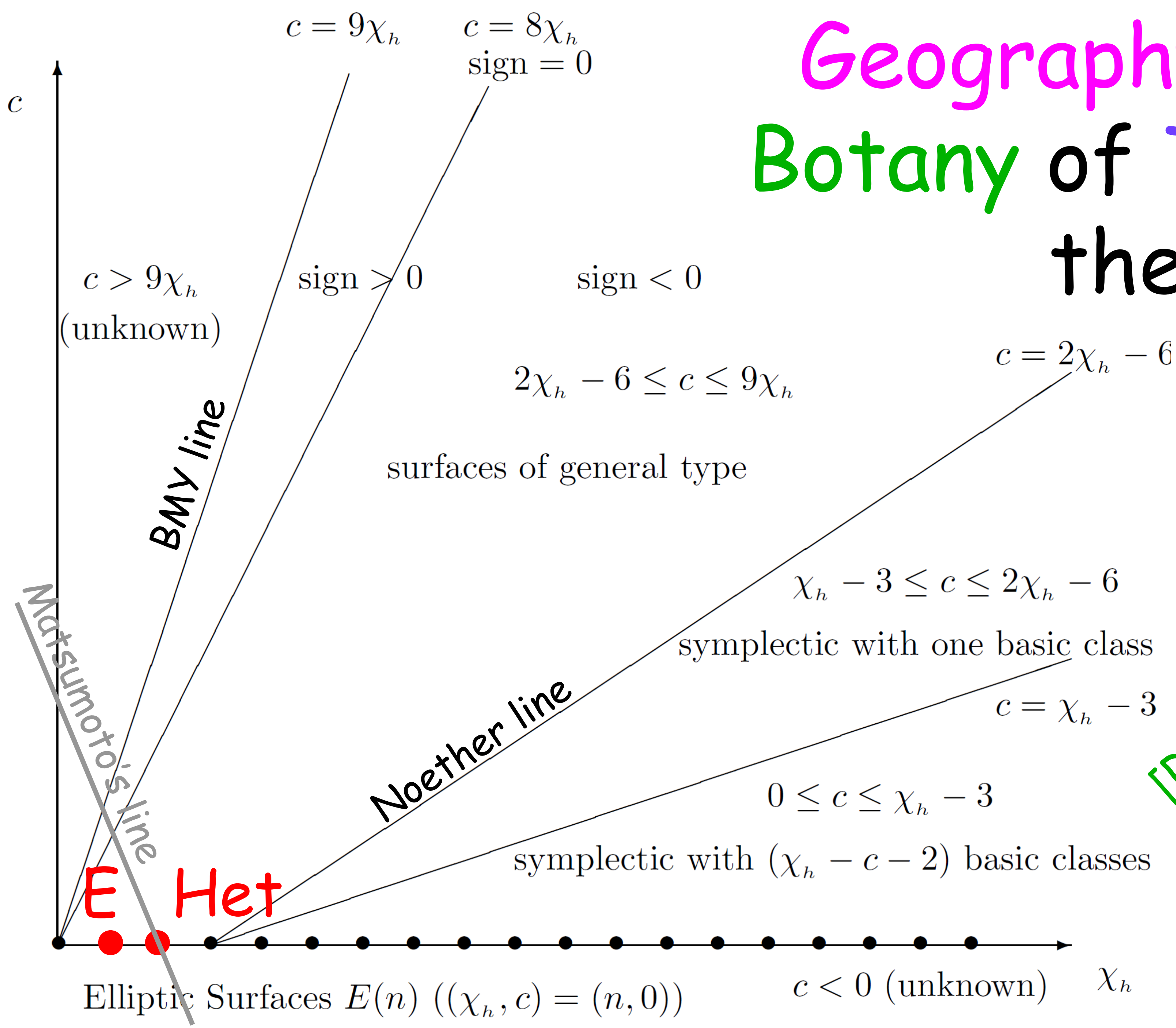
moduli space
of instantons

$$G_{\text{flavor}}(T[M_4]) = U(1)^{b_2}$$

$F_A^+ = 0$

modular !

Geography and Botany of $T[M_4]$ theories



Instantons on ALE spaces

$M_4 = A_K$ ALE space, bounded by $M_3 = L(k+1,1)$

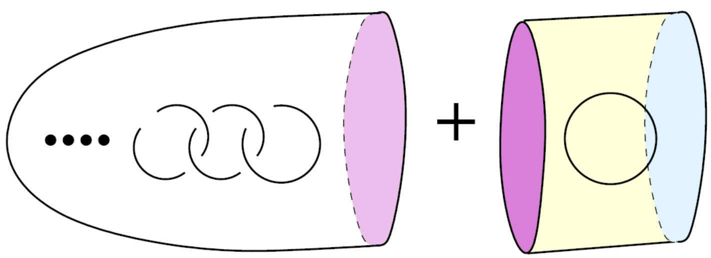
elliptic genus of $T[A_k, U(N)]$

$\sum_n x^c q^n \chi(\mathcal{M}_{n,c})$ = character of a level N representation of affine SU(k+1) labeled by flat connection ρ on $L(k+1,1)$

[H.Nakajima]

First New Results

M_4 = cobordism between L(k,1) and L(k+1,1)



$\sum_n x^c q^n \chi(\mathcal{M}_{n,c})$ = branching function of a G/H coset

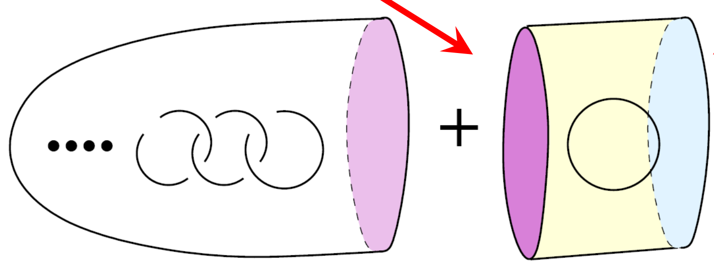
$$\chi^G_\rho = \sum_{\rho'} C^{\rho'}_\rho \chi^H_{\rho'}$$

First New Results

$$\chi^G_\rho = \sum_{\rho'} C^{\rho'}_{\rho}\, \chi^H_{\rho'}$$

flat connection on L(k,1)

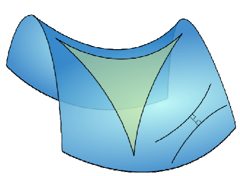
flat connection on L(k+1,1)



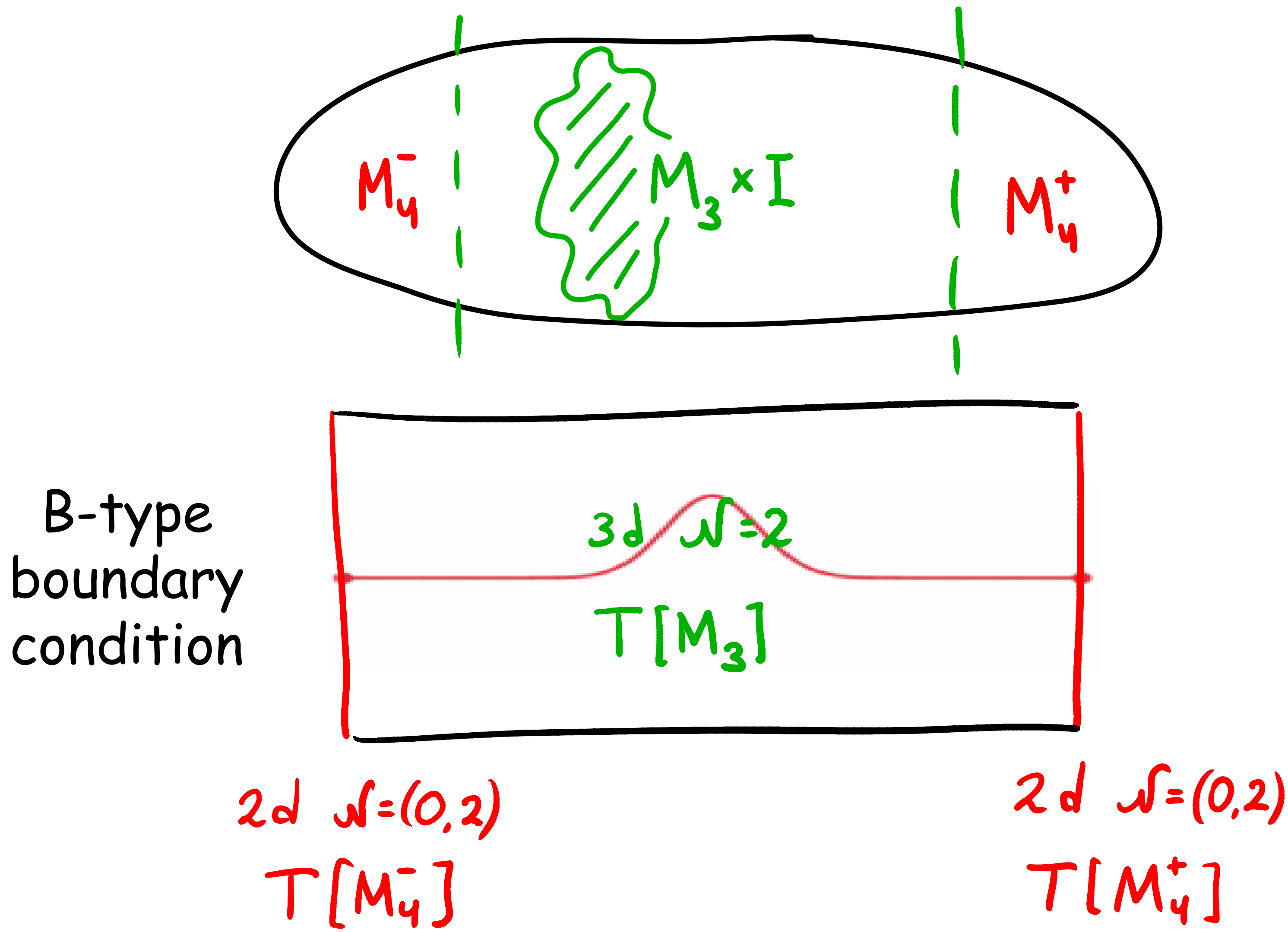
New 4-manifold invariants from 2d $\mathcal{N} = (0,2)$ theories

M_4 ➡ $T[M_4; G]$ ➡ $Z_{T[M_4;G]} =$ 4-manifold invariant

- equivariant elliptic genus = $\chi(\mathcal{M}_{\text{inst}})$
- Q-cohomology = Donaldson invariants
- moduli space of marginal couplings, etc.

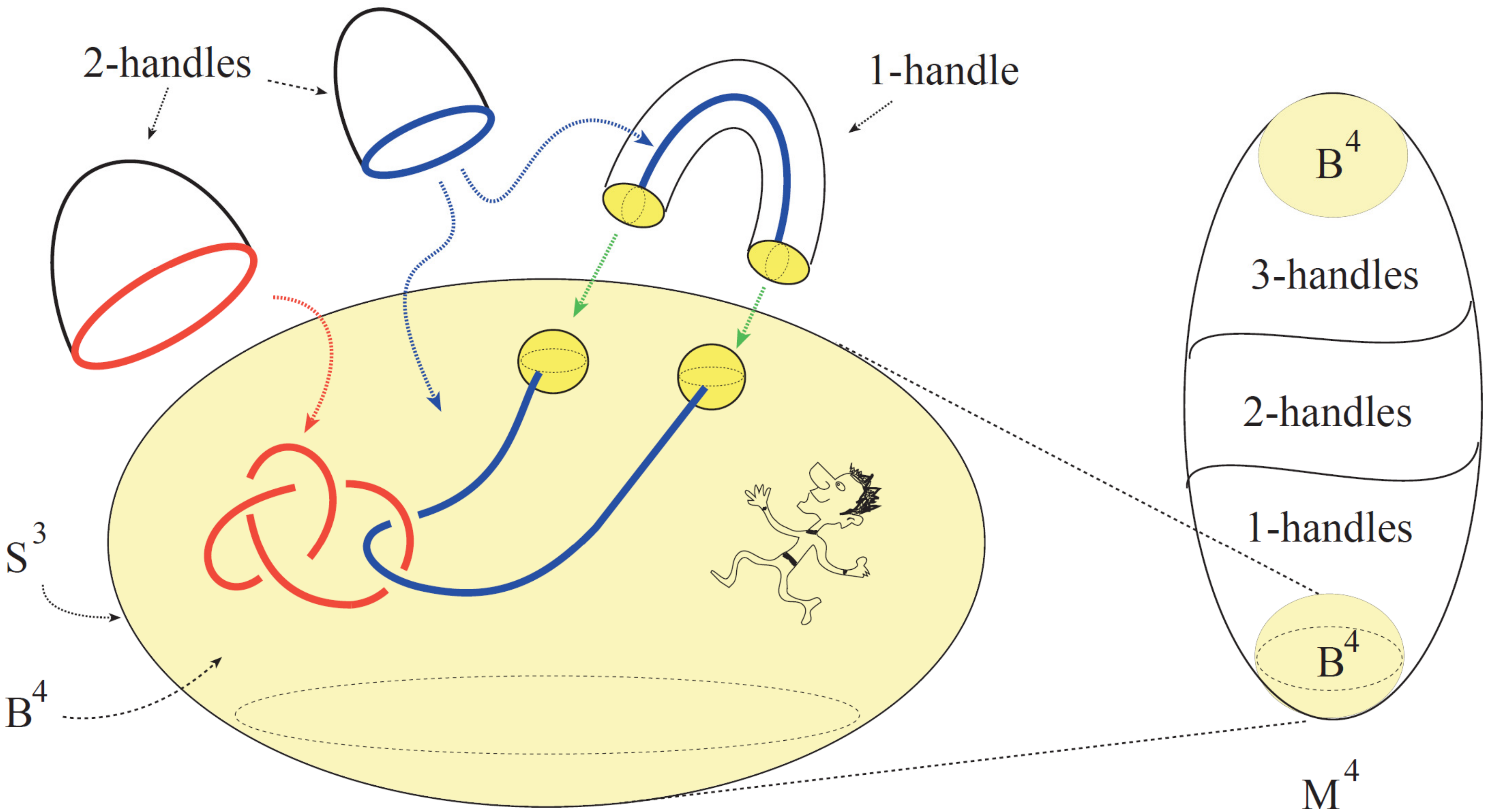


Gluing



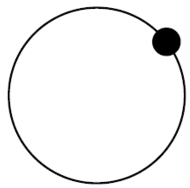
M_4^-
$M_3 \times I$
M_4^+
B-type boundary condition
3d $\mathcal{N}=2$
$T[M_3]$
2d $\mathcal{N}=(0,2)$
$T[M_4^-]$
2d $\mathcal{N}=(0,2)$
$T[M_4^+]$

Building blocks

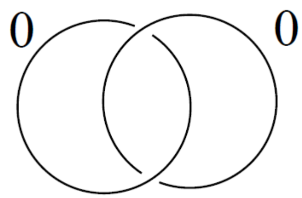


S. Akbulut, 2012

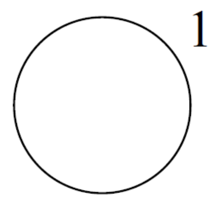
Kirby diagrams



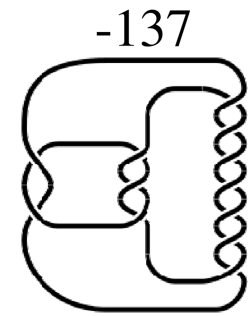
$S^1 \times S^3$



$S^2 \times S^2$



$\mathbb{CP}^2$

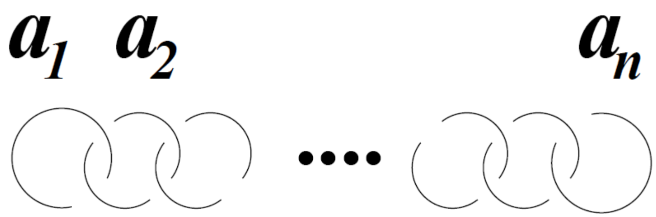


$$M_4 \ : \qquad K_1^{a_1}\, K_2^{a_2}\, \ldots\, K_n^{a_n}$$

<u>Intersection form on $H_2(M_4;\mathbb{Z})$:</u>

$$Q_{ij} = \begin{cases} \mathrm{lk}(K_i, K_j), & \text{if } i \neq j \\ a_i, & \text{if } i = j \end{cases}$$

Plumbing graphs



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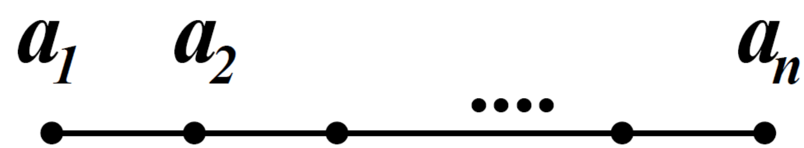
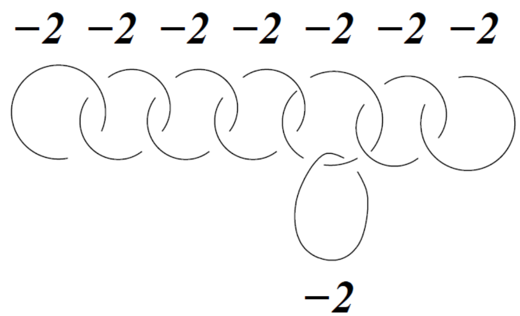


E_8 =



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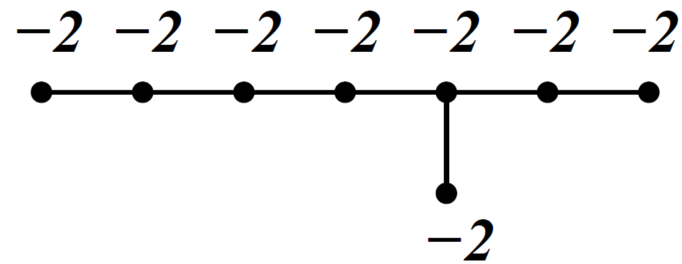
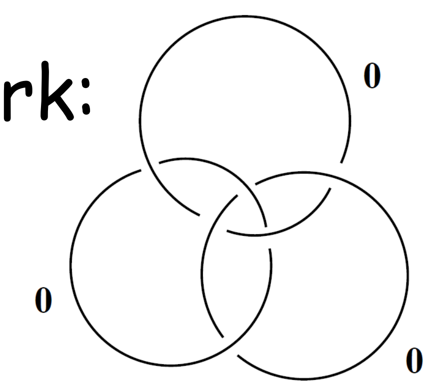
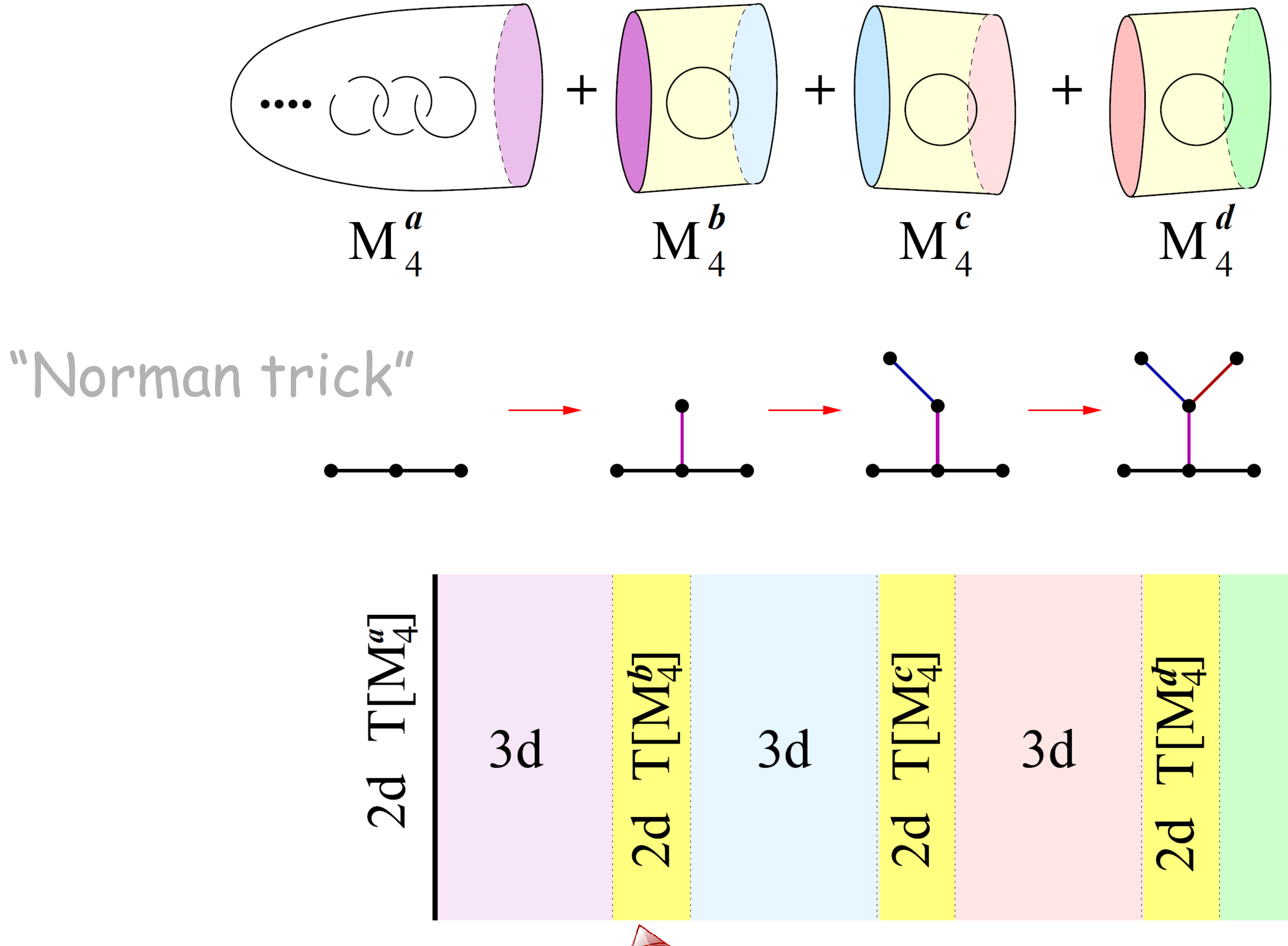


does not always work:



4-manifold bounded by a 3-torus



$$\chi_\rho^G = \sum_{\rho'} C_\rho^{\rho'} \chi_{\rho'}^H$$

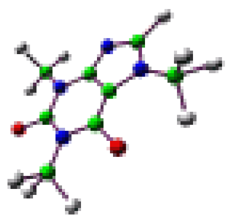
coset branching function

4-manifold M_4	**2d** $(0,2)$ **theory** $T[M_4]$
handle slides	dualities of $T[M_4]$
boundary conditions	vacua of $T[M_3]$
3d Kirby calculus	dualities of $T[M_3]$
cobordism from M_3^- to M_3^+	domain wall (interface) between $T[M_3^-]$ and $T[M_3^+]$
gluing	fusion
Vafa-Witten partition function	flavored (equivariant) elliptic genus
Z_{VW}(cobordism)	branching function
instanton number	L_0
embedded surfaces	chiral operators
Donaldson polynomials	chiral ring relations

Quiver Chern-Simons theory

vertex a ⟷ **U(1)** Chern-Simons at level a

$$S = \frac{a}{4\pi}\int (A \wedge dA + \ldots)$$

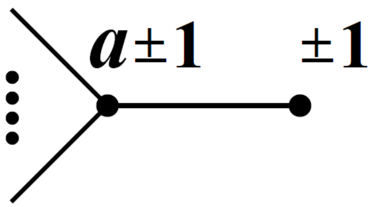


edge a_i — a_j ⟷

$$S = \frac{1}{2\pi}\int (A_i \wedge dA_j + \ldots)$$

cf. [D.Belov, G.Moore]
[A.Kapustin, N.Saulina]
[J.Fuchs, C.Schweigert, A.Valentino]
:

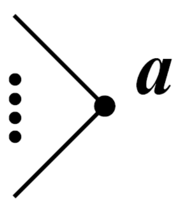
Quiver Chern-Simons theory



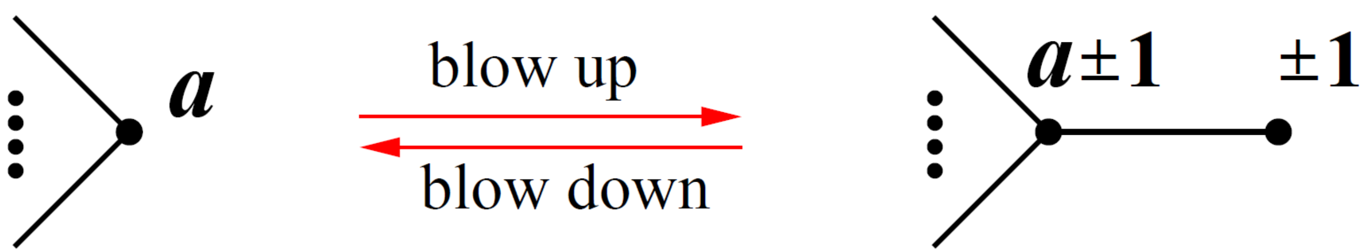
$$= \frac{1}{4\pi}\int \Big(\pm A\wedge dA + 2B\wedge dA + (a\pm 1)B\wedge dB + \dots \Big)$$

integrate out A

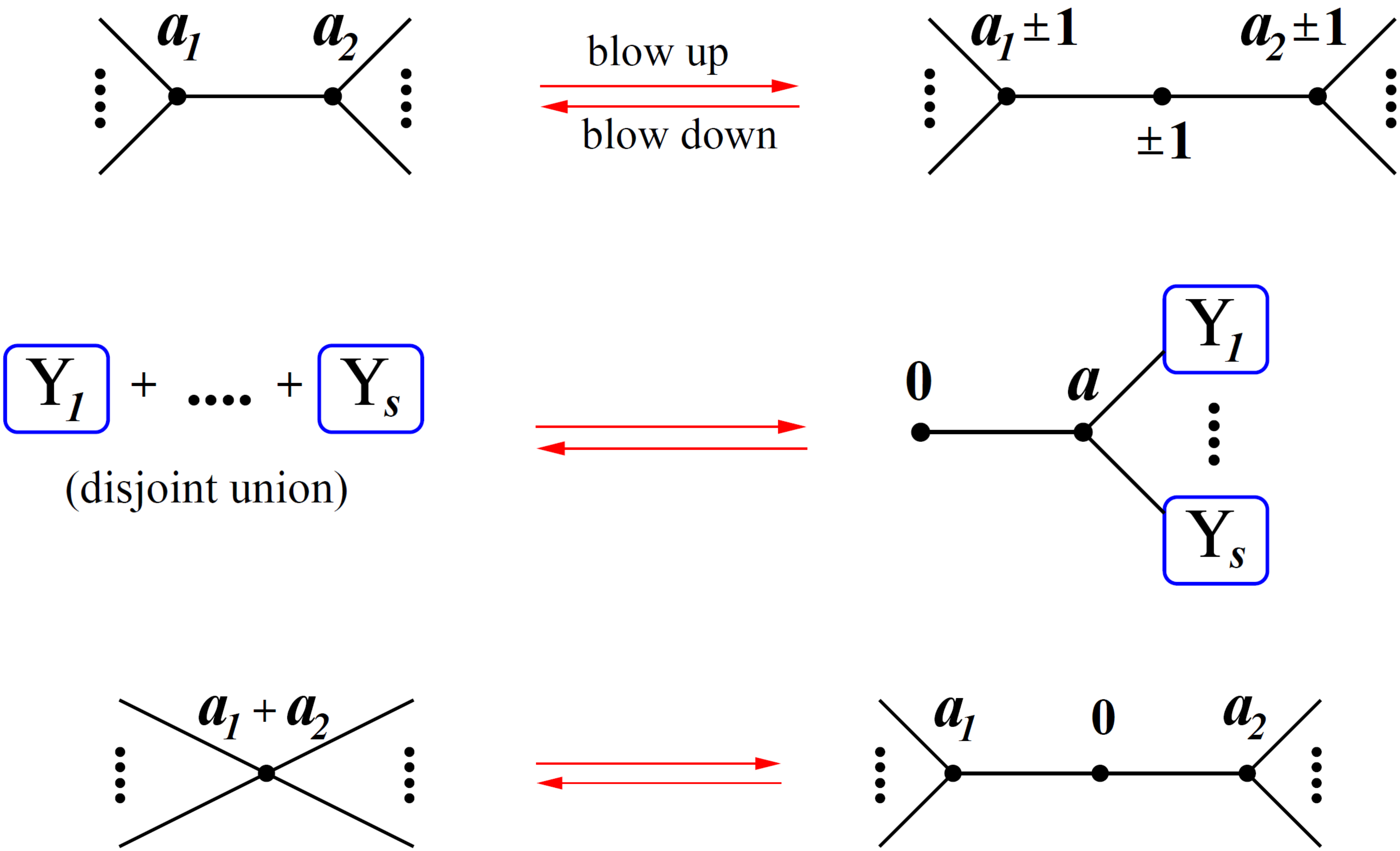
$$= \frac{1}{4\pi}\int \Big(\pm B\wedge dB \mp 2B\wedge dB + (a\pm 1)B\wedge dB + \dots \Big)$$



$$= \frac{1}{4\pi}\int (aB\wedge dB + \dots)$$



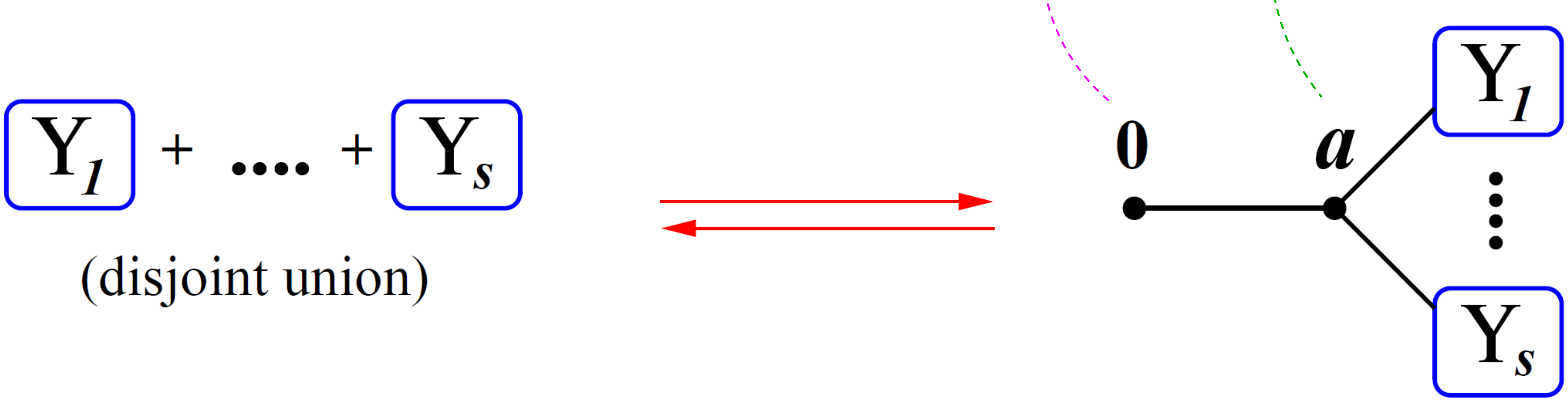
3d Kirby moves



3d Kirby moves

$$\mathcal{L} = \frac{1}{4\pi} \int (2A \wedge dB + aB \wedge dB + \ldots)$$

A is Lagrange multiplier

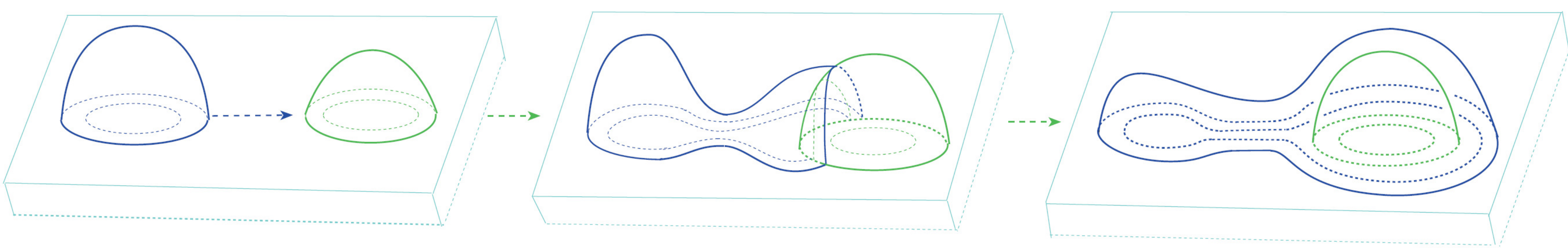


Integrating out A makes B pure gauge
and removes all its Chern-Simons couplings

4d Kirby moves

- identity for instanton partition function (= equivariant elliptic genus):

$$(q;q)_\infty \int dz \frac{1}{\theta(z^{-1}w)} \prod_{i=1}^{N_f} \theta(q^{\frac{1}{2}} z x_i) = \prod_{i=1}^{N_f} \theta(q^{\frac{1}{2}} x_i w)$$



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2d $\mathcal{N} = (0,2)$ SQED

	Φ	$\Psi_{i=1,\ldots,N_f}$
$U(1)_{\text{gauge}}$	-1	$+1$
$U(1)_{\text{flavor}}$	$+1$	0

$\cong$ $\left\{ \begin{array}{l} N_f \ \text{ Fermi multiplets } \Psi'_{i=1,\ldots,N_f} \\ \text{with charge } +1 \text{ under } U(1)_{\text{flavor}} \end{array} \right\}$

2d (0,2) "twisted superpotential"

$$\widetilde{J} = -\frac{i}{8\pi}(N_f - 1)\log(\Phi)$$

"mesons" $\Psi'_i = \Phi\Psi_i$

New 2d $\mathcal{N} = (0,2)$ dualities from 4-manifolds

2d $\mathcal{N} = (0,2)$ SQCD

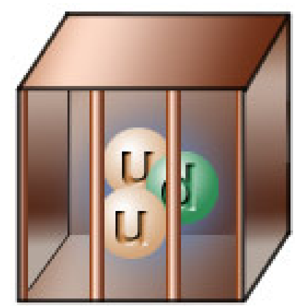
	Φ	Ψ	P	Γ
$U(N_c)_{\text{gauge}}$	$\square$	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$
$U(N_b)_{\text{flavor}}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$
$U(N_f)_{\text{flavor}}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$U(N_f - N_b + 2N_c)$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\overline{\square}$

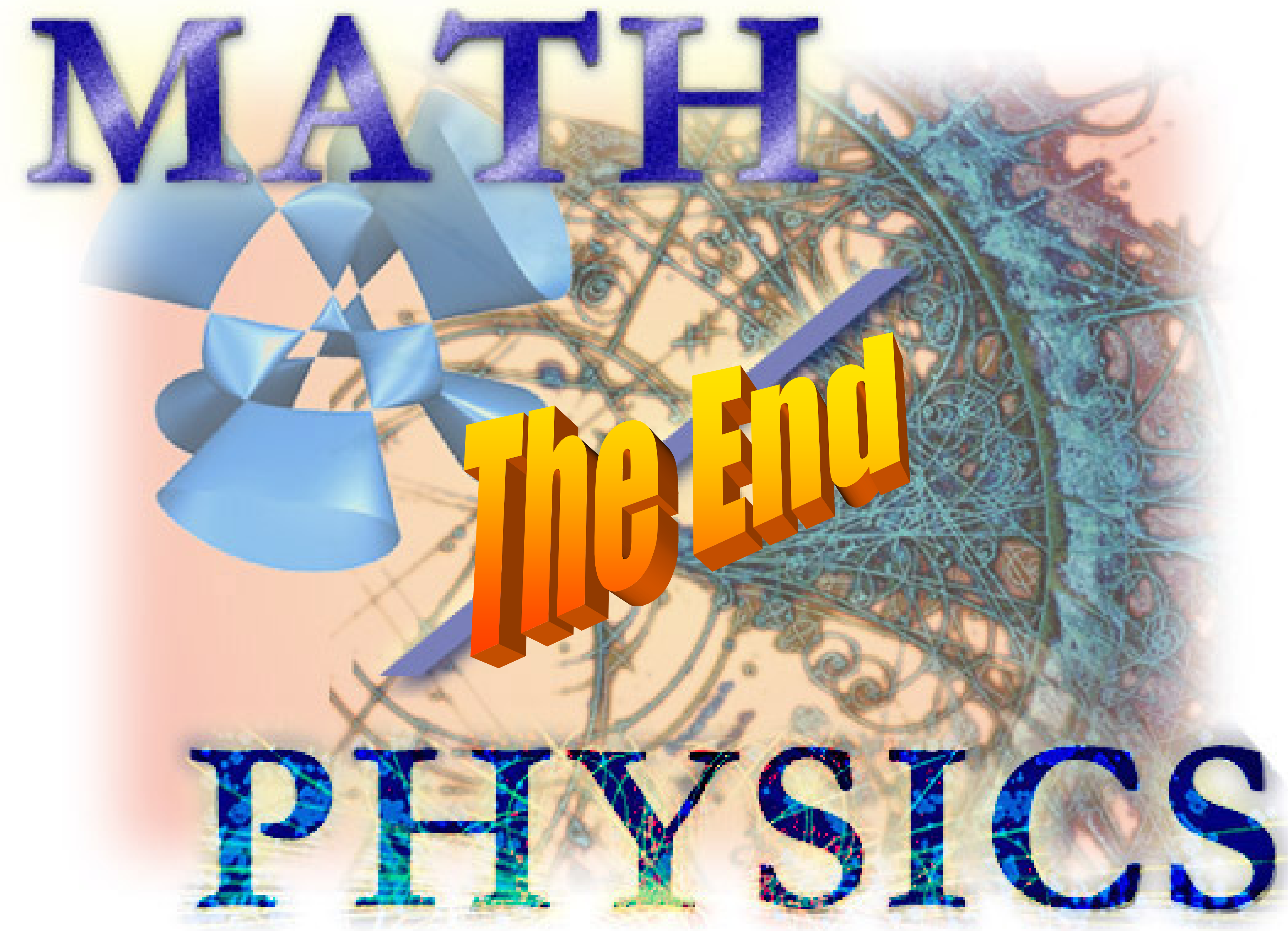
$\cong$

dual 2d $\mathcal{N} = (0,2)$ SQCD

	Φ'	Ψ'	P'	Γ'
$U(N_b - N_c)_{\text{gauge}}$	$\square$	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$
$U(N_b)$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$
$U(N_f - N_b + 2N_c)$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$U(N_f)$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\overline{\square}$

- 2d (0,2) analogue of Seiberg duality
- the very first non-abelian 2d (0,2) duality

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