







Beam instabilities (II)

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in CERN Accelerator School, Advanced Level, Warsaw
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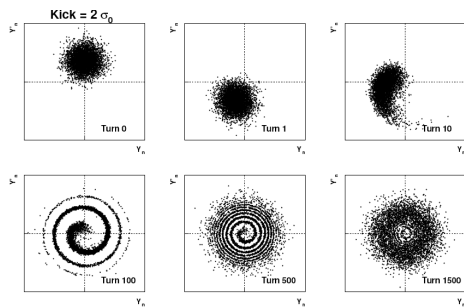
Summary of the first part

- What is a beam instability?
 - A beam becomes unstable when a moment of its distribution exhibits an exponential growth (e.g. mean positions $\langle x \rangle$, $\langle y \rangle$, $\langle z \rangle$, standard deviations σ_x , σ_y , σ_z , etc.) – resulting into beam loss or emittance growth!
- Instabilities are caused by the electro-magnetic fields trailing behind charged particles moving at the speed of light
 - Origin: discontinuities, lossy materials
 - Described through wake functions and beam coupling impedances

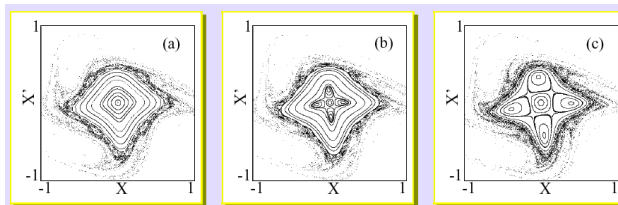
⇒ Longitudinal plane


- Energy loss and potential well distortion
 - Synchronous phase shift
 - Bunch lengthening/shortening, synchrotron tune shift
- Instabilities
 - Robinson instability (dipole mode)
 - Coupled bunch instabilities
 - Single bunch instabilities

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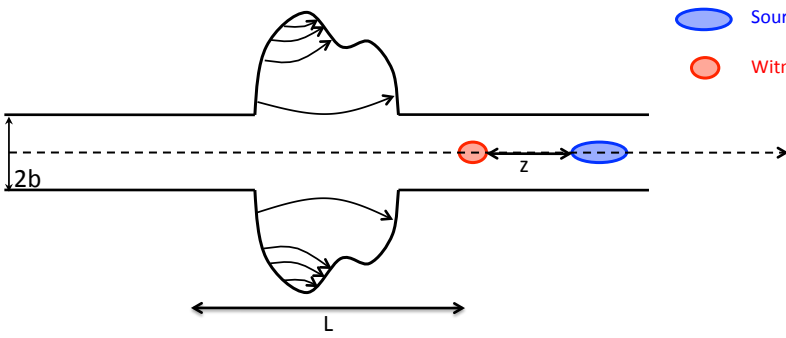



2. The transverse plane





Transverse wake function: definition





- In an axisymmetric structure (or simply with a top-bottom and left-right symmetry) a source particle traveling on axis cannot induce net transverse forces on a witness particle also following on axis
- At the zero-th order, there is no transverse effect
- We need to introduce a breaking of the symmetry to drive transverse effect, but at the first order there are two possibilities, i.e. offset the source or the witness

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Transverse dipolar wake function: definition

Source, q_1
Witness, q_2

Δx_1 (or Δy_1)

z

$2b$

L

$$\int_0^L F_x(s, z) ds = -q_1 q_2 W_x(z) \Delta x_1$$

$$\int_0^L F_y(s, z) ds = -q_1 q_2 W_y(z) \Delta y_1$$

$$F_{x,y}(s, z) = q_2 \left(\vec{E} + \vec{v} \times \vec{B} \right)_{x,y}$$

$\Delta E_{2x,y} \Rightarrow \frac{\Delta E_{2x,y}}{E_0} = \Delta x'_2, \Delta y'_2$

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Transverse quadrupolar wake function: definition

Source, q_1
Witness, q_2

Δx_2 (or Δy_2)

z

$2b$

L



$$\int_0^L F_x(s, z) ds = -q_1 q_2 W_{Qx}(z) \Delta x_2$$

$$\int_0^L F_y(s, z) ds = -q_1 q_2 W_{Qy}(z) \Delta y_2$$

$$F_{x,y}(s, z) = q_2 \left(\vec{E} + \vec{v} \times \vec{B} \right)_{x,y}$$

$\Delta E_{2x,y} \Rightarrow \frac{\Delta E_{2x,y}}{E_0} = \Delta x'_2, \Delta y'_2$

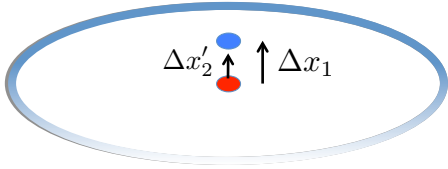
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Transverse dipolar wake function

$$W_x(z) = -\frac{E_0}{q_1 q_2} \frac{\Delta x'_2}{\Delta x_1} \quad z \rightarrow 0 \quad W_x(0) = 0$$

- The value of the transverse dipolar wake functions in 0, $W_{x,y}(0)$, vanishes because source and witness particles are traveling parallel and they can only – mutually – interact through space charge, which is not included in this framework
- $W_{x,y}(0^-) < 0$ since trailing particles are deflected toward the source particle (Δx_1 and Δx_2 have the same sign)


 $\frac{\Delta x'_2}{\Delta x_1} > 0$

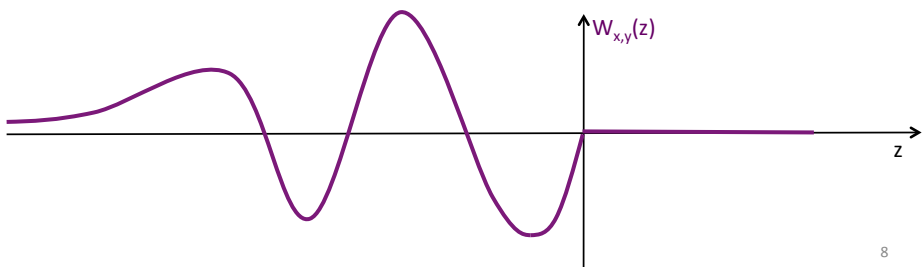
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Transverse dipolar wake function

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- $W_{x,y}(0^-) < 0$ since trailing particles are deflected toward the source particle (Δx_1 and Δx_2 have the same sign)
- $W_{x,y}(z)$ has a discontinuous derivative in $z=0$ and it vanishes for all $z>0$ because of the ultra-relativistic approximation



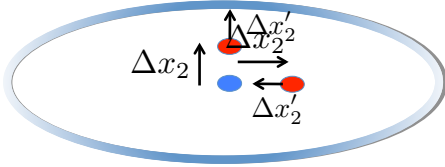
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Transverse quadrupolar wake function

$$W_{Qx}(z) = -\frac{E_0}{q_1 q_2} \frac{\Delta x'_2}{\Delta x_2} \quad z \rightarrow 0 \quad W_{Qx}(0) = 0$$

- The value of the transverse quadrupolar wake functions in 0, $W_{Qx,y}(0)$, vanishes because source and witness particles are traveling parallel and they can only – mutually – interact through space charge, which is not included in this framework
- $W_{Qx,y}(0^-)$ can be of either sign since trailing particles can be either attracted or deflected even more off axis (depends on geometry and boundary conditions)



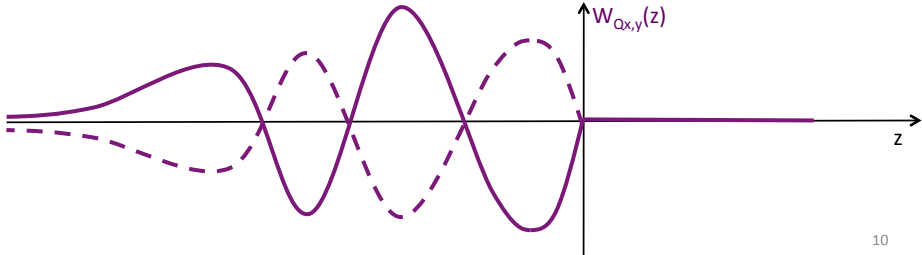
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Transverse quadrupolar wake function

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Transverse impedance

- The transverse wake function of an accelerator component is basically its Green function in time domain (i.e., its response to a pulse excitation)
 \Rightarrow Very useful for macroparticle models and simulations, because it relates source perturbations to the associated kicks on trailing particles!
- We can also describe it as a transfer function in frequency domain
- This is the definition of **transverse beam coupling impedance** of the element under study

$$Z_{\perp}^{\text{dip}}(\omega) = \begin{cases} \Delta x_2 \int_{-\infty}^{\infty} W_{\perp}(z) \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c} & [\Omega/\text{m/s}] \\ \Delta y_2' = -\left(\frac{q_1 q_2}{E_0}\right) \int_{-\infty}^{\infty} W_{\perp}(z) \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c} & [\Omega/\text{m}] \end{cases}$$

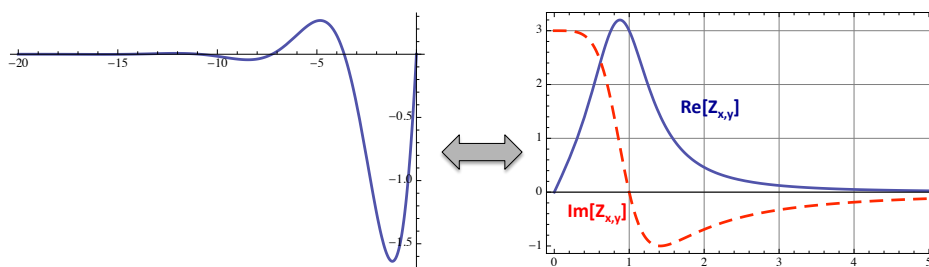
* linear terms retained, however coupling terms are neglected (usually small in symmetric structures)
 ** m^{-1} refers then to a transverse offset and does not represent a normalization per unit length of the structure

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Transverse impedance: resonator

$$Z_{\perp}^{\text{dip}}(\omega) = i \int_{-\infty}^{\infty} W_{\perp}(z) \exp\left(-\frac{i\omega z}{c}\right) \frac{dz}{c}$$




- Shape of wake function can be similar to that in longitudinal plane, determined by the oscillations of the trailing electromagnetic fields
- Contrary to longitudinal impedances, $\text{Re}[Z_{xy}]$ is an odd function of frequency, while $\text{Im}[Z_{xy}]$ is an even function

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Transverse wake & impedance

Equations of the resonator




$$Z_{x,y}^{\text{Res}}(\omega) = \frac{\omega_r}{\omega} \frac{R_{s(x,y)}}{1 + iQ \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega} \right)}$$


$$W_{x,y}^{\text{Res}}(z) = \begin{cases} \frac{R_{s(x,y)} \omega_r^2}{Q \bar{\omega}} \exp\left(\frac{\alpha_t z}{c}\right) \sin\left(\frac{\bar{\omega} z}{c}\right) & \text{if } z < 0 \\ 0 & \text{if } z \geq 0 \end{cases}$$

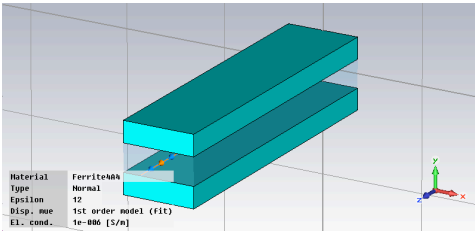
$$\alpha_t = \frac{\omega_r}{2Q} \quad \bar{\omega} = \sqrt{\omega_r^2 - \alpha_t^2}$$

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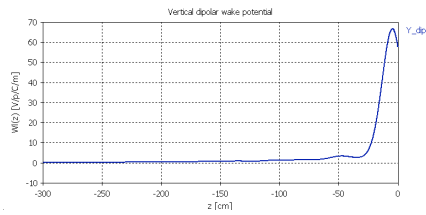
Transverse impedance: kicker





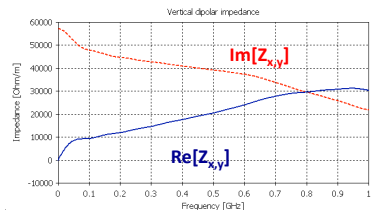
Material	Ferriteksh
Type	Normal
Epsilon	12
Disp. me	1st order model (fit)
El. cond.	1e-086 [S/m]

- An example: magnetic kickers are usually large contributors to the transverse impedance of a machine
- It is a broad band contribution
 - No trapped modes
 - Losses both in vacuum chamber and ferrite (kicker heating and outgassing)



Vertical dipolar wake potential

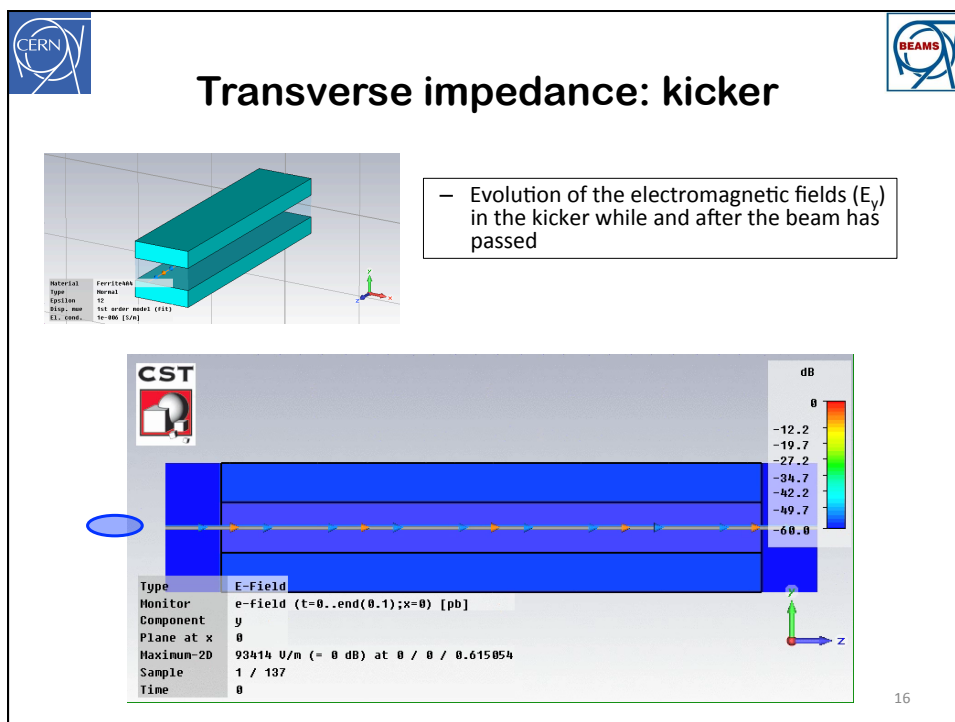
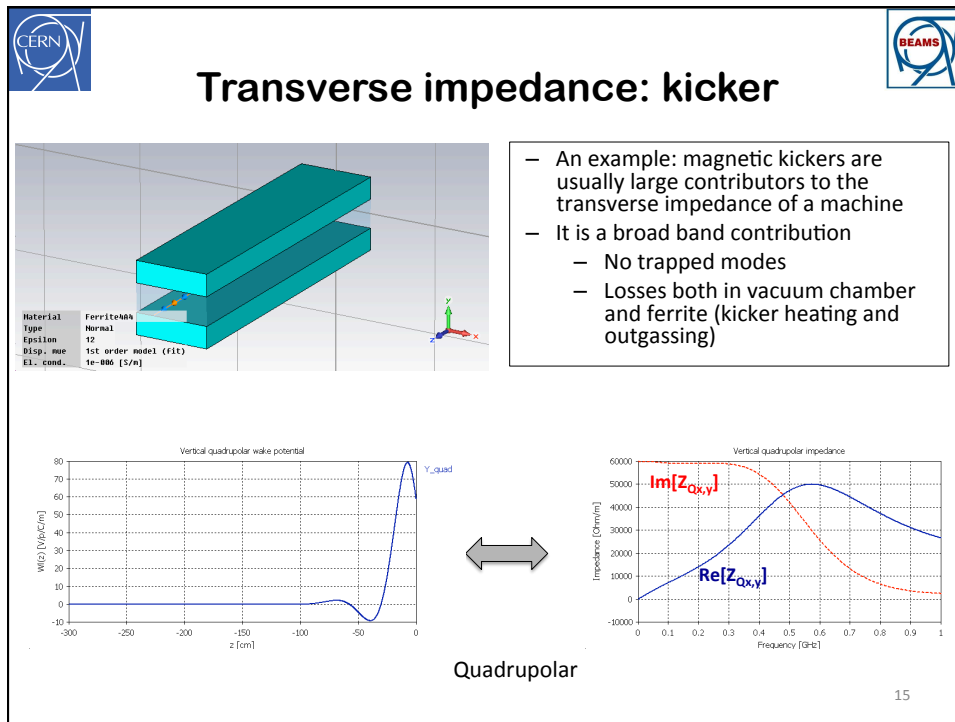
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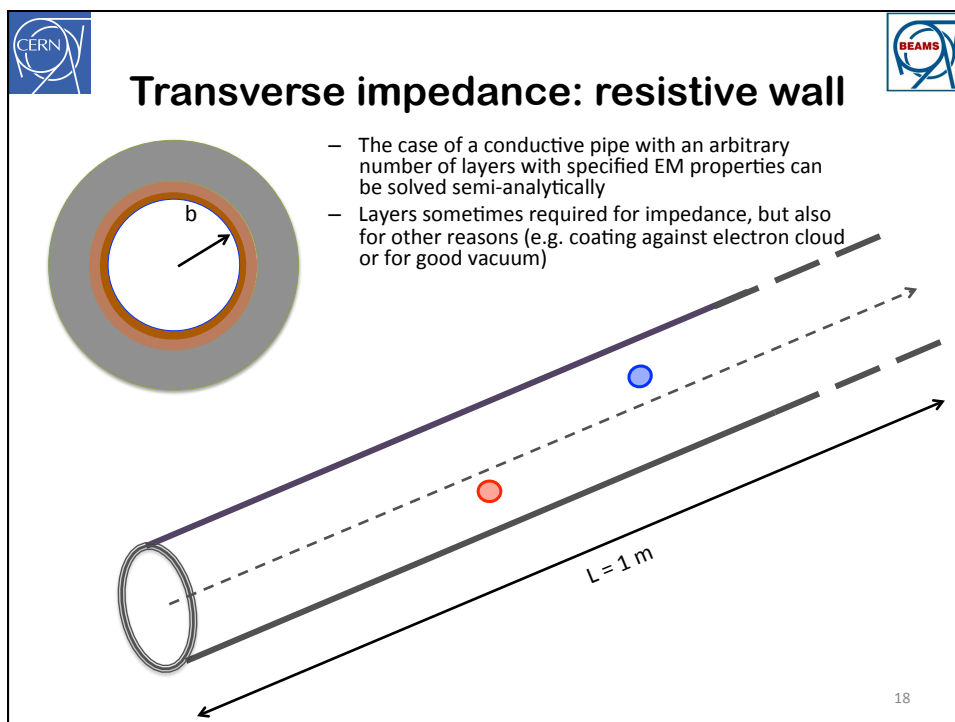
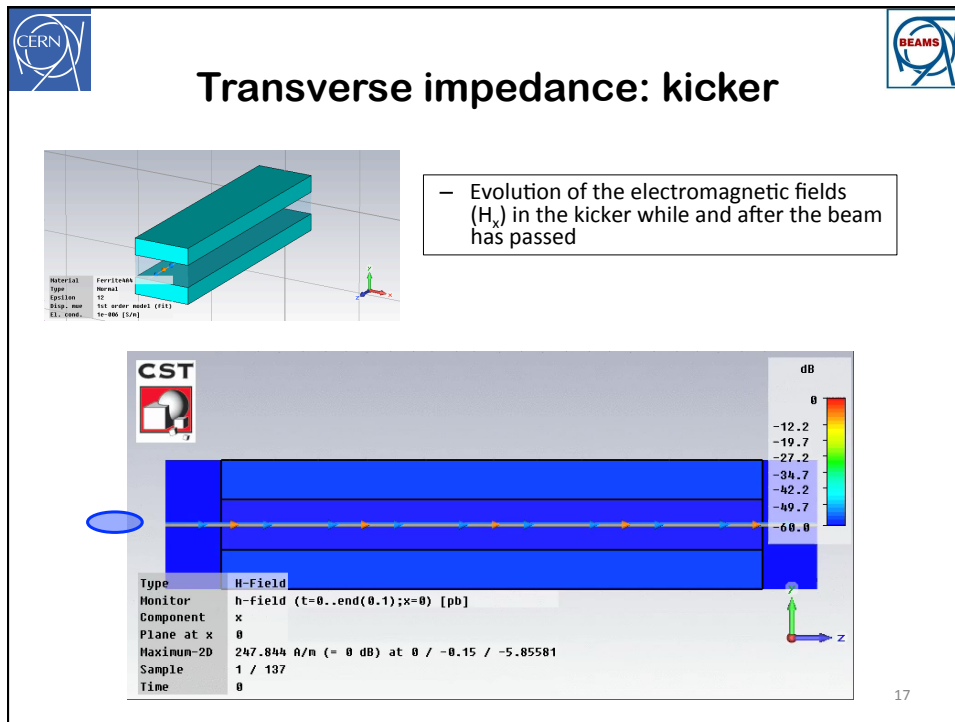




Vertical dipolar impedance

Dipolar

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Transverse impedance: resistive wall



$$\nabla \cdot \vec{E} = \frac{\tilde{\rho}}{\epsilon_0 \epsilon_1(\omega)} \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \mu_1(\omega) \vec{J} + i\omega \frac{\mu_1(\omega) \epsilon_1(\omega)}{c^2} \vec{E} \quad \nabla \times \vec{E} = -i\omega \vec{B}$$

Source terms (displaced point charge traveling along s with speed v) in cylindrical coordinates and frequency domain:

$$\begin{aligned} \tilde{\rho}(r, \theta, s, \omega) &= \frac{q_1}{r_1 v} \delta(r - r_1) \delta_P(\theta) \exp\left(-\frac{i\omega s}{v}\right) = \\ &= \frac{q_1}{r_1 v} \int_{-\infty}^{\infty} dk' \exp(-ik' s) \delta\left(k' - \frac{\omega s}{v}\right) \sum_{m=0}^{\infty} \frac{\cos m\theta}{\pi(1 + \delta_{m0})} \delta(r - r_1) \\ \vec{J}(r, \theta, s, \omega) &= \tilde{\rho}(r, \theta, s, \omega) \vec{v} \end{aligned}$$

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Transverse impedance: resistive wall

$$\nabla \cdot \vec{E} = \frac{\tilde{\rho}}{\epsilon_0 \epsilon_1(\omega)} \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \mu_1(\omega) \vec{J} + i\omega \frac{\mu_1(\omega) \epsilon_1(\omega)}{c^2} \vec{E} \quad \nabla \times \vec{E} = -i\omega \vec{B}$$

Source terms (displaced point charge traveling along s with speed v) in cylindrical coordinates

Expansion in longitudinal modes

$$\begin{aligned} \tilde{\rho}(r, \theta, s, \omega) &= \frac{q_1}{r_1 v} \delta(r - r_1) \delta_P(\theta) \exp\left(-\frac{i\omega s}{v}\right) = \\ &= \frac{q_1}{r_1 v} \int_{-\infty}^{\infty} dk' \exp(-ik' s) \delta\left(k' - \frac{\omega s}{v}\right) \sum_{m=0}^{\infty} \frac{\cos m\theta}{\pi(1 + \delta_{m0})} \delta(r - r_1) \\ \vec{J}(r, \theta, s, \omega) &= \tilde{\rho}(r, \theta, s, \omega) \vec{v} \end{aligned}$$

Expansion in azimuthal modes

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Transverse impedance: resistive wall



Maxwell's equations combine into the wave equations:

$$\nabla^2 \vec{E} + \omega^2 \frac{\epsilon_1(\omega) \mu_1(\omega)}{c^2} \vec{E} = \frac{1}{\epsilon_0 \epsilon_1(\omega)} \nabla \tilde{\rho} + i\omega \mu_0 \mu_1(\omega) \tilde{\rho} \vec{v}$$

We can seek solutions as expansions of longitudinal and azimuthal modes (for both E and B)

$$\vec{E}(r, \theta, s, \omega) =$$

$$\int_{-\infty}^{\infty} dk' \exp(-ik's) \left(\sum_{m=0}^{\infty} \frac{\vec{E}^{(m,c)}(r, k', \omega)}{1 + \delta_{m0}} \cos m\theta + \sum_{m=1}^{\infty} \vec{E}^{(m,s)}(r, k', \omega) \sin m\theta \right)$$

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Transverse impedance: resistive wall



It can be demonstrated that the components of the force on the witness charge only depends on E_s

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial s^2} + \frac{\omega^2}{c^2} \epsilon_1(\omega) \mu_1(\omega) \right] E_s = \frac{1}{\epsilon_0 \epsilon_1(\omega)} \frac{\partial \tilde{\rho}}{\partial s} + i\omega \mu_0 \mu_1(\omega) \tilde{\rho} v$$

$$\left\{ \begin{aligned} \frac{d^2 E_s^{(m,c)}}{dr^2} + \frac{1}{r} \frac{dE_s^{(m,c)}}{dr} - \left(\frac{m^2}{r^2} + k'^2 - \frac{\omega^2}{c^2} \epsilon_1(\omega) \mu_1(\omega) \right) E_s^{(m,c)} &= \\ &= \frac{j q_1 \delta(r - r_1) \delta(k' - \frac{\omega}{v})}{\pi r_1 (1 + \delta_{m0})} \left[-\frac{k'}{\epsilon_0 \epsilon_1(\omega)} + \omega \mu_0 \mu_1(\omega) \right] \\ \frac{d^2 E_s^{(m,s)}}{dr^2} + \frac{1}{r} \frac{dE_s^{(m,s)}}{dr} - \left(\frac{m^2}{r^2} + k'^2 - \frac{\omega^2}{c^2} \epsilon_1(\omega) \mu_1(\omega) \right) E_s^{(m,s)} &= 0 \end{aligned} \right.$$

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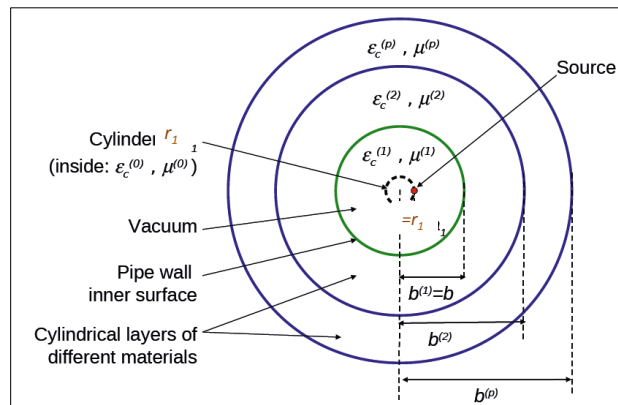


Transverse impedance: resistive wall



The equations for the coefficients of the azimuthal modes of E_z must be solved in all the media and the conservation of the tangential components of the fields is applied at the boundaries between different layers

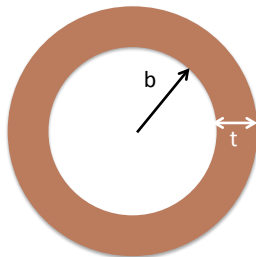
→ E.g. **ImpedanceWake2D** code (N. Mounet) can calculate impedances and then wakes by inverse Fourier transform. It can deal with both round and flat structures



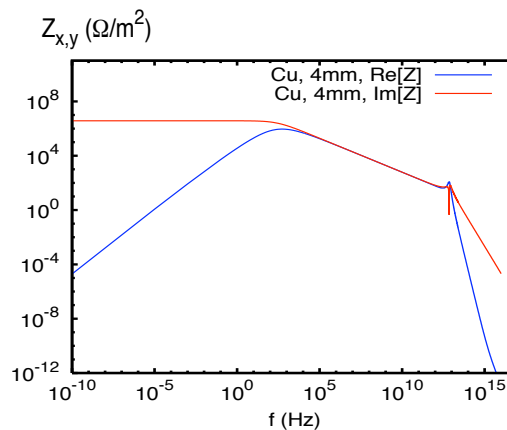
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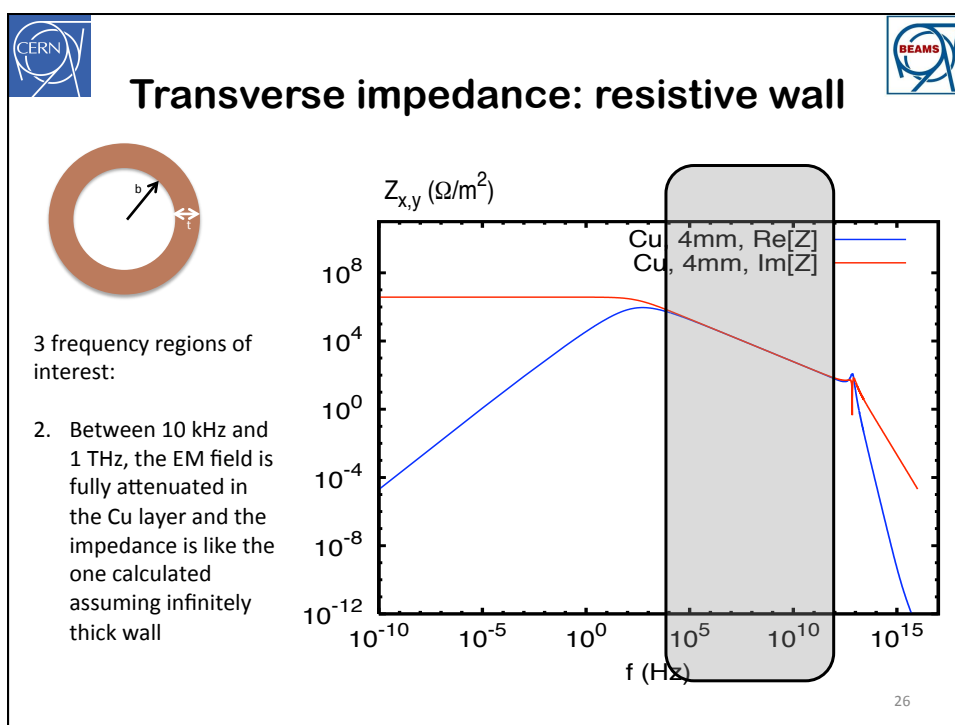
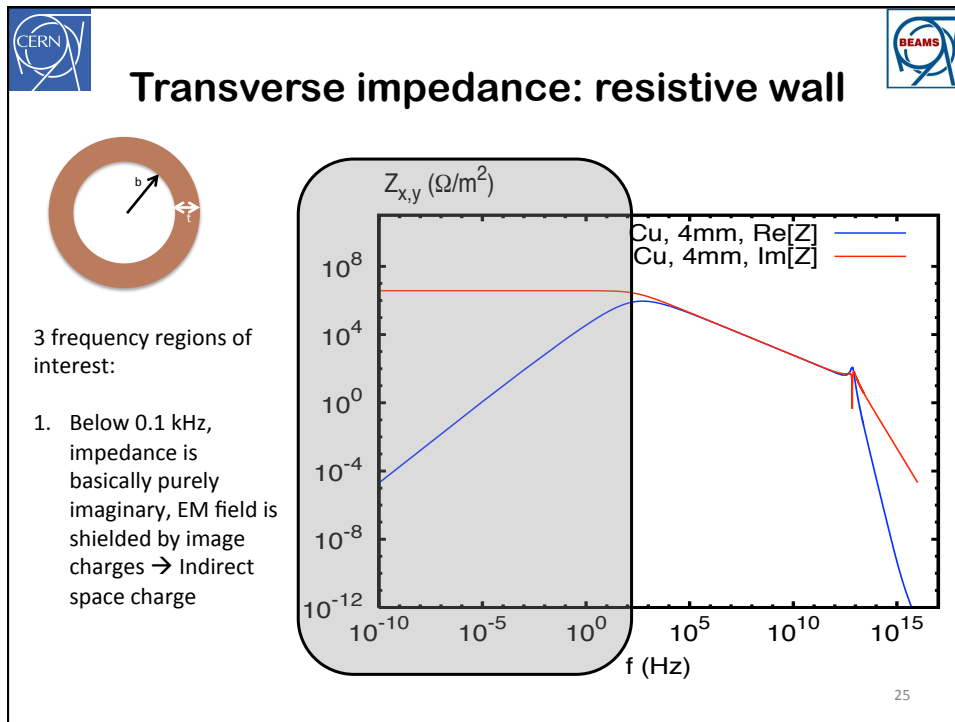
Transverse impedance: resistive wall

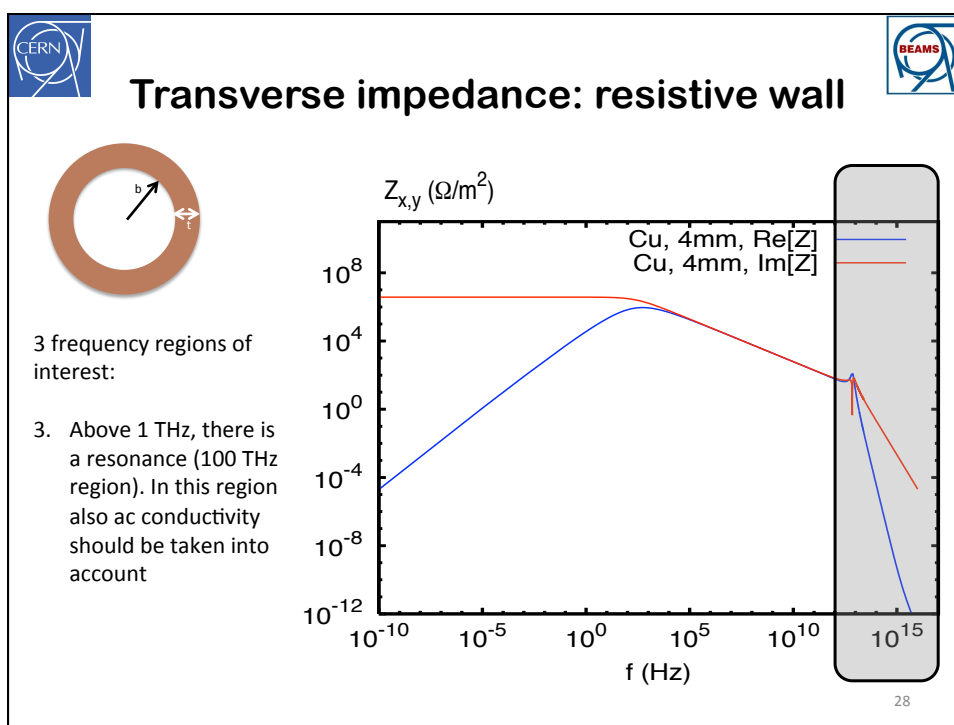
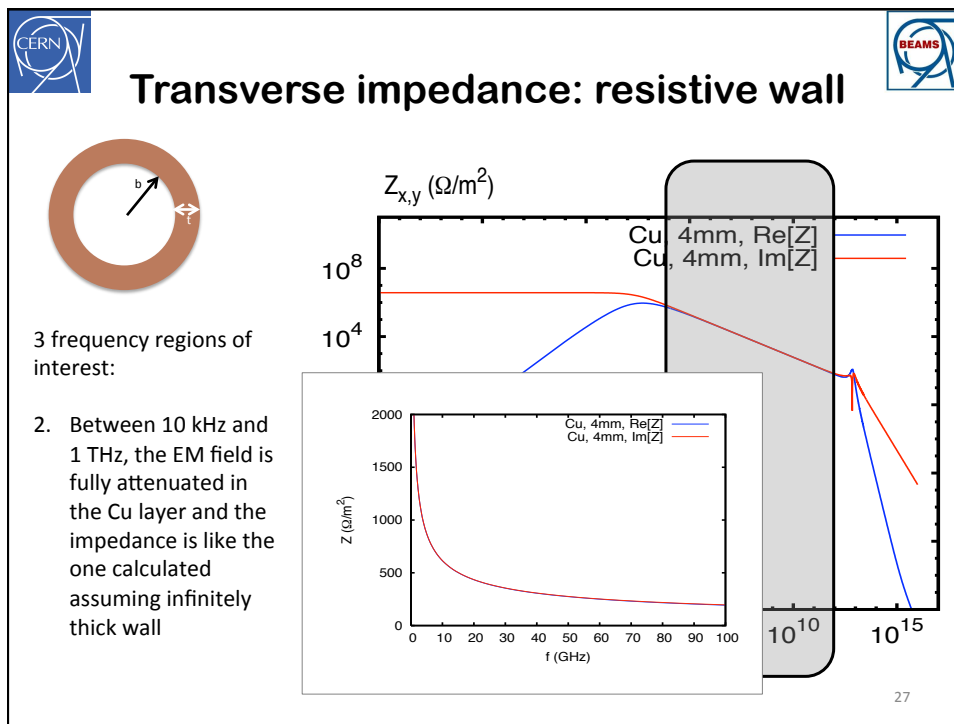


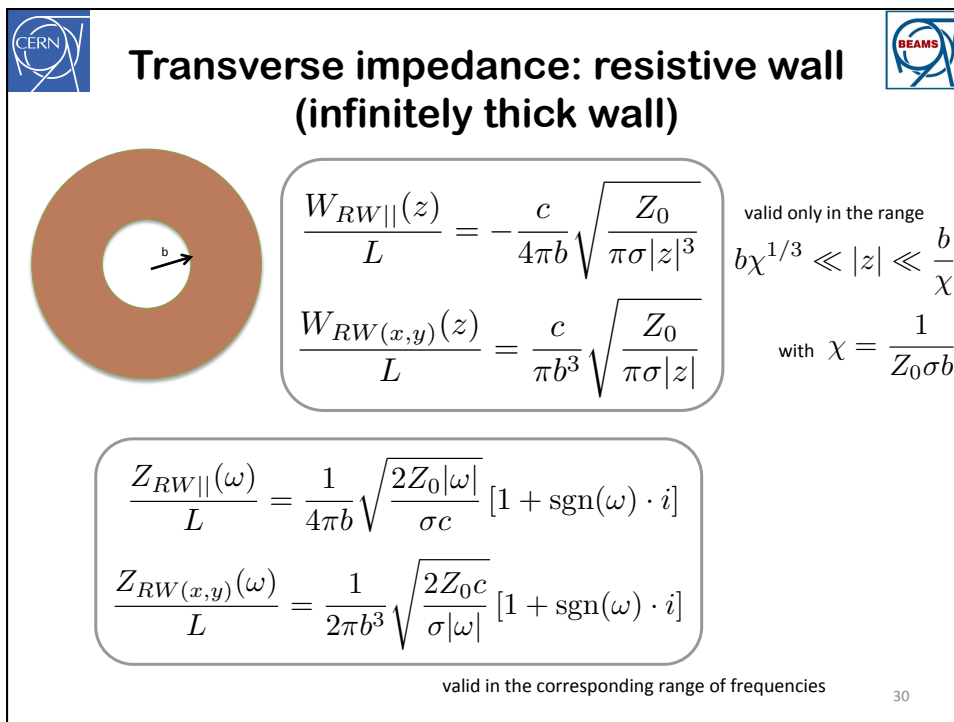
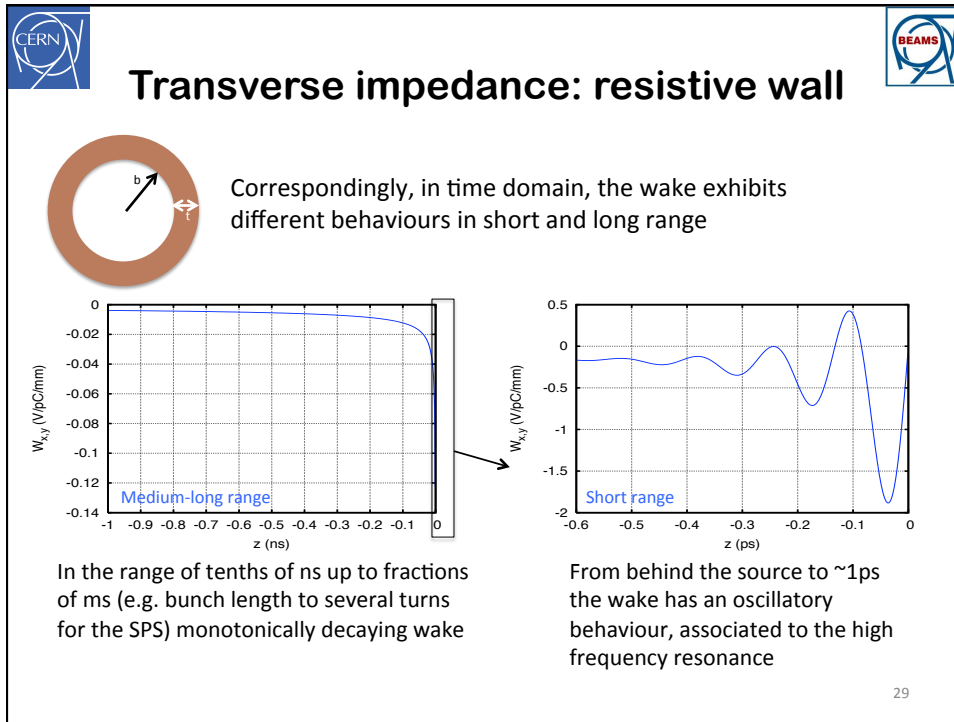
- An interesting example: a 1 m long Cu pipe with radius $b=2$ cm and thickness $t = 4$ mm in vacuum



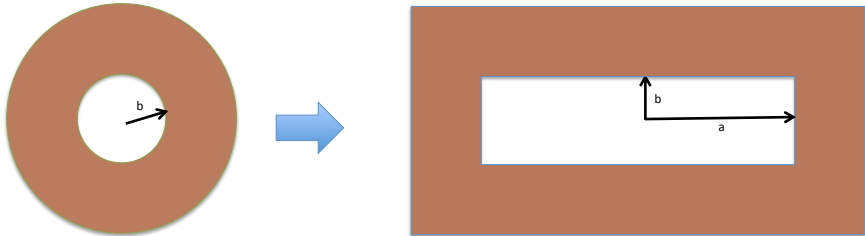
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**Transverse impedance: resistive wall
(infinitely thick wall)**

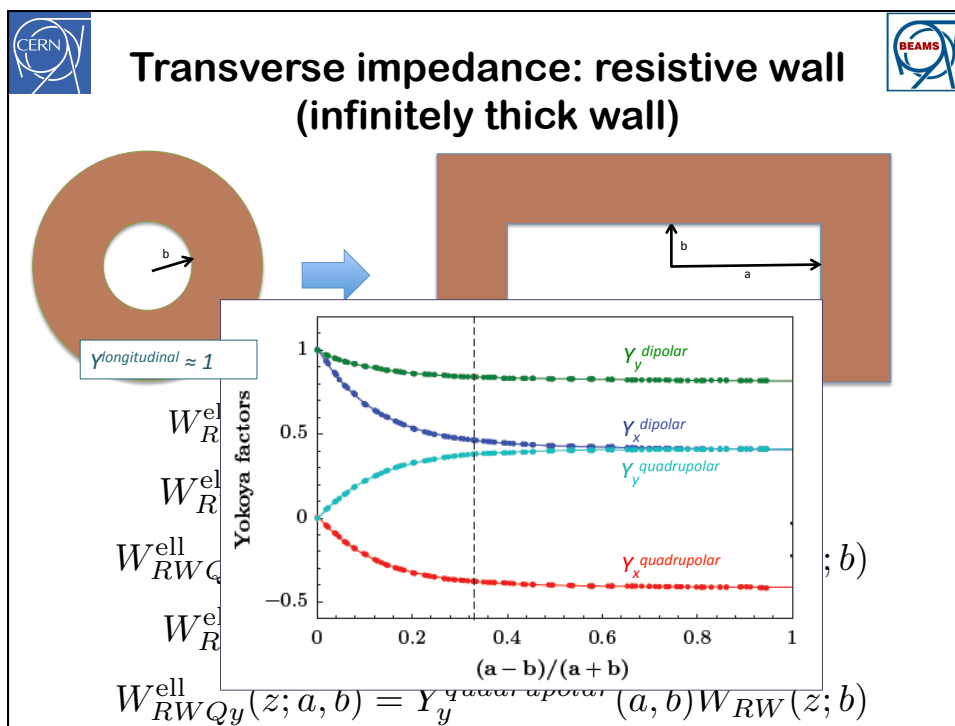



$$W_{RW||}^{\text{ell}}(z; a, b) = Y^{\text{longitudinal}}(a, b) W_{RW}(z; b)$$

$$W_{RWx}^{\text{ell}}(z; a, b) = Y_x^{\text{dipolar}}(a, b) W_{RW}(z; b)$$


$$W_{RWQx}^{\text{ell}}(z; a, b) = Y_x^{\text{quadrupolar}}(a, b) W_{RW}(z; b)$$

$$W_{RWy}^{\text{ell}}(z; a, b) = Y_y^{\text{dipolar}}(a, b) W_{RW}(z; b)$$

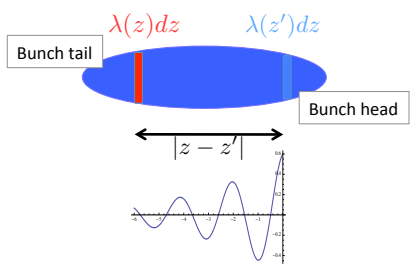
$$W_{RWQy}^{\text{ell}}(z; a, b) = Y_y^{\text{quadrupolar}}(a, b) W_{RW}(z; b)$$




Single particle equations of the transverse motion in presence of dipolar wake fields



- The single particle in the witness slice $\lambda(z)dz$ will feel the external focusing forces and that associated to the wake in s_0
- Space charge here neglected
- The wake contribution can extend to several turns




$$\frac{d^2x}{ds^2} + K_x(s)x = - \left(\frac{e^2}{m_0c^2} \right) \sum_{k=-\infty}^{\infty} \frac{N}{\gamma C} \int_{-\infty}^{\infty} \lambda(z' + kC) \langle x \rangle(s_0, z' + kC) W_x(s_0, z - z' - kC) dz'$$

$$\frac{d^2y}{ds^2} + K_y(s)y = - \left(\frac{e^2}{m_0c^2} \right) \sum_{k=-\infty}^{\infty} \frac{N}{\gamma C} \int_{-\infty}^{\infty} \lambda(z' + kC) \langle y \rangle(s_0, z' + kC) W_y(s_0, z - z' - kC) dz'$$


External Focusing

Wake fields

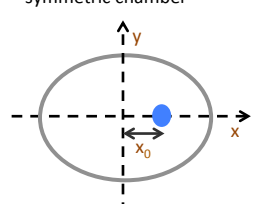
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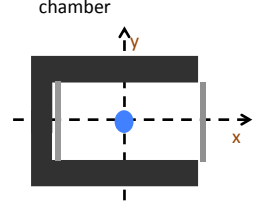
Beam deflection kick



Off-axis traversal of symmetric chamber



Traversal of asymmetric chamber



$$\Delta x'(z) = - \frac{e^2 x_0}{E_0} \int_{-\hat{z}}^{\hat{z}} \lambda(z') [W_x(z - z') + W_{Qx}(z - z')] dz'$$

⇓



$$\langle \Delta x' \rangle = - \frac{e^2 x_0}{2\pi N E_0} \int_0^\infty |\tilde{\lambda}(\omega)|^2 \text{Im}[Z_x(\omega) + Z_{Qx}(\omega)] d\omega$$

or

$$\langle \Delta x' \rangle = - \frac{e^2 x_0 \omega_0}{2\pi N E_0} \sum_{p=-\infty}^{\infty} |\tilde{\lambda}(p\omega_0)|^2 \text{Im}[Z_x(p\omega_0) + Z_{Qx}(p\omega_0)]$$

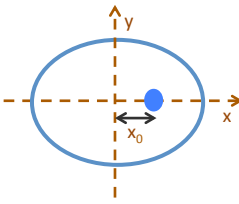
$$\langle \Delta x' \rangle = - \frac{e^2 \omega_0}{2\pi N E_0} \sum_{p=-\infty}^{\infty} |\tilde{\lambda}(p\omega_0)|^2 \text{Im}[Z_{Cx}(p\omega_0)]$$

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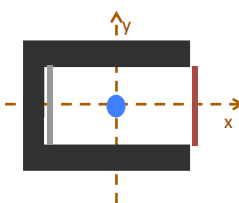



Beam deflection kick

Off-axis traversal of symmetric chamber



Traversal of asymmetric chamber





$$\Delta x'(z) = -\frac{e^2 x_0}{E_0} \int_{-\hat{z}}^{\hat{z}} \lambda(z') [W_x(z - z') + W_{Qx}(z - z')] dz'$$

The beam deflection kicks

- ⇒ Are the transverse equivalent of the energy loss in the longitudinal plane
- ⇒ Can be responsible for intensity dependent orbit variations
- ⇒ Cause z-dependent orbits and can determine tilted equilibrium bunch distributions for long bunches

$$\langle \Delta x' \rangle = -\frac{e^2 \omega_0}{2\pi N E_0} \sum_{p=-\infty}^{\infty} |\tilde{\lambda}(p\omega_0)|^2 \text{Im}[Z_{Cx}(p\omega_0)]$$

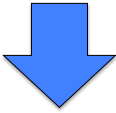
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The Rigid Bunch Instability

– To illustrate the rigid bunch instability we will use some simplifications:

- ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear betatron oscillations in absence of the wake forces)
- ⇒ Longitudinal motion is neglected
- ⇒ Smooth approximation → constant focusing + distributed wake



– In a similar fashion as was done for the Robinson instability in the longitudinal plane we want to

- ⇒ Calculate the betatron tune shift due to the wake
- ⇒ Derive possible conditions for the excitation of an unstable motion

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The Rigid Bunch Instability

- To illustrate the rigid bunch instability we will use some simplifications:
 - ⇒ The bunch is point-like and feels an external linear force (i.e. it would execute linear betatron oscillations in absence of the wake forces)
 - ⇒ Longitudinal motion is neglected
 - ⇒ Smooth approximation → constant focusing + distributed wake

$$\frac{d^2 y}{ds^2} + \left(\frac{\omega_\beta}{c}\right)^2 y = - \left(\frac{e^2}{m_0 c^2}\right) \frac{N}{\gamma C} \sum_{k=-\infty}^{\infty} y(s - kC) W_y(kC)$$

$$y \propto \exp\left(\frac{-i\Omega s}{c}\right) \Rightarrow \Omega^2 - \omega_\beta^2 = \frac{Ne^2}{m_0 \gamma C} \sum_{k=-\infty}^{\infty} \exp(ik\Omega T_0) W_y(kC)$$

$$= -i \frac{Ne^2}{m_0 \gamma C T_0} \sum_{p=-\infty}^{\infty} Z_y(p\omega_0 + \Omega)$$

Comes from the definition of Z_y

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The Rigid Bunch Instability

- ⇒ We assume a small deviation from the betatron tune
 - ⇒ $\text{Re}(\Omega - \omega_\beta) \rightarrow$ Betatron tune shift
 - ⇒ $\text{Im}(\Omega - \omega_\beta) \rightarrow$ Growth/damping rate, if it is positive there is an instability!

$$\Omega^2 - \omega_\beta^2 \approx 2\omega_\beta \cdot (\Omega - \omega_\beta)$$

$$\frac{1}{4\pi} \left[\beta_y \frac{e I_b \text{Im}(Z_y^{\text{eff}})}{E} \right] = \frac{1}{4\pi} \oint \beta_y(s) \Delta k(s) ds$$

$$\frac{\text{Re}(\Omega - \omega_\beta)}{\omega_0} = \Delta\nu_y \approx \frac{Ne^2 \beta_y}{4\pi m_0 \gamma c C} \sum_{p=-\infty}^{\infty} \text{Im}[Z_y(p\omega_0 + \omega_\beta)]$$

$$\text{Im}(\Omega - \omega_\beta) = \tau_y^{-1} \approx -\frac{Ne^2 \beta_y}{2m_0 \gamma C^2} \sum_{p=-\infty}^{\infty} \text{Re}[Z_y(p\omega_0 + \omega_\beta)]$$

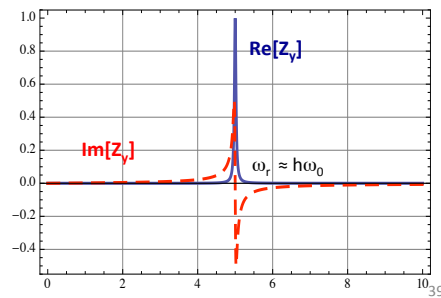
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The Rigid Bunch Instability

$$\text{Im}(\Omega - \omega_\beta) = \tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2} \sum_{p=-\infty}^{\infty} \text{Re}[Z_y(p\omega_0 + \omega_\beta)]$$

⇒ We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0$ (e.g. RF cavity fundamental mode or HOM)



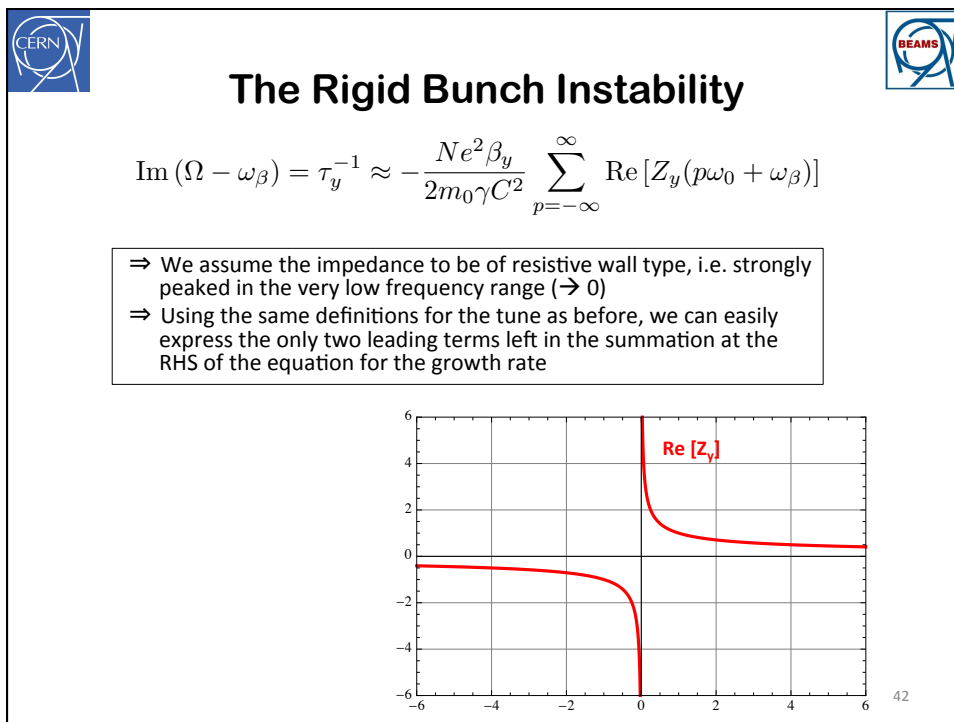
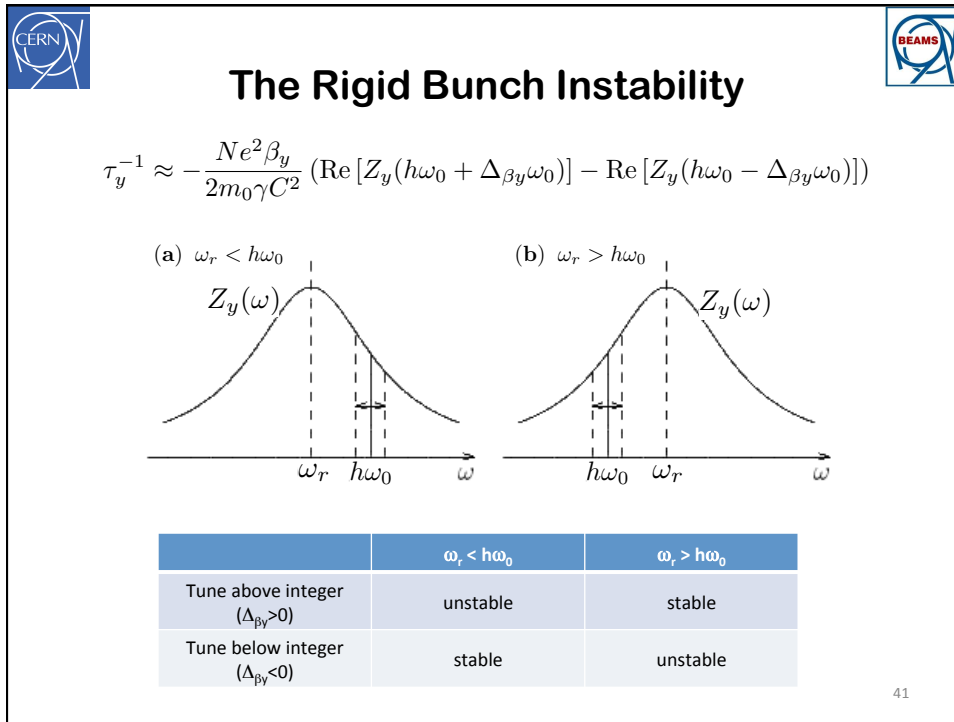
The Rigid Bunch Instability

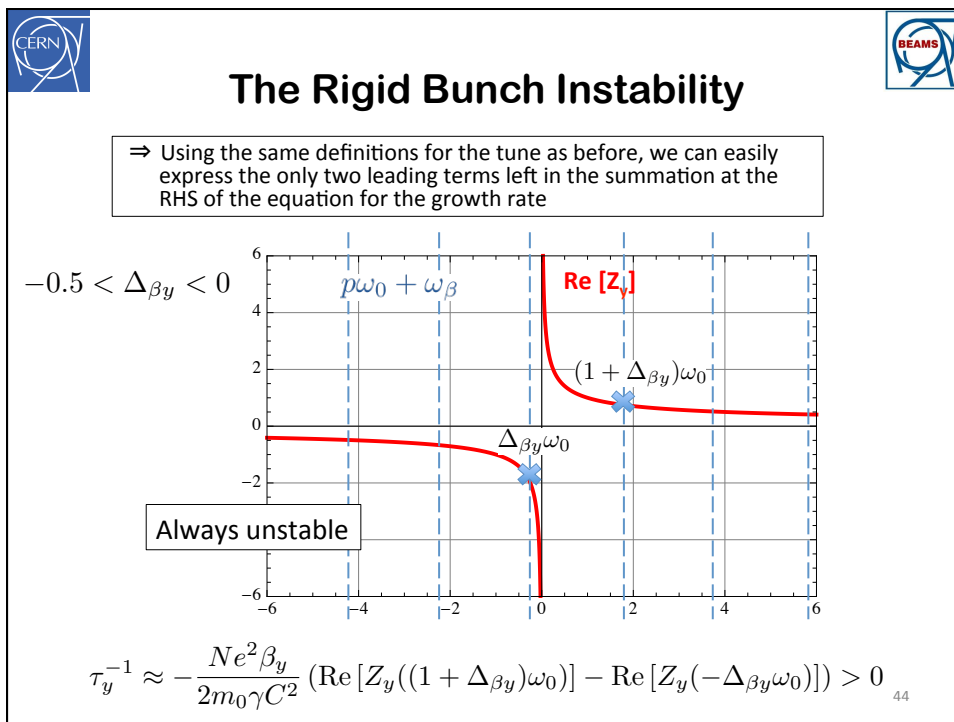
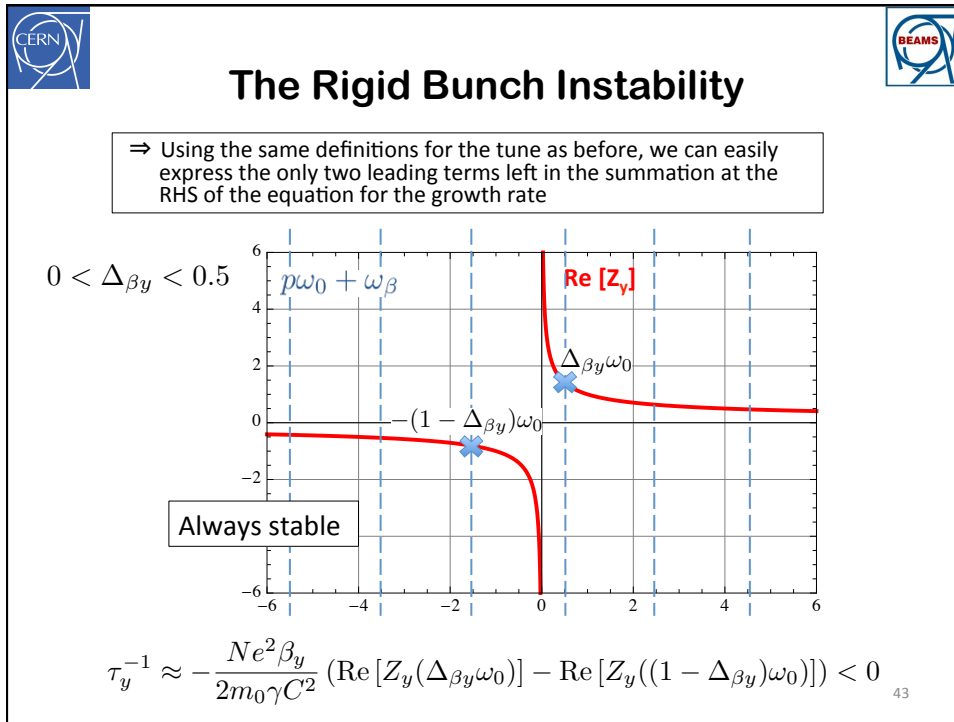
$$\text{Im}(\Omega - \omega_\beta) = \tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2} \sum_{p=-\infty}^{\infty} \text{Re}[Z_y(p\omega_0 + \omega_\beta)]$$

⇒ We assume the impedance to be peaked at a frequency ω_r close to $h\omega_0$ (e.g. RF cavity fundamental mode or HOM)

⇒ Defining the tune $\nu_y = n_y + \Delta_{\beta y}$ with $-0.5 < \Delta_{\beta y} < 0.5$, we can easily express the only two leading terms left in the summation at the RHS of the equation for the growth rate

$$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2} (\text{Re}[Z_y(h\omega_0 + \Delta_{\beta y}\omega_0)] - \text{Re}[Z_y(h\omega_0 - \Delta_{\beta y}\omega_0)])$$





$-0.5 < \Delta\beta_y < 0$

$p\omega_0 + \omega_\beta$



$\text{Re}[Z_y]$

This is the reason why most of the running machines are usually operated with a fractional part of the tunes below 0.5!

In practice, tunes above the half integer can be used, if the resistive wall instability is Landau damped (refer to W. Herr's lectures!) or efficiently suppressed with a feedback system

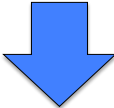
Always unstable

$\tau_y^{-1} \approx -\frac{Ne^2\beta_y}{2m_0\gamma C^2} (\text{Re}[Z_y((1 + \Delta\beta_y)\omega_0)] - \text{Re}[Z_y(-\Delta\beta_y\omega_0)]) > 0$





The Strong Head Tail Instability (aka Transverse Mode Coupling Instability)

- To illustrate TMCI we will need to make use of some simplifications:
 - ⇒ The bunch is represented through two particles carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
 - ⇒ They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing).
 - ⇒ Zero chromaticity ($Q'_{x,y}=0$)
 - ⇒ Constant transverse wake left behind by the leading particle
 - ⇒ Smooth approximation → constant focusing + distributed wake



- We will
 - ⇒ Calculate a stability condition (threshold) for the transverse motion
 - ⇒ Have a look at the excited oscillation modes of the centroid

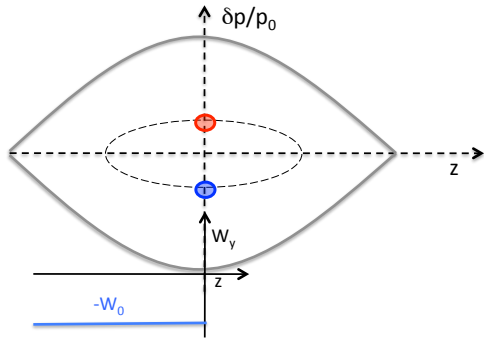
46

The Strong Head Tail Instability (aka Transverse Mode Coupling Instability)

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● Particle 1 (+Ne/2)

● Particle 2 (+Ne/2)

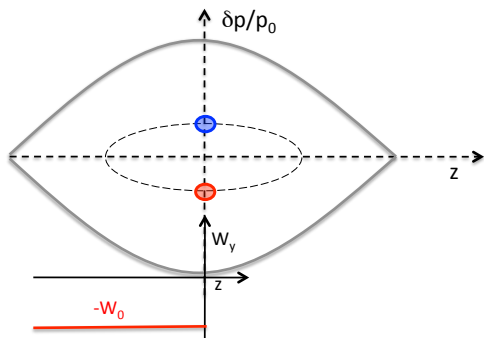
47

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● Particle 1 (+Ne/2)

● Particle 2 (+Ne/2)

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The Strong Head Tail Instability (aka Transverse Mode Coupling Instability)




⇒ During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1

$$\left\{ \begin{array}{l} \frac{d^2 y_1}{ds^2} + \left(\frac{\omega_\beta}{c} \right)^2 y_1 = 0 \\ \frac{d^2 y_2}{ds^2} + \left(\frac{\omega_\beta}{c} \right)^2 y_2 = \left(\frac{e^2}{m_0 c^2} \right) \frac{N W_0}{2 \gamma C} y_1(s) \end{array} \right. \quad 0 < s < \frac{\pi c}{\omega_s}$$

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The Strong Head Tail Instability (aka Transverse Mode Coupling Instability)




⇒ During the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1


⇒ During the second half of the synchrotron period, the situation is reversed

$$\left\{ \begin{array}{l} \frac{d^2 y_1}{ds^2} + \left(\frac{\omega_\beta}{c} \right)^2 y_1 = \left(\frac{e^2}{m_0 c^2} \right) \frac{N W_0}{2 \gamma C} y_2(s) \\ \frac{d^2 y_2}{ds^2} + \left(\frac{\omega_\beta}{c} \right)^2 y_2 = 0 \end{array} \right. \quad \frac{\pi c}{\omega_s} < s < \frac{2\pi c}{\omega_s}$$

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The Strong Head Tail Instability (aka Transverse Mode Coupling Instability)



⇒ We solve with respect to the complex variables defined below during the first half of synchrotron period

⇒ $y_1(s)$ is a free betatron oscillation

⇒ $y_2(s)$ is the sum of a free betatron oscillation plus a driven oscillation with $y_1(s)$ being its driving term


$$\tilde{y}_{1,2}(s) = y_{1,2}(s) + i \frac{c}{\omega_\beta} y'_{1,2}(s)$$

$$\tilde{y}_1(s) = \tilde{y}_1(0) \exp\left(\frac{-i\omega_\beta s}{c}\right)$$


$$\tilde{y}_2(s) = \underbrace{\tilde{y}_2(0) \exp\left(\frac{-i\omega_\beta s}{c}\right)}_{\text{Free oscillation term}} + \underbrace{i \frac{Ne^2 W_0}{4m_0 \gamma c \omega_\beta} \left[\frac{c}{\omega_\beta} \tilde{y}_1^*(0) \sin\left(\frac{\omega_\beta s}{c}\right) + \tilde{y}_1(0) s \exp\left(\frac{-i\omega_\beta s}{c}\right) \right]}_{\text{Driven oscillation term}}$$

Free oscillation term
Driven oscillation term

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The Strong Head Tail Instability Transfer map



$$\tilde{y}_1\left(\frac{\pi c}{\omega_s}\right) = \tilde{y}_1(0) \exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right)$$

$$\tilde{y}_2\left(\frac{\pi c}{\omega_s}\right) = \tilde{y}_2(0) \exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right) + i \frac{Ne^2 W_0}{4m_0 \gamma c \omega_\beta} \left[\frac{c}{\omega_\beta} \tilde{y}_1^*(0) \sin\left(\frac{\pi\omega_\beta}{\omega_s}\right) + \tilde{y}_1(0) \left(\frac{\pi c}{\omega_s}\right) \exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right) \right]$$


⇒ Second term in RHS equation for $y_2(s)$ negligible if $\omega_s \ll \omega_\beta$

⇒ We can now transform these equations into linear mapping across half synchrotron period


$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=\pi c/\omega_s} = \exp\left(-\frac{i\pi\omega_\beta}{\omega_s}\right) \cdot \begin{pmatrix} 1 & 0 \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi Ne^2 W_0}{4m_0 \gamma c \omega_\beta \omega_s}$$

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The Strong Head Tail Instability Transfer map



⇒ In the second half of synchrotron period, particles 1 and 2 exchange their roles

⇒ We can therefore find the transfer matrix over the full synchrotron period for both particles


⇒ We can analyze the eigenvalues of the two particle system

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s}$$


$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=2\pi c/\omega_s} = \exp\left(-\frac{i 2\pi \omega_\beta}{\omega_s}\right) \cdot \begin{pmatrix} 1 & i\Upsilon \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=2\pi c/\omega_s} = \exp\left(-\frac{i 2\pi \omega_\beta}{\omega_s}\right) \cdot \begin{pmatrix} 1 - \Upsilon^2 & i\Upsilon \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

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The Strong Head Tail Instability Stability condition



⇒ Since the product of the eigenvalues is 1, the only condition for stability is that they both be purely imaginary exponentials

⇒ From the second equation for the eigenvalues, it is clear that this is true only when $\sin(\phi/2) < 1$


⇒ This translates into a condition on the beam/wake parameters

$$\lambda_1 \cdot \lambda_2 = 1 \quad \Rightarrow \quad \lambda_{1,2} = \exp(\pm i\phi)$$

$$\lambda_1 + \lambda_2 = 2 - \Upsilon^2 \quad \Rightarrow \quad \sin\left(\frac{\phi}{2}\right) = \frac{\Upsilon}{2}$$


$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s} \leq 2$$

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The Strong Head Tail Instability

Stability condition



$$N \leq N_{\text{threshold}} = \frac{8}{\pi e^2} \frac{p_0 \omega_s}{\beta_y} \left(\frac{C}{W_0} \right)$$


⇒ Proportional to p_0 → bunches with higher energy tend to be more stable

⇒ Proportional to ω_s → the quicker is the longitudinal motion within the bunch, the more stable is the bunch

⇒ Inversely proportional to β_y → the effect of the impedance is enhanced if the kick is given at a location with large beta function


⇒ Inversely proportional to the wake per unit length along the ring, W_0/C → a large integrated wake (impedance) lowers the instability threshold

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The Strong Head Tail Instability

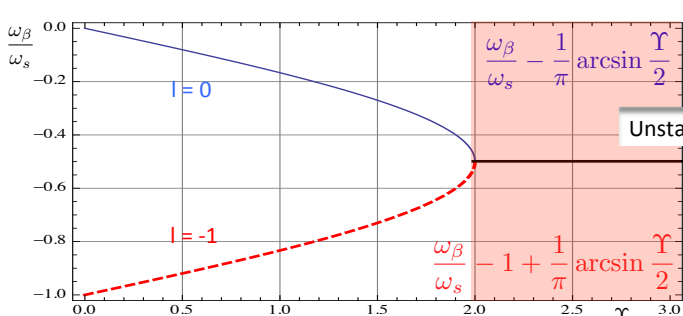
Mode frequencies

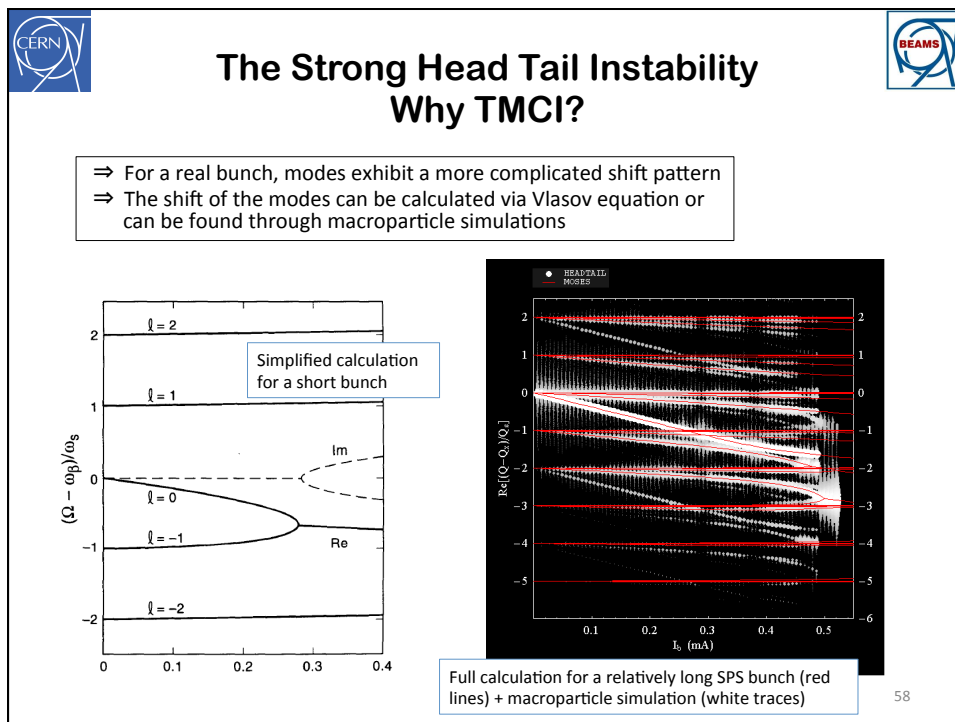
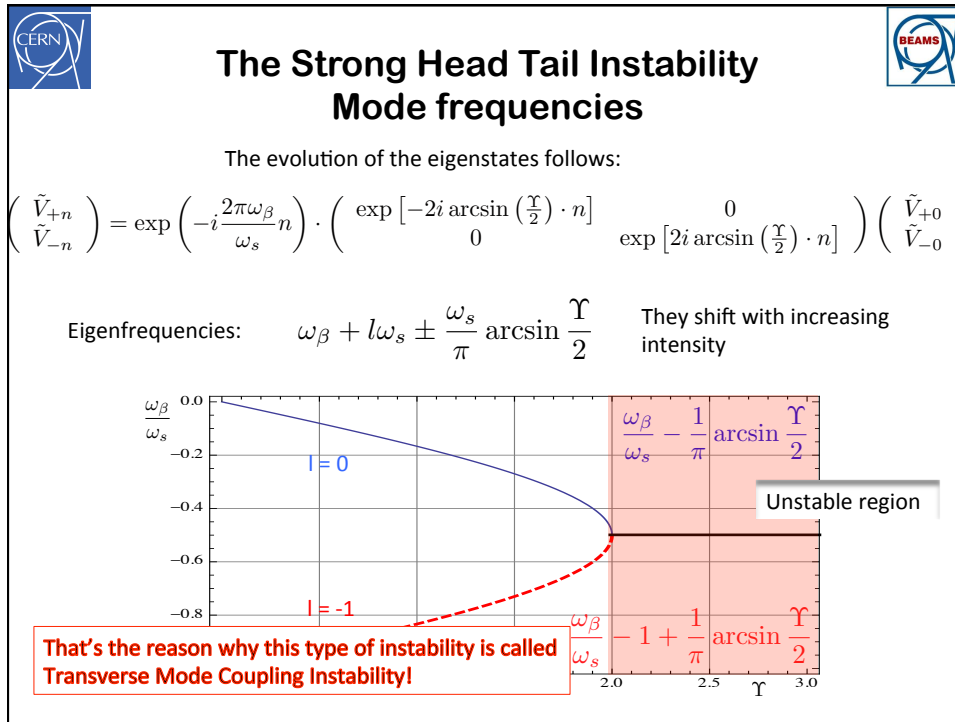


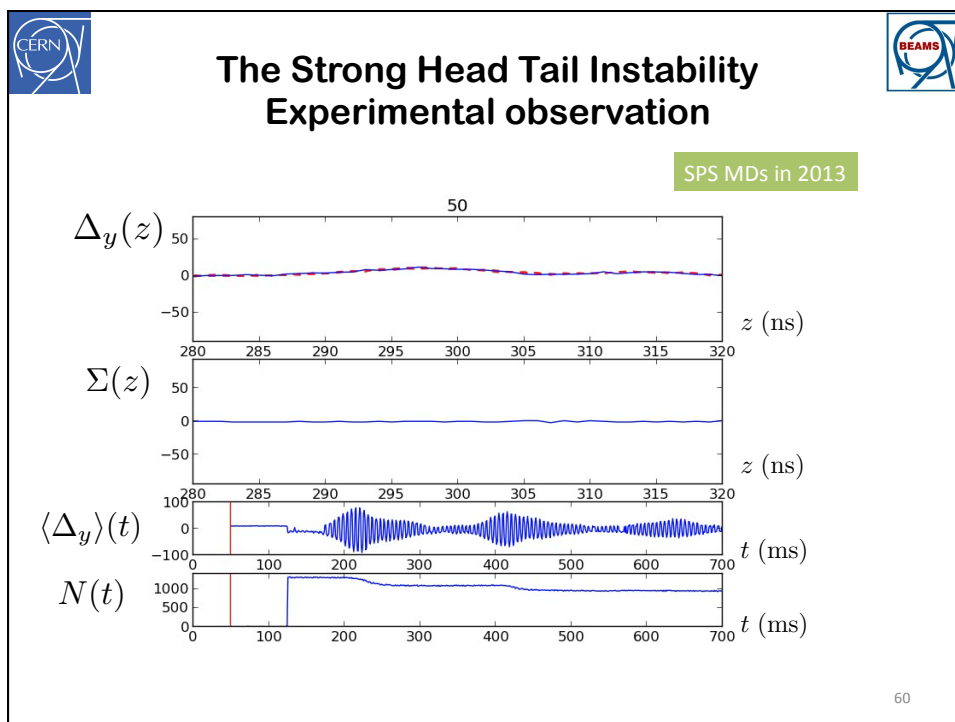
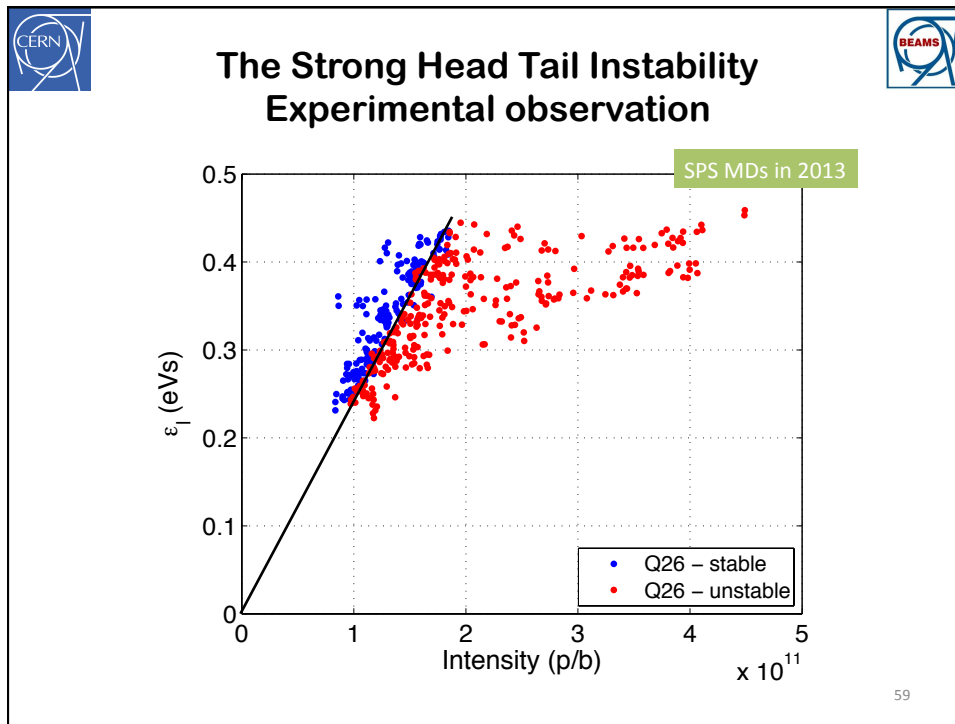
The evolution of the eigenstates follows:


$$\begin{pmatrix} \tilde{V}_{+n} \\ \tilde{V}_{-n} \end{pmatrix} = \exp \left(-i \frac{2\pi\omega_\beta}{\omega_s} n \right) \cdot \begin{pmatrix} \exp \left[-2i \arcsin \left(\frac{\gamma}{2} \right) \cdot n \right] & 0 \\ 0 & \exp \left[2i \arcsin \left(\frac{\gamma}{2} \right) \cdot n \right] \end{pmatrix} \begin{pmatrix} \tilde{V}_{+0} \\ \tilde{V}_{-0} \end{pmatrix}$$

Eigenfrequencies: $\omega_\beta + l\omega_s \pm \frac{\omega_s}{\pi} \arcsin \frac{\gamma}{2}$ They shift with increasing intensity











The Head Tail Instability




- To illustrate the head-tail instability we will need to make use of some simplifications:
 - ⇒ The bunch is represented through two particles carrying half the total bunch charge and placed in opposite phase in the longitudinal phase space
 - ⇒ They both feel external linear focusing in all three directions (i.e. linear betatron focusing + linear synchrotron focusing).
 - ⇒ **Chromaticity is different from zero ($Q'_{x,y} \neq 0$)**
 - ⇒ Constant transverse wake left behind by the leading particle
 - ⇒ Smooth approximation → constant focusing + distributed wake




- We can
 - ⇒ Show that this system is intrinsically unstable
 - ⇒ Calculate the growth time of the excited oscillation modes

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The Head Tail Instability Equations of motion




- ⇒ As for the TMCI, during the first half of the synchrotron motion, particle 1 is leading and executes free betatron oscillations, while particle 2 is trailing and feels the defocusing wake of particle 1
- ⇒ During the second half of the synchrotron period, the situation is reversed

$$\left\{ \begin{array}{l} \frac{d^2 y_1}{ds^2} + \left[\frac{\omega_\beta (1 + \xi_y \delta(s))}{c} \right]^2 y_1 = 0 \\ \frac{d^2 y_2}{ds^2} + \left[\frac{\omega_\beta (1 + \xi_y \delta(s))}{c} \right]^2 y_2 = \left(\frac{e^2}{m_0 c^2} \right) \frac{N W_0}{2 \gamma C} y_1(s) \end{array} \right. \quad 0 < s < \frac{\pi c}{\omega_s}$$




Difference! → now the frequency of free oscillation is modulated by the momentum spread, $\delta(s)$

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The Head Tail Instability Oscillation modes




⇒ Similarly to the solution for the Strong Head Tail Instability, we obtain the transport map

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=2\pi c/\omega_s} = \begin{pmatrix} i\Upsilon & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=\pi c/\omega_s} = \begin{pmatrix} 1 - \Upsilon^2 & i\Upsilon \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s} \left(1 + i \frac{4 \xi_y \omega_\beta \hat{z}}{\pi c \eta} \right) \quad \text{Complex number!}$$

Weak beam intensity:

$|\Upsilon| \ll 1$




$\lambda_{\pm} \approx \exp(\pm i\Upsilon)$


+ mode is “in-phase” mode → the two particles oscillate in phase (ω_p)

- mode is “out-phase” mode → the two particles oscillate in opposition of phase ($\omega_p \pm \omega_s$)

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The Head Tail Instability Growth/damping time



$$\tau^{-1} = \text{Im} \left(\pm \Upsilon \cdot \frac{\omega_s}{2\pi} \right) = \mp \frac{e^2}{2\pi} \cdot \frac{N \xi_y \hat{z}}{p_0 \eta} \left(\frac{W_0}{C} \right)$$

⇒ Inversely proportional to p_0 → bunches with higher energy tend to be less affected by impedances

⇒ Proportional to N → the more intense is the bunch, the more sensitive it is


⇒ Proportional to bunch length → this depends on the chosen shape of the wake

⇒ Proportional to ξ_y → higher chromaticity enhances the head-tail effect

⇒ Inversely proportional to η → faster synchrotron motion stabilizes (lowest rise times close to transition crossing!)


⇒ Proportional to the wake per unit length along the ring, W_0/C → a large integrated wake (impedance) gives a stronger effect

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The Head Tail Instability

Growth/damping time



$$\tau^{-1} = \text{Im} \left(\pm \Upsilon \cdot \frac{\omega_s}{2\pi} \right) = \mp \frac{e^2}{2\pi} \cdot \frac{N \xi_y \hat{z}}{p_0 \eta} \left(\frac{W_0}{C} \right)$$


Mode 0 (+)

	$\xi_y > 0$	$\xi_y < 0$
Above transition ($\eta > 0$)	damped	unstable
Below transition ($\eta < 0$)	unstable	damped


Mode 1 (−)

	$\xi_y > 0$	$\xi_y < 0$
Above transition ($\eta > 0$)	unstable	damped
Below transition ($\eta < 0$)	damped	unstable

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The Head Tail Instability



- The head-tail instability is unavoidable in the two-particle model
 - Either mode 0 or mode 1 is unstable
 - Growth/damping times are in all cases identical
- Fortunately, the situation is less dramatic in reality
 - The number of modes increases with the number of particles we consider in the model (and becomes infinite in the limit of a continuous bunch)
 - The instability conditions for mode 0 remain unchanged, but all the other modes become unstable with much longer rise times when mode 0 is stable

Mode 0

	$\xi_y > 0$	$\xi_y < 0$
Above transition ($\eta > 0$)	damped	unstable
Below transition ($\eta < 0$)	unstable	damped

All modes > 0

	$\xi_y > 0$	$\xi_y < 0$
Above transition ($\eta > 0$)	unstable	damped
Below transition ($\eta < 0$)	damped	unstable

$$\sum_{l=-\infty}^{\infty} \frac{1}{\tau_l} = 0$$



The Head Tail Instability



- The head-tail instability is unavoidable in the two-particle model
 - Either mode 0 or mode 1 is unstable
 - Growth/damping times are in all cases identical
- Fortunately, the situation is less dramatic in reality
 - The number of modes increases with the number of particles we consider in the model (and becomes infinite in the limit of a continuous bunch)
 - The instability conditions for mode 0 remain unchanged, but all the other modes become unstable with much longer rise times when mode 0 is stable
 - Therefore, the bunch can be in practice stabilized by using the settings that make mode 0 stable ($\xi < 0$ below transition and $\xi > 0$ above transition) and relying on feedback or Landau damping (refer to W. Herr's lectures) for the other modes
- To be able to study these effects we would need to resort to a more detailed description of the bunch
 - Vlasov equation (kinetic model)
 - Macroparticle simulations

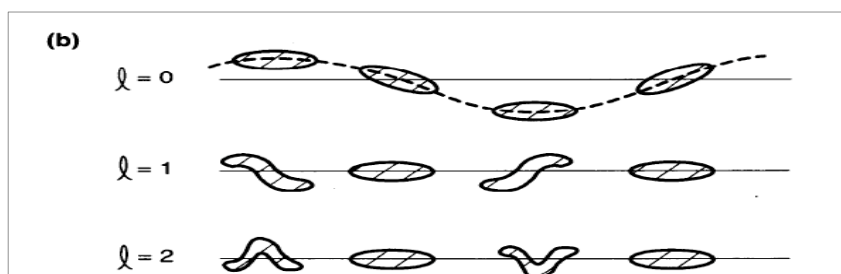
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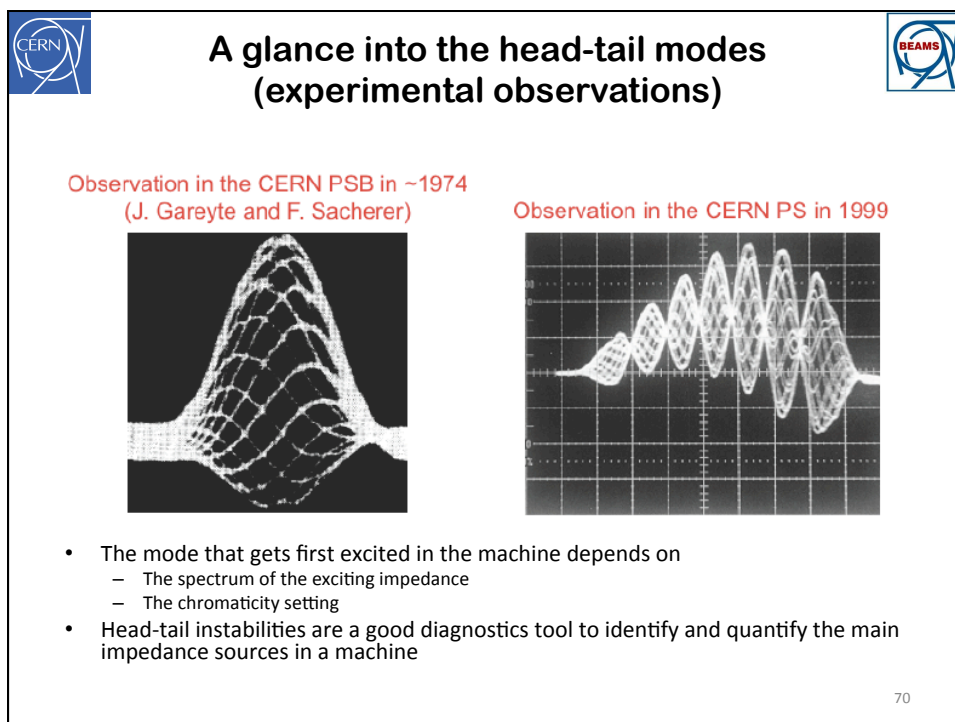
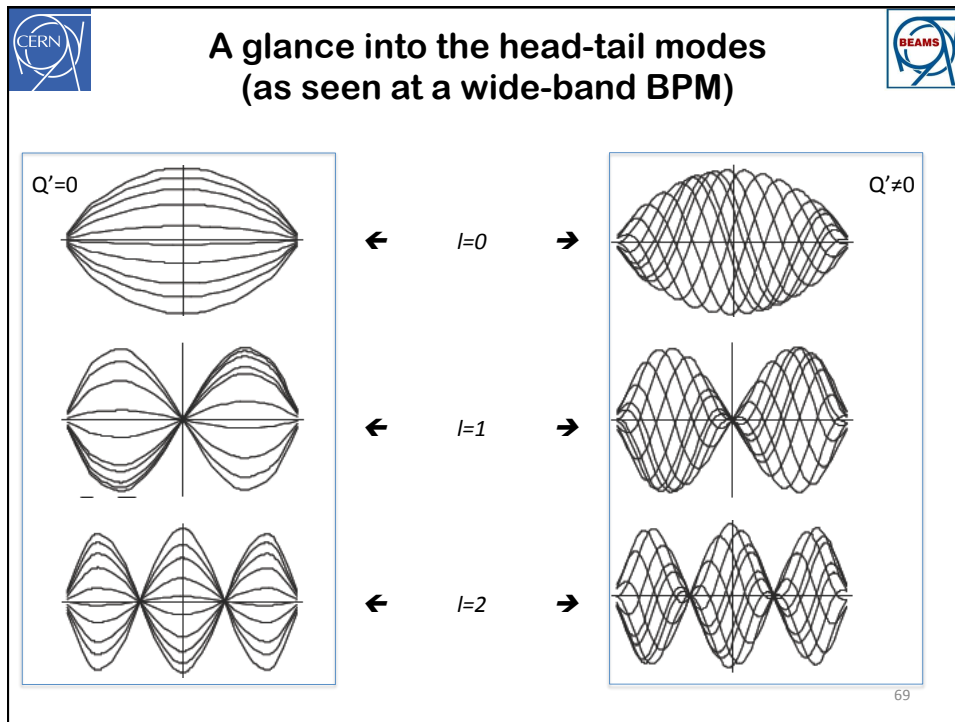
A glance into the head-tail modes

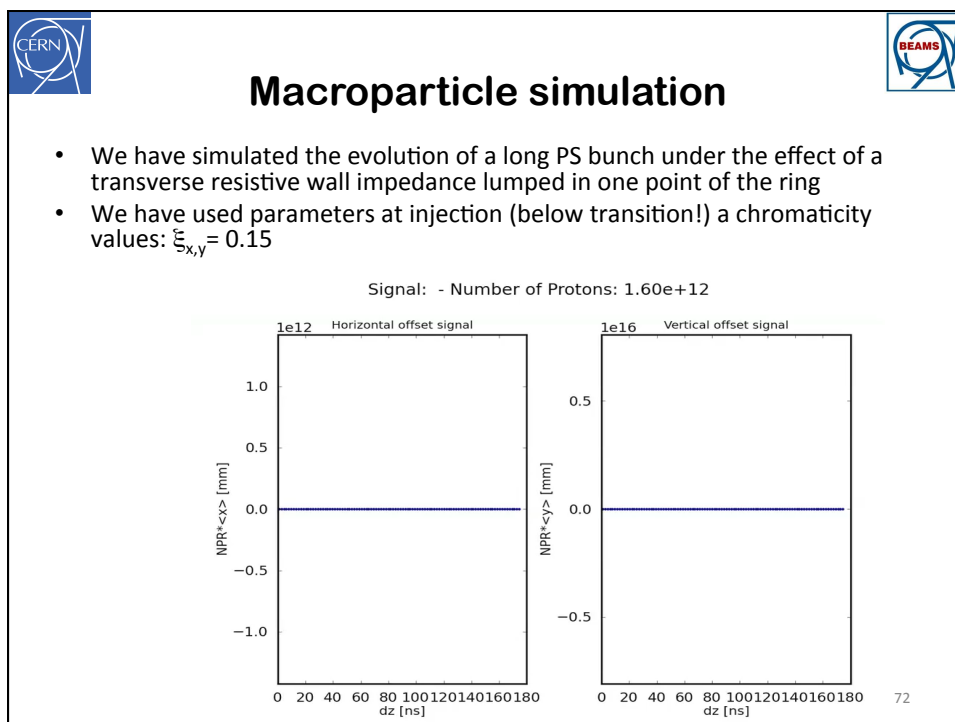
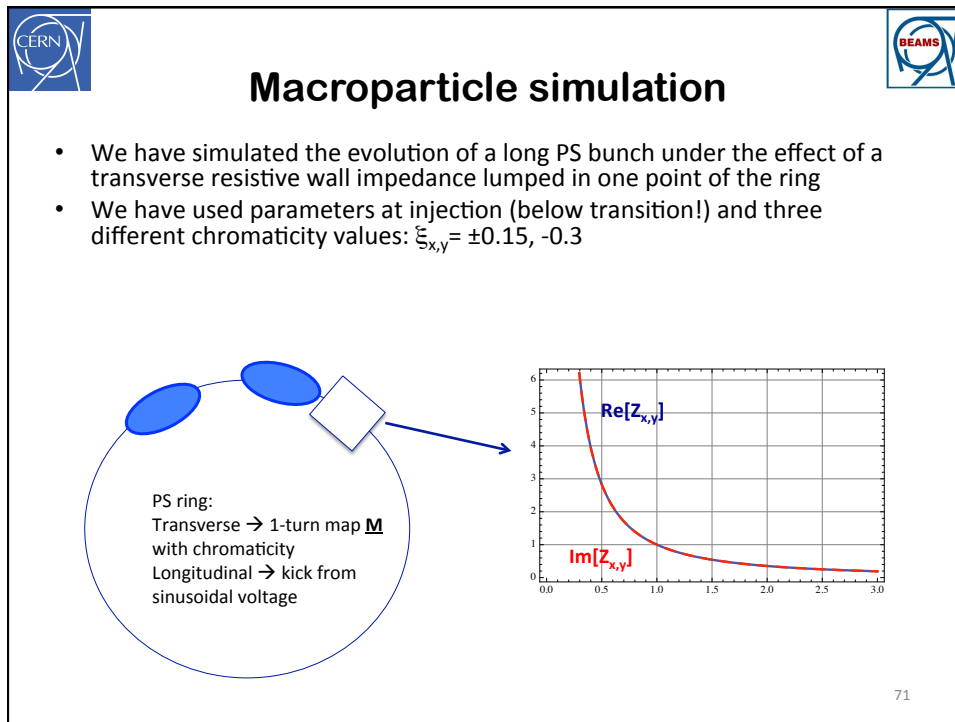


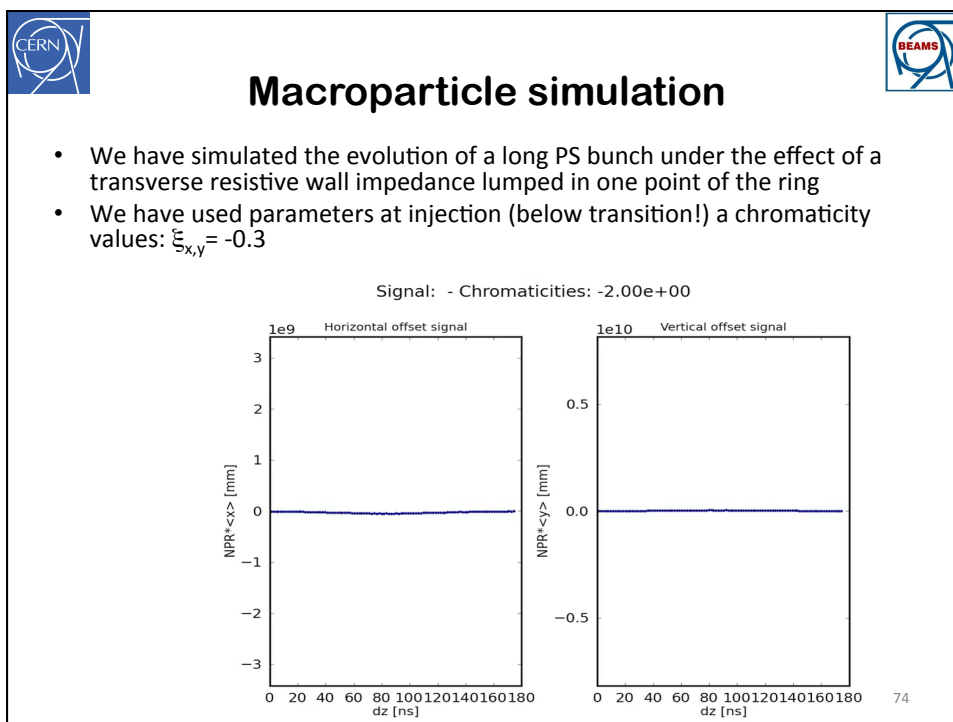
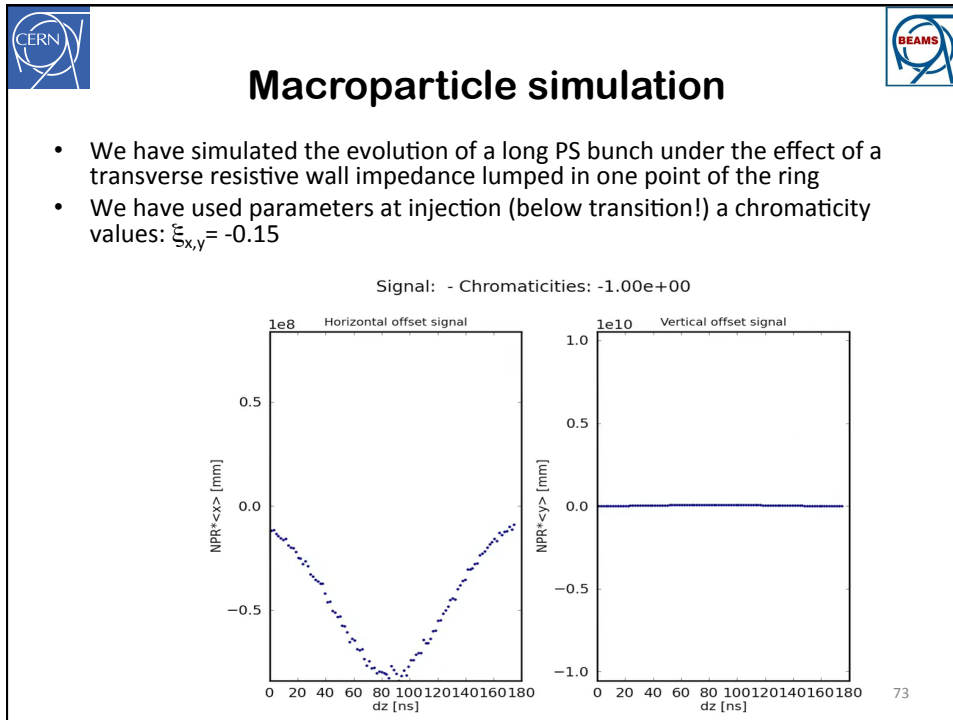
- Different transverse head-tail modes correspond to different parts of the bunch oscillating with relative phase differences. E.g.
 - Mode 0 is a rigid bunch mode
 - Mode 1 has head and tail oscillating in counter-phase
 - Mode 2 has head and tail oscillating in phase and the bunch center in opposition



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Conclusions

- A particle beam can be driven **unstable** by its interaction with its own induced EM fields
 - Longitudinal, transverse
 - Multi-bunch, single bunch
- Simplified models within the **wake/impedance framework** can be adopted to explain the mechanism of the instability
 - Stability criteria involving beam/machine parameters
 - Growth/damping times
- More sophisticated tools are necessary to describe in deeper detail the beam instabilities (kinetic theory, macroparticle simulations)

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Fortunately

- ⇒ In real life **beam stability** is eased by some mechanisms so far not included in our simplified linearized models
- **Spreads and nonlinearities** stabilize (Landau damping, refer to W. Herr's lecture)
 - Longitudinal: momentum spread, synchrotron frequency spread
 - Transverse: chromaticity, betatron tune spreads (e.g from machine nonlinearities, enhanced with purposely higher order fields)
 - **Active feedback systems** are routinely employed to control/suppress instabilities
 - Coherent motion is detected (pick-up) and damped (kicker) before it can degrade the beam
 - Sometimes bandwidth/power requirements can be very stringent
 - **Impedance localization and reduction techniques** are applied to old accelerators as well as for the design of new accelerators to extend their performance reach!

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Thank you for your attention

Many thanks to H. Bartosik, G. Iadarola, K. Li, N. Mounet,
B. Salvant, R. Tomás, C. Zannini
for material, discussions, suggestions, inspiration, review, help
& support and to A. Chao for his book!



The Head Tail Instability Equations of motion


- ⇒ Let's first write the solution without wake field assuming a linear synchrotron motion and particles in opposite phase ($z_1 = -z_2$)
- ⇒ It is already clear that head and tail of the bunch exhibit a phase difference given by the chromatic term

$$\tilde{y}_1(0) \exp \left[-i\omega_\beta \frac{s}{c} + i \frac{\xi_y \omega_\beta}{c\eta} \hat{z} \sin \left(\frac{\omega_s s}{c} \right) \right]$$

$$\tilde{y}_2(0) \exp \left[-i\omega_\beta \frac{s}{c} - i \frac{\xi_y \omega_\beta}{c\eta} \hat{z} \sin \left(\frac{\omega_s s}{c} \right) \right]$$


$$\frac{\xi_y \omega_\beta \hat{z}}{c\eta} \text{ is the head-tail phase shift}$$

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The Head Tail Instability

Equations of motion



⇒ The free oscillation is the correct solution for $y_1(s)$ in the first half synchrotron period

⇒ For $y_2(s)$ we assume a similar type of solution, allowing for a slowly time varying coefficient

⇒ Substituting into the equation of motion this yields


$$\tilde{y}_1(0) \exp \left[-i\omega_\beta \frac{s}{c} + i \frac{\xi_y \omega_\beta}{c\eta} \hat{z} \sin \left(\frac{\omega_s s}{c} \right) \right]$$

$$\tilde{y}_2(s) \exp \left[-i\omega_\beta \frac{s}{c} + i \frac{\xi_y \omega_\beta}{c\eta} \hat{z} \sin \left(\frac{\omega_s s}{c} \right) \right]$$

➔


$$\tilde{y}_2'(s) \approx \left(\frac{e^2}{m_0 c} \right) \frac{NW_0}{4\gamma C \omega_\beta} \tilde{y}_1(0) \exp \left[2i \frac{\xi_y \omega_\beta}{c\eta} \hat{z} \sin \left(\frac{\omega_s s}{c} \right) \right]$$

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The Head Tail Instability

Transfer map



⇒ For small head-tail shifts, we can expand the exponential in Taylor series and find an expression for $y_2(s)$

⇒ We can write a transfer map over the first half of synchrotron period in the same form as was done for the study of the TMCI

⇒ This time Υ is a complex parameter!

$$\tilde{y}_2(s) \approx \tilde{y}_2(0) + \left(\frac{e^2}{m_0 c} \right) \frac{NW_0}{4\gamma C \omega_\beta} \tilde{y}_1(0) \left[s + i \frac{2\xi_y \omega_\beta \hat{z}}{\eta \omega_s} \left(1 - \cos \frac{\omega_s s}{c} \right) \right]$$

$$\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=\pi c/\omega_s} = \begin{pmatrix} 1 & 0 \\ i\Upsilon & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}_{s=0}$$

$$\Upsilon = \frac{\pi N e^2 W_0}{4 m_0 \gamma C \omega_\beta \omega_s} \left(1 + i \frac{4 \xi_y \omega_\beta \hat{z}}{\pi c \eta} \right)$$

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