Holographic Entanglement Entropy and Thermalization

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Plan of the talk

- Entanglement entropy and n-partite information
- Holographic description
- Scaling behaviors during a time evolution

References

M. A, M. R. M. Mozaffar and M. R. Tanhayi, “Evolution of Holographic n-partite Information,” [arXiv:1406.7677 [hep-th]].

M. A, A. F. Astaneh and M. R. M. Mozaffar, “Thermalization in backgrounds with hyperscaling violating factor,” [arXiv:1401.2807 [hep-th]].

P. Fonda, L. Franti, V. Kernen, E. Keski-Vakkuri, L. Thorlacius and E. Tonni, “Holographic thermalization with Lifshitz scaling and hyperscaling violation,” [[arXiv:1401.6088 [hep-th]].

H. Liu and S. J. Suh, “Entanglement growth during thermalization in holographic systems,” [arXiv:1311.1200 [hep-th]].

Entanglement entropy

Consider a generic quantum system with a Hilbert space $\mathcal{H}$. For a pure state $|\psi\rangle$ in the system which evolves in time by its Hamiltonian H the density matrix is given by

$$\rho_{\text{total}} = |\psi\rangle\langle\psi|.$$

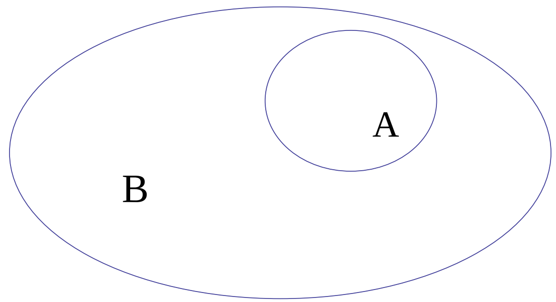
Physical quantities are computed as expectation values of operators as follows

$$\langle O\rangle = \langle\psi|O|\psi\rangle = \text{Tr}(\rho_{\text{total}} O)$$

In mixed states, the system is described by a density matrix ρ. An example of a mixed state is the canonical distribution

$$\rho = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}$$

Assume that the quantum system has multiple degrees of freedom and so one can decompose the total system into two subsystems A and B



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

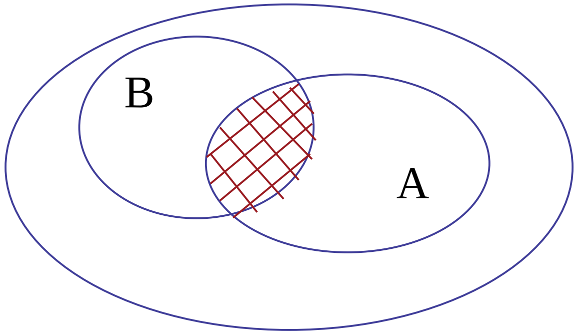
The reduced density matrix of the subsystem A

$$\rho_A = \mathrm{Tr}_B(\rho_{\mathrm{total}})$$

Then the entanglement entropy is defined as the von Neumann entropy for A

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

Properties of Entanglement entropy



1. For pure state

$$S_A = A_B$$

2. For two subspace A and B, the strong subadditivity is

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

3. Leading divergence term is proportional to the area of the boundary ∂A

$$S_A = c_0 \frac{\text{Area}}{\epsilon^{d-1}} + O(\epsilon^{-(d-2)}), \qquad S_A = \frac{c}{3} \ln \frac{\ell}{\epsilon} \qquad \text{for } 2D$$

where c_0 is a numerical constant; ϵ is the ultra-violet(UV) cut off in quantum field theories.

Rényi entropies

It is also useful to compute Rényi entropies

$$S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho^n$$

Then the entanglement entropy is given by

$$S_E = \lim_{n \to 1} S_n$$

Practically one may first compute $\mathrm{Tr}(\rho^n)$ by making use the replica trick and then

$$S_E = -\partial_n \mathrm{Tr} \rho^n |_{n=1}$$

Mutual information

One may study entanglement entropy for two disjoint regions. For two disjoint regions A and B, it is more natural to compute the amount of correlations (both classical and quantum) between these two regions which is given by the mutual information.

It is actually a quantity which measures the amount of information that A and B can share which in terms of the entanglement entropy is given by

$$I(A,B) = S(A) + S(B) - S(A \cup B),$$

Although the entanglement entropy is UV divergent, the mutual information is finite. Moreover by making use of the subadditivity property of the entanglement entropy, it is evident that the mutual information is always non-negative and it is zero for two uncorrelated systems.

n-partite information

More generally one may want to compute entanglement entropy for a subsystem consists of n disjoint regions A_i, $i = 1, \cdots, n$.

Following the notion of mutual information for a system of two disjoint regions, it is natural to define a quantity, n-partite information, which could measure the amount of information or correlations (both classical and quantum) between them. Intuitively, one would expect that for n un-correlated systems the n-partite information must be zero. Moreover, for n disconnected systems it should be finite.

Actually for a given n disjoint regions, there is no a unique way to define n-partite information and indeed, it can be defined in different ways. In particular in terms of entanglement entropy one may define the n-partite information as follows

$$
\begin{aligned}
I^{[n]}(A_{\{i\}}) \;=\; & \sum_{i=1}^{n} S(A_i) - \sum_{i<j}^{n} S(A_i \cup A_j) + \sum_{i<j<k}^{n} S(A_i \cup A_j \cup A_k) - \cdots\cdots \\
& -(-1)^n S(A_1 \cup A_2 \cup \cdots \cup A_n),
\end{aligned}
$$

In terms of the mutual information, this n-partite information may be recast into the following form

$$
\begin{aligned}
I^{[n]}(A_{\{i\}}) \;=\; & \sum_{i=2}^{n} I^{[2]}(A_1, A_i) - \sum_{i=2<j}^{n} I^{[2]}(A_1, A_i \cup A_j) \\
& + \sum_{i=2<j<k}^{n} I^{[2]}(A_1, A_i \cup A_j \cup A_k) - \cdots \\
& +(-1)^n I^{[2]}(A_1, A_2 \cup A_2 \cdots \cup A_n).
\end{aligned}
$$

It is worth mentioning that although the mutual information is always non-negative, the n-partite information $I^{[n]}$ could have either signs.

It may be reexpressed in terms of $(n-1)$-partite information as follows

$$\begin{aligned} I^{[n]}(A_{\{i\}}) \;=\; & I^{[n-1]}(A_{\{1,\cdots,n-2\}}, A_{n-1}) + I^{[n-1]}(A_{\{1,\cdots,n-2\}}, A_n) \\ & - I^{[n-1]}(A_{\{1,\cdots,n-2\}}, A_{n-1} \cup A_n). \end{aligned}$$

n-partite information $I^{[n]}$ may be thought of a quantity which measures the degree of extensivity of the $(n-1)$-partite information.

In the literature of information theory for a subsystem consisting of n disjoint regions, one may define another quantity which, indeed, is a direct generalization of mutual information known as multi-partite entanglement defined as follows

$$J^{[n]}(A_{\{i\}}) = \sum_{i}^{n} S(A_i) - S(A_1 \cup A_2 \cup \cdots \cup A_n),$$

In terms of the mutual information it may be recast into the following form

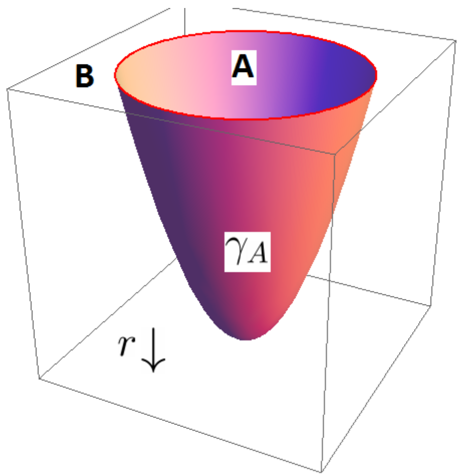
$$J^{[n]}(A_{\{i\}}) = I^{[2]}(A_1, A_2) + I^{[2]}(A_1 \cup A_2, A_3) + \cdots + I^{[2]}(A_1 \cup A_2 \cdots \cup A_{n-1}, A_n).$$

Note that this quantity is finite for a system with n disjoint regions and is zero for n un-correlated regions. It is always non-negative.

In general, for a generic quantum system it is difficult to compute enat-nglemenet entropy and n partite information. We note, however, that for those strongly coupled systems which have gravitational duals, in order to compute the entanglement entropy one may employ its holographic description.

Holographic Formula for Entanglement Entropy

For static background and fixed time divide the boundary into A and B. Extend this division $A \cup B$ to the bulk spacetime. Extend ∂A to a surface γ_A in the entire spacetime such that $\partial \gamma_A = \partial A$.



$$S_A = \left. \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}} \right|_{\text{min}}$$

S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT," Phys. Rev. Lett. **96**, 181602 (2006) [hep-th/0603001].

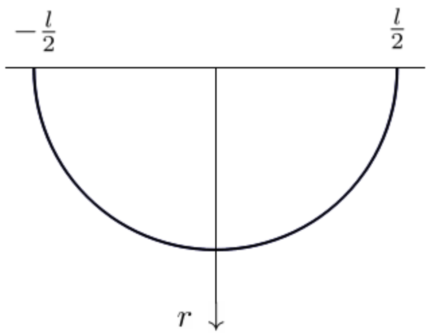
Static solutions

Let's compute the holographic entanglement entropy for a strip in a static asymptotically AdS geometry.

$$dS^2 = \frac{L^2}{r^2}\left(-f(r)dt^2 + g(r)dr^2 + dx_1^2 + dx_{d-2}^2\right),$$

For black hole solution

$$f(r) = g(r)^{-1} = 1 - mr^d = 1 - \frac{r^d}{r_H^d}$$



Consider an entangling region in the shape of a strip with the width of ℓ given by

$$-\frac{\ell}{2} \leq x_1 \leq \frac{\ell}{2}, \qquad 0 \leq x_i \leq L, \qquad i = 2, \cdots, d-2.$$

The holographic entanglement entropy may be computed by minimizing a codimension two hypersurface in the bulk geometry whose intersection with the boundary coincides with the above strip.

Assuming that the profile of the hypersurface in the bulk is parameterized by $x_1 = x(r)$, the corresponding induced metric is

$$dS^2_{\text{ind}} = \frac{R^2}{r^2}\Big[\Big(g(r) + x'^2\Big)\, dr^2 + d\vec{x}^2\Big].$$

Therefore the area A reads

$$A = L^{d-2} R^{d-1} \int dr \frac{\sqrt{g + x'^2}}{r^{d-1}},$$

$$\frac{\ell}{2} = \int_0^{r_t} dr \frac{\sqrt{g(r)}\left(\frac{r}{r_t}\right)^{d-1}}{\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}, \qquad S = \frac{L^{d-2} R^{d-1}}{2G_N} \int_\epsilon^{r_t} \frac{\sqrt{g(r)}dr}{r^{d-1}\sqrt{1 - \left(\frac{r}{r_t}\right)^{2(d-1)}}}$$

where r_t is a turning point and ϵ is a UV cut-off.

For the vaccum state where $f(r) = g(r) = 1$ (AdS solution) one gets

$$S = \begin{cases} \frac{L^{d-2}R^{d-1}}{2G}\left(-\frac{1}{(d-1)\epsilon^{d-2}} + \frac{c_0}{\ell^{d-2}}\right) & \text{for } d \neq 2, \\ \\ \frac{R}{2G}\ln\frac{\ell}{\epsilon} = \frac{c}{3}\ln\frac{\ell}{\epsilon}, & \text{for } d = 2, \end{cases}$$

with c_0 being a numerical factor

$$c_0 = \frac{2^{d-2}\pi^{\frac{d-1}{2}}}{d-2}\left(\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right)^{d-1}$$

When $f \neq 1$, in general, it is not possible to find an explicit expression for the entanglement entropy, though in certain limits one may extract a general behavior of the entanglement entropy.

In particular in the limit of $ml^d \ll 1$, one finds

$$\Delta A = \frac{L^{d-2}}{2} \int d\rho \;\; \delta_f \left(\frac{\sqrt{f^{-1} + x'^2}}{\rho^{d-1}} \right) \Big|_{f=1} \Delta f,$$

which leads to the following expression for the entanglement entropy

$$S_{\mathrm{BH}} = S_{\mathrm{vac}} + \frac{L^{d-2}}{4G_N} c_1 m \ell^2,$$

where S_{vac} is the entanglement entropy of the vacuum solution, and

$$c_1 = \frac{1}{16(d+1)\sqrt{\pi}} \frac{\Gamma(\frac{1}{2(d-1)})^2 \Gamma(\frac{1}{d-1})}{\Gamma(\frac{d}{2(d-1)})^2 \Gamma(\frac{1}{2} + \frac{1}{d-1})}.$$

For $m\ell^d \gg 1$ the main contributions to the entanglement entropy comes from the limit where the minimal surface is extended all the way to the horizon so that $\rho_t \sim \rho_H$. Setting $u = \frac{\rho}{\rho_t}$ one gets

$$\frac{\ell}{2} \approx \rho_H \int_0^1 \frac{u^{d-1} du}{\sqrt{(1-u^d)\left(1-u^{2(d-1)}\right)}},$$

$$S_{\text{BH}} \approx \frac{L^{d-2}}{4G_N \rho_H^{d-2}} \int_{\frac{\epsilon}{\rho_H}}^1 \frac{du}{u^{d-1}\sqrt{(1-u^d)\left(1-u^{2(d-1)}\right)}}.$$

Note that apart from the UV divergent term in S_{BH}, due to the double zero in the square roots, the main contributions in the above integrals come from $u = 1$ point. Indeed around $u = 1$ it may be recast to the following form

$$S_{\text{BH}} \approx \frac{L^{d-2}}{4G_N \rho_H^{d-2}} \left(\int_0^1 \frac{u^{d-1} du}{\sqrt{(1-u^d)\left(1-u^{2(d-1)}\right)}} + \int_{\frac{\epsilon}{\rho_H}}^1 du \, \frac{\sqrt{1-u^{2(d-1)}}}{u^{d-1}\sqrt{1-u^d}} \right).$$

Therefore one arrives at

$$S_{\mathrm{BH}} \approx \frac{L^{d-2}}{4G_N}\left(\frac{1}{(d-2)\epsilon^{d-2}}+\frac{\ell}{2\rho_H^{d-1}}-\frac{c_2}{\rho_H^{d-2}}\right).$$

where c_2 is a positive number. For example for $d = 3, 4$ one gets $c_2 = 0.88, 0.33$, respectively.

Note that the first finite term in the above expression is proportional to the volume which is indeed the thermal entropy, while the second finite term is proportional to the area of the entangling region.

W. Fischler and S. Kundu, "Strongly Coupled Gauge Theories: High and Low Temperature Behavior of Non-local Observables," [arXiv:1212.2643 [hep-th]].

Time-dependent backgrounds

So far we have considered static case where we have a time slice on which we can define minimal surfaces. In the time-dependent case there is no a natural choice of the time-slices.

In Lorentzian geometry there is no minimal area surface. In order to resolve this issue we use the covariant holographic entanglement entropy which is

$$S_A(t) = \frac{\text{Area}(\gamma_A(t))}{4G_N^{(d+2)}}$$

where $\gamma_A(t)$ is the extremal surface in the bulk Lorentzian spacetime with the boundary condition $\partial\gamma_A(t) = \partial A(t)$.

Strong subadditivity?

V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP **0707**, 062 (2007) [arXiv:0705.0016 [hep-th]].

Example of time-dependent case: Black hole formation or Thermalization

$$\begin{aligned} \text{Geometry} &\Longleftrightarrow \text{State} \\ AdS \text{ solution} &\Longleftrightarrow \text{Vaccum state} \\ \text{Black hole} &\Longleftrightarrow \text{Excited state; thermal} \end{aligned}$$

Let us perturbe a system so that the end point of the time evolution would be a thermal state. This might be done by a global quantum quench. Typically during evolution the system is out of equilibrium.

The thermalization process after a global quantum quench may be map to a black hole formation due to a gravitational collapse.

A quantum quench in the field theory may occurs due to a sudden change in the system which might be caused by turning on the source of an operator in an interval $\delta t \to 0$.

This change can excite the system to an excited state with non-zero energy density that could eventually thermalize to an equilibrium state.

P. Calabrese and J. L. Cardy, "Evolution of Entanglement Entropy in One-Dimensional Systems," arXiv:cond-mat/0503393 [cond-mat].

From gravity point of view this might be described by a gravitational collapse of a thin shell of matter which can be modelled by an AdS-Vaidya metric.

$$dS^2 = \frac{R^2}{r^2}[f(r,v)dv^2 - 2drdv + d\vec{x}^2], \quad f(r,v) = 1 - m\theta(v)r^d$$

where r is the radial coordinate, x_is are spatial boundary coordinates and v is the null coordinate. Here $\theta(v)$ is the step function and therefore for $v < 0$ the geometry is an AdS metric while for $v > 0$ it is an AdS-Schwarzschild black hole.

General solution with hyperscaling factor

$$S = -\frac{1}{16\pi G_N}\int d^{D+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{N_g} e^{\lambda_i\phi}F^{(i)2}\right],$$

where $V(\phi) = V_0 e^{\gamma\phi}$, G is the Newton constant, γ, V_0 and λ_i are free parameters of the model.

One of the gauge field is required to produce an anisotropy while the above particular form of the potential is needed to get hyperscaling violating factor. The other gauge fields make the background charged. In what follows we will consider $N_g = 2$.

K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. P. Trivedi and A. Westphal, "Holography of Dyonic Dilaton Black Branes," [arXiv:1007.2490 [hep-th]].

The model admits solutions with hyperscaling violating factor

$$ds^2 = r^{-2\frac{\theta}{D}}\Big(- r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2\Big),$$

Under scaling

$$t \to \xi^z t, \qquad x_i \to \xi x, \qquad r \to \xi^{-1} r$$

the metric scales $ds \to \xi^{\theta/D} ds$.

$$S \sim T^{(D-\theta)/z}$$

C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, "Effective Holographic Theories for low-temperature condensed matter systems," [arXiv:1005.4690 [hep-th]].

X. Dong, S. Harrison, S. Kachru, G. Torroba and H. Wang, "Aspects of holography for theories with hyperscaling violation," [arXiv:1201.1905 [hep-th]].

It has exact charged black hole solutions as follows

$$ds^2 = r^{-2\frac{\theta}{D}}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2\right), \quad \phi = \beta \ln r,$$

$$A_t^{(1)} = \sqrt{\frac{2(z-1)}{D-\theta+z}}\, r^{D-\theta+z}, \qquad A_t^{(2)} = \sqrt{\frac{2(D-\theta)}{D-\theta+z-2}}\, \frac{Q}{r^{D-\theta+z-2}},$$

with $\beta = \sqrt{2(D-\theta)(z-1-\theta/D)}$ and

$$f(r) = 1 - \frac{m}{r^{D-\theta+z}} + \frac{Q^2}{r^{2(D-\theta+z-1)}}.$$

where z is the dynamical exponent and θ is the hyperscaling violation exponent.

M. A., E. O Colgain and H. Yavartanoo, “Charged Black Branes with Hyperscaling Violating Factor,” [arXiv:1209.3946 [hep-th]].

We can also find a Vaidya metric with hyperscaling violating factor

$$ds^2 = r^{-2\frac{\theta}{D}}\left(-r^{2z}f(r,v)dv^2 + 2r^{z-1}drdv + r^2 d\vec{x}^2\right), \quad \phi = \beta \ln r,$$

$$A_v^{(1)} = \sqrt{\frac{2(z-1)}{D-\theta+z}}\, r^{D-\theta+z}, \qquad A_v^{(2)} = \sqrt{\frac{2(D-\theta)}{D-\theta+z-2}}\, \frac{Q(v)}{r^{D-\theta+z-2}}.$$

$$f(r,v) = 1 - \frac{m(v)}{r^{D-\theta+z}} + \frac{Q(v)^2}{r^{2(D-\theta+z-1)}},$$

The energy momentum and current density of the charged infalling matter are given by $T_{\mu\nu} = \varrho U_\mu U_\nu$ and $J^{(2)}_\mu = \varrho_e U_\mu$ with $U_\mu = \delta_{\mu v}$, and

$$\varrho = \frac{\theta - D}{2}\frac{\partial f(r,v)}{\partial v}\, r^z, \qquad \varrho_e = \frac{\partial Q(v)}{\partial v}\sqrt{2(D-\theta)(D-\theta+z-2)}\, r^{\theta - D}.$$

Note that the null energy condition requires $\varrho > 0$.

In what follows $d = D - \theta + 1$ is the effective dimension. We will also set $Q = 0$.

Entnaglement entropy for a strip

To compute the entanglement entropy for a strip with width ℓ, let us consider the following strip

$$-\frac{\ell}{2} \le x_1 = x \le \frac{\ell}{2}, \qquad 0 \le x_a \le L, \quad \text{for } a = 2, \cdots, D.$$

Since the metric is not static one needs to use the covariant proposal for the holographic entanglement entropy. Therefore the corresponding co-dimension two hypersurface in the bulk may be parametrized by $v(x)$ and $r(x)$. Then the induced metric on the hypersurface, setting $r = \frac{1}{\rho}$, is

$$ds^2_{\text{ind}} = \rho^{2\frac{1-d}{D}}\left[\left(1 - \rho^{2-2z} f(\rho, v) v'^2 - 2\rho^{1-z} v'\rho'\right) dx^2 + dx_a^2\right),$$

The area of the hypersurface reads

$$\mathcal{A} = \frac{L^{D-1}}{2} \int_{-\ell/2}^{\ell/2} dx \, \frac{\sqrt{1 - 2\rho^{1-z} v' \rho' - \rho^{2-2z} v'^2 f}}{\rho^{d-1}}$$

We note, however, that since the action is independent of x the corresponding Hamiltonian is a constant of motion

$$\rho^n \mathcal{L} = H = \text{constant}.$$

Moreover we have two equations of motion for v and ρ. Indeed, by making use of the above conservation law the corresponding equations of motion read

$$\partial_x P_v = \frac{P_\rho^2}{2} \frac{\partial f}{\partial v}, \qquad \partial_x P_\rho = \frac{P_\rho^2}{2} \frac{\partial f}{\partial \rho} + \frac{n}{\rho^{2n+1}} H^2 + \frac{1-z}{\rho^{2-z}} P_\rho P_v,$$

where

$$P_v = \rho^{1-z}(\rho' + \rho^{1-z} v' f), \qquad P_\rho = \rho^{1-z} v',$$

are the momenta conjugate to v and ρ up to a factor of H^{-1}, respectively.

These equations have to be supplemented by the following boundary conditions

$$\rho(\frac{\ell}{2}) = 0, \qquad v(\frac{\ell}{2}) = t, \qquad \rho'(0) = 0, \qquad v'(0) = 0,$$

and

$$\rho(0) = \rho_t, \qquad v(0) = v_t,$$

where (ρ_t, v_t) is the coordinate of the extremal hypersurface turning point in the bulk.

In what follows we will consider the case of $\ell \gg \rho_H$

i) $v < 0$ region

In this case which corresponds to the vacuum solution one has

$$P_{(i)v} = \rho' + \rho^{1-z} v' = 0,$$

which together with the conservation law yields to

$$v(\rho) = v_t + \frac{1}{z}(\rho_t^z - \rho^z), \qquad x(\rho) = \int_\rho^{\rho_t} \frac{d\xi \; \xi^n}{\sqrt{\rho_t^{2n} - \xi^{2n}}}.$$

Note also that at the null shell where $v = 0$, from the above equation, one gets

$$\rho_c^z = \rho_t^z + z v_t$$

which, indeed, gives the point where the extremal hypersurface intersects the null shell. Moreover, from the conservation law in the initial phase one finds

$$\rho'_{(i)} = -\rho_c^{1-z} v'_{(i)} = -\sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2n} - 1}$$

ii) $v>0$ region

In this case which the corresponding geometry is a the black hole, using the conservation law one arrives at

$$\rho'^2 = \frac{P^2_{(f)v}}{\rho^{2-2z}} + \left(\left(\frac{\rho_t}{\rho} \right)^{2n} - 1 \right) f(\rho) \equiv V_{eff}(\rho),$$

which can also be used to find

$$\frac{dv}{d\rho} = -\frac{1}{\rho^{2(1-z)}\tilde{f}(\rho)} \left(\rho^{1-z} + \frac{P_{(f)v}}{\sqrt{V_{eff}(\rho)}} \right).$$

Here $V_{eff}(\rho)$ might be thought of as an effective potential for a one dimensional dynamical system whose dynamical variable is ρ. In particular the turning point of the potential can be found by setting $V_{eff}(\rho) = 0$.

iii) Matching at the null shell

Since ρ and v are the coordinates of the space time they should be continuous across the null shell.

We note. however, that since one is injecting matters along the null direction v, one would expect that its corresponding momentum conjugate jumps once one moves from $v < 0$ region to $v > 0$ region.

Therefore by integrating the equations of motion across the null shell one arrives at

$$\rho'_{(f)} = \left(1 - \frac{1}{2} g(\rho_c)\right) \rho'_{(i)}, \qquad \mathcal{L}_{(f)} = \mathcal{L}_{(i)}, \qquad v'_{(f)} = v'_{(i)}.$$

It is, then, easy to read the momentum conjugate of v in $v > 0$ region

$$P_{(f)v} = \frac{1}{2} \rho_c^{1-z} g(\rho_c) \rho'_{(i)} = -\frac{1}{2} \rho_c^{1-z} g(\rho_c) \sqrt{\left(\frac{\rho_t}{\rho_c}\right)^{2n} - 1}.$$

Now we have all ingredients to find the area of the corresponding extremal hypersurface in the bulk. In general the extremal hypersurface could extend in both $v<0$ and $v>0$ regions of space-time. Therefore the width ℓ and the boundary time are found

$$\frac{\ell}{2}=\int_{\rho_c}^{\rho_t}\frac{d\rho\,\rho^{d-1}}{\sqrt{\rho_t^{2(d-1)}-\rho^{2(d-1)}}}+\int_0^{\rho_c}\frac{d\rho}{\sqrt{V_{eff}(\rho)}},\qquad t=\int_0^{\rho_c}\frac{\rho^{z-1}d\rho}{f(\rho)}\left(1+\frac{\rho^{z-1}E}{\sqrt{V_{eff}(\rho)}}\right),$$

where $E=P_{(f)v}$.

Finally the entanglement reads

$$S=\frac{L^{D-2}}{2G}\left[\int_{\rho_c}^{\rho_t}\frac{\rho_t^{d-1}d\rho}{\rho^{d-1}\sqrt{\rho_t^{2(d-1)}-\rho^{2(d-1)}}}+\rho_t^{d-1}\int_0^{\rho_c}\frac{d\rho}{\rho^{2(d-1)}\sqrt{V_{eff}(\rho)}}\right].$$

The behavior of entanglement entropy has been numerically studied in several papers including

J. Abajo-Arrastia, J. Aparicio and E. Lopez, "Holographic Evolution of Entanglement Entropy," [arXiv:1006.4090 [hep-th]].

T. Albash and C. V. Johnson, "Evolution of Holographic Entanglement Entropy after Thermal and Electromagnetic Quenches," [arXiv:1008.3027 [hep-th]]

V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri, B. Muller and A. Schafer *et al.*, "Thermalization of Strongly Coupled Field Theories," [arXiv:1012.4753 [hep-th]].

In what follows we will proceed with semi-analytic results following H. Liu and S. J. Suh, [arXiv:1311.1200 [hep-th]].

Early time

At the early time where $t \ll \rho_H$ the crossing point of the hypersurfaces is very close to the boundary, $\frac{\rho_c}{\rho_H} \ll 1$. Therefore one may expand t, and A leading to

$$
\begin{aligned}
t &\approx \frac{\rho_c^z}{z}\left(1+\frac{1}{d+1}\left(\frac{\rho_c}{\rho_H}\right)^d+\frac{1}{2d+1}\left(\frac{\rho_c}{\rho_H}\right)^{2d}+\dots\right), \\
A &\approx \frac{L^{d-2}}{(d-2)}\left(\frac{1}{\epsilon^{d-2}}-c\frac{1}{\rho_t^{d-2}}\right)+\frac{L^{d-2}m}{2(z+1)}\rho_c^{1+z}\left(1+\frac{1}{2d}\left(\frac{\rho_c}{\rho_t}\right)^{2(d-1)}+\dots\right),
\end{aligned}
$$

where $c=\sqrt{\pi}\frac{\Gamma(\frac{d}{2(d-1)})}{\Gamma(\frac{1}{2(d-1)})}$. So that at leading order one finds

$$
S \approx S_{\text{vac}}+\frac{L^{D-1}m}{4G(z+1)}(zt)^{1+\frac{1}{z}}.
$$

Intermediate time interval

In the intermediate time interval where $\rho_H^z \ll t \ll \rho_H^{z-1}\frac{\ell}{2}$, the entanglement entropy growth linearly with time. Indeed, there is a critical extremal surface which is responsible for the linear growth in this time interval.

$V_{eff}(\rho)$ might be thought of as an effective potential for a one dimensional dynamical system whose dynamical variable is ρ.

For a fixed extremal hypersurface turning point in the bulk, ρ_t, there is a free parameter in the effective potential given by ρ_c which may be tuned to a particular value $\rho_c = \rho_c^*$ such that the minimum of the effective potential becomes zero

$$\left.\frac{\partial V_{eff}(\rho)}{\partial \rho}\right|_{\rho_m,\rho_c^*} = 0, \qquad\qquad V_{eff}(\rho)|_{\rho_m,\rho_c^*} = 0.$$

If the hypersurface intersects the null shell at the critical point it remains fixed at ρ_m.

Therefore in the intermediate time interval the main contributions to ℓ, t and A come from a hypersurface which is closed to the critical extremal hypersurface.

In this case assuming $\rho_c = \rho_c^*(1-\delta)$ for $\delta \ll 1$ in the limit of $\rho \to \rho_m$ and with the conditions $\frac{\rho_c^*}{\rho_t}, \frac{\rho_m}{\rho_t} \ll 1$ one finds

$$t \approx -\frac{\rho_m^{2(z-1)} E^*}{f(\rho_m)\sqrt{\frac{1}{2}V''_{eff}}} \log\delta, \qquad \frac{\ell}{2} \approx c\rho_t + \frac{f(\rho_m)}{E^*} t$$

$$A \approx \frac{L^{d-2}}{(d-2)}\left(\frac{1}{\epsilon^{d-2}} - c\frac{1}{\rho_t^{d-2}}\right) - \frac{L^{d-2}\rho_t^{d-1}}{\rho_m^{2(d-1)}\sqrt{\frac{1}{2}V''_{eff}}} \log\delta$$

where $E^* \equiv E(\rho_c^*)$. So that

$$S \approx S_{\text{vac}} + L^{D-1} S_{\text{th}}\, v_E\, \rho_H^{1-z} t.$$

The scaling behaviours of entanglement entropy

- Early times growth where $t \ll \rho_H^z$

$$\Delta S \approx \frac{L^{D-1} m}{4G(z+1)} (zt)^{1+\frac{1}{z}},$$

- The intermediate region where $\rho_H^{z-1} \frac{\ell}{2} \gg t \gg \rho_H^z$

$$\Delta S \approx L^{D-1} S_{\text{th}}\, v_E\, \rho_H^{1-z} t,$$

where

$$v_E = \left(\frac{d+z-3}{2(d+z-2)}\right)^{\frac{d+z-2}{d+z-1}} \sqrt{\frac{d+z-1}{d+z-3}}, \qquad S_{\text{th}} = \frac{1}{4G\rho_H^{d-1}}$$

- Late time saturation $t \sim \rho_H^{z-1} \frac{\ell}{2}$

$$\Delta S \approx \frac{L^{D-1}\ell}{4G\rho_H^{d-1}} = L^{D-1} \ell S_{\text{th}} \ .$$

P. Calabrese and J. L. Cardy, “Evolution of Entanglement Entropy in One-Dimensional Systems,” arXiv:cond-mat/0503393 [cond-mat].

To summarize

The system has to scales: the size of enatngling region ℓ and the radius of horizon ρ_H.

Thereore we have two time scales

$$t \sim \rho_H \qquad \text{local equilibrium,}$$

$$t \sim \frac{\ell}{2} \qquad \text{saturation on entanglement emtropy.}$$

When $\rho_H > \frac{\ell}{2}$ the entanglement entropy saturates at $t \sim \frac{\ell}{2}$ before the system reaches a local equilibrium, whereas for $\rho_H < \frac{\ell}{2}$ the entanglement entropy is far from its equilibrium value even though the system is locally equilibrated.

For $\frac{\ell}{2} < \rho_H$ one has ($z = 1$)

$$\text{Early times} \qquad S \sim S_{\text{vac}} + V_{d-1}\,\mathcal{E}\,t^2,$$

$$\text{Saturation} \qquad S \sim S_{\text{vac}} + V_{d-1}\,\mathcal{E}\,\frac{\ell^2}{4}.$$

Here $\mathcal{E}$ is the energy density. Sinice the system has not reached a local equilibrium, this is the quantity one may define.

It is consistent with the first law of entnaglement entropy

$$\Delta E = T_E \Delta S_E, \qquad T_E \sim \frac{1}{\ell}.$$

For $\frac{\ell}{2} < \rho_H$ one has

$$\text{Early times} \qquad S \sim S_{\text{vac}} + V_{d-1}\,\mathcal{E}\,t^2,$$

$$\text{Intermediate} \qquad S \sim S_{\text{vac}} + V_{d-1}\,S_{th}\,t,$$

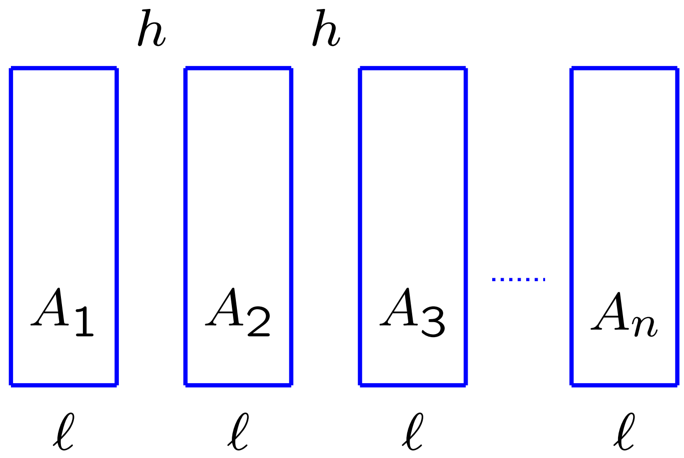
$$\text{Saturation} \qquad S \sim S_{\text{vac}} + V_{d-1}\,S_{th}\,\frac{\ell}{2} + \frac{V_{d-1}}{\rho_H^{d-2}}.$$

The intermediate region is $\rho_H < t < \frac{\ell}{2}$. So that at the early times the system is out of equilibrium, though the system reaches a local equilbrium while the enatnglement entropy still grows with time.

After the local equilibrium the enatnglement entropy may be given in terms of the thermal entropy.

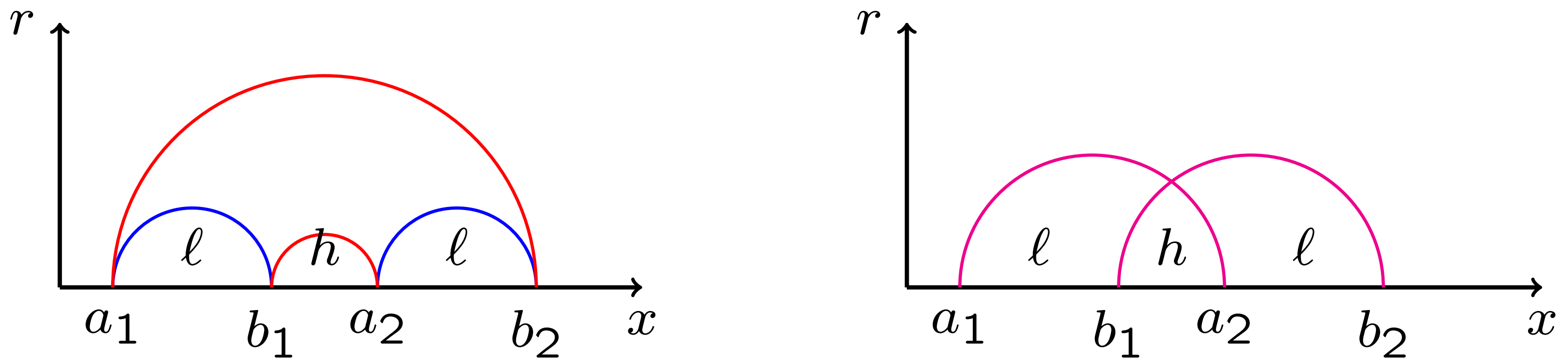
The entanglement entropy at the early times is sensitive to the state, while in the intermediate region it always grows linearly.

Holographic n-partite information



We will study n-partite information of a subsystem consists of n disjoint regions A_i, $i = 1, \cdots, n$ in a d-dimensional CFT for the vacuum and thermal states whose gravity duals are provided by the AdS and AdS black brane geometries. The n disjoint regions are given by n parallel infinite strips of equal width ℓ separated by $n-1$ regions of width h.

$$
\begin{aligned}
I^{[n]}(A_{\{i\}}) &= \sum_{i=1}^{n} S(A_i) - \sum_{i<j}^{n} S(A_i \cup A_j) + \sum_{i<j<k}^{n} S(A_i \cup A_j \cup A_k) - \cdots\cdots \\
&- (-1)^n S(A_1 \cup A_2 \cup \cdots \cup A_n).
\end{aligned}
$$



The main subtlety in evaluating the above quantity is the computation of entanglement entropy of union of subsystem.

For a given two strips with the widths ℓ and distance h, there are two minimal hypersurfaces associated with the entanglement entropy $S(A \cup B)$ and thus the corresponding entanglement entropy behaves differently.

$$S(A \cup B) = \begin{cases} S(2\ell + h) + S(h) & h \ll \ell, \\ 2S(\ell) & h \gg \ell, \end{cases}$$

Therefore the mutual information becomes

$$I(A \cup B) = \begin{cases} 2S(\ell) - S(2\ell + h) - S(h) & h \ll \ell, \\ 0 & h \gg \ell, \end{cases}$$

The holographic mutual information undergoes a first order phase transition as one increases the distance between two strips. Indeed, there is a critical value of $\frac{h}{\ell}$ above which the mutual information vanishes. As we just observed, this peculiar behavior has to do with the definition of entanglement entropy of the union $A \cup B$.

M. Headrick, [arXiv:1006.0047 [hep-th]].

V. E. Hubeny and M. Rangamani, [arXiv:0711.4118 [hep-th]].

E. Tonni, [arXiv:1011.0166 [hep-th]].

n-disjoint regions

Following the same procedure for $h \ll \ell$ one has

$$S(A_i \cup A_{i+j}) = \begin{cases} S(2\ell + h) + S(h) & j = 1, \\ 2S(\ell) & j > 1, \end{cases} .$$

Similarly for the union of three regions one uses

$$S(A_i \cup A_{i+j} \cup A_{i+j+k}) = \begin{cases} S(3\ell + 2h) + 2S(h) & j = 1, k = 1 \\ S(2\ell + h) + 2S(h) & j = 1, k > 1, \text{ or } j > 1, k = 1, \\ 3S(\ell) & j > 1, k > 1 \end{cases}$$

and more generally for arbitrary integer numbers k, m and $j > 1$ one has

$$\begin{aligned} & S(A_i \cup A_{i+1} \cdots \cup A_{i+k} \cup A_{i+k+j} \cup A_{i+k+j+1} \cdots \cup A_{i+k+j+m}) \\ & \qquad = S(A_i \cup A_{i+1} \cdots \cup A_{i+k}) + S(A_{i+k+j} \cup \cdots \cup A_{i+k+j+m}) \\ & \qquad = S(k\ell + (k-1)h) + (k-1)S(h) + S(m\ell + (m-1)h) \\ & \qquad \quad +(m-1)S(h). \end{aligned}$$

By making use of these expressions, the n partitie information for our system may be simplified significantly as follows

$$\begin{aligned} I^{[n]}(A_{\{i\}}) &= (-1)^n \Big[2S\Big((n-1)\ell + (n-2)h\Big) - S\Big(n\ell + (n-1)h\Big) \\ &\quad -S\Big((n-2)\ell + (n-3)h\Big)\Big] \\ &\equiv (-1)^n \tilde{I}^{[n]}. \end{aligned}$$

Interestingly enough, one observes that among various co-dimension two hypersurfaces only three of them corresponding to $(n-1)\ell + (n-2)h$, $n\ell + (n-1)$ and $(n-2)\ell + (n-3)h$ contribute to the n-partite information.

Static solutions

n-partite information for the vacuum state of a CFT whose gravity dual is given by an AdS background

$$\tilde{I}^{[n]}_{\text{vac}} = \frac{L^{d-2}c_0}{4G_N}\bigg(-\frac{2}{((n-1)\ell+(n-2)h)^{d-2}}+\frac{1}{(n\ell+(n-1)h)^{d-2}} +\frac{1}{((n-2)\ell+(n-3)h)^{d-2}}\bigg).$$

For a thermal state whose gravity dual is provided by an AdS black brane geometry, and in the limit of $\ell \ll \rho_H$, one finds

$$\tilde{I}^{[n]}_{\text{BH}} = \tilde{I}^{[n]}_{\text{vac}} - \frac{L^{d-2}}{2G_N}c_1\frac{(\ell+h)^2}{\rho_H^d}.$$

while for $\rho_H \ll \ell$ it vanishes. It is becuase

$$-2[(n-1)\ell+(n-2)h]+[(n-2)\ell+(n-3)h]+[n\ell+(n-1)h]=0,$$

W. Fischler, A. Kundu and S. Kundu, "Holographic Mutual Information at Finite Temperature," [arXiv:1212.4764 [hep-th]].

Time dependent behavior of mutual information and tripartite information have been studied numerically literature. See for example

V. Balasubramanian, A. Bernamonti, N. Copland, B. Craps and F. Galli, "Thermalization of mutual and tripartite information in strongly coupled two dimensional conformal field theories," [arXiv:1110.0488 [hep-th]].

A. Allais and E. Tonni, "Holographic evolution of the mutual information," [arXiv:1110.1607 [hep-th]].

R. Callan, J. -Y. He and M. Headrick, "Strong subadditivity and the covariant holographic entanglement entropy formula," [arXiv:1204.2309 [hep-th]].

They have definite sign if the solution satisfies null energy condition.

Time dependent solutions

For the system we are considering there are four time scales given by the radius of the horizon ρ_H and the three entangling regions: $(n-2)\ell+(n-3)h$, $(n-1)\ell+(n-2)h$ and $n\ell+(n-1)h$.

Assuming in $h \ll \ell$, one recognizes four possibilities for the order of these scales as follows

$$
\begin{aligned}
&2\rho_H \ll (n-2)\ell+(n-3)h < (n-1)\ell+(n-2)h < n\ell+(n-1)h,\\
&(n-2)\ell+(n-3)h < 2\rho_H < (n-1)\ell+(n-2)h < n\ell+(n-1)h,\\
&(n-2)\ell+(n-3)h < (n-1)\ell+(n-2)h < 2\rho_H < n\ell+(n-1)h,\\
&(n-2)\ell+(n-3)h < (n-1)\ell+(n-2)h < n\ell+(n-1)h < 2\rho_H,
\end{aligned}
$$

• First case: $2\rho_H \ll (n-2)\ell + (n-3)h < (n-1)\ell + (n-2)h < n\ell + (n-1)h$

At the early times one finds

$$\tilde{I}^{[n]} = \tilde{I}^{[n]}_{\text{vac}} + \mathcal{O}(t^{2d}),$$

The system reaches a local equilibrium at $t \sim \rho_H$ after which it does not produce thermal entropy, though the entanglement entropy associated with the entangling regions appearing in the n-partite information still increasing with time.

$$\begin{aligned}\tilde{I}^{[n]} &= \tilde{I}^{[n]}_{\text{vac}} + \frac{L^{d-2}}{4G_N}\left(\frac{c_2}{\rho_H^{d-2}} - \frac{c_0}{((n-2)\ell + (n-3)h)^{d-2}}\right) \\ &\quad + \frac{L^{d-2}}{4G_N\rho_H^{d-1}}\left(v_E t - \frac{(n-2)\ell + (n-3)h}{2}\right),\end{aligned}$$

It reaches a maximum at the decreases with time

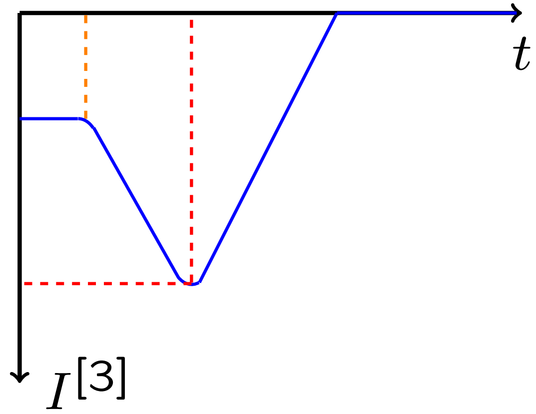
$$\tilde{I}^{[n]} \approx \tilde{I}^{[n](1)}_{\max} + \frac{L^{d-2}}{4G_N}\left(\frac{c_0}{((n-1)\ell+(n-2)h)^{d-2}} - \frac{c_2}{\rho_H^{d-2}}\right) + \frac{L^{d-2}}{4G_N\rho_H^{d-1}}\left(\frac{n-1}{2}\ell + \frac{n-2}{2}h - v_E t\right).$$

Finally it saturates at

$$v_E t_s \approx \frac{n}{2}\ell + \frac{n-1}{2}h - c_2\rho_H.$$

To be compared with the saturation time of the entanglement entropy of a strip with the width $n\ell + (n-1)h$

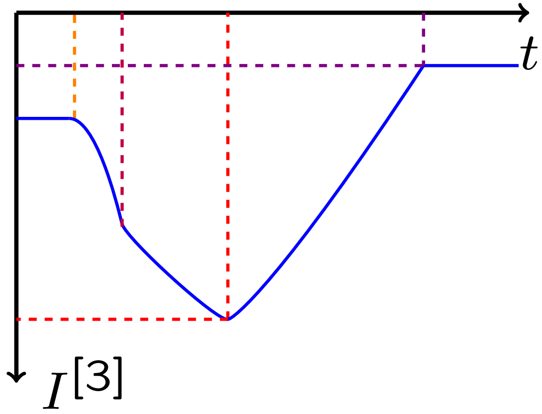
$$v_E t_s \approx \frac{n}{2}\ell + \frac{n-1}{2}h.$$



One should consider the factor of $(-1)^n$ in the expression.

For example for $n = 3$ one finds that in the case of $\rho_H \ll l_i$, the 3-partite information starts from its value in the vacuum and remains almost constant up to $t \sim \frac{1}{2}\ell$, then it decreases linearly with time till it reaches its minimum value. After that it increases linearly with time till it becomes zero at the saturation time given by $t_s \sim \frac{3}{2}\ell + h - c_2\rho_H$.

- Second case: $(n-2)\ell+(n-3)h < 2\rho_H < (n-1)\ell+(n-2)h < n\ell+(n-1)h$

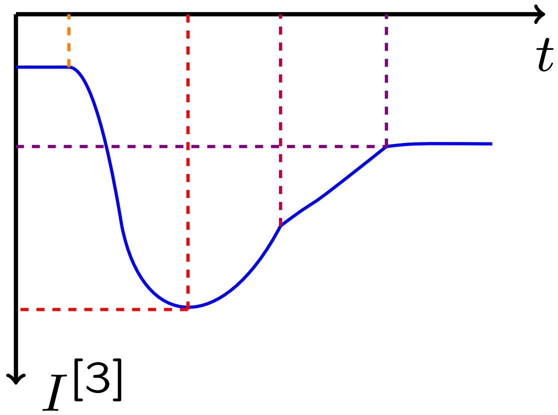


$$\tilde{I}^{[n]} \approx \tilde{I}^{[n]}_{\text{vac}} + \frac{L^{d-2}}{4G_N\rho_H^d}\left(\frac{t^2}{4} - c_1\Big((n-2)\ell+(n-3)h\Big)^2\right),$$

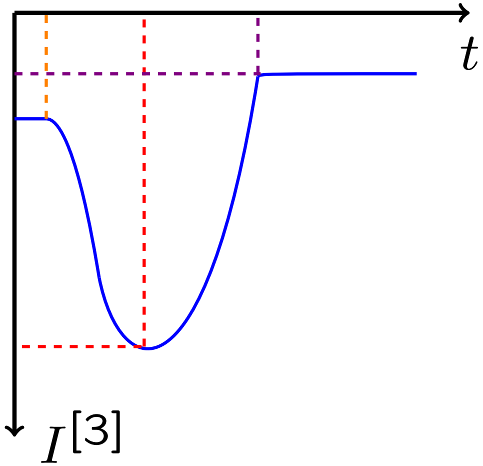
Finally it saturates at

$$v_E\, t_s \approx \frac{n}{2}\ell + \frac{n-1}{2}h - c_2\rho_H,$$

- Third case: $(n-2)\ell+(n-3)h<(n-1)\ell+(n-2)h<2\rho_H<n\ell+(n-1)h$



- Fourth case: $(n-2)\ell+(n-3)h<(n-1)\ell+(n-2)h<n\ell+(n-1)h<2\rho_H$,



Within the context of the AdS/CFT correspondence we have computed $\tilde{I}^{[n]}$; it is always positive. In other words the holographic n-partite information has definite sign: for even n it is positive and for odd n it is negative, though for a generic field theory it could have either signs.

One may suspect that having definite sign for the n-partite information is, indeed, an intrinsic property of a field theory which has gravity dual.

Although we have considered special case, the general feature remains the same for general case.

P. Hayden, M. Headrick and A. Maloney, "Holographic Mutual Information is Monogamous," [arXiv:1107.2940 [hep-th]].