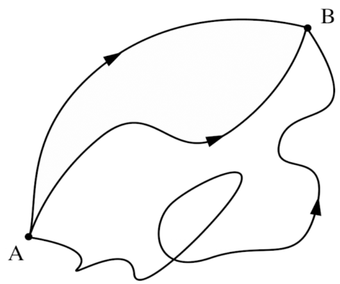
Construction of a relativistic field theory

Lagrangian $L = T - V$ (Nonrelativistic mechanics)

Action $S = \int_{t_1}^{t_2} L\, dt$



- Classical path … minimises action

- Quantum mechanics … sum over all paths with amplitude $\propto e^{iS/\hbar}$

Lagrangian invariant under all the symmetries of nature

-makes it easy to construct viable theories

Lagrangian formulation of the Klein Gordon equation

$$L = \int \mathfrak{L}\, d^3x, \quad \mathfrak{L} \quad \text{lagrangian density}$$

Klein Gordon field $\phi(x)$

$$\mathfrak{L} = \underbrace{\left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x)}_{T} - \underbrace{m^2 \phi(x)^\dagger \phi(x)}_{V}$$

Manifestly Lorentz invariant

Classical path :

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial \mathfrak{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)} = 0$$

Euler Lagrange equation

⇨

$$\boxed{(\partial_\mu \partial^\mu + m^2)\phi = 0}$$

Klein Gordon equation

New symmetries

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \rightarrow e^{i\alpha} \phi(x)$ …an Abelian (U(1)) gauge symmetry

A symmetry implies a conserved current and charge.

e.g. Translation ⇒ Momentum conservation

Rotation ⇒ Angular momentum conservation

What conservation law does the U(1) invariance imply?

Noether current

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \to e^{i\alpha}\phi(x)$ …an Abelian (U(1)) gauge symmetry

$$0 = \delta\mathfrak{L} = \frac{\partial \mathfrak{L}}{\partial \phi}\delta\phi + \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\delta(\partial^\mu \phi) + (\phi \leftrightarrow \phi^\dagger)$$

($\delta\phi = i\alpha\phi$, $\delta(\partial^\mu\phi) = i\alpha\partial_\mu\phi$)

$$= i\alpha\left[\frac{\partial \mathfrak{L}}{\partial \phi} - \partial^\mu\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\right)\right]\phi + i\alpha\partial^\mu\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\phi\right) - (\phi \leftrightarrow \phi^\dagger)$$

0 (Euler lagrange eqs.)

$$\Rightarrow \quad \partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\phi - \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi^\dagger)}\phi^\dagger\right)$$

Noether current

The Klein Gordon current

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\psi(x) \to e^{i\alpha}\psi(x)$ …an Abelian (U(1)) gauge symmetry

$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi)}\phi - \frac{\partial \mathfrak{L}}{\partial(\partial^\mu \phi^\dagger)}\phi^\dagger \right)$$

$$j_\mu^{KG} = -ie\left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right)$$

This is of the form of the electromagnetic current we used for the KG field

The Klein Gordon current

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This is of the form of the electromagnetic current we used for the KG field

$Q = \int d^3x \; j^0$ is the associated conserved charge

Suppose we have two fields with different U(1) charges :

$$\phi_{1,2}(x) \rightarrow e^{i\alpha Q_{1,2}} \phi_{1,2}(x)$$

$$\mathfrak{L} = \left(\partial_\mu \phi_1(x)\right)^\dagger \partial^\mu \phi_1(x) - m^2 \phi_1(x)^\dagger \phi_1(x)$$
$$+ \left(\partial_\mu \phi_2(x)\right)^\dagger \partial^\mu \phi_2(x) - m^2 \phi_2(x)^\dagger \phi_2(x)$$

..no cross terms possible (corresponding to charge conservation)

Additional terms

Terms allowed by U(1) symmetry

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) + \lambda \left|\phi\right|^4 + \frac{\lambda'}{M^2}\left|\phi\right|^6 + \ldots$$

Renormalisable $D \leq 4$

If $M \gg 10^3 GeV$, "Effective" Field theory approximately renormalisable

U(1) local gauge invariance and QED

$$\phi(x) \rightarrow e^{i\alpha(x)Q}\phi(x)$$

$$\mathfrak{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$ not invariant due to derivatives

$$\partial_\mu \phi \rightarrow \partial_\mu e^{i\alpha(x)Q}\phi = e^{i\alpha(x)Q}\partial_\mu \phi + iQe^{i\alpha(x)Q}\phi\, \partial_\mu \alpha(x)$$

To obtain invariant Lagrangian look for a modified derivative transforming covariantly

$$D_\mu \phi \rightarrow e^{i\alpha(x)} D_\mu \phi$$

U(1) local gauge invariance and QED

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$$\partial_\mu \phi - iQA_\mu \phi \to \partial_\mu e^{i\alpha(x)Q}\phi = e^{i\alpha(x)Q}(\partial_\mu \phi - iQA_\mu \phi) + iQe^{i\alpha(x)Q}\phi\, \partial_\mu \alpha(x) - iQe^{i\alpha(x)Q}\phi\, \partial_\mu \alpha(x)$$

To obtain invariant Lagrangian look for a modified derivative transforming covariantly

$$D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$$

Need to introduce a new vector field $A_\mu \to A_\mu + \partial_\mu \alpha$

$$D_\mu = \partial_\mu - iQA_\mu$$

$$\phi(x) \to e^{iQ\alpha(x)}\phi(x)$$

$$D_\mu\phi \to e^{i\alpha(x)}D_\mu\phi$$

$$A_\mu \to A_\mu + \partial_\mu\alpha$$

Yang-Mills (+Shaw)

$$\mathfrak{L} = \left(D_\mu\phi(x)\right)^\dagger D^\mu\phi(x) - m^2\phi(x)^\dagger\phi(x)$$ is invariant under local U(1)

Note : $\partial_\mu \to D_\mu = \partial_\mu - iQA_\mu$ is equivalent to $p^\mu \to p^\mu + eA^\mu$

universal coupling of electromagnetism *follows* from local gauge invariance

The Euler lagrange equation give the KG equation:

$$(\partial_\mu\partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu\partial_\mu) - e^2A^2$$

$$\phi(x) \to e^{iQ\alpha(x)}\phi(x)$$

$$D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$$

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

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universal coupling of electromagnetism *follows* from local gauge invariance

$$\text{i.e.} \quad \mathfrak{L} = \mathfrak{L}^{\mathrm{KG}} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) - j_\mu^{KG} A^\mu + O(e^2)$$

The electromagnetic Lagrangian

$$\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} \to F_{\mu\nu}, \qquad A_\mu \to A_\mu + \partial_\mu \alpha$$

$$\mathfrak{L}^{EM} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$M^2 A^\mu A_\mu$ Forbidden by gauge invariance

The Euler-Lagrange equations give Maxwell equations !

$$\frac{\partial \mathfrak{L}}{\partial A^\nu} - \partial^\mu \frac{\partial \mathfrak{L}}{\partial(\partial^\mu A^\nu)} = 0$$

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \equiv \quad \begin{array}{ll} \nabla.\mathbf{E} = \rho, & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla.\mathbf{B} = 0, & \nabla \times \mathbf{B} - \frac{\partial E}{\partial t} = \mathbf{j} \end{array}$$

EM dynamics follows from a local gauge symmetry!!

The photon propagator

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial^\mu A_\mu) = j^\nu$$

The Klein Gordon propagator (reminder)

$$(\partial_\mu \partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

$$(\partial_\mu \partial^\mu + m^2)\Delta_F(x'-x) = \delta^4(x'-x)$$

In momentum space:

$$\widetilde{\Delta}_F(p) = \frac{i}{-p^2 + m^2 \pm i\varepsilon}$$

With normalisation convention used in Feynman rules = inverse of momentum space operator multiplied by -i

The photon propagator

- The propagators determined by terms quadratic in the fields, using the Euler Lagrange equations.

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial^\mu A_\mu) = j^\nu$$

Choose as

$$-\frac{1}{\xi}\partial^\mu A_\mu$$

(gauge fixing)

Gauge ambiguity

$$A_\mu \to A_\mu + \partial_\mu \alpha \qquad \partial^\mu A_\mu \to \partial^\mu A_\mu + \partial^2 \alpha$$

i.e. with suitable "gauge" choice of α ("ξ" gauge) want to solve

$$\partial_\mu \partial^\mu A^\nu - (1-\frac{1}{\xi})\partial^\nu (\partial_\mu A^\mu) \equiv (g^{\nu\mu}\partial^2 - (1-\frac{1}{\xi})\partial^\nu \partial_\mu) A^\mu = j^\nu$$

In momentum space the photon propagator is

$$-i\left(g^{\mu\nu} p^2 - (1-\frac{1}{\xi}) p^\mu p^\nu \right)^{-1} = \frac{i}{p^2}\left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2} \right)$$

('t Hooft Feynman gauge ξ=1)

Extension to non-Abelian symmetry

(*The Standard Model* $SU(3)\otimes SU(2)\otimes U(1)$)

$SU(2)$ local gauge invariance

$$Q=\begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R$$

$$Q\rightarrow e^{ig_2\alpha(x).\frac{\sigma}{2}}Q$$

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\quad \sigma_2=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathfrak{L}=i\overline{Q}D_\mu\gamma^\mu Q+i\overline{u}_R\partial_\mu\gamma^\mu u_R+i\overline{d}_R\partial_\mu\gamma^\mu d_R$$

$$\mathrm{D}_\mu=\partial_\mu+ig_2\frac{\sigma_i}{2}W_\mu^i$$

$$D_\mu Q\rightarrow e^{ig_2\alpha(x).\frac{\sigma}{2}}D_\mu Q$$

where

$$W_{\mu,i}\rightarrow W_{\mu,i}-\partial_\mu\alpha_i-g_2\varepsilon_{ijk}\alpha_j W_{\mu,k}$$

$$\left(\left[\frac{\sigma_i}{2},\frac{\sigma_j}{2}\right]=i\varepsilon_{ijk}\frac{\sigma_k}{2}\right)$$

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(The Standard Model $SU(3)\otimes SU(2)\otimes U(1)$*)*

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$$D_\mu=\partial_\mu+ig_2\frac{\sigma_i}{2}W_\mu^i$$

Need 3 gauge bosons

W^+, W^-, W^3

where

$$W_{\mu,i}\to W_{\mu,i}-\partial_\mu\alpha_i-g_2\varepsilon_{ijk}\alpha_j W_{\mu,k}$$

$$\left(\left[\frac{\sigma_i}{2},\frac{\sigma_j}{2}\right]=i\varepsilon_{ijk}\frac{\sigma_k}{2}\right)$$

$SU(3)$ local gauge invariance

$$\begin{pmatrix} q \\ q \\ q \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_8 = \sqrt{\frac{1}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots$$

Symmetry :

Local conservation of
3 strong colour charges

$$\Psi_a \rightarrow \left(e^{ig_3\alpha(x).\lambda}\right)^a_b \Psi_a$$

$$G^r_\mu \rightarrow G^r_\mu - \partial_\mu \alpha^r - g_3 f^{rst} \alpha^s G^t_\mu$$

$$\overline{\Psi}(\partial_\mu - ig_3 \lambda_r G^r_\mu)\gamma^\mu \Psi$$

QCD : a non-Abelian (SU(3))
local gauge field theory

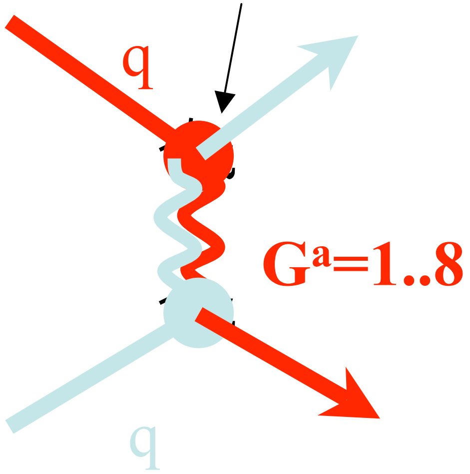
The strong interactions

QCD Quantum Chromodynamics

SU(3)

$$\begin{pmatrix} q \\ q \\ q \end{pmatrix}$$

Strong coupling, α_3



Gauge boson
(J=1)
“Gluons”

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Local conservation of
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$$G^r_\mu \to G^r_\mu - \frac{1}{g_3}\partial_\mu \alpha^r - f^{rst}\alpha^s G^t_\mu$$

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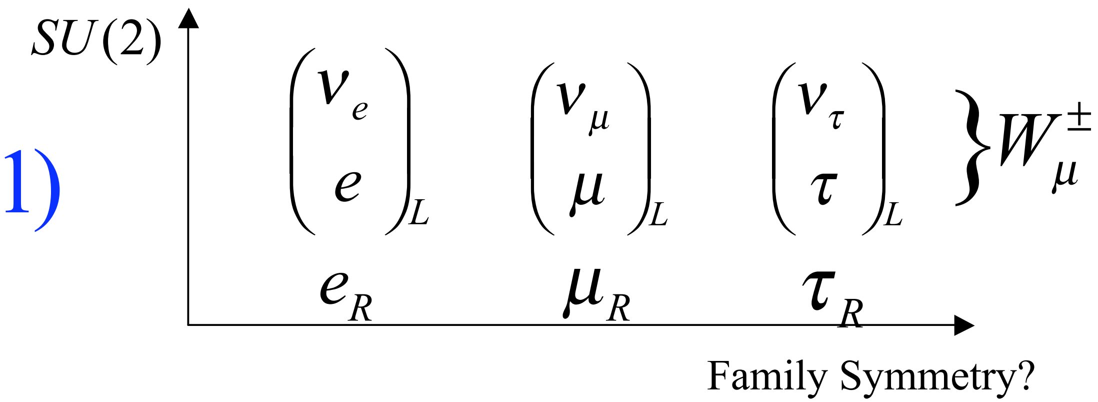
QCD : a non-Abelian (SU(3))
local gauge field theory

Partial Unification

$$SU(3) \otimes SU(2) \otimes U(1)$$



Matter Sector "chiral"