Saturation Physics on the Energy Frontier

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with Kazuhiro Watanabe, Bo-Wen Xiao, Feng Yuan

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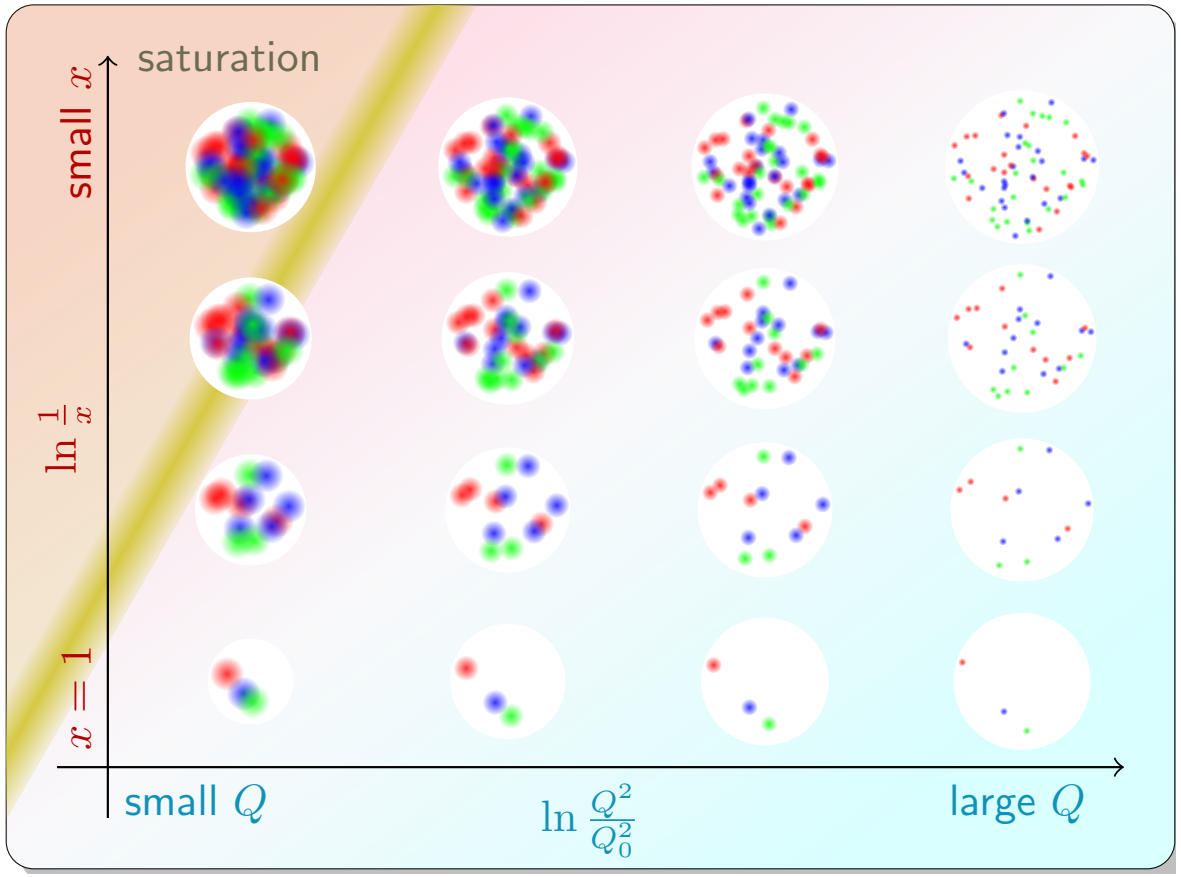
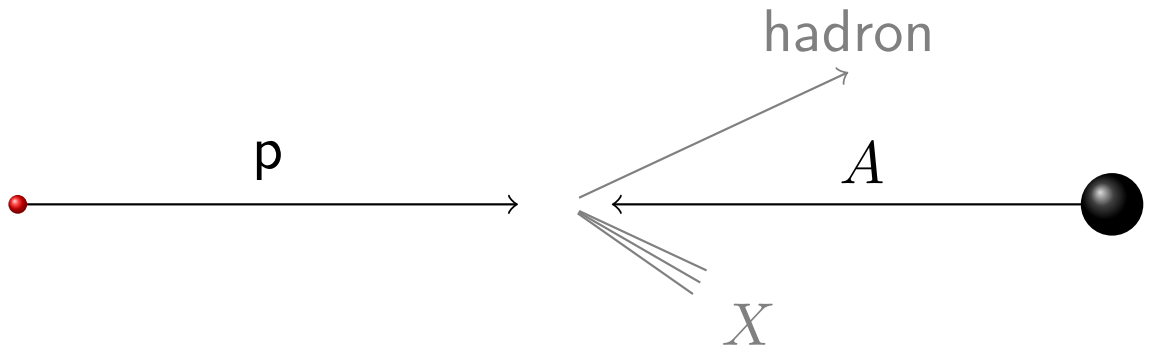
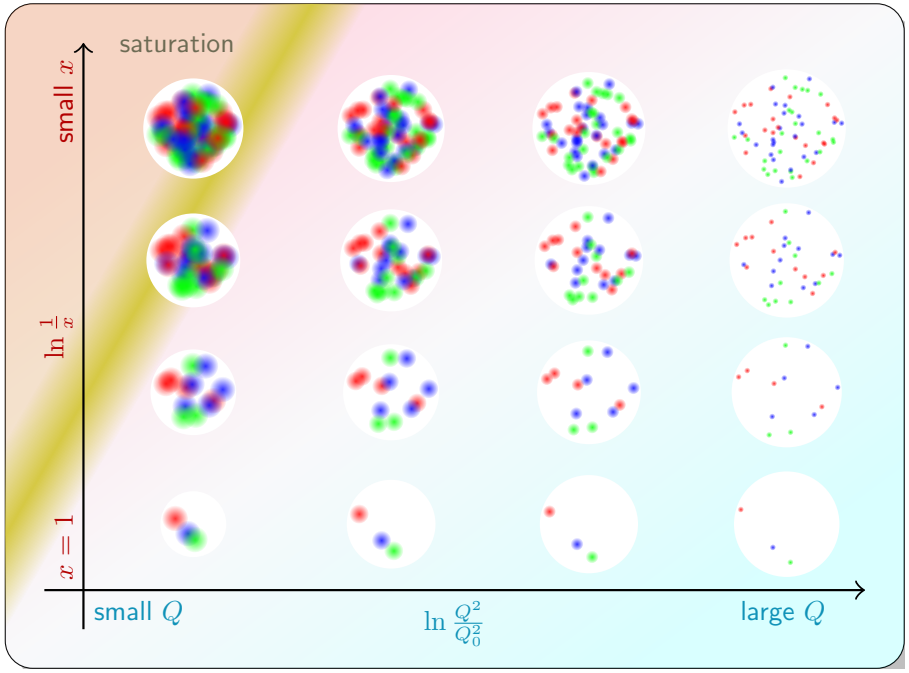
Saturation



Advantages of pA

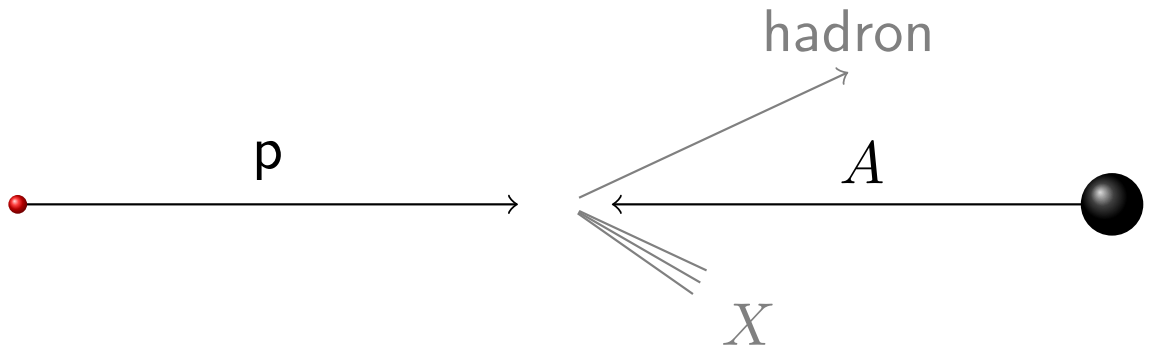


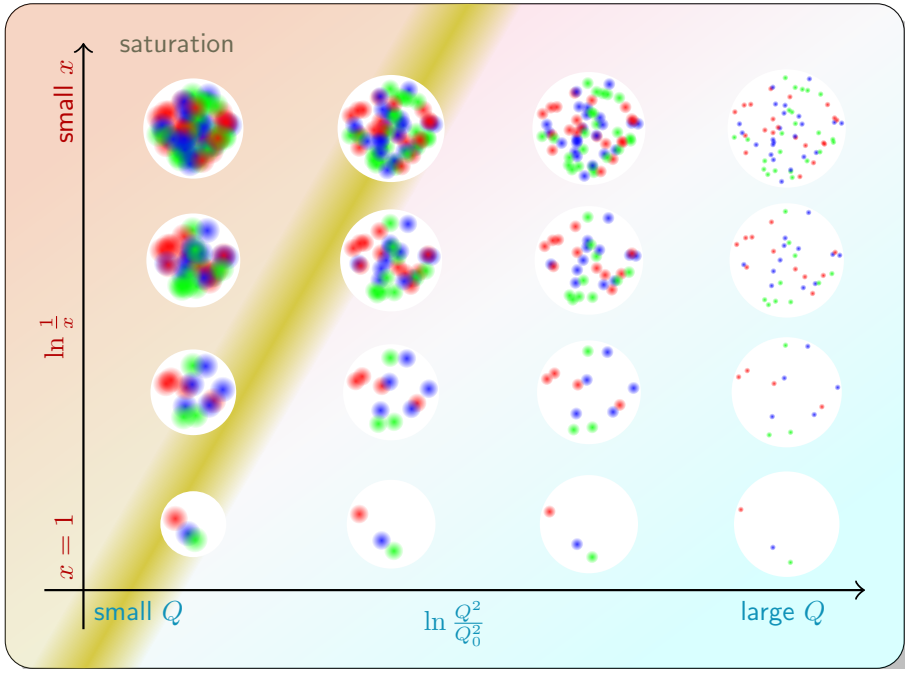


$$Q^2 \lesssim Q_s^2 = cA^{1/3} Q_0^2 \left(\frac{x_0}{x} \right)^{\lambda}$$

- Heavy target: large A
- Light projectile: no medium

Advantages of pA





$$Q^2 \lesssim Q_s^2 = cA^{1/3} Q_0^2 \left(\frac{x_0}{x} \right)^{\lambda}$$

- Heavy target: large A
- Light projectile: no medium

Hybrid Model

Cross section in the hybrid formalism:

$$\frac{\mathrm{d}^3\sigma}{\mathrm{d}Y\,\mathrm{d}^2\vec{p}_\perp} = \sum_i \int \frac{\mathrm{d}z}{z^2}\frac{\mathrm{d}x}{x} x f_i(x,\mu) D_{h/i}(z,\mu) F\left(x, \frac{p_\perp}{z}\right) \mathcal{P}(\xi)(\ldots)$$

- Parton distribution (initial state projectile)
- Dipole gluon distribution (initial state target)
- Fragmentation function (final state)
- Perturbative factors

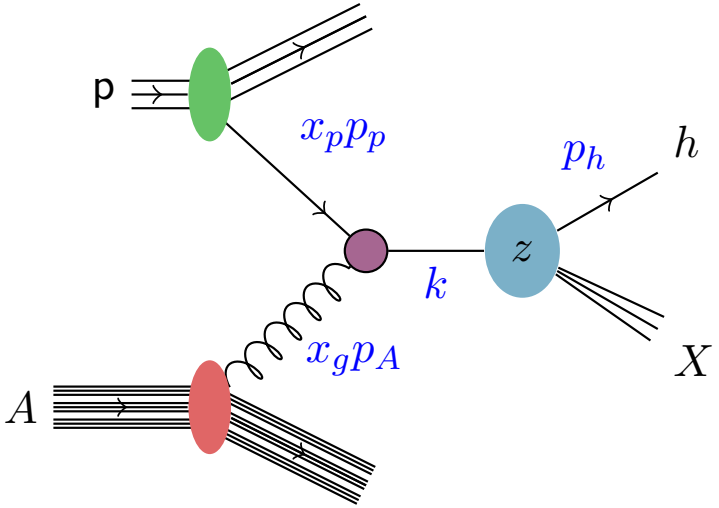


figure adapted from Dominguez 2011.

History of the pA Calculation

- Dumitru and Jalilian-Marian (2002)
- Dumitru, Hayashigaki, et al. (2006)
- Fujii et al. (2011)
- Albacete et al. (2013)
- Rezaeian (2013)
- Staśto, Xiao, and Zaslavsky (2014)
- Kang et al. (2014)
- Staśto, Xiao, Yuan, et al. (2014)
- Altinoluk et al. (2014)
- Watanabe et al. (2015)

Gluon dist

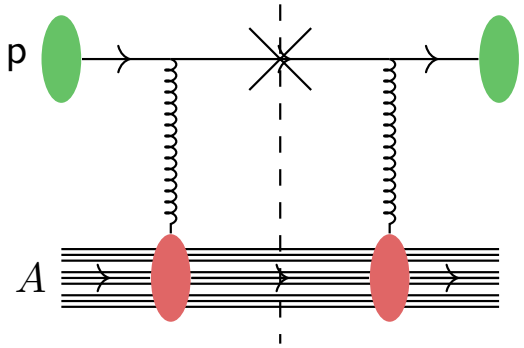
- GBW
- MV/AAMQS
- LO BK
- rcBK
- b-CGC
- NLO BK

Cross section

- LO
- inel NLO
- other NLO
- rapidity NLO
- splitting NLO

First Calculation

Dumitru and Jalilian-Marian (2002)



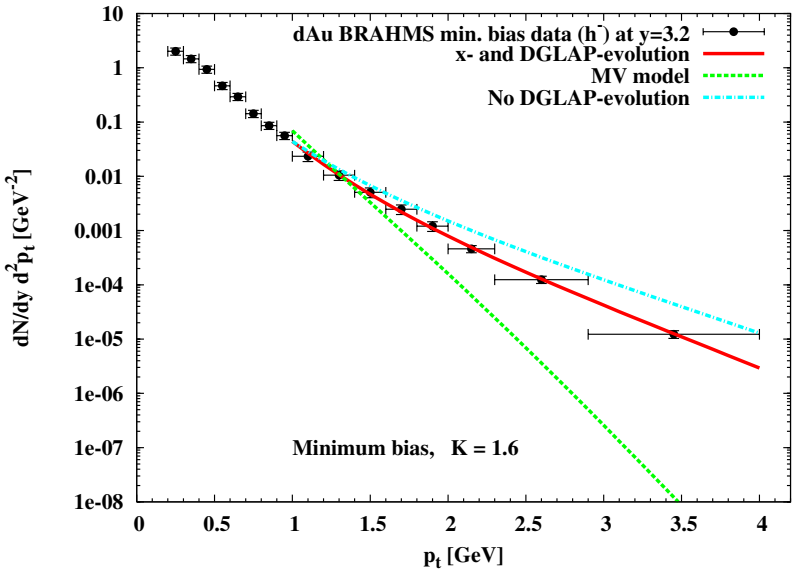
No numerical results

Gluon dist
- GBW
- MV/AAMQS
- LO BK
- rcBK
- b-CGC
- NLO BK

Cross section
- LO
- inel NLO
- other NLO
- rapidity NLO
- splitting NLO

First Numerical Results

Dumitru, Hayashigaki, et al. (2006)



Prefactor $K = 1.6$

Gluon dist	
GBW	
MV/AAMQS	
LO BK	
rcBK	
b-CGC	
NLO BK	

Cross section	
LO	
inel NLO	
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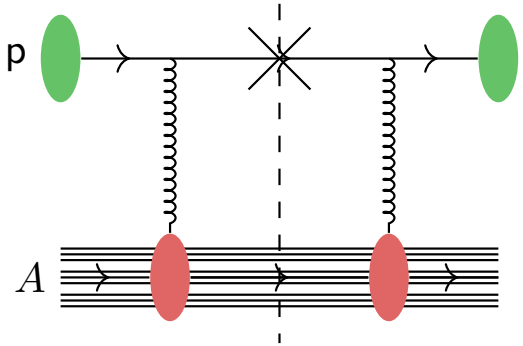
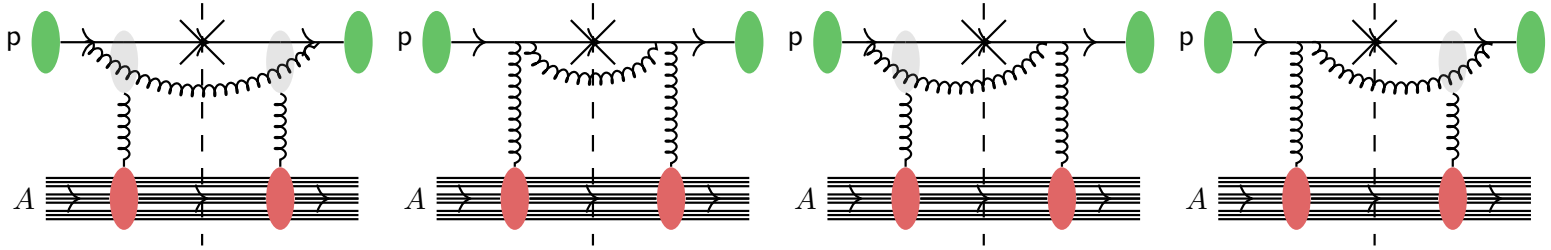
rapidity NLO	
splitting NLO	

Inelastic Diagrams

Leading:

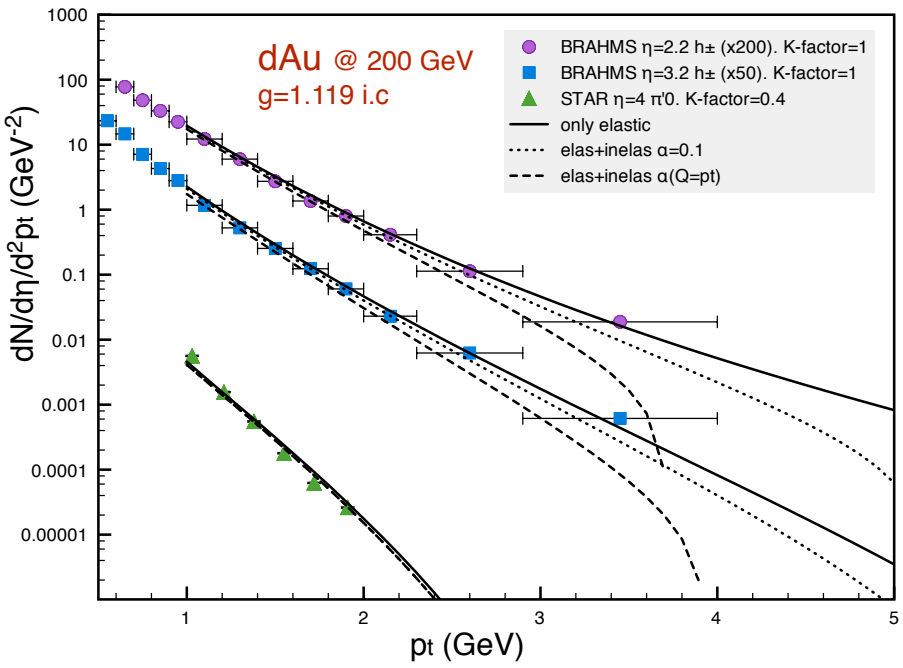


Next-to-leading:



Inelastic NLO Terms

Albacete et al. (2013)



Prefactor

$K = 1$ for charged hadrons

$K = 0.4$ for neutral hadrons

Gluon dist	
GBW	
MV/AAMQS	
LO BK	
rcBK	
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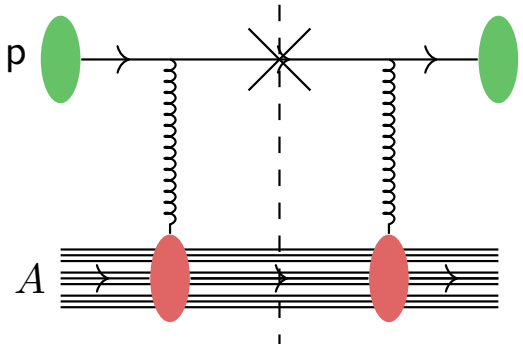
NLO BK	

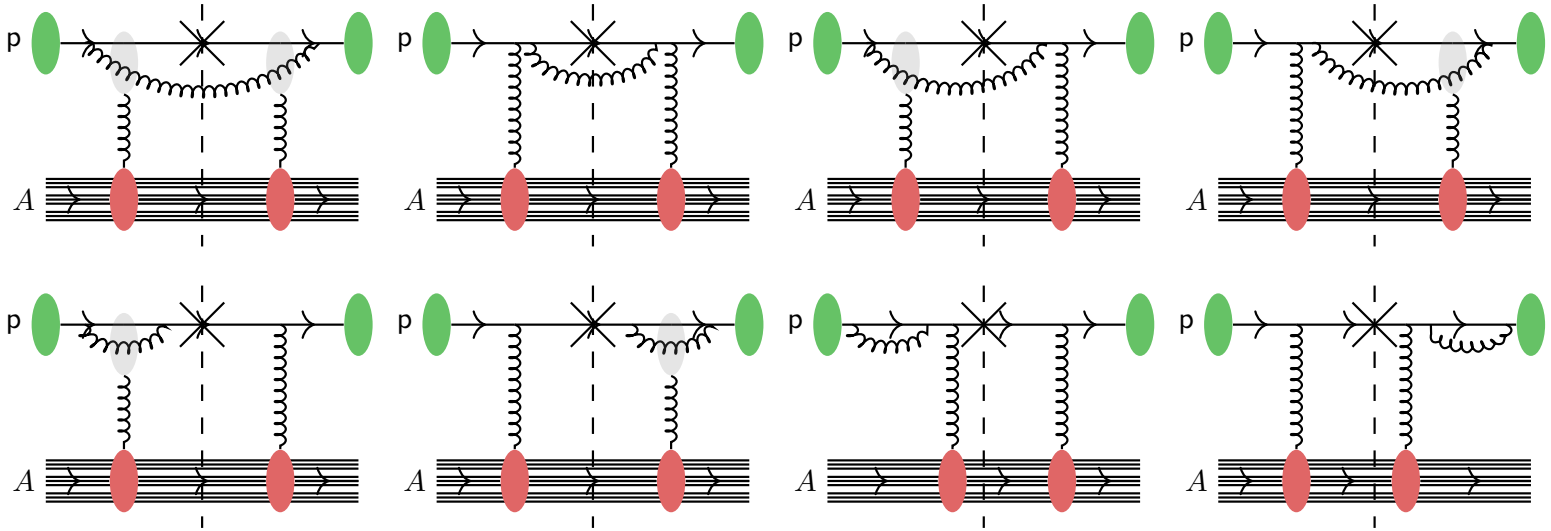
Cross section	
LO	
inel NLO	
other NLO	
rapidity NLO	
splitting NLO	

NLO Diagrams

Leading:



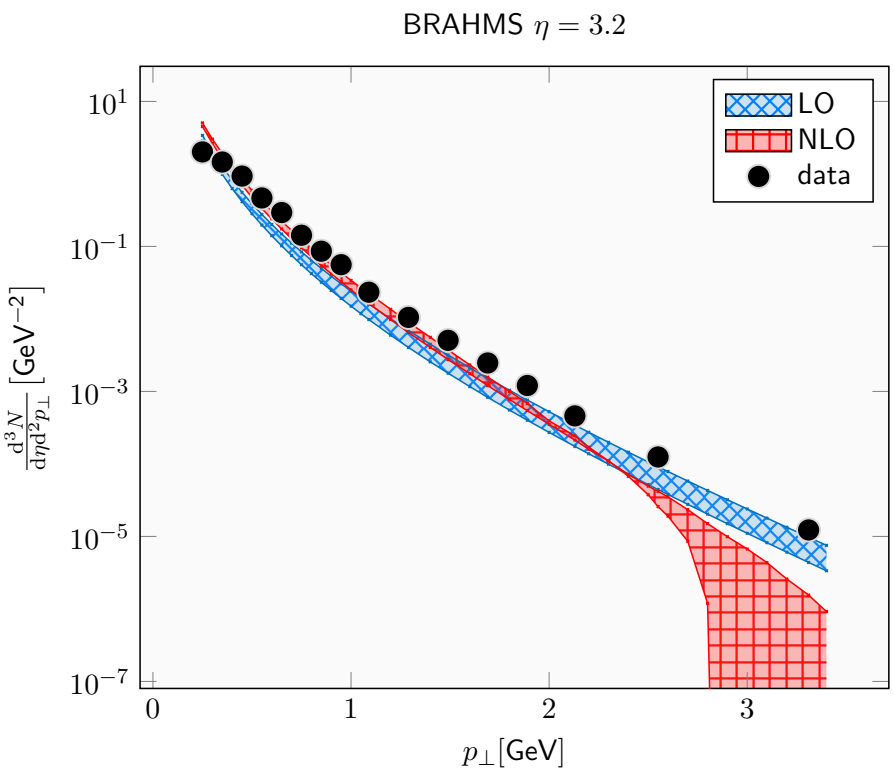
Next-to-leading:



Chirilli et al. 2012, 1203.6139.

NLO Numerical Result

Staśto, Xiao, and Zaslavsky (2014)



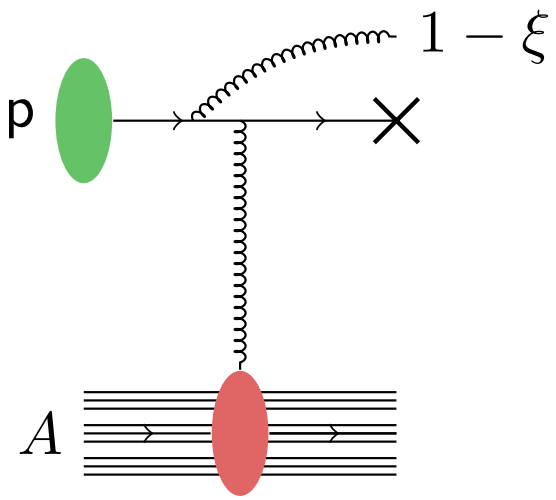
Gluon dist	Cross section
GBW	LO
MV/AAMQS	inel NLO
LO BK	other NLO
rcBK	rapidity NLO
b-CGC	splitting NLO
NLO BK	

Includes virtual corrections

$K = 1$

Kinematical Constraint

Watanabe et al. (2015)



First LHC numerical results

Gluon dist	Cross section
GBW	LO
MV/AAMQS	inel NLO
LO BK	other NLO
rcBK	rapidity NLO
b-CGC	splitting NLO
NLO BK	

Alternate derivation: Altinoluk et al. 2014, 1411.2869.

Challenges for Numerical Calculation

Singularities

$$\int_{\tau}^{1} \mathrm{d}z \int_{\frac{\tau}{z}}^{1} \mathrm{d}\xi \left[\frac{F_s(z,\xi)}{(1-\xi)_+} + F_n(z,\xi) + F_d(z,\xi)\delta(1-\xi) \right]$$

Watanabe et al. 2015, 1505.05183.

Challenges for Numerical Calculation

Singularities

$$\int_{\tau}^{1} \mathrm{d}z \int_{\frac{\tau}{z}}^{1} \mathrm{d}\xi \left[\frac{F_s(z,\xi)}{(1-\xi)_+} + F_n(z,\xi) + F_d(z,\xi)\delta(1-\xi) \right]$$

Fourier integrals

$$\int \mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp e^{i\vec{l}_\perp \cdot \vec{s}_\perp} e^{i\vec{l}'_\perp \cdot \vec{t}_\perp} (\ldots)$$

Watanabe et al. 2015, 1505.05183.

Challenges for Numerical Calculation

Singularities

$$\int_{\tau}^{1} \mathrm{d}z \int_{\frac{\tau}{z}}^{1} \mathrm{d}\xi \left[\frac{F_s(z,\xi)}{(1-\xi)_+} + F_n(z,\xi) + F_d(z,\xi)\delta(1-\xi) \right]$$

Fourier integrals

$$\int \mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp e^{i\vec{l}_\perp\cdot\vec{s}_\perp} e^{i\vec{l}'_\perp\cdot\vec{t}_\perp}(\ldots)$$

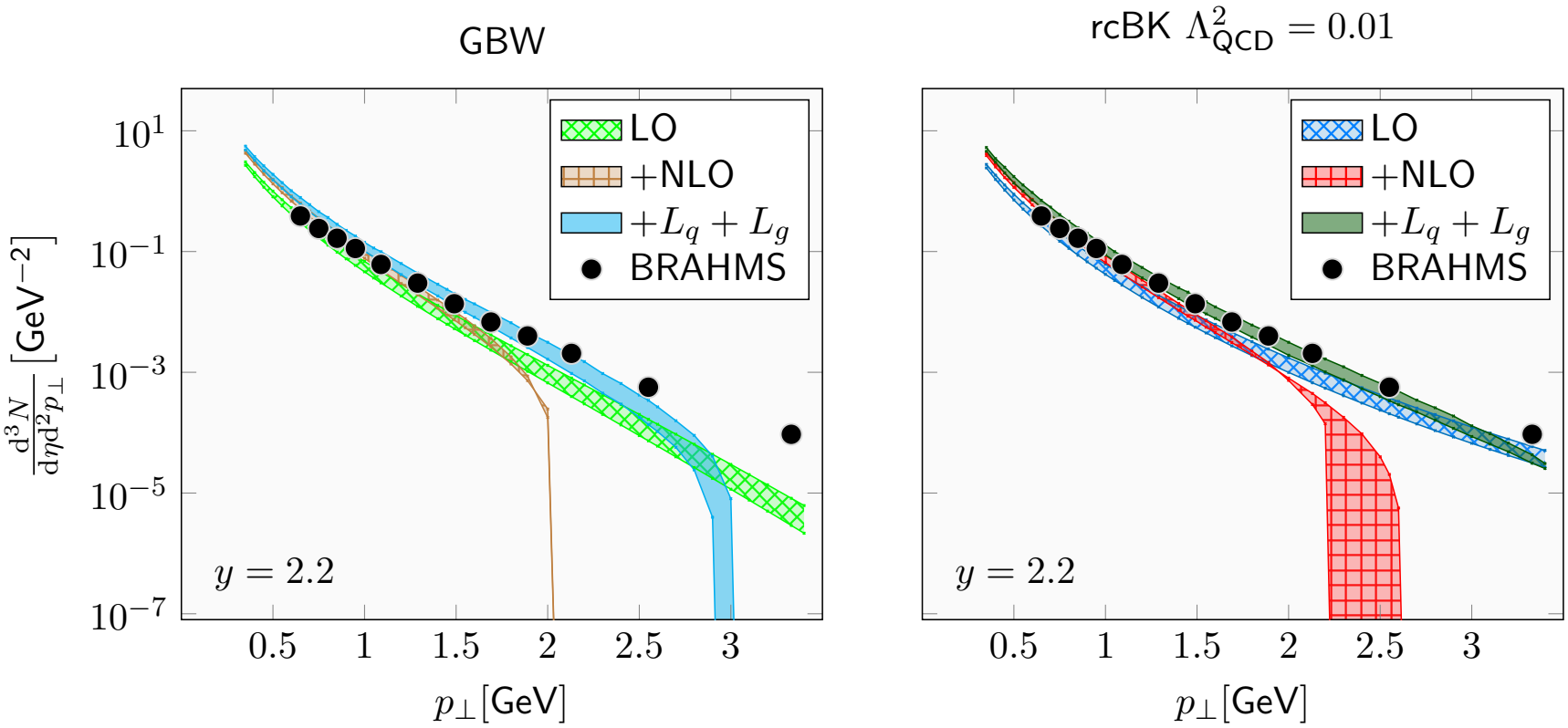
Leading Order Cancellations

$$\mathcal{O}(k_\perp^{-2}) - \mathcal{O}(k_\perp^{-2}) \to \mathcal{O}(k_\perp^{-4})$$

...plus Monte Carlo statistical error

Watanabe et al. 2015, 1505.05183.

RHIC Results

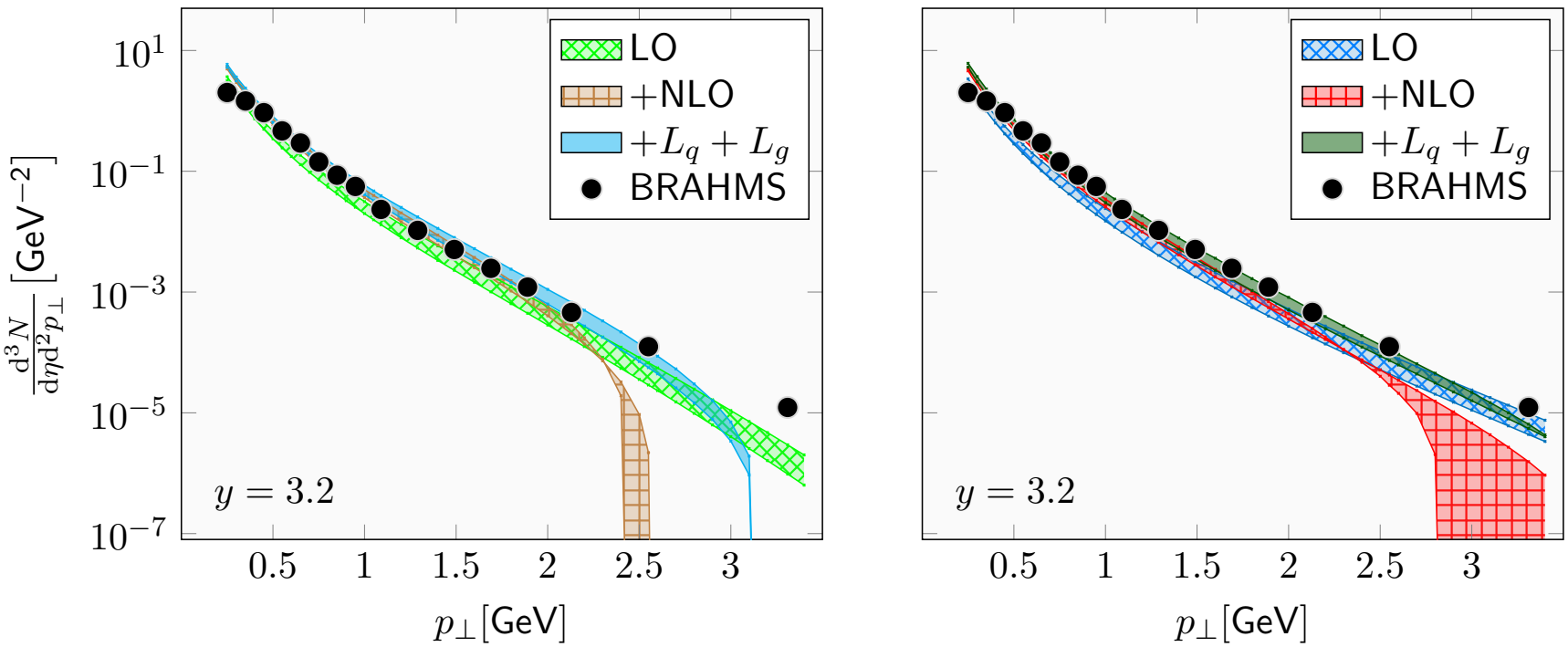


New terms improve matching at low $p_\perp$

data: Arsene et al. 2004, nucl-ex/0403005.

plots: Watanabe et al. 2015, 1505.05183.

RHIC Results

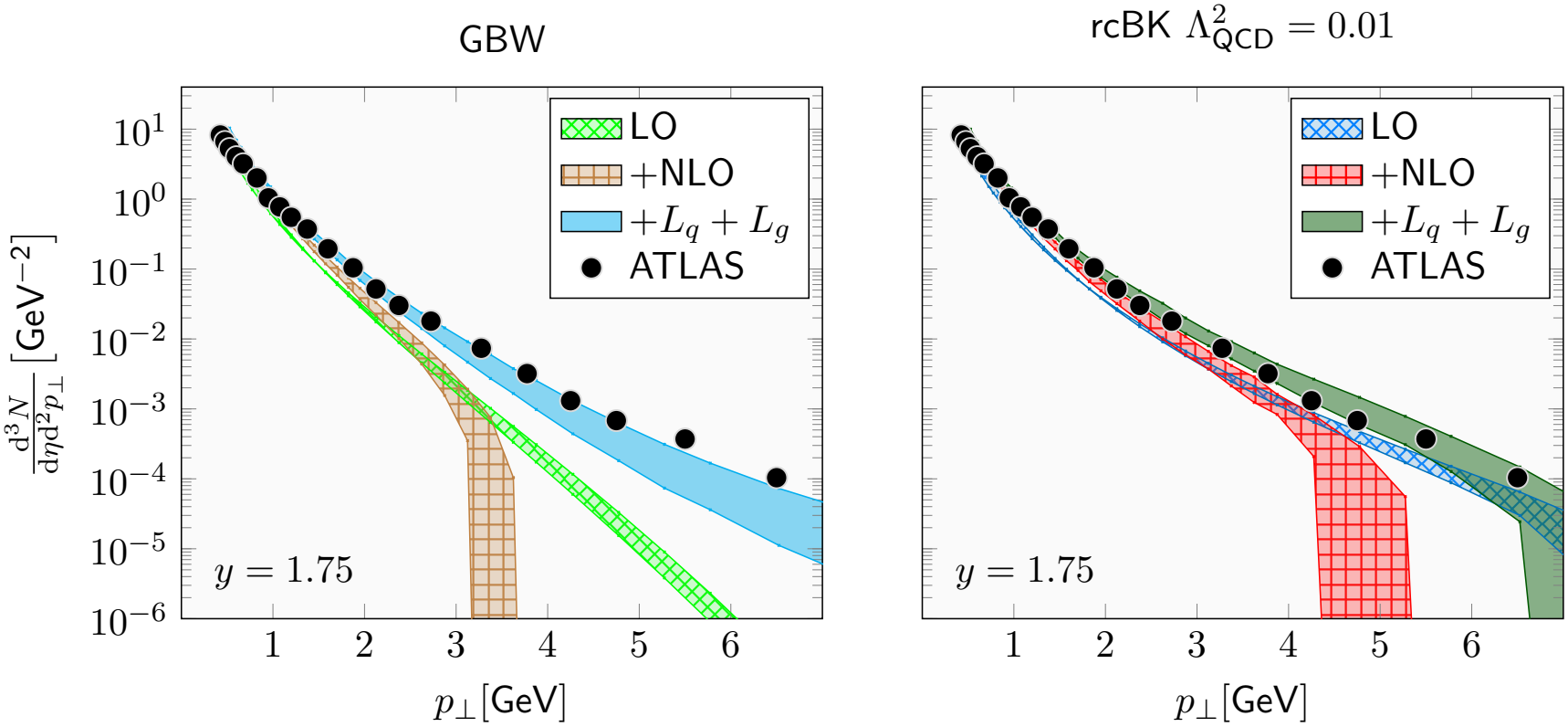


New terms improve matching at low $p_\perp$

data: Arsene et al. 2004, `nucl-ex/0403005`.

plots: Watanabe et al. 2015, `1505.05183`.

LHC Results

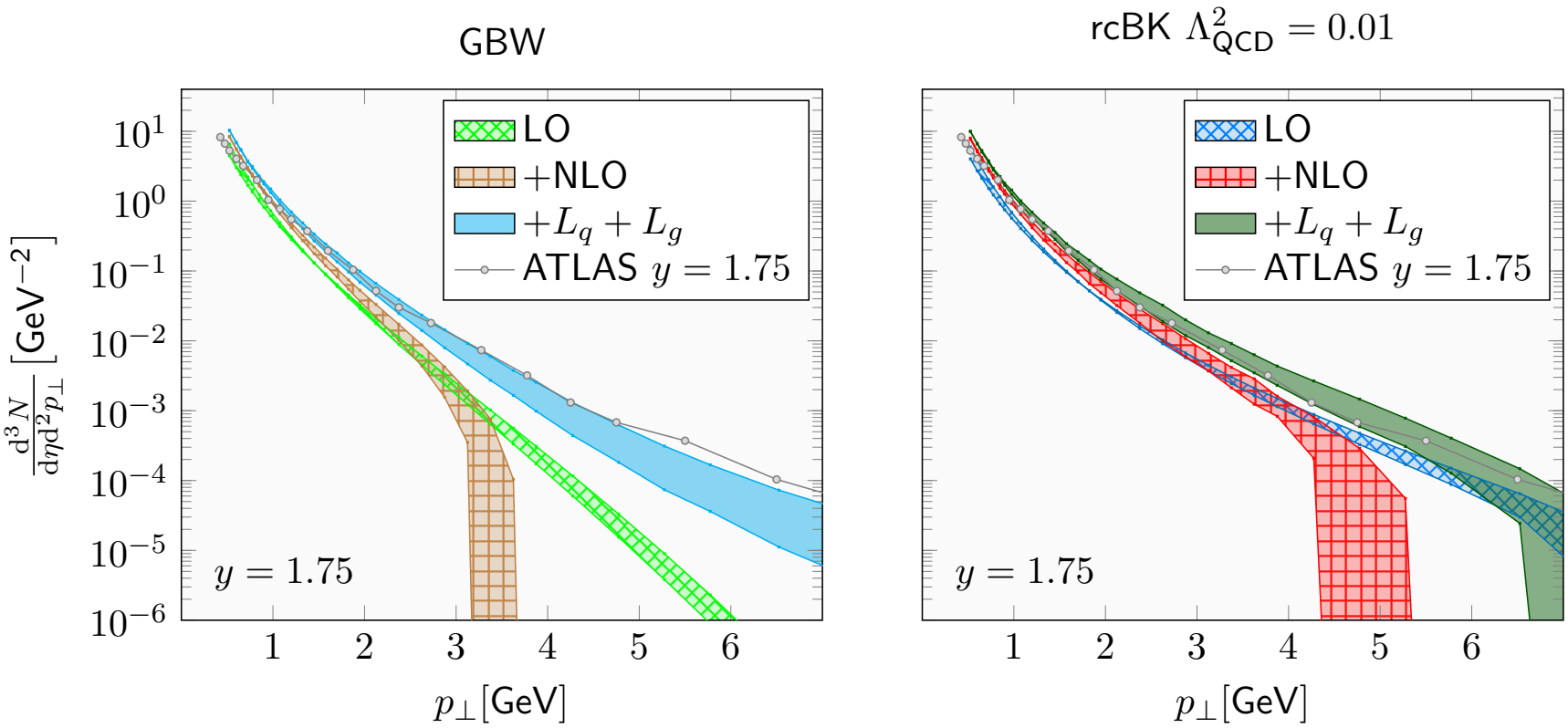


rcBK calculation matches neatly up to $p_\perp \approx 6$ GeV

data: Milov 2014, 1403.5738.

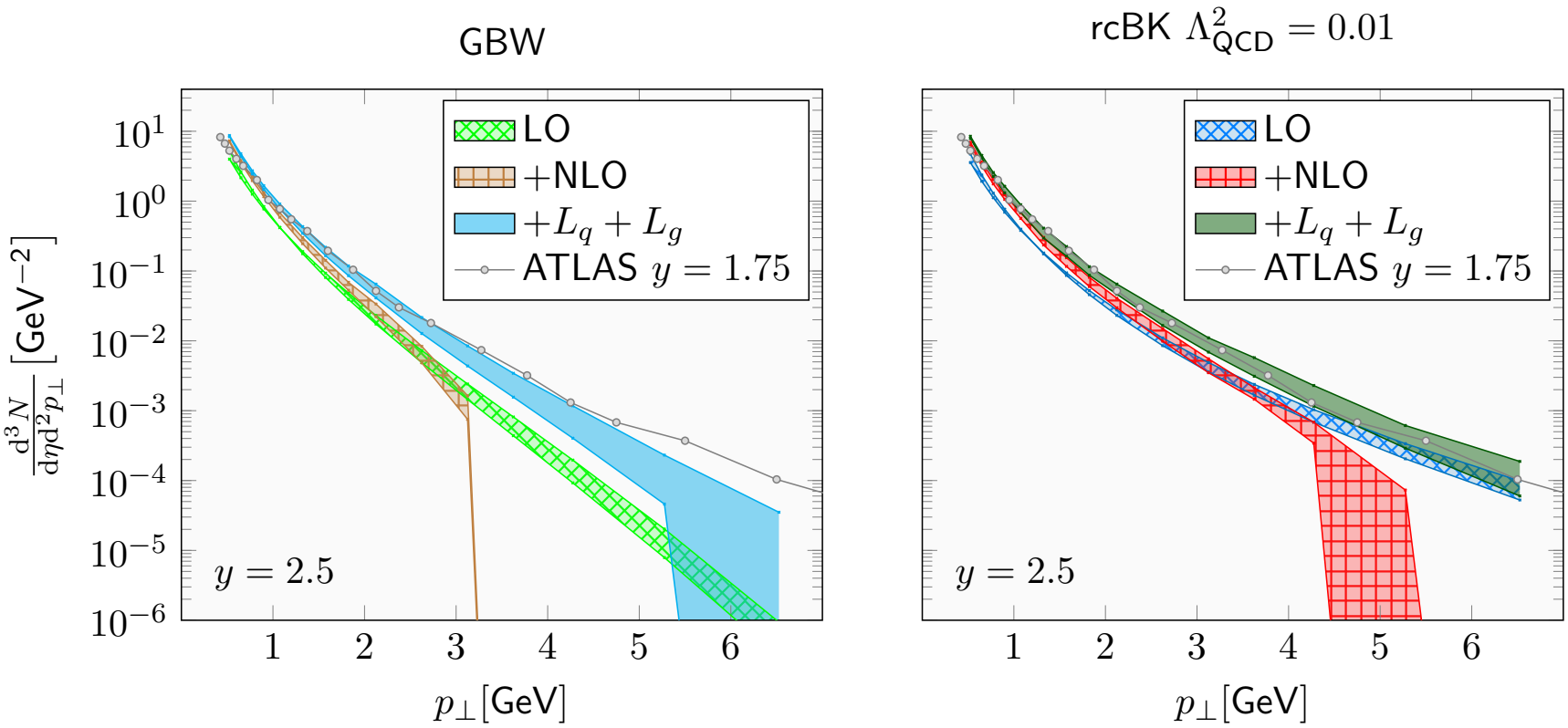
plots: Watanabe et al. 2015, 1505.05183.

Importance of Higher Rapidity



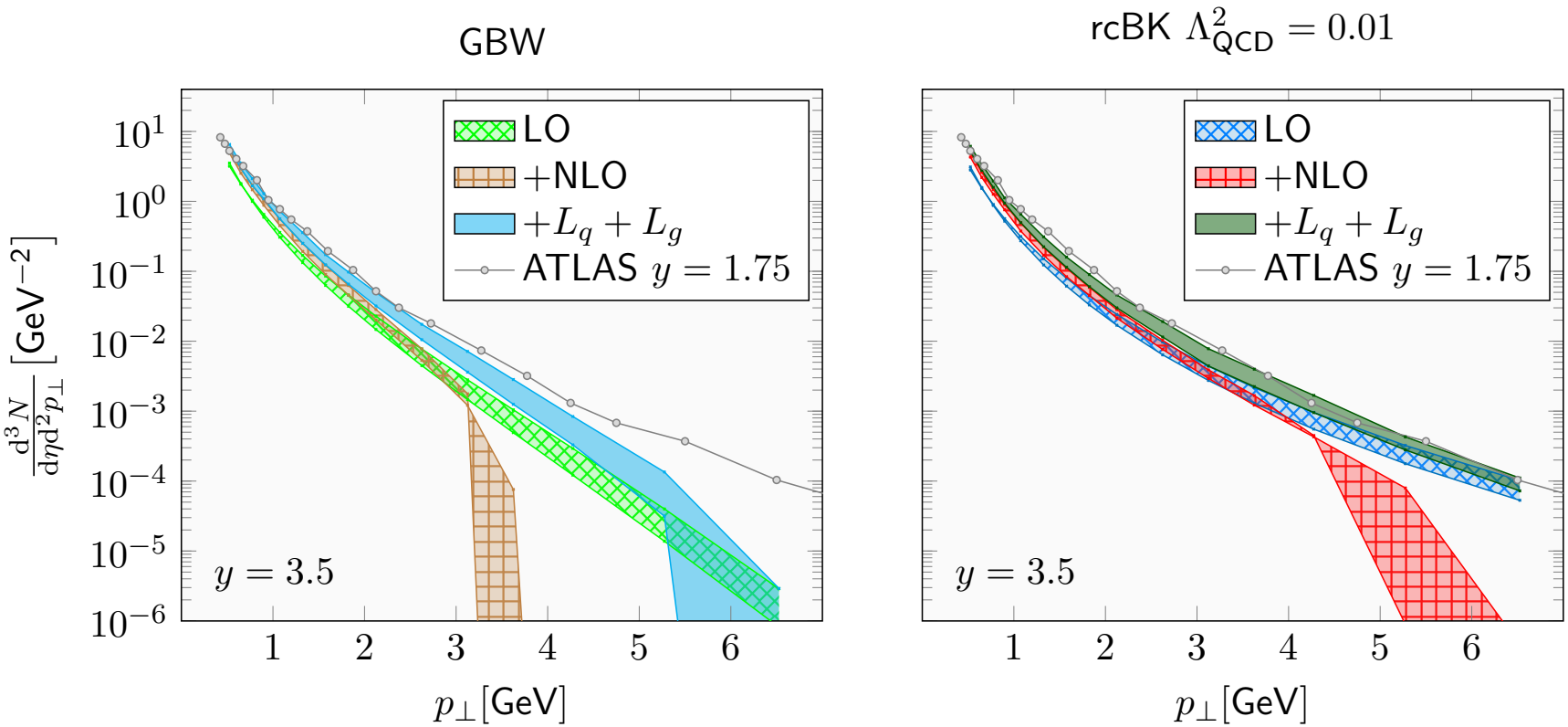
Higher rapidity alters low-$p_\perp$ result

Importance of Higher Rapidity



Higher rapidity alters low-$p_\perp$ result

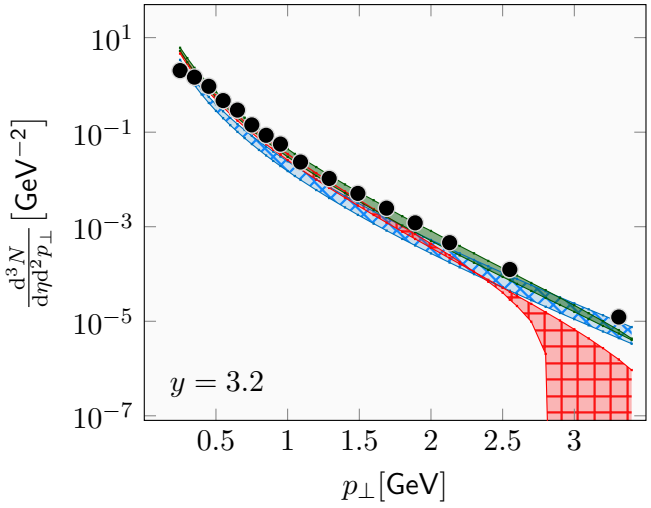
Importance of Higher Rapidity



Higher rapidity alters low-$p_\perp$ result

Summary

$$\frac{\mathrm{d}^5\sigma^{pA\to hX}}{\mathrm{d}Y\mathrm{d}^2p_\perp\mathrm{d}^2b_\perp} = \int \frac{\mathrm{d}z\mathrm{d}x}{z^2} q(x,Q_f^2) D_{q/h}(z,Q_f^2) \frac{\mathrm{d}^5\sigma_{qA}^{\mathrm{tot}}}{\mathrm{d}Y_q\mathrm{d}^2q_\perp\mathrm{d}^2b_\perp}$$



Complete numerical implementation of NLO $pA \to h + X$

Critical step

More forward-rapidity data from LHC experiments

Supplemental Slides

- full expressions
- additional history
  - rcBK
  - rapidity divergence
  - collinear matching
  - Ioffe time
- numerical challenges
  - singularities
  - Fourier integrals
  - new Fourier transforms
  - other numerical errors
- sources of negativity
- kinematical constraint
- beam direction
- LHC results

Full NLO Cross Section

Complete NLO corrections to the cross section for $\mathrm{p}A \to h + X$:

$$\frac{\mathrm{d}^3\sigma}{\mathrm{d}Y\,\mathrm{d}^2\vec{p}_\perp} = \int \frac{\mathrm{d}z\mathrm{d}\xi}{z^2} \begin{bmatrix} xq_i(x,\mu) & xg(x,\mu) \end{bmatrix} \begin{bmatrix} S_{qq} & S_{qg} \\ S_{gq} & S_{gg} \end{bmatrix} \begin{bmatrix} D_{h/q_i}(z,\mu) \\ D_{h/g}(z,\mu) \end{bmatrix}$$

$$\begin{aligned} S_{jk} = & \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp)\mathcal{H}_{2jk}^{(0)} && \text{LO dipole} \\ & + \frac{\alpha_s}{2\pi}\int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp)\mathcal{H}_{2jk}^{(1)} && \text{NLO dipole} \\ & + \frac{\alpha_s}{2\pi}\int \frac{\mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp}{(2\pi)^2} S_Y^{(4)}(r_\perp, s_\perp, t_\perp)\mathcal{H}_{4jk}^{(1)} && \text{NLO quadrupole} \\ & + \cdots && \text{etc.} \end{aligned}$$

Note: we also use $S_Y^{(4)}(r_\perp, s_\perp, t_\perp) \to S_Y^{(2)}(s_\perp)S_Y^{(2)}(t_\perp)$

Chirilli et al. 2012, 1203.6139.

Quark-Quark Channel

$$S_{qq} = \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp)\mathcal{H}_{2qq}^{(0)} + \frac{\alpha_s}{2\pi}\int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp)\mathcal{H}_{2qq}^{(1)}$$
$$+ \frac{\alpha_s}{2\pi}\int \frac{\mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp}{(2\pi)^2} S_Y^{(4)}(r_\perp, s_\perp, t_\perp)\mathcal{H}_{4qq}^{(1)}$$

$$\mathcal{H}_{2qq}^{(0)} = e^{-i\vec{k}_\perp\cdot\vec{r}_\perp}\delta(1-\xi)$$

$$\mathcal{H}_{2qq}^{(1)} = C_F\mathcal{P}_{qq}(\xi)\left(e^{-i\vec{k}_\perp\cdot\vec{r}_\perp} + \frac{1}{\xi^2}e^{-i\vec{k}_\perp\cdot\vec{r}_\perp/\xi}\right)\ln\frac{c_0^2}{r_\perp^2\mu^2} - 3C_F e^{-i\vec{k}_\perp\cdot\vec{r}_\perp}\delta(1-\xi)\ln\frac{c_0^2}{r_\perp^2 k_\perp^2}$$

$$\mathcal{H}_{4qq}^{(1)} = -4\pi N_c e^{-ik_\perp\cdot r_\perp}\left\{e^{-i\frac{1-\xi}{\xi}k_\perp\cdot(x_\perp-b_\perp)}\frac{1+\xi^2}{(1-\xi)_+}\frac{1}{\xi}\frac{x_\perp-b_\perp}{(x_\perp-b_\perp)^2}\cdot\frac{y_\perp-b_\perp}{(y_\perp-b_\perp)^2}\right.$$
$$\left. - \delta(1-\xi)\int_0^1 \mathrm{d}\xi'\frac{1+\xi'^2}{(1-\xi')_+}\left[\frac{e^{-i(1-\xi)k_\perp\cdot(y_\perp-b_\perp)}}{(b_\perp-y_\perp)^2} - \delta^{(2)}(b_\perp-y_\perp)\int \mathrm{d}^2r'_\perp\frac{e^{-ik_\perp\cdot r'_\perp}}{r'^2_\perp}\right]\right\}$$

where

$$\mathcal{P}_{qq}(\xi) = \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi)$$

Gluon-Gluon Channel

$$S_{gg} = \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) S_Y^{(2)}(r_\perp) \mathcal{H}_{2gg}^{(0)} + \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) \mathcal{H}_{2gg}^{(1)} + \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) \mathcal{H}_{2q\bar{q}}^{(1)}$$
$$+ \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) S_Y^{(2)}(s_\perp) S_Y^{(2)}(t_\perp) \mathcal{H}_{6gg}^{(1)}$$

$$\mathcal{H}_{2gg}^{(0)} = e^{-i\vec{k}_\perp \cdot \vec{r}_\perp} \delta(1-\xi)$$
$$\mathcal{H}_{2gg}^{(1)} = N_c \left[\frac{2\xi}{(1-\xi)_+} + \frac{2(1-\xi)}{\xi} + 2\xi(1-\xi) + \left(\frac{11}{6} - \frac{2N_f T_R}{3N_c} \right) \delta(1-\xi) \right]$$
$$\times \ln \frac{c_0^2}{\mu^2 r_\perp^2} \left(e^{-i\vec{k}_\perp \cdot \vec{r}_\perp} + \frac{1}{\xi^2} e^{-i\frac{\vec{k}_\perp}{\xi} \cdot \vec{r}_\perp} \right) - \left(\frac{11}{3} - \frac{4N_f T_R}{3N_c} \right) N_c \delta(1-\xi) e^{-i\vec{k}_\perp \cdot \vec{r}_\perp} \ln \frac{c_0^2}{r_\perp^2 k_\perp^2}$$
$$\mathcal{H}_{2q\bar{q}}^{(1)} = 8\pi N_f T_R e^{-i\vec{k}_\perp \cdot (\vec{y}_\perp - \vec{b}_\perp)} \delta(1-\xi)$$
$$\times \int_0^1 \mathrm{d}\xi' [\xi'^2 + (1-\xi')^2] \left[\frac{e^{-i\xi' \vec{k}_\perp \cdot (\vec{x}_\perp - \vec{y}_\perp)}}{(\vec{x}_\perp - \vec{y}_\perp)^2} - \delta^{(2)}(\vec{x}_\perp - \vec{y}_\perp) \int \mathrm{d}^2\vec{r}'_\perp \frac{e^{i\vec{k}_\perp \cdot \vec{r}'_\perp}}{r'^2_\perp} \right]$$

Gluon-Gluon Channel

$$S_{gg} = \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) S_Y^{(2)}(r_\perp) \mathcal{H}_{2gg}^{(0)} + \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) \mathcal{H}_{2gg}^{(1)} + \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) \mathcal{H}_{2q\bar{q}}^{(1)}$$
$$+ \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp) S_Y^{(2)}(s_\perp) S_Y^{(2)}(t_\perp) \mathcal{H}_{6gg}^{(1)}$$

$$\mathcal{H}_{6gg}^{(1)} = -16\pi N_c e^{-i\vec{k}_\perp \cdot \vec{r}_\perp} \Bigg\{ e^{-i\frac{\vec{k}_\perp}{\xi}\cdot(\vec{y}_\perp - \vec{b}_\perp)} \frac{[1-\xi(1-\xi)]^2}{(1-\xi)_+} \frac{1}{\xi^2} \frac{\vec{x}_\perp - \vec{y}_\perp}{(\vec{x}_\perp - \vec{y}_\perp)^2} \cdot \frac{\vec{b}_\perp - \vec{y}_\perp}{(\vec{b}_\perp - \vec{y}_\perp)^2}$$
$$- \delta(1-\xi) \int_0^1 \mathrm{d}\xi' \left[\frac{\xi'}{(1-\xi')_+} + \frac{1}{2}\xi'(1-\xi') \right] \left[\frac{e^{-i\xi'\vec{k}_\perp \cdot (\vec{y}_\perp - \vec{b}_\perp)}}{(\vec{b}_\perp - \vec{y}_\perp)^2} \right.$$
$$\left. - \delta^{(2)}(\vec{b}_\perp - \vec{y}_\perp) \int \mathrm{d}^2\vec{r}'_\perp \frac{e^{i\vec{k}_\perp \cdot \vec{r}'_\perp}}{r'^2_\perp} \right] \Bigg\}$$

Quark-Gluon Channel

$$S_{gq} = \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp)[\mathcal{H}_{2gq}^{(1,1)} + S_Y^{(2)}(r_\perp)\mathcal{H}_{2gq}^{(1,2)}]$$
$$+ \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp}{(2\pi)^2} S_Y^{(4)}(r_\perp, s_\perp, t_\perp)\mathcal{H}_{4gq}^{(1)}$$

$$\mathcal{H}_{2gq}^{(1,1)} = \frac{N_c}{2}\frac{1}{\xi^2} e^{-i\vec{k}_\perp\cdot\vec{r}_\perp/\xi}\frac{1}{\xi}\left[1+(1-\xi)^2\right]\ln\frac{c_0^2}{r_\perp^2\mu^2}$$
$$\mathcal{H}_{2gq}^{(1,2)} = \frac{N_c}{2} e^{-i\vec{k}_\perp\cdot\vec{r}_\perp}\frac{1}{\xi}\left[1+(1-\xi)^2\right]\ln\frac{c_0^2}{r_\perp^2\mu^2}$$
$$\mathcal{H}_{4gq}^{(1)} = 4\pi N_c e^{-i\vec{k}_\perp\cdot\vec{r}_\perp/\xi - i\vec{k}_\perp\cdot\vec{t}_\perp}\frac{1}{\xi}\left[1+(1-\xi)^2\right]\frac{\vec{r}_\perp}{r_\perp^2}\cdot\frac{\vec{t}_\perp}{t_\perp^2}$$

Gluon-Quark Channel

$$S_{qg} = \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{r}_\perp}{(2\pi)^2} S_Y^{(2)}(r_\perp)[\mathcal{H}_{2qg}^{(1,1)} + S_Y^{(2)}(r_\perp)\mathcal{H}_{2qg}^{(1,2)}]$$
$$+ \frac{\alpha_s}{2\pi} \int \frac{\mathrm{d}^2\vec{s}_\perp \mathrm{d}^2\vec{t}_\perp}{(2\pi)^2} S_Y^{(4)}(r_\perp, s_\perp, t_\perp)\mathcal{H}_{4qg}^{(1)}$$

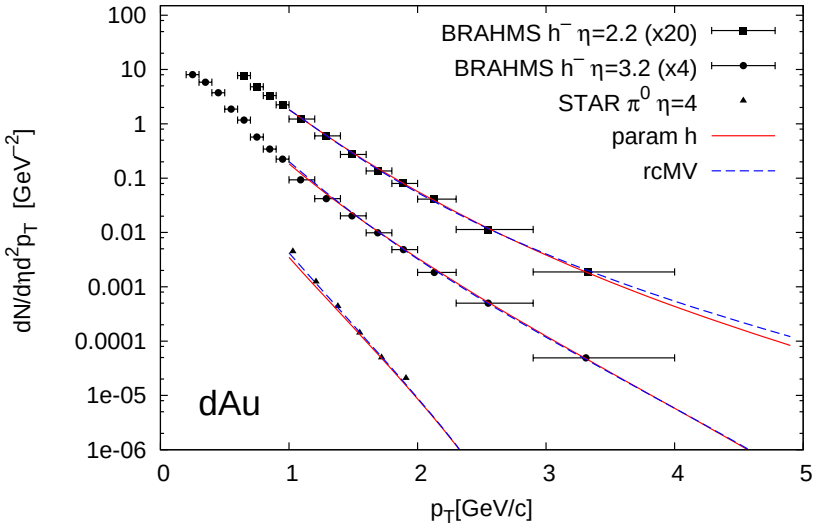
$$\mathcal{H}_{2qg}^{(1,1)} = \frac{1}{2} e^{-i\vec{k}_\perp\cdot\vec{r}_\perp}\left[(1-\xi)^2 + \xi^2\right]\left(\ln\frac{c_0^2}{r_\perp^2\mu^2} - 1\right)$$

$$\mathcal{H}_{2qg}^{(1,2)} = \frac{1}{2\xi^2} e^{-i\vec{k}_\perp\cdot\vec{r}_\perp/\xi}\left[(1-\xi)^2 + \xi^2\right]\left(\ln\frac{c_0^2}{r_\perp^2\mu^2} - 1\right)$$

$$\mathcal{H}_{4qg}^{(1)} = 4\pi e^{-i\vec{k}_\perp\cdot\vec{r}_\perp - i\vec{k}_\perp\cdot\vec{t}_\perp/\xi}\frac{(1-\xi)^2+\xi^2}{\xi}\frac{\vec{r}_\perp}{r_\perp^2}\cdot\frac{\vec{t}_\perp}{t_\perp^2}$$

Incorporating rcBK

Fujii et al. (2011)



Prefactor $K = 1.5$ for charged particles
$K = 0.5$ for neutral particles

Gluon dist

- GBW
- MV/AAMQS
- LO BK
- rcBK
- b-CGC
- NLO BK

Cross section

- LO
- inel NLO
- other NLO
- rapidity NLO
- splitting NLO

Rapidity Divergence

Rapidity divergence in gluon distribution[1]

$$\mathcal{F}(x_g, k_\perp) = \mathcal{F}^{(0)}(x_g, k_\perp) - \frac{\alpha_s N_c}{2\pi^2} \int_0^1 \frac{\mathrm{d}\xi}{1-\xi} \times \int \frac{\mathrm{d}^2\vec{x}_\perp \mathrm{d}^2\vec{y}_\perp \mathrm{d}^2\vec{b}_\perp}{(2\pi)^2} e^{-i\vec{k}_\perp \cdot (\vec{x}_\perp - \vec{y}_\perp)} \frac{(\vec{x}_\perp - \vec{y}_\perp)^2}{(\vec{x}_\perp - \vec{b}_\perp)^2 (\vec{y}_\perp - \vec{b}_\perp)^2} \times \left[S_Y^{(2)}(\vec{x}_\perp, \vec{y}_\perp) - S_Y^{(4)}(\vec{x}_\perp, \vec{b}_\perp, \vec{y}_\perp) \right]$$

Upper limit:

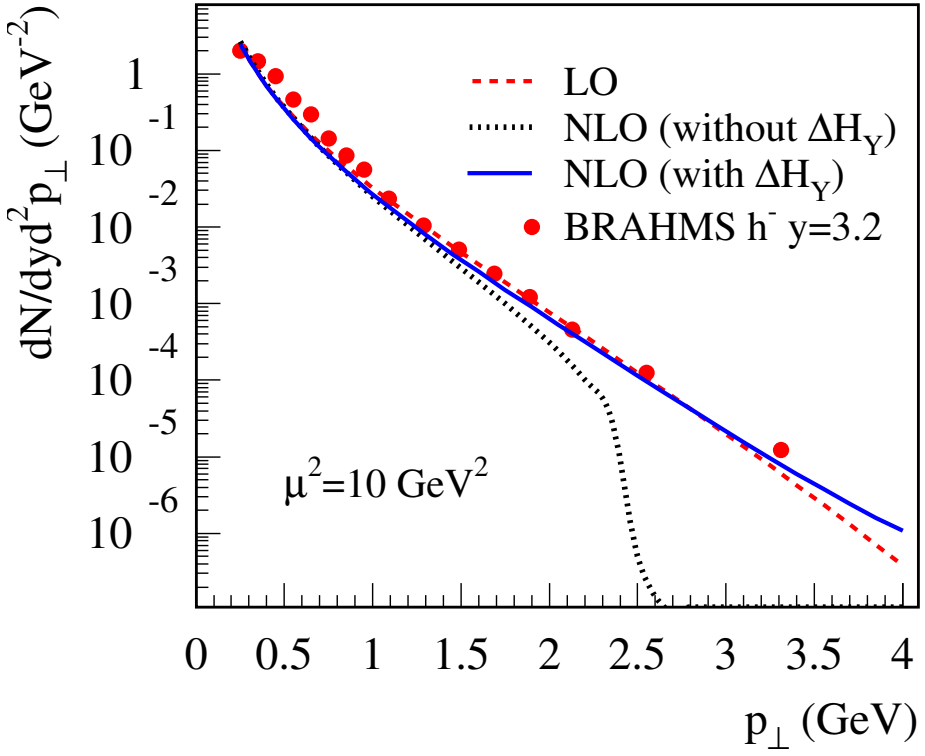
- $\xi_{\text{max}} = 1$ in $\sqrt{s} \to \infty$ limit
- $\xi_{\text{max}} = 1 - \frac{k_\perp}{\sqrt{s}} e^{-y} = 1 - e^{-Y}$ in exact kinematics
- $\xi_{\text{max}} = 1 - e^{-Y_0}$ with rapidity cutoff?[2]

[1] Chirilli et al. 2012, 1203.6139, eq. (21).

[2] Kang et al. 2014, 1403.5221.

Rapidity Correction

Kang et al. (2014)



Rapidity correction (believed unphysical) (by us)

Gluon dist

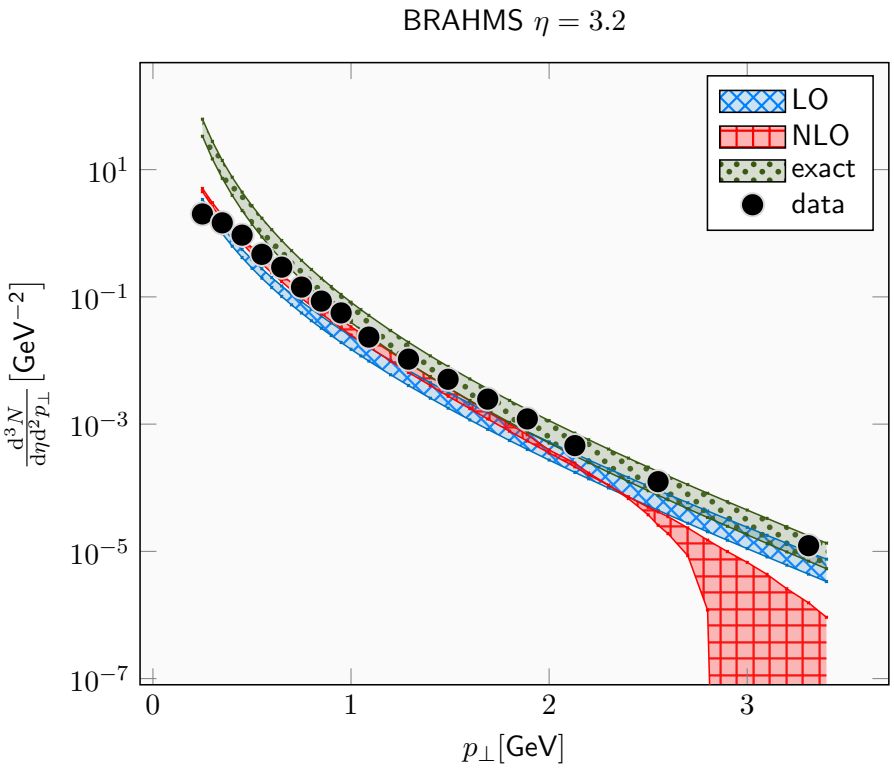
- GBW
- MV/AAMQS
- LO BK
- rcBK
- b-CGC
- NLO BK

Cross section

- LO
- inel NLO
- other NLO
- rapidity NLO
- splitting NLO

Matching to Collinear

Staśto, Xiao, Yuan, et al. (2014)



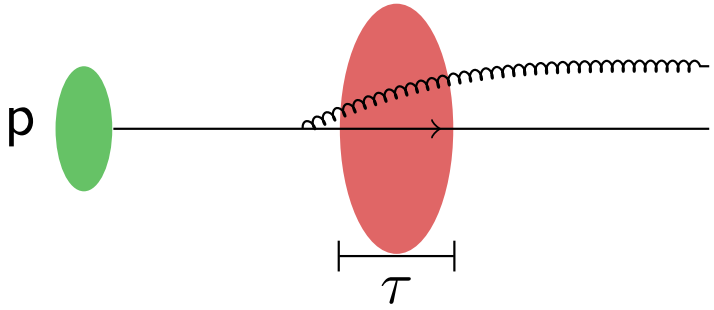
Primitive kinematical constraint

Gluon dist	Cross section
GBW	LO
MV/AAMQS	inel NLO
LO BK	other NLO
rcBK	rapidity NLO
b-CGC	splitting NLO
NLO BK	

Beuf:2014uia.

Ioffe Time

Altinoluk et al. (2014)



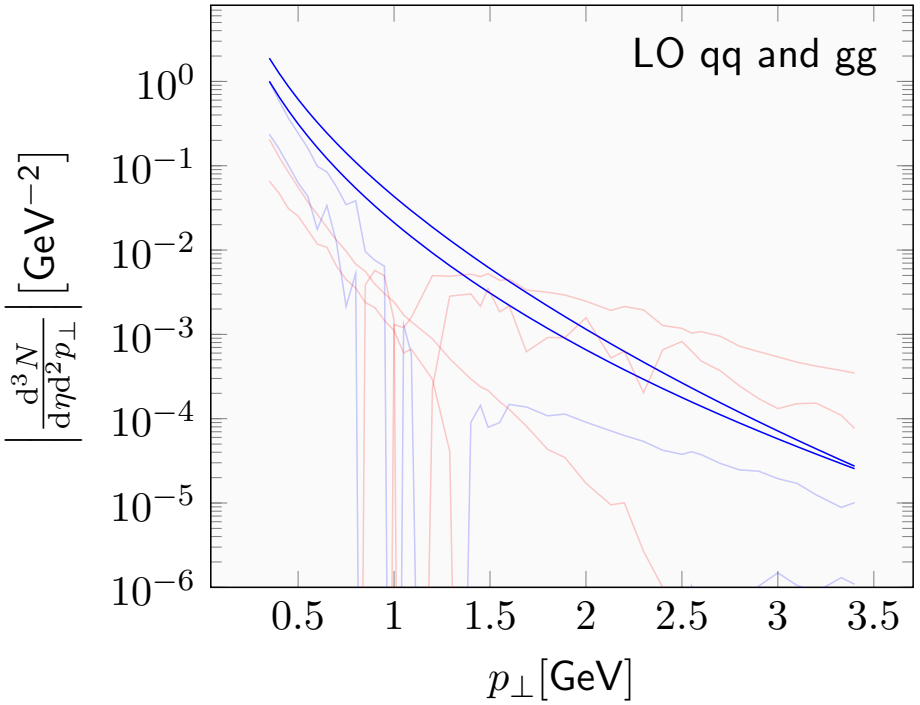
$$\frac{2(1-\xi)\xi x_g P^+}{k_\perp^2} > \tau$$

No numerical results

Gluon dist
GBW
MV/AAMQS
LO BK
rcBK
b-CGC
NLO BK

Cross section
LO
inel NLO
other NLO
rapidity NLO
splitting NLO

Breakdown by Channel

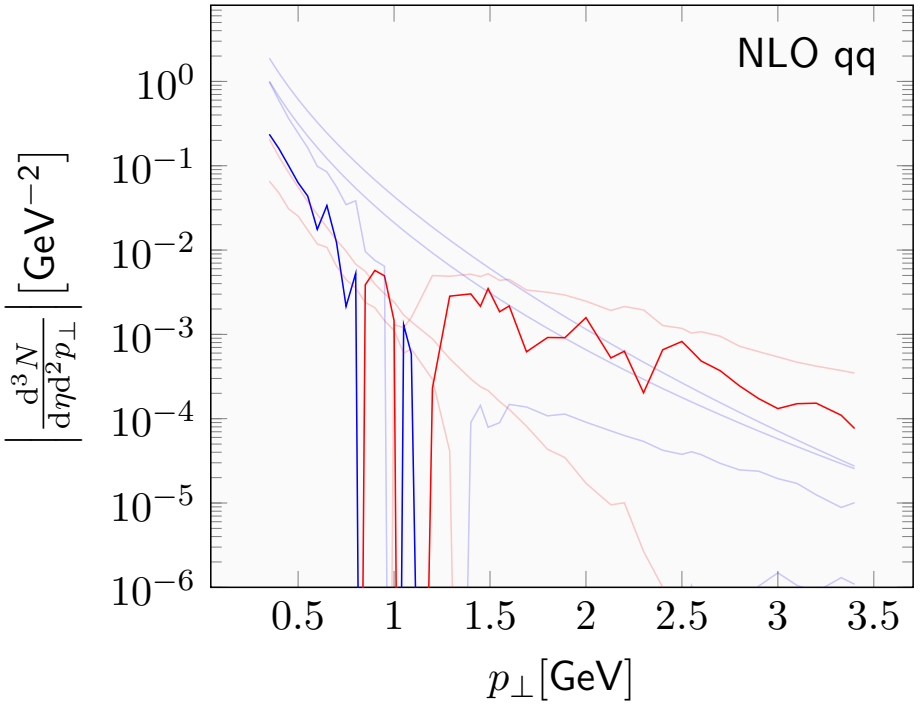


Plot shows magnitude of channel contribution

Coloring indicates where value is
Negative
Positive

Negativity comes from NLO diagonal channels: qq and gg

Breakdown by Channel



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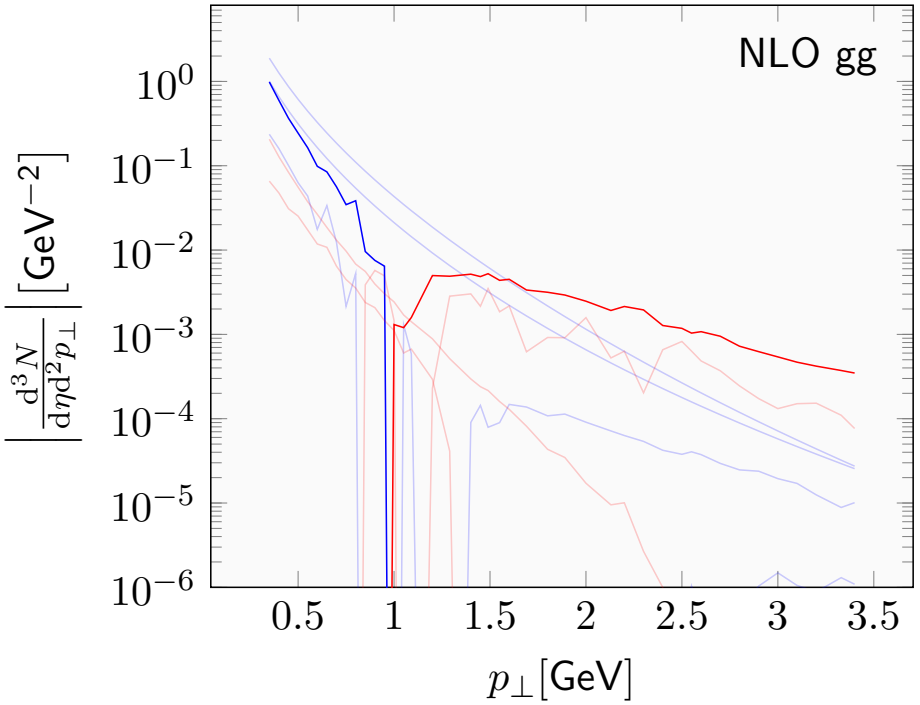
Breakdown by Channel



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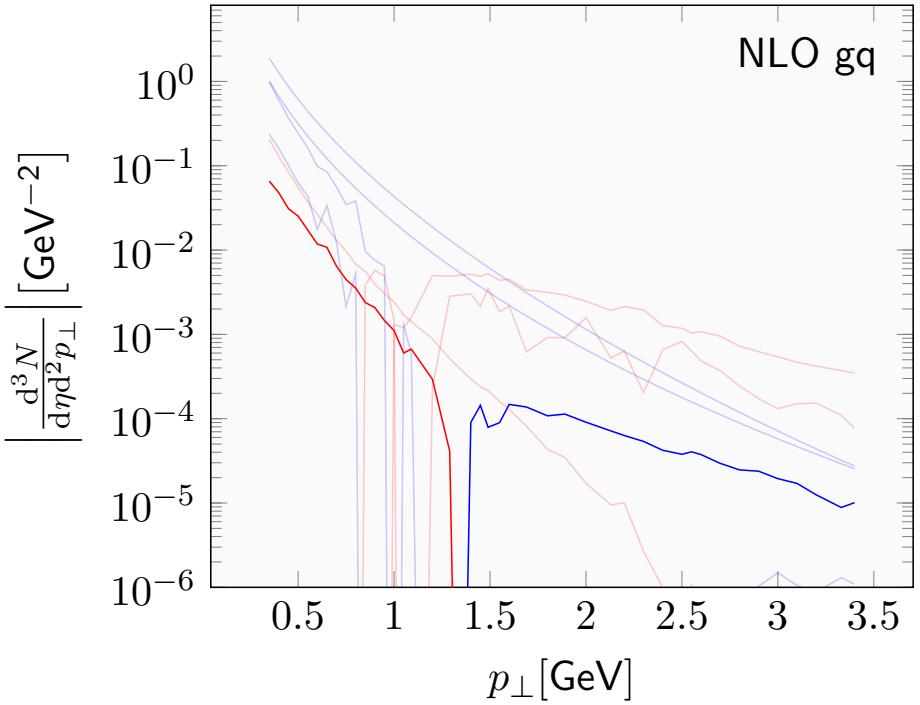
Breakdown by Channel



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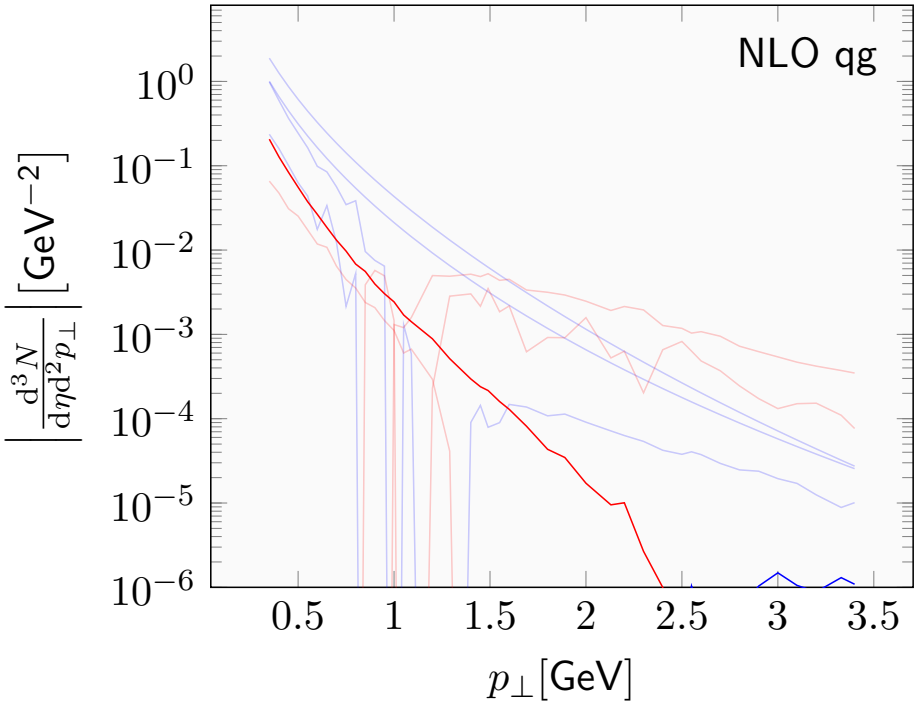
Breakdown by Channel



Plot shows magnitude of channel contribution

Coloring indicates where value is
Negative
Positive

Negativity comes from NLO diagonal channels: qq and gg

Breakdown by Term

Negativity comes from plus prescription

$$\int_{\tau/z}^{1} \frac{\mathrm{d}\xi}{(1-\xi)_+} f(\xi) = \int_{\tau/z}^{1} \mathrm{d}\xi \frac{f(\xi) - f(1)}{1-\xi} + f(1) \ln\left(1 - \frac{\tau}{z}\right)$$

- First term negative because $f(\xi) < f(1)$
- Second term negative because $\frac{\tau}{z} < 1$

qq	$f(\xi) \sim 1 + \xi^2$
gg	$f(\xi) \sim \xi$
gg	$f(\xi) \sim (1 - \xi + \xi^2)^2$

Removing Singularities

Eliminate delta functions and plus prescriptions

$$\int_\tau^1 \mathrm{d}z \int_{\frac{\tau}{z}}^1 \mathrm{d}\xi \left[\frac{F_s(z,\xi)}{(1-\xi)_+} + F_n(z,\xi) + F_d(z,\xi)\delta(1-\xi) \right]$$
$$= \int_\tau^1 \mathrm{d}z \int_\tau^1 \mathrm{d}y \frac{z-\tau}{z(1-\tau)} \left[\frac{F_s(z,\xi) - F_s(z,1)}{1-\xi} + F_n(z,\xi) \right]$$
$$+ \int_\tau^1 \mathrm{d}z \left[F_s(z,1) \ln\left(1 - \frac{\tau}{z}\right) + F_d(z,1) \right]$$

$$\delta^2(\vec{r}_\perp) \int \frac{\mathrm{d}^2\vec{r}'_\perp}{r'^2_\perp} e^{i\vec{k}_\perp \cdot \vec{r}'_\perp} - \frac{1}{r_\perp^2} e^{-i\xi' \vec{k}_\perp \cdot \vec{r}_\perp}$$
$$= \frac{1}{4\pi} \int \mathrm{d}^2\vec{k}'_\perp e^{-i\vec{k}'_\perp \cdot \vec{r}_\perp} \ln \frac{(\vec{k}'_\perp - \xi' \vec{k}_\perp)^2}{k_\perp^2}$$

Fourier Integrals

Fourier integrals are highly imprecise

$$\int \mathrm{d}^2\vec{r}_\perp S_Y^{(2)}(r_\perp) e^{i\vec{k}_\perp\cdot\vec{r}_\perp}(\ldots)$$

$$\int \mathrm{d}^2\vec{s}_\perp S_Y^{(4)}(r_\perp, s_\perp, t_\perp) e^{i\vec{k}_\perp\cdot\vec{r}_\perp}(\ldots)$$

Easiest solution: transform to momentum space

$$\begin{aligned} F(k_\perp) &= \frac{1}{(2\pi)^2}\iint \mathrm{d}^2\vec{r}_\perp S_Y^{(2)}(r_\perp) e^{i\vec{k}_\perp\cdot\vec{r}_\perp} \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{d}r_\perp S_Y^{(2)}(r_\perp) J_0(k_\perp r_\perp) \end{aligned}$$

and compute F directly

Fourier Integrals

Fourier integrals are highly imprecise

$$\int \mathrm{d}^2\vec{r}_\perp S_Y^{(2)}(r_\perp) e^{i\vec{k}_\perp\cdot\vec{r}_\perp}(\ldots)$$

$$\int \mathrm{d}^2\vec{s}_\perp S_Y^{(4)}(r_\perp, s_\perp, t_\perp) e^{i\vec{k}_\perp\cdot\vec{r}_\perp}(\ldots)$$

Alternate solution: algorithms for direct evaluation of multidimensional Fourier integrals (not explored)

New Fourier Transforms

$$\int \frac{\mathrm{d}^2 x_\perp}{(2\pi)^2} S(x_\perp) \ln \frac{c_0^2}{x_\perp^2 \mu^2} e^{-ik_\perp \cdot x_\perp}$$
$$= \frac{1}{\pi} \int \frac{\mathrm{d}^2 \vec{l_\perp}}{l_\perp^2} \left[F(\vec{k_\perp} + \vec{l_\perp}) - J_0\left(\frac{c_0}{\mu} l_\perp\right) F(k_\perp) \right]$$

$$\int \frac{\mathrm{d}^2 r_\perp}{(2\pi)^2} S(r_\perp) \left(\ln \frac{r_\perp^2 k_\perp^2}{c_0^2} \right)^2 e^{-ik_\perp \cdot r_\perp}$$
$$= \frac{2}{\pi} \int \frac{\mathrm{d}^2 \vec{l_\perp}}{l_\perp^2} \ln \frac{k_\perp^2}{l_\perp^2} \left[\theta(k_\perp - l_\perp) F(k_\perp) - F(\vec{k_\perp} + \vec{l_\perp}) \right]$$

Watanabe et al. 2015, 1505.05183.

Remaining Evaluation Errors

- Inaccuracy of Fourier integrals
- Monte Carlo statistical error
- Cancellation of large terms

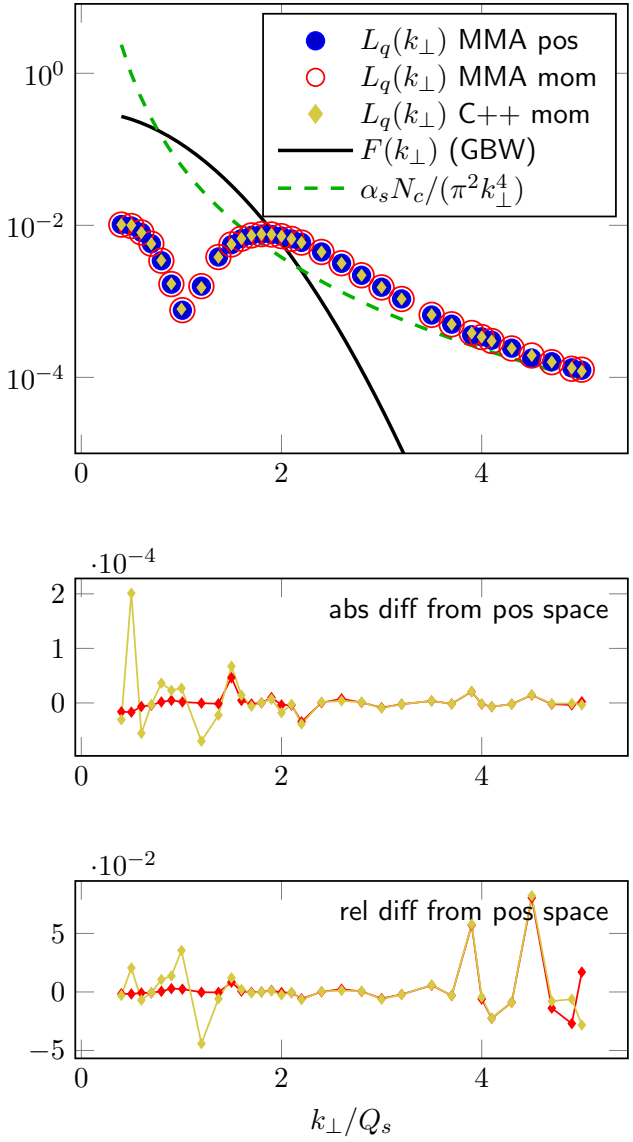
Multiple runs to improve statistics

Remaining Evaluation Errors

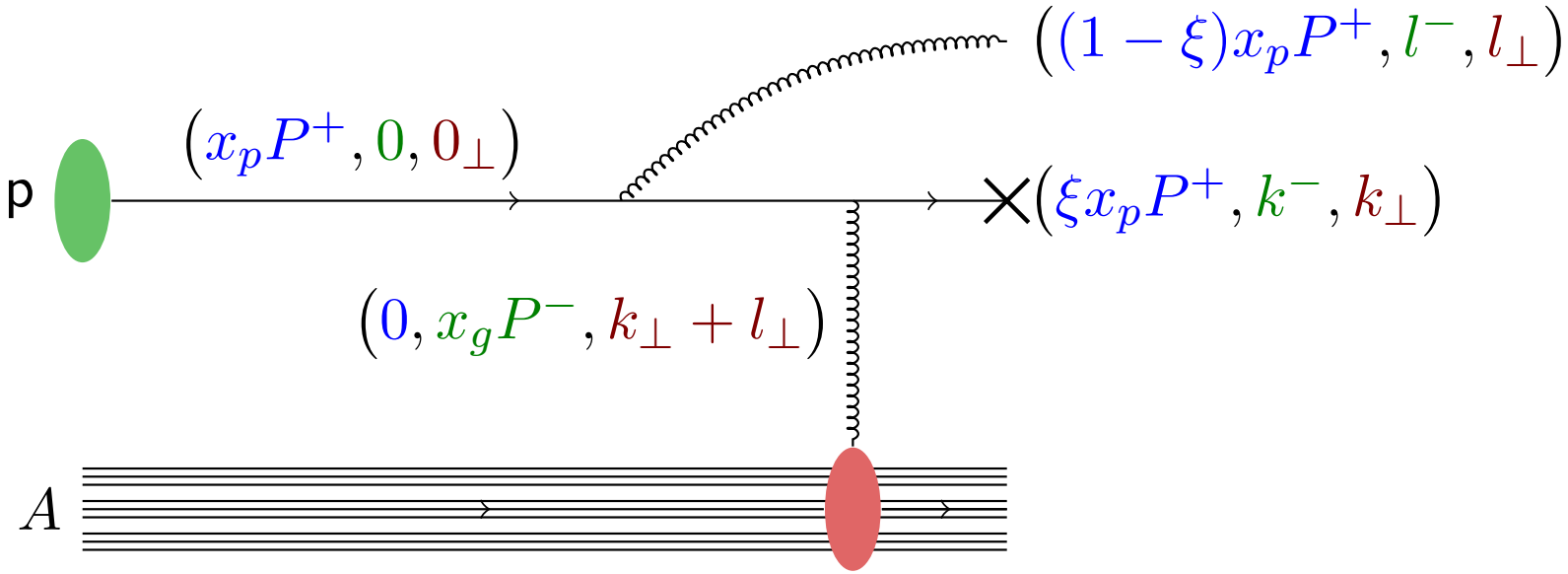
- Inaccuracy of Fourier integrals
- Monte Carlo statistical error
- Cancellation of large terms

Two parallel implementations of selected parts:

- Mathematica, for rapid prototyping
- C++, for execution speed



Derivation of the Kinematical Constraint



$$x_gP^- = \frac{l_\perp^2}{2(1-\xi)x_pP^+} + \frac{k_\perp^2}{2\xi x_pP^+} \leq P^-$$

$$x_g \leq 1$$

$$\xi \leq 1 - \frac{l_\perp^2}{x_ps}$$

figure adapted from Watanabe et al. 2015, 1505.05183.

The Beam Direction Problem

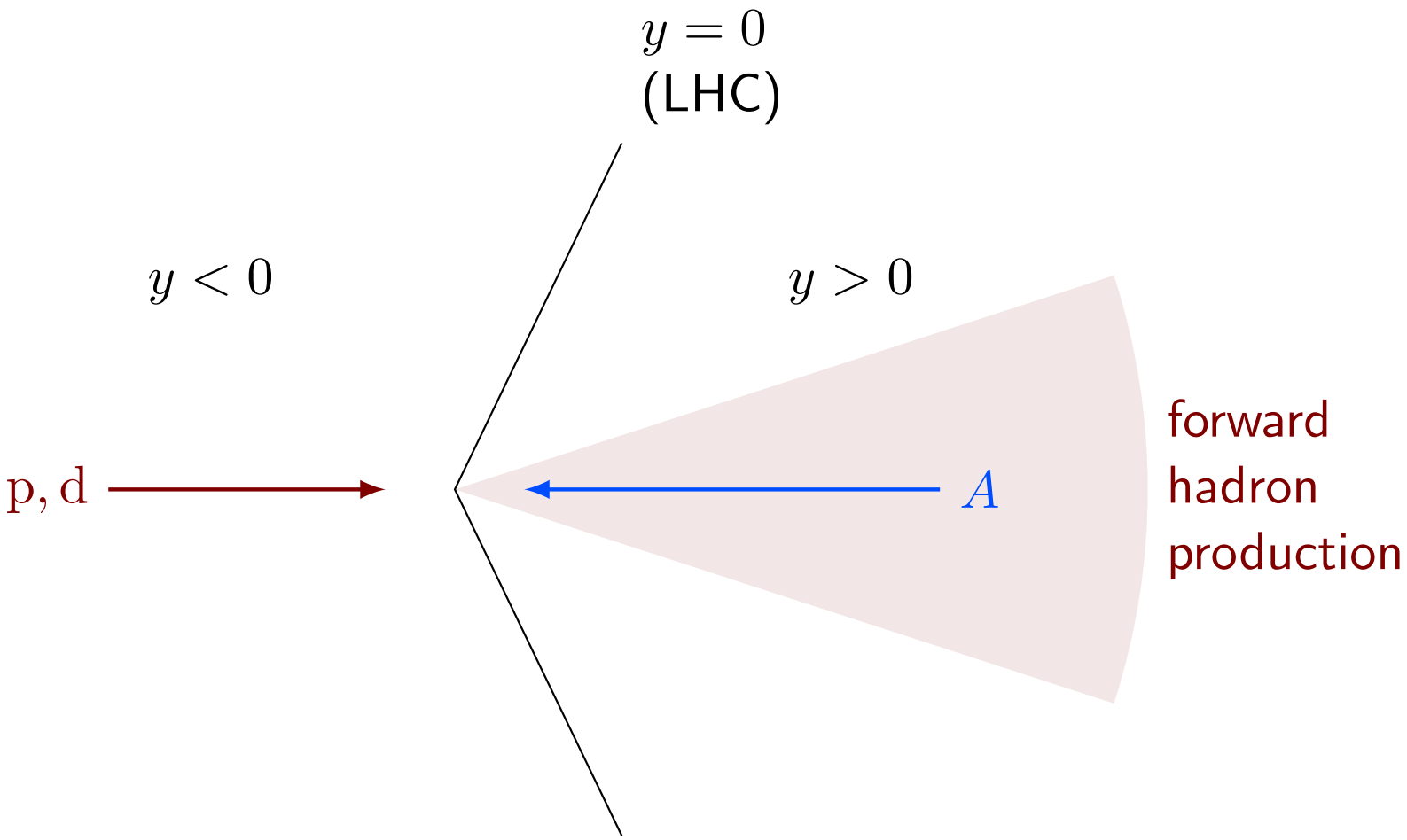
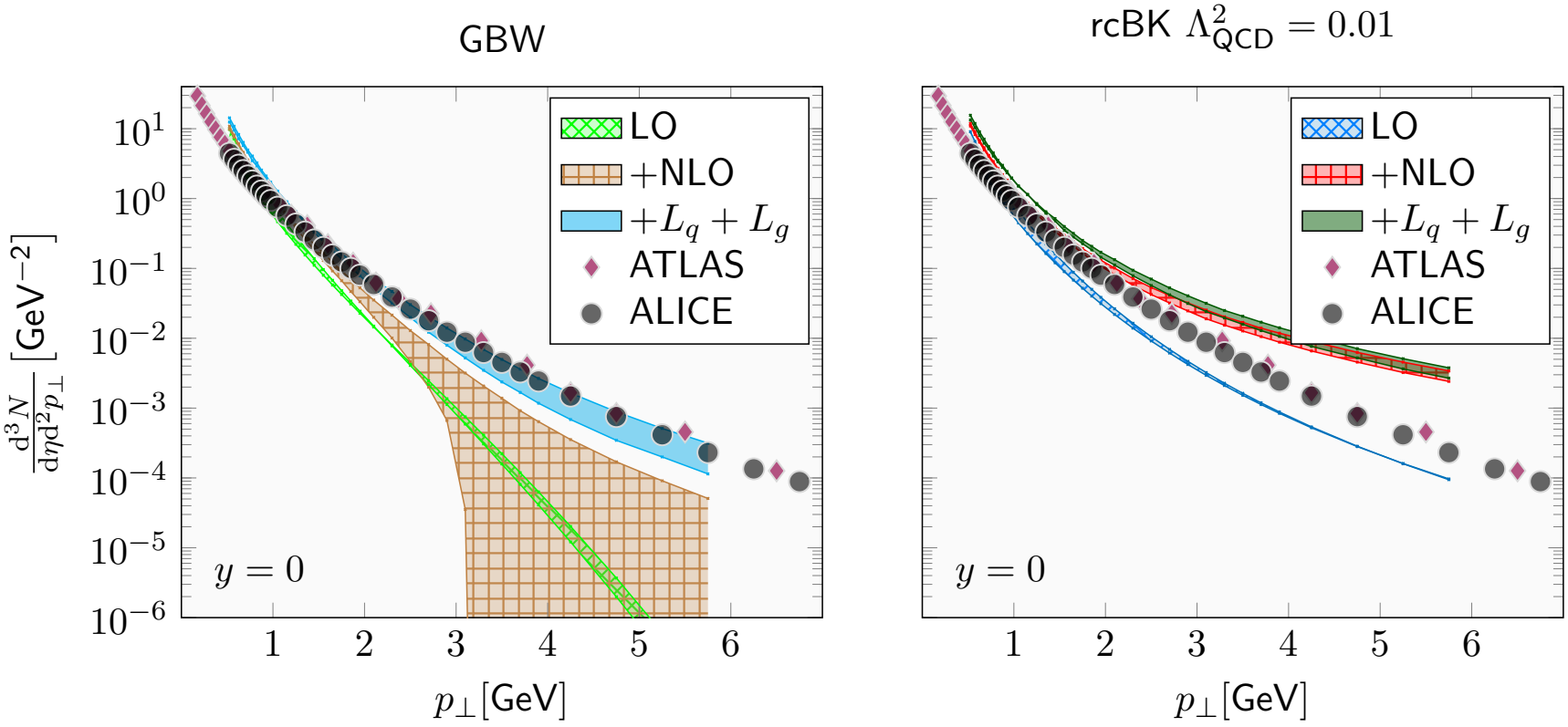


figure adapted from Watanabe et al. 2015, 1505.05183.

LHC Results at Central Rapidity



ALICE:2012mj; data: Milov 2014, 1403.5738.

plots: Watanabe et al. 2015, 1505.05183.

LHC Predictions for Run II