Inverse scattering methods in STU supergravity

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Joint work with Despoina Katsimpouri and Amitabh Virmani

[KKV1, KKV2, KKV3: JHEP and 1211.3044, 1311.7018, 1409.6471]

Context and Plan

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Plan

- Integrability of gravity-matter systems
- Solution generating techniques
- Applications in gravity and STU supergravity
- Outlook

Symmetries in gravity (I)

Standard $D = 4$ gravity: $R^{(4)}_{MN} = 0$

<u>Assume:</u> One Killing vector:

$$g^{(4)}_{MN} = \begin{pmatrix} e^{-\phi} g^{(3)}_{mn} + e^{\phi} A_m A_n & e^{\phi} A_m \\ e^{\phi} A_n & e^{\phi} \end{pmatrix}$$

Effective dynamics in $D = 3$ for $g^{(3)}_{mn}$, ϕ and A_m.
On-shell duality relation

$$F_{mn} = e^{-2\phi} \epsilon_{mn}{}^{p} \partial_p \chi.$$

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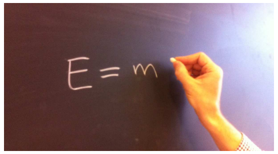
χ and ϕ parametrise $SL(2,\mathbb{R})/SO(2)$ coset space [Ehlers '52]

$$V = \begin{pmatrix} e^{-\phi/2} & e^{-\phi/2}\chi \\ 0 & e^{\phi/2} \end{pmatrix} \Rightarrow P_m := \frac{1}{2}\left(\partial_m V V^{-1} + (\partial_m V V^{-1})^T\right)$$



Symmetries in gravity (II)

Standard $D = 4$ gravity with one Killing vector. $g^{(3)}_{mn}$, V.

$$R^{(3)}_{mn} = \mathrm{Tr}\,(P_m P_n)$$

$$D_m \left(\sqrt{g_{(3)}} P^m \right) = 0$$

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Comment: For maximal supergravity $SL(2,\mathbb{R}) \to E_{8(8)}$

Symmetries in gravity (III)

Now assume two (commuting) Killing vectors (e.g., ∂_ϕ, ∂_t).
$\Rightarrow$ Effective reduction to $D = 2$

$$g^{(3)}_{mn} = \begin{pmatrix} f^2 g^{(2)}_{\mu\nu} & 0 \\ 0 & \rho^2 \end{pmatrix}$$

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Symmetries of this system (beyond $SL(2,\mathbb{R})$)?
Dualisations of what??

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- $\Rightarrow$ ∞ number of conserved quantities $\Rightarrow$ Integrability

[Ernst; Belinski, Zakharov; Breitenlohner, Maison; Julia; Nicolai]

Integrability (I)

Go back to ‘Ehlers’ formulation

$$\partial_\mu \left(\rho P^\mu\right) = \rho \left[Q_\mu, P^\mu\right]$$

$$\pm i f^{-1} \partial_\pm f = \frac{\rho}{2} \mathrm{Tr}\left(P_\pm P_\pm\right)$$

Get f by direct integration if $P_\pm \in \text{Lie}(SL(2, \mathbb{R}))$ known.

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$$\partial_\mu M_\nu(t) - \partial_\mu M_\nu(t) = [M_\mu(t), M_\nu(t)]$$

Often useful to introduce spectral parameter t into the problem (complex analysis and spectrum of conserved quantities).

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Linear system for gravity [Breitenlohner, Maison 1985]

$$\partial_\pm \mathcal{V}(t) \cdot \mathcal{V}(t)^{-1} = \frac{1 \mp it}{1 \pm it} P_\pm + Q_\pm$$

Compatibility yields Einstein equation iff

$$t^{-1} \partial_\pm t = \frac{1 \mp it}{1 \pm it} \rho^{-1} \partial_\pm \rho$$

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constant spectral parameter

$\Rightarrow$ t lives on two-sheeted Riemann surface.

Integrability (III)

Basic objects:

- Vielbein: $\mathcal{V}(t) = V + tV_{(1)} + t^2 V_{(2)} + \ldots$
- Monodromy matrix: $\mathcal{M}(w) = \mathcal{V}^T(-1/t)\mathcal{V}(t)$

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Group theoretic structure

- Transition $V \in SL(2,\mathbb{R}) \to \mathcal{V}(t)$ related to $SL(2,\mathbb{R})$ loop group; in fact whole affine Kac–Moody group [Julia 1980]

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- ∞-ly many conserved quantities from global KM action

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These properties are hallmarks of an integrable system.

$\Rightarrow$ Gravity reduced to $D = 2$ is integrable!

Similar results for many other (super-)gravities [Breitenlohner, Gibbons, Maison; Julia; Nicolai; KKV]

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Many techniques developed for integrable models:

- Inverse scattering method
- Bäcklund transformations
- ...

$\Rightarrow$ Use integrability to construct solutions of gravity!

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Sketch of generating technique $V \to V^{\text{new}}$ with affine $g(w)$

$$V \quad \to \quad \mathcal{V}(t) \quad \to \quad \mathcal{M}(w)$$

Seed solution

Group transformation

$$\mathcal{M}^{\text{new}}(w) = g^T(w)\mathcal{M}(w)g(w)$$

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Hard step: Factorisation of $\mathcal{M}^{\text{new}}(w)$ into $\mathcal{V}^{\text{new}}(t)$

$\Rightarrow$ Riemann–Hilbert problem

Solution generation (II)

Restrict attention to soliton sector: [BZ=Belinski-Zakharov 1978]

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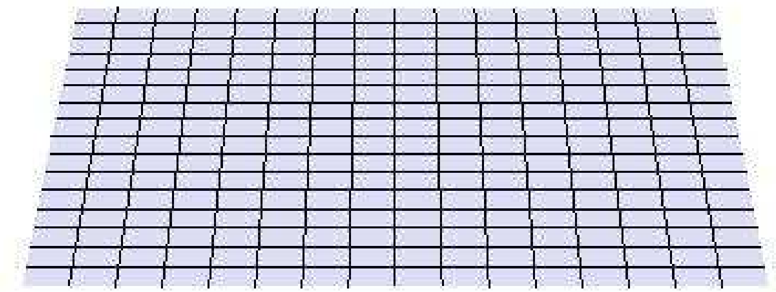
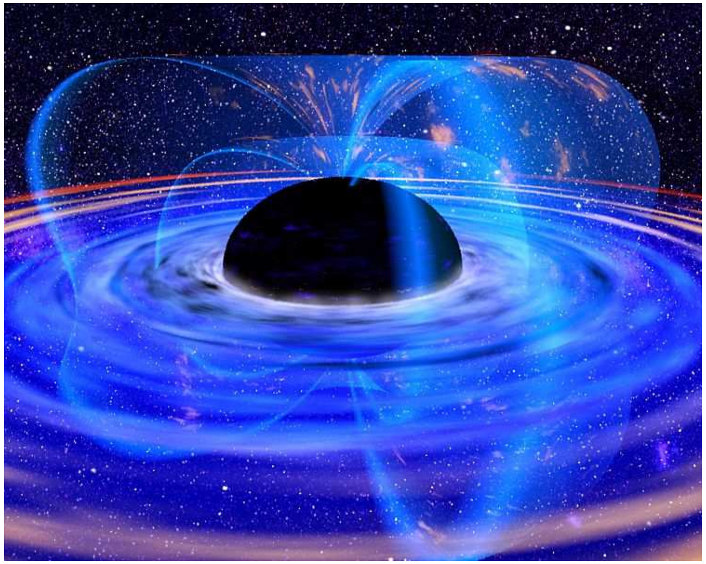
Includes many interesting solutions.



No pole



Two poles

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$D = 4$ gravity: Form of $\mathcal{M}(w)$ [drop $^{\text{new}}$]

$$\mathcal{M}(w) = \mathbb{1} + \sum_{k=1}^{N} \frac{A_k}{w - w_k}$$

with

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In this case, problem becomes an exercise in linear algebra!

[Belinski, Zakharov; Breitenlohner, Maison]

Solution generation (IV)

$$\mathcal{M}(w) = \mathbb{1} + \sum_{k=1}^{N} \frac{A_k}{w - w_k} \quad \Rightarrow \quad \mathcal{M}(w)^{-1} = \mathbb{1} - \sum_{k=1}^{N} \frac{B_k}{w - w_k}$$

Parametrise (rank one!)

$$A_k = \alpha_k a_k a_k^T \,, \quad B_k = \beta_k b_k b_k^T$$

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To obtain factorisation $\mathcal{M}(w) = \mathcal{V}^T(-1/t)\mathcal{V}(t)$

1. Make similar factorized ansatz for $\mathcal{V}(t)$
2. Analyse carefully residues of $\mathcal{M}(w)\mathcal{M}^{-1}(w)$ etc.

Solution generation (V)

Upshot is

$$M = V^T V = \mathbb{1} + \sum_{k,l=1}^{N} b_k t_k^{-1} \left(\Gamma^{-1}\right)_{kl} a_l^T$$

with

$$\Gamma_{kl} = \begin{cases} \frac{\gamma_k}{t_k} & \text{for} \quad k = l \\ \frac{a_k^T b_l}{t_k - t_l} & \text{for} \quad k \neq l \end{cases}$$

$$t_k = \frac{1}{\rho}\left((z - w_k) + \sqrt{(z - w_k)^2 + \rho^2}\right)$$

Conformal factor also simple: $f^2 = c \det(\Gamma) \prod_{k=1}^{N} (\nu_k t_k)$

[Expressions for γ_k and ν_k slightly more involved $\Rightarrow$ didn't fit.]

Solution examples

Schwarzschild [Breitenlohner, Maison]

$$\mathcal{M}(w) = \begin{pmatrix} \frac{w+m}{w-m} & 0 \\ 0 & \frac{w-m}{w+m} \end{pmatrix}$$

Kerr–NUT [KKV1]

$$\mathcal{M}(w) = \frac{1}{w^2 - c^2} \begin{pmatrix} (w+m)^2 + (n+a)^2 & 2(am - wn) \\ 2(am - wn) & (w-m)^2 + (n-a)^2 \end{pmatrix}$$

with $c^2 = m^2 + n^2 - a^2$.

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- Formalism above has the potential to generalise to other groups/theories. E.g.
  - E_8 for maximal supergravity
  - $SO(4,4)$ for STU-gravity
  - G_2 for minimal $D = 5$ supergravity

Other theories

$D = 4$ gravity well-studied, also within inverse scattering approach of Belinski and Zakharov.

- BZ inverse scattering approach not phrased group-theoretically, secretly strongly modelled on $GL(2, \mathbb{R})$. Does not generalise immediately to other theories ([Pomeransky] for $D = 5$ gravity).
- Formalism above has the potential to generalise to other groups/theories. E.g.
  - E_8 for maximal supergravity
  - $SO(4, 4)$ for STU-gravity
  - G_2 for minimal $D = 5$ supergravity
- However, some important technical differences arise...

STU model (I)

$\mathcal{N} = 2$ supergravity in $D = 4$ coupled to three vector multiplets with cubic prepotential [Cremmer et al. 1985]

$$\mathcal{F} = -\frac{X^1 X^2 X^3}{X^0}$$

Dimensionally reduce to $D = 3$. Bosonic fields $g^{(3)}_{mn}$ and V.

V now in $SO(4,4)/SO(2,2) \times SO(2,2)$, total of 16 scalars. Otherwise set-up identical to before.

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Technical differences in

- Rank of residue of $\mathcal{M}(w)$
- Properties of $SO(4,4)$

STU model (II)

Example: 4D Kerr [KKV2]

$$\mathcal{M}(w) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a^2+(w-m)^2}{w^2-c^2} & 0 & 0 & 0 & 0 & -\frac{2am}{w^2-c^2} \\ 0 & 0 & 0 & \frac{a^2+(w-m)^2}{w^2-c^2} & 0 & 0 & \frac{2am}{w^2-c^2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{2am}{w^2-c^2} & 0 & 0 & \frac{a^2+(m+w)^2}{w^2-c^2} & 0 \\ 0 & 0 & -\frac{2am}{w^2-c^2} & 0 & 0 & 0 & 0 & \frac{a^2+(m+w)^2}{w^2-c^2} \end{pmatrix}$$

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Two solitons at $w = \pm c = \sqrt{m^2 - a^2}$. Residues of rank two!
Need to generalise formalism for solution generation.

Generalised solution generation [KKV2]

rank one	rank r
soliton w_k; $k = 1, \ldots N$	soliton w_k; $k = 1, \ldots N$
vectors a_k	vectors a_k^{α}; $\alpha = 1, \ldots, r$
matrix Γ_{kl}	matrix $\Gamma_{kl}^{\alpha\beta}$
conf. $f \propto \det(\Gamma_{kl}) \prod_{k=1}^{N} \tilde{t}_k$	$f \propto \det(\Gamma_{kl}^{\alpha\beta}) \left(\prod_{k=1}^{N} \tilde{t}_k \right)^r$

New formula for asymptotically 4D solution roughly

$$M = \mathbb{1} + \sum_{k,l=1}^{N} \sum_{\alpha,\beta=1}^{r} b_k^{\alpha} t_k^{-1} (\Gamma^{-1})_{kl}^{\alpha\beta} (a_l^{\beta})^T$$

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[Effectively everything boosted up to an rN-dim'l space.]

STU model in various dimensions

STU supergravity can be lifted to $D = 5$ and $D = 6$.

$$\mathcal{L}_{(6)} = R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{12}e^{-\sqrt{2}\Phi}H_{MNP}H^{MNP}$$

What about generating black holes in $D > 4$?

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Important for

- Exploring zoo of higher-dimensional black objects
- Mathur's fuzzball proposal, e.g, understanding the D1-D5-P system.

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Example: JMaRT solution [hep-th/0504181]. Smooth non-SUSY 5D solution with two or three e-m charges.

5D asymptotics

Solutions above had 4D-asymptotics and $\lim_{w\to\infty}\mathcal{M}(w) = \mathbb{1}$

For 5D-asymptotics ($\mathbb{R}^{1,4}$) get (see also [Giusto, Saxena 2007])

$$\lim_{w\to\infty}\mathcal{M}(w) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =: Y$$

Technical changes in inverse scattering procedure: Charging transformations have to preserve these asymptotics but otherwise very similar.

JMaRT from inverse scattering [KKV3]

Original JMaRT construction from over-rotating charged Myers-Perry black hole [Cvetič–Youm 96].

Over-rotating in inverse scattering $\Longrightarrow$ complex poles $\Longrightarrow$ complex residues for $SO(4,4)$ $\mathcal{M}(w)$

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Way around: Use embedding into Lorentzian 6D and use Euclidean 5D solution

$$ds_6^2 = -dt^2 + ds_5^2$$

In Euclidean 5D have instanton similar to over-rotating MP but with real poles.

JMaRT from inverse scattering (II)

Strategy

1. Construct 5D instanton from inverse scattering (c real)

$$\mathcal{M}(w) = Y + \frac{A_1}{w-c} + \frac{A_2}{w+c}$$

   and lift to Lorentzian 6D

2. Charge up 6D solution with appropriate $SO(4,4)$ element

3. Analyse solution and recognise JMaRT

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Strategy works but details are a bit lengthy $\Longrightarrow$ [KKV3].

Outlook

- ‘BM method’ intertwines nicely with finite group methods. Wide applicability
- Fermions can be included [Nicolai 1990]
- Characterisation of physically interesting solutions in terms of group data?
- Study of thermodynamics?
- Useful for systematic study of zoo of solutions?
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Thank you for your attention!