

Introduction to Transverse Beam Dynamics

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The Ideal World

I.) Magnetic Fields and Particle Trajectories



Luminosity Run of a typical storage ring:

LHC Storage Ring: *Protons* accelerated and *stored for 12 hours*
distance of particles travelling at about $v \approx c$
 $L = 10^{10}-10^{11}$ km

... several times Sun - Pluto and back ♪

intensity (10^{11})



- guide the particles on a well defined orbit („design orbit“)
- focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.

Transverse Beam Dynamics:

0.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine“
 → need transverse deflecting force

Lorentz force $F = q (\vec{E} + \vec{v} \times \vec{B})$

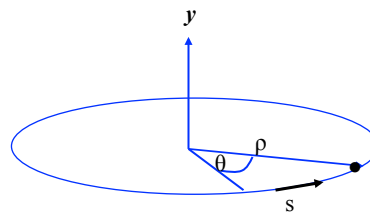
typical velocity in high energy machines: $v \approx c \approx 3 \cdot 10^8 \frac{m}{s}$

old greek dictum of wisdom:

if you are clever, you use magnetic fields in an accelerator wherever it is possible.

But remember: magn. fields act allways perpendicular to the velocity of the particle
 → only bending forces, → no „beam acceleration“

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force

$$F_L = e v B$$

centrifugal force

$$F_{centr} = \frac{\gamma m_0 v^2}{\rho}$$

$$\frac{\gamma m_0 v}{\rho} = e v B$$

$$\frac{p}{e} = B \rho$$

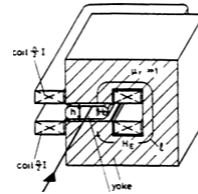
$B \rho = \text{"beam rigidity"}$

1.) The Magnetic Guide Field

Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 n I}{h}$$

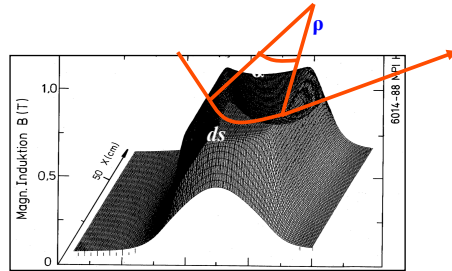


convenient units:

$$B = [T] = \left[\frac{Vs}{m^2} \right] \quad p = \left[\frac{GeV}{c} \right]$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B \rho \quad \longrightarrow \quad \frac{1}{\rho} = \frac{e B}{p}$$



field map of a storage ring dipole magnet

Example LHC:

$$B = 8.3 T$$

$$p = 7000 \frac{GeV}{c}$$

The Magnetic Guide Field



$$\frac{1}{\rho} = e \frac{8.3 \frac{Vs}{m^2}}{7000 * 10^9 \frac{eV}{c}} = \frac{8.3 s \cdot 3 * 10^8 \frac{m}{s}}{7000 * 10^9 m^2}$$

$$\frac{1}{\rho} = 0.3 \frac{8.3}{7000} \frac{1}{m}$$

$$\rho = 2.53 km \quad \longrightarrow \quad 2\pi\rho = 17.6 km$$

≈ 66%

$$B \approx 1 \dots 8 T$$

rule of thumb:

$$\frac{1}{\rho} \approx 0.3 \frac{B[T]}{p[GeV/c]}$$

„normalised bending strength“

2.) Quadrupole Magnets:

required: **focusing forces** to keep trajectories in vicinity of the ideal orbit

linear increasing Lorentz force

linear increasing magnetic field

$$B_y = g x \quad , \quad B_x = g y$$

normalised quadrupole field:

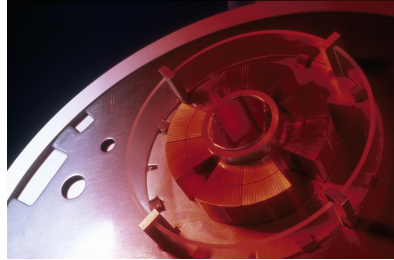
gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 n I}{r^2}$$

$$k = \frac{g}{p/e}$$

simple rule:

$$k = 0.3 \frac{g(T/m)}{p(GeV/c)}$$



LHC main quadrupole magnet

$$g \approx 25 \dots 220 \text{ T/m}$$

what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} = 0$$

$$\Rightarrow \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}$$

3.) The equation of motion:

Linear approximation:

* ideal particle \rightarrow design orbit

* any other particle \rightarrow coordinates x, y **small quantities**
 $x, y \ll \rho$

\rightarrow magnetic guide field: only linear terms in x & y of B
have to be taken into account

Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx} x + \frac{1}{2!} \frac{d^2 B_y}{dx^2} x^2 + \frac{1}{3!} \frac{d^3 B_y}{dx^3} x^3 + \dots$$

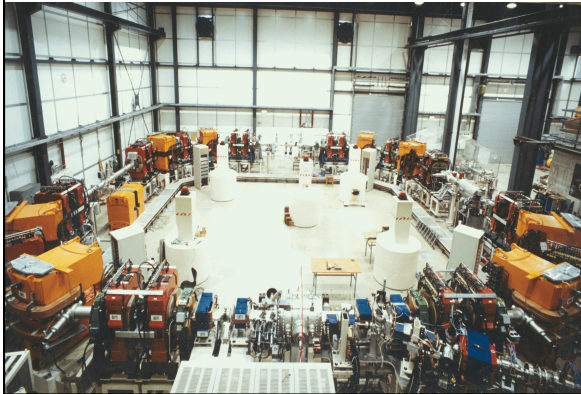
normalise to momentum
 $p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0 \rho} + \frac{g^* x}{p/e} + \frac{1}{2!} \frac{eg'}{p/e} + \frac{1}{3!} \frac{eg''}{p/e} + \dots$$

The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + kx + \frac{1}{2!} \cancel{m} x^2 + \frac{1}{3!} \cancel{n} x^3 + \dots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* man sieht nur
dipole und quads → linear

Equation of Motion:

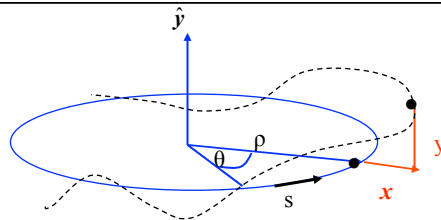
Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2 \rho}{dt^2} - \rho \left(\frac{d\theta}{dt} \right)^2$$

general trajectory: $\rho \rightarrow \rho + x$

$$F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y v$$



Ideal orbit: $\rho = \text{const}, \quad \frac{d\rho}{dt} = 0$

$$\text{Force: } F = m \rho \left(\frac{d\theta}{dt} \right)^2 = m \rho \omega^2$$

$$F = mv^2 / \rho$$

develop for small x :

$$x \ll \rho$$

guide field in linear approx.

$$B_z = B_0 + x \frac{\partial B_z}{\partial x}$$

independent variable: $t \rightarrow s$

$$\frac{dx}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt}$$

$$x' = \frac{dx}{ds}$$

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} \left(1 - \frac{x}{\rho}\right) = eB_z v$$

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} \left(1 - \frac{x}{\rho}\right) = ev \left\{ B_0 + x \frac{\partial B_z}{\partial x} \right\}$$

$$x'' - \frac{1}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{eB_0}{mv} + \frac{exg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = -\frac{1}{\rho} + kx$$

$$x'' + x \left(\frac{1}{\rho^2} - k \right) = 0$$

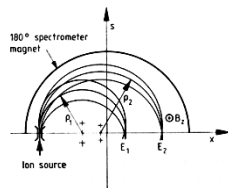
Remarks:

* *The Weak Focusing Term*

$$x'' + \left(\frac{1}{\rho^2} - k \right) x = 0$$

... there seems to be a focusing even without a quadrupole gradient ... **but it is WEAK !**

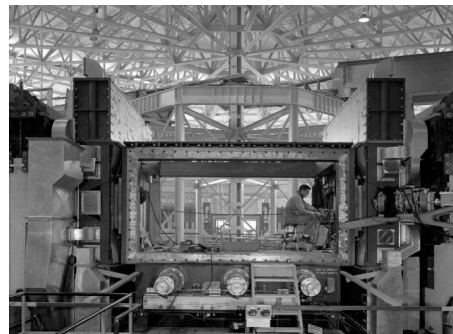
„weak focusing of dipole magnets“



Mass spectrometer: particles are separated according to their energy and focused due to the $1/\rho$ effect of the dipole

Don Edwards: ... This circumstance is illustrated in Fig. 4, in which an engineer is sitting at a desk within the vacuum chamber. The problem was a result of the weak focusing provided by the magnet systems.

The **higher the energy**, the larger ρ and the **weaker the dipole focusing**

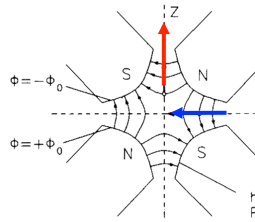


Bevatron, Berkeley

*** *vertical plane*

Equation for the **vertical motion**:

$$z'' + k \cdot z = 0$$



*** *keep it linear*

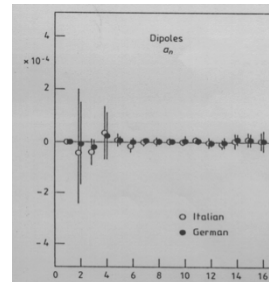
Taylor Expansion of the B field:

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*divide by the main field
to get the relative error contribution*

→ *definition of multipole coefficients.*

*Multipole contributions to the
HERA s.c. dipole field*



4.) *Solution of Trajectory Equations*

$$\left. \begin{array}{l} \text{Define ... hor. plane: } K = 1/\rho^2 - k \\ \text{... vert. Plane: } K = k \end{array} \right\} x'' + K x = 0$$

Differential Equation of harmonic oscillator ... with spring constant K

Ansatz: $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s)$$

$$x''(s) = -a_1 \omega^2 \cos(\omega s) - a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s)$$

determine a_1, a_2 by boundary conditions:

$$s = 0 \quad \longrightarrow \quad \begin{cases} x(0) = x_0 & , \quad a_1 = x_0 \\ x'(0) = x'_0 & , \quad a_2 = \frac{x'_0}{\sqrt{K}} \end{cases}$$

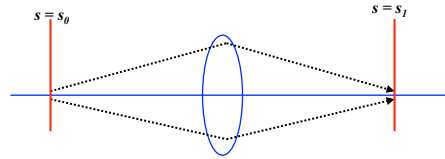
Hor. Focusing Quadrupole $K > 0$:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x'_0 \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x'_0 \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

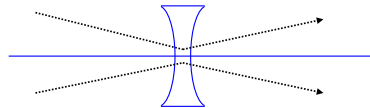
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s) \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole: $K < 0$

$$M_{defoc} = \begin{pmatrix} \cosh \sqrt{|K|}l & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|}l \\ \sqrt{|K|} \sinh \sqrt{|K|}l & \cosh \sqrt{|K|}l \end{pmatrix}$$



drift space: $K = 0$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



! with the assumptions made, the motion in the horizontal and vertical planes are independent „... the particle motion in x & z is uncoupled“

**!! for all magnet matrices the condition $\det(M) = 1$ is fulfilled
which means we are dealing with a conservative system**

Thin Lens Approximation:

$$\text{matrix of a quadrupole lens} \quad M = \begin{pmatrix} \cos \sqrt{|k|}l & \frac{1}{\sqrt{|k|}} \sin \sqrt{|k|}l \\ -\sqrt{|k|} \sin \sqrt{|k|}l & \cos \sqrt{|k|}l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{kl_q} \gg l_q \quad \dots \text{focal length of the lens is much bigger than the length of the magnet}$$

limes: $l_q \rightarrow 0$ while keeping $k l_q = \text{const}$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$M_z = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

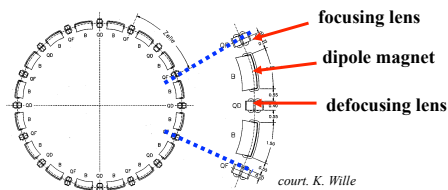
... useful for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

Transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{\text{total}} = M_{QF} * M_D * M_{QD} * M_{\text{Bend}} * M_D * \dots$$

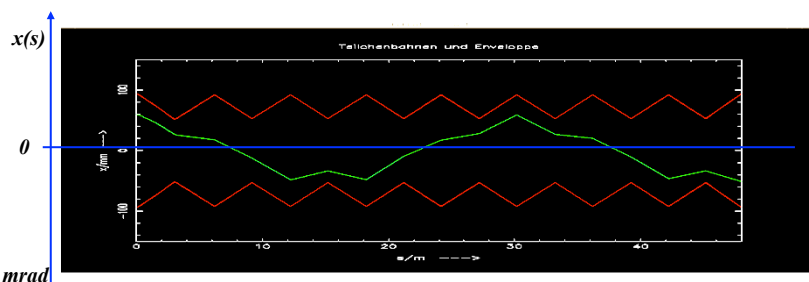
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



„C“ and „S“ = sin- and cos- like trajectories of the lattice structure, in other words the two independent solutions of the homogeneous equation of motion

typical values
in a strong
foc. machine:

$x \approx \text{mm}, x' \leq \text{mrad}$



5.) Orbit & Tune:

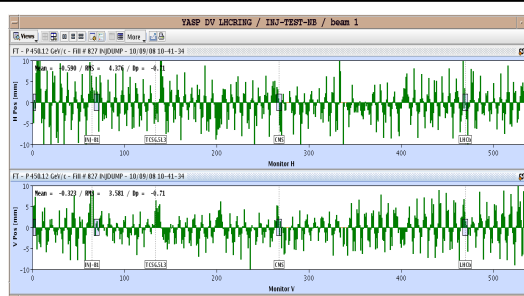
Tune: number of oscillations per turn

64.31

59.32

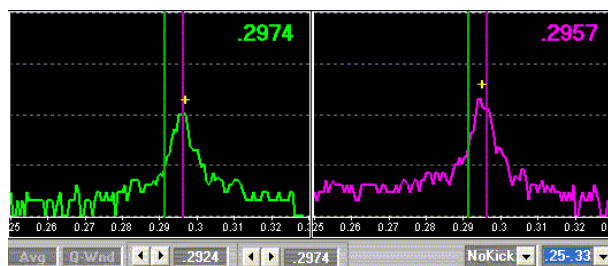
Relevant for beam stability:

non integer part



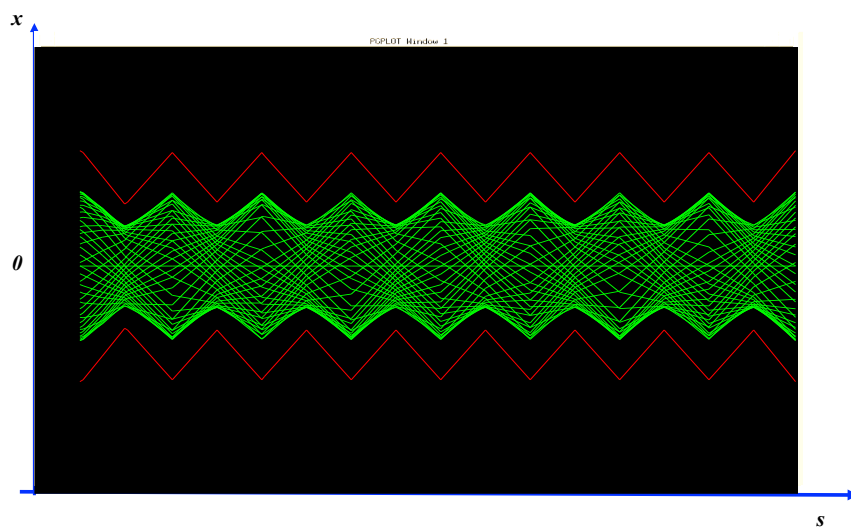
LHC revolution frequency: 11.3 kHz

$$0.31 * 11.3 \text{ kHz} = 3.5 \text{ kHz}$$



Question: what will happen, if the particle performs a second turn ?

... or a third one or ... 10^{10} turns



Astronomer Hill:

*differential equation for motions with periodic focusing properties
„Hill's equation“*

*Example: particle motion with
periodic coefficient*



equation of motion: $x''(s) - k(s)x(s) = 0$

*restoring force $\neq \text{const}$,
 $k(s)$ = depending on the position s
 $k(s+L) = k(s)$, periodic function*



*we expect a kind of quasi harmonic
oscillation: amplitude & phase will depend
on the position s in the ring.*

6.) The Beta Function

General solution of Hill's equation:

(i) $x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)$

ε, Φ = integration **constants** determined by initial conditions

$\beta(s)$ **periodic function** given by **focusing properties** of the lattice \leftrightarrow quadrupoles

$$\beta(s+L) = \beta(s)$$

Inserting (i) into the equation of motion ...

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

$\Psi(s)$ = „**phase advance**“ of the oscillation between point „0“ and „ s “ in the lattice.
For one complete revolution: number of oscillations per turn „**Tune**“

$$Q_y = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) & x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \\ (2) & x'(s) = -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}} \{ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \} \end{cases}$$

from (1) we get

$$\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\epsilon} \sqrt{\beta(s)}}$$

Insert into (2) and solve for ϵ

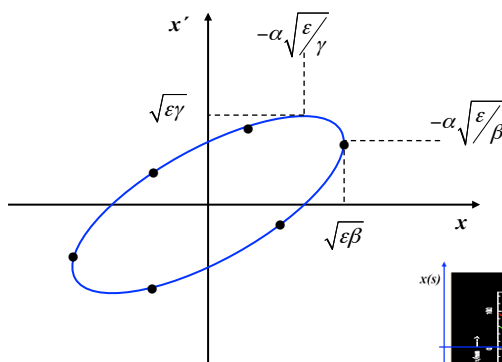
$$\begin{aligned} \alpha(s) &= \frac{-1}{2} \beta'(s) \\ \gamma(s) &= \frac{1 + \alpha(s)^2}{\beta(s)} \end{aligned}$$

$$\epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$

- * ϵ is a constant of the motion ... it is independent of „s“
- * parametric representation of an ellipse in the x, x' space
- * shape and orientation of ellipse are given by α, β, γ

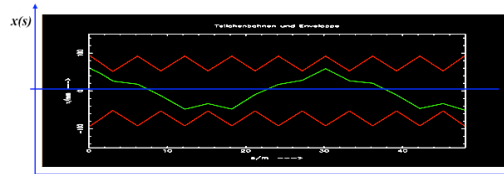
Beam Emittance and Phase Space Ellipse

$$\epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$



Liouville: in reasonable storage rings area in phase space is constant.

$$A = \pi \epsilon = \text{const}$$



ϵ beam emittance = **woozilycity** of the particle ensemble, **intrinsic beam parameter**, cannot be changed by the foc. properties.

Scientifiquely speaking: area covered in transverse x, x' phase space ... and it is constant !!!

Phase Space Ellipse

particle trajectory: $x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos\{\psi(s) + \phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ \longrightarrow x' at that position ...?

... put $\hat{x}(s)$ into $\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$ and solve for x'

$$\varepsilon = \gamma \cdot \varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta} \cdot x' + \beta x'^2 \longrightarrow x' = -\alpha \cdot \sqrt{\varepsilon/\beta}$$

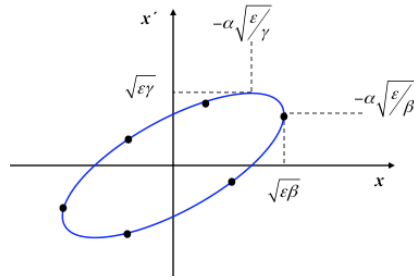
and in the same way we obtain:

$$\hat{x}' = \sqrt{\varepsilon\gamma} \quad x = \pm \alpha \sqrt{\varepsilon/\gamma}$$

* A high β -function means a large beam size and a small beam divergence. !

... et vice versa !!!

* In the middle of a quadrupole β = maximum, α = zero $x' = 0$... and the ellipse is flat

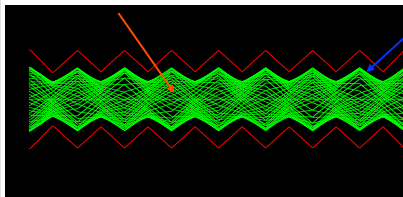


shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\Psi(s) + \phi)$$

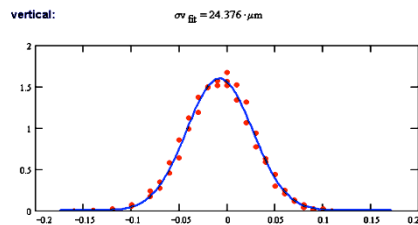
$$\hat{x}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)}$$



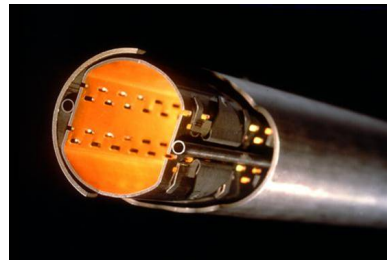
single particle trajectories, $N \approx 10^{11}$ per bunch

Gauß Particle Distribution: $\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$

particle at distance 1σ from centre \leftrightarrow 68.3 % of all beam particles

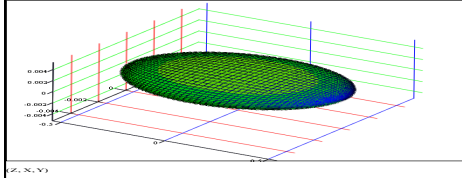


LHC: $\sigma = \sqrt{\varepsilon \cdot \beta} = \sqrt{5 \cdot 10^{-10} \text{ m} \cdot 180 \text{ m}} = 0.3 \text{ mm}$



aperture requirements: $r_0 = 10 \cdot \sigma$

Emittance of the Particle Ensemble:



particle bunch

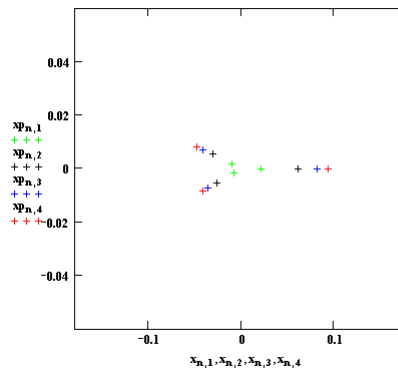
Example: HERA

beam parameters in the arc

$$\beta(x) \approx 80 \text{ m}$$

$$\varepsilon \approx 7 \cdot 10^{-9} \text{ rad} \cdot \text{m} \quad (\Leftrightarrow 1 \sigma)$$

$$\sigma = \sqrt{\varepsilon \beta} \approx 0.75 \text{ mm}$$



8.) Transfer Matrix M ... yes we had the topic already

$$\left. \begin{array}{l} \text{general solution} \\ \text{of Hill's equation} \end{array} \right\} \begin{cases} x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos\{\psi(s) + \phi\} \\ x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}} [\alpha(s) \cos\{\psi(s) + \phi\} + \sin\{\psi(s) + \phi\}] \end{cases}$$

remember the trigonometrical gymnastics: $\sin(a+b) = \dots$ etc

$$\begin{aligned} x(s) &= \sqrt{\varepsilon} \sqrt{\beta_s} (\cos \psi_s \cos \phi - \sin \psi_s \sin \phi) \\ x'(s) &= \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}} [\alpha_s \cos \psi_s \cos \phi - \alpha_s \sin \psi_s \sin \phi + \sin \psi_s \cos \phi + \cos \psi_s \sin \phi] \end{aligned}$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\left. \begin{array}{l} \cos \phi = \frac{x_0}{\sqrt{\varepsilon \beta_0}} \\ \sin \phi = -\frac{1}{\sqrt{\varepsilon}} (x'_0 \sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}) \end{array} \right\} \text{inserting above ...}$$

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}} \{ \cos \psi_s + \alpha_0 \sin \psi_s \} \underline{x_0} + \{ \sqrt{\beta_s \beta_0} \sin \psi_s \} \underline{x'_0}$$

$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s \beta_0}} \{ (\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s \} \underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}} \{ \cos \psi_s - \alpha_s \sin \psi_s \} \underline{x'_0}$$

which can be expressed ... for convenience ... *in matrix form*

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$

* we can calculate *the single particle trajectories* between two locations in the ring,
if we know the α β γ at these positions.

* and nothing but the α β γ at these positions.

* ... !

* Äquivalenz der Matrizen

11.) Résumé:

beam rigidity:

$$B \cdot \rho = \frac{p}{q}$$

bending strength of a dipole:

$$\frac{1}{\rho} [m^{-1}] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$$

focusing strength of a quadrupole:

$$k [m^{-2}] = \frac{0.2998 \cdot g}{p(GeV/c)}$$

focal length of a quadrupole:

$$f = \frac{1}{k \cdot l_q}$$

equation of motion:

$$x'' + Kx = \frac{1}{\rho} \frac{\Delta p}{p}$$

matrix of a foc. quadrupole:

$$x_{s2} = M \cdot x_{s1}$$

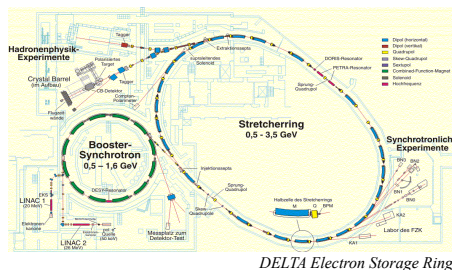
$$M = \begin{pmatrix} \cos \sqrt{|K|} l & \frac{1}{\sqrt{|K|}} \sin \sqrt{|K|} l \\ -\sqrt{|K|} \sin \sqrt{|K|} l & \cos \sqrt{|K|} l \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

12.) Bibliography

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- 6.) Bernhard Holzer: *Lattice Design*, CERN Acc. School: Interm. Acc. phys course, <http://cas.web.cern.ch/cas/ZEUTHEN/lectures-zeuthen.htm>
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- 10.) D. Edwards, M. Syphers : *An Introduction to the Physics of Particle Accelerators*, SSC Lab 1990

9.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$



„This rather formidable looking matrix simplifies considerably if we consider one complete revolution ...“

$$M(s) = \begin{pmatrix} \cos \psi_{\text{turn}} + \alpha_s \sin \psi_{\text{turn}} & \beta_s \sin \psi_{\text{turn}} \\ -\gamma_s \sin \psi_{\text{turn}} & \cos \psi_{\text{turn}} - \alpha_s \sin \psi_{\text{turn}} \end{pmatrix}$$

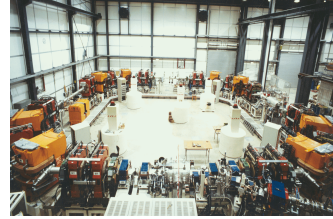
$$\psi_{\text{turn}} = \int_s^{s+L} \frac{ds}{\beta(s)} \quad \psi_{\text{turn}} = \text{phase advance per period}$$

Tune: Phase advance per turn in units of 2π

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

Stability Criterion:

Question: what will happen, if we do not make too many mistakes and your **particle performs one complete turn** ?



Matrix for 1 turn:

$$M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \underbrace{\cos\psi}_{I} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\sin\psi}_{J} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

Matrix for N turns:

$$M^N = (I \cdot \cos\psi + J \cdot \sin\psi)^N = I \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = \text{real} \quad \Leftrightarrow \quad |\cos\psi| \leq 1 \quad \Leftrightarrow \quad \text{Tr}(M) \leq 2$$

stability criterion proof for the disbelieving colleagues !!

Matrix for 1 turn: $M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \underbrace{\cos\psi}_{I} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\sin\psi}_{J} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$

Matrix for 2 turns:

$$M^2 = (I \cos\psi_1 + J \sin\psi_1)(I \cos\psi_2 + J \sin\psi_2) \\ = I^2 \cos\psi_1 \cos\psi_2 + IJ \cos\psi_1 \sin\psi_2 + JI \sin\psi_1 \cos\psi_2 + J^2 \sin\psi_1 \sin\psi_2$$

now ...

$$I^2 = I$$

$$\left. \begin{aligned} IJ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \\ JI &= \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \end{aligned} \right\} IJ = JI$$

$$J^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$M^2 = I \cos(\psi_1 + \psi_2) + J \sin(\psi_1 + \psi_2)$$

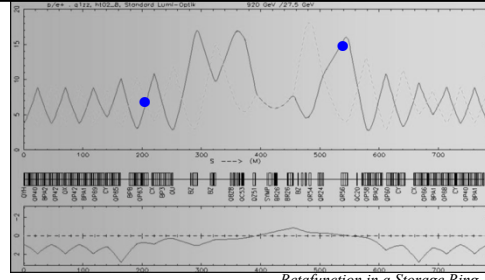
$$M^2 = I \cos(2\psi) + J \sin(2\psi)$$

10.) Transformation of α, β, γ

consider two positions in the storage ring: s_0, s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M^* \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$



Betafunction in a Storage Ring

since $\epsilon = \text{const}$ (Liouville):

$$\epsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2$$

$$\epsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

... remember $W = CS^{-1}SC^T = 1$

$$\left. \begin{aligned} \begin{pmatrix} x \\ x' \end{pmatrix}_0 &= M^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ M^{-1} &= \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \end{aligned} \right\} \rightarrow \begin{aligned} x_0 &= m_{22}x - m_{12}x' \\ x_0' &= -m_{21}x + m_{11}x' \end{aligned} \quad \dots \text{inserting into } \epsilon$$

$$\epsilon = \beta_0 (m_{11}x' - m_{21}x)^2 + 2\alpha_0 (m_{22}x - m_{12}x')(m_{11}x' - m_{21}x) + \gamma_0 (m_{22}x - m_{12}x')^2$$

sort via x, x' and compare the coefficients to get

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\beta(s) = m_{11}^2 \beta_0 - 2m_{11}m_{12} \alpha_0 + m_{12}^2 \gamma_0$$

$$\alpha(s) = -m_{11}m_{21} \beta_0 + (m_{12}m_{21} + m_{11}m_{22}) \alpha_0 - m_{12}m_{22} \gamma_0$$

$$\gamma(s) = m_{21}^2 \beta_0 - 2m_{21}m_{22} \alpha_0 + m_{22}^2 \gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{11}m_{22} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_1}$$



- 1.) this expression is important
- 2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.
- 3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.
- 4.) go back to point 1.)

II.) Acceleration and Momentum Spread

The „ not so ideal world “

Remember:

Beam Emittance and Phase Space Ellipse:

equation of motion: $x''(s) - k(s) x(s) = 0$

general solution of Hills equation: $x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \varphi)$

beam size: $\sigma = \sqrt{\epsilon \beta} \approx "mm"$

$$\epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$

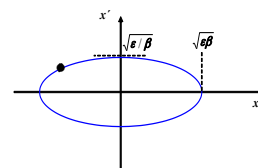
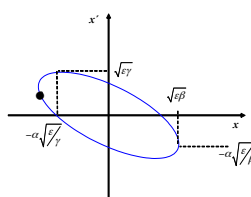
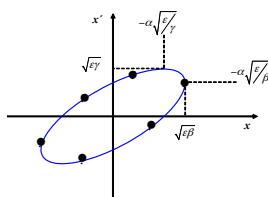
$$\alpha(s) = \frac{-1}{2} \beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

* ϵ is a **constant** of the motion ... it is independent of „s“

* parametric representation of an **ellipse** in the $x x'$ space

* shape and orientation of ellipse are given by α, β, γ

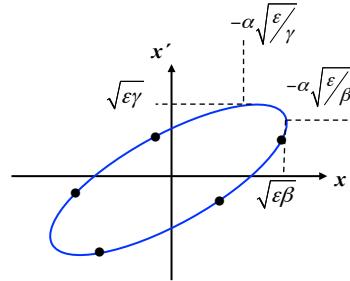


11.) Liouville during Acceleration

$$\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$

Beam Emittance corresponds to the area covered in the x, x' Phase Space Ellipse

Liouville: Area in phase space is constant.



But so sorry ... $\varepsilon \neq \text{const}$!

Classical Mechanics:

phase space = diagram of the two canonical variables
position & **momentum**

x p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} ; \quad L = T - V = \text{kin. Energy} - \text{pot. Energy}$$

According to Hamiltonian mechanics:

phase space diagram relates the variables q and p

$$q = \text{position} = x$$

$$p = \text{momentum} = \gamma m v = mc\gamma\beta_x$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouville's Theorem: $\int p dq = \text{const}$

for convenience (i.e. **because we are lazy bones**) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{\beta_x}{\beta} \quad \text{where } \beta_x = v_x / c$$

$$\int p dq = mc \int \gamma \beta_x dx$$

$$\int p dq = mc \gamma \beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\Rightarrow \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

**the beam emittance
shrinks during
acceleration $\varepsilon \sim 1/\gamma$**

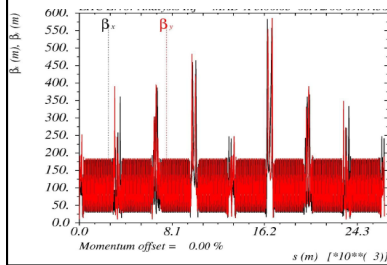
Nota bene:

- 1.) A proton machine ... or an electron linac ... needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

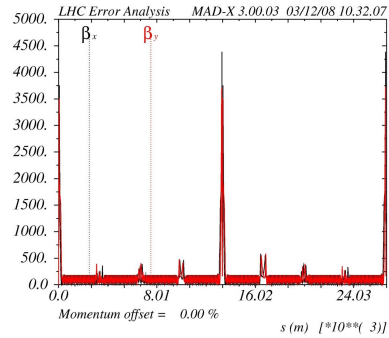
$$\sigma = \sqrt{\varepsilon \beta}$$

- 2.) At lowest energy the machine will have the major aperture problems,
→ here we have to **minimise** $\hat{\beta}$

- 3.) we need **different beam optics** adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC injection
optics at 450 GeV

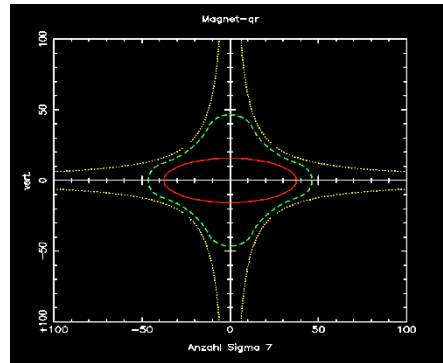
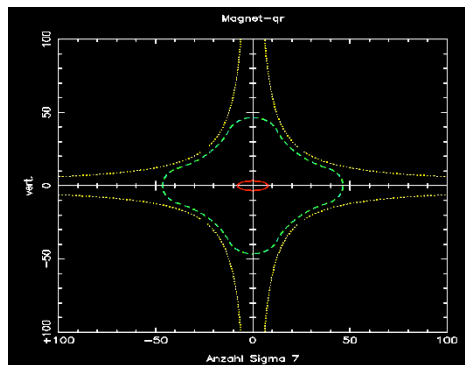


LHC mini beta
optics at 7000 GeV

Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε (40GeV) = $1.2 \cdot 10^{-7}$
 ε (920GeV) = $5.1 \cdot 10^{-9}$



7 σ beam envelope at E = 40 GeV

... and at E = 920 GeV

12.) The „ $\Delta p / p \neq 0$ “ Problem

A kind of ideal machine ...

the Tandem Van-de Graaf

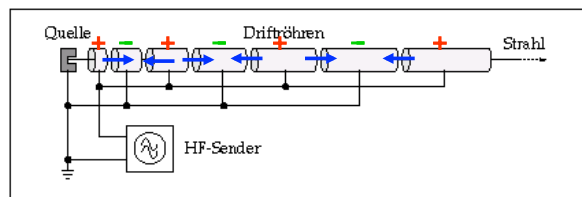


12.) The „ $\Delta p / p \neq 0$ “ Problem

Linear Accelerator

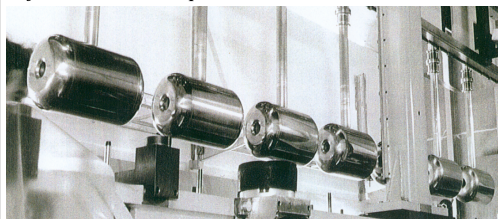
Energy Gain per „Gap“:

$$W = q U_0 \sin \omega_{RF} t$$



drift tube structure at a proton linac

1928, Wideroe



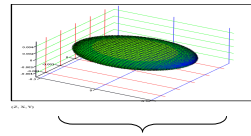
500 MHz cavities in an electron storage ring



* **RF Acceleration:** multiple application of the same acceleration voltage;
brilliant idea to gain higher energies
... but **changing acceleration voltage**

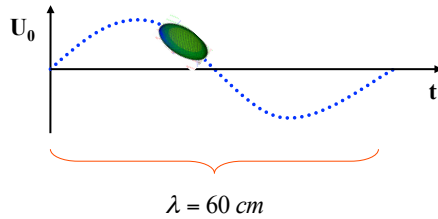
Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Example: HERA RF:

Bunch length of Electrons $\approx 1\text{cm}$



$$\left. \begin{array}{l} \nu = 500\text{MHz} \\ c = \lambda \nu \end{array} \right\} \lambda = 60\text{ cm}$$

$$\begin{array}{l} \sin(90^\circ) = 1 \\ \sin(84^\circ) = 0.994 \end{array} \quad \frac{\Delta U}{U} = 6.0 \cdot 10^{-3}$$

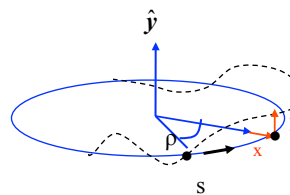
typical momentum spread of an electron bunch:

$$\frac{\Delta p}{p} \approx 1.0 \cdot 10^{-3}$$

13.) Dispersion: trajectories for $\Delta p / p \neq 0$

Force acting on the particle

$$F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y v$$



remember: $x \approx \text{mm}$, $\rho \approx \text{m}$... \rightarrow develop for small x

$$m \frac{d^2 x}{dt^2} - \frac{mv^2}{\rho} \left(1 - \frac{x}{\rho}\right) = e B_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$ $B_y = B_0 + x \frac{\partial B_y}{\partial x}$

$$x'' - \frac{1}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{e B_0}{mv} + \frac{e x g}{mv}$$

$p = p_0 + \Delta p$

... but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2} e B_0 + \underbrace{\frac{x e g}{p_0}}_{k * x} - \underbrace{x e g \frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-e B_0)}{p_0}}_{\frac{1}{\rho}} + k * x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k * x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \quad \rightarrow$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} \frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
 \rightarrow inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to $\Delta p/p$:

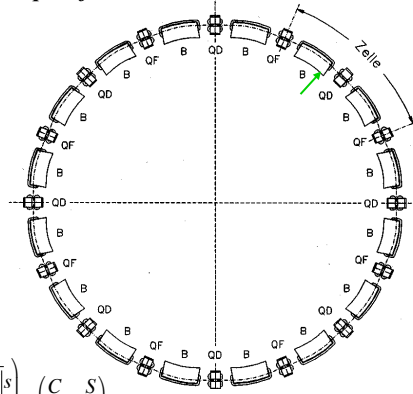
$$D(s) = \frac{x_i(s)}{\Delta p/p}$$

Dispersion function D(s)

- * is that **special orbit**, an **ideal particle** would have for $\Delta p/p = 1$
- * the **orbit of any particle** is the **sum** of the well known x_β and the **dispersion**
- * as **D(s)** is just **another orbit** it will be subject to the focusing properties of the lattice

Dispersion:

Example: homogenous dipole field



bit for $\Delta p/p > 0$

$$= D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

e.g. matrix for a quadrupole lens:

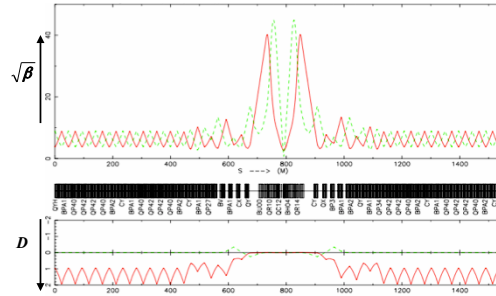
$$M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s) \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\left. \begin{aligned} x(s) &= x_\beta(s) + D(s) \cdot \frac{\Delta p}{p} \\ x(s) &= C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p} \end{aligned} \right\}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example HERA

$$\left. \begin{aligned} x_\beta &= 1 \dots 2 \text{ mm} \\ D(s) &\approx 1 \dots 2 \text{ m} \\ \Delta p/p &\approx 1 \cdot 10^{-3} \end{aligned} \right\}$$

Amplitude of Orbit oscillation

contribution due to Dispersion \approx beam size

\rightarrow Dispersion must vanish at the collision point



Calculate D, D'

$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}$$

(proof: see appendix)

Example: Drift

$$M_{\text{Drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$M_{\text{Drift}} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(s) = S(s) \underbrace{\int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s}}_{=0} - C(s) \underbrace{\int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}}_{=0}$$

Example: Dipole

$$M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s) \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

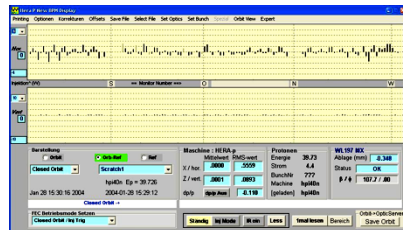
$$K = \frac{1}{\rho^2} \quad \text{---} \quad s = l_B$$

$$M_{\text{Dipole}} = \begin{pmatrix} \cos \frac{l}{\rho} & \rho \sin \frac{l}{\rho} \\ -\frac{1}{\rho} \sin \frac{l}{\rho} & \cos \frac{l}{\rho} \end{pmatrix} \rightarrow$$

$$D(s) = \rho \cdot (1 - \cos \frac{l}{\rho})$$

$$D'(s) = \sin \frac{l}{\rho}$$

Dispersion is visible



HERA Standard Orbit

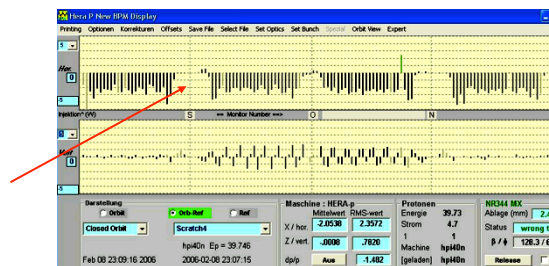
dedicated energy change of the stored beam

→ closed orbit is moved to a dispersions trajectory

$$x_p = D(s) * \frac{\Delta p}{p}$$

Attention: at the Interaction Points we require $D=D'=0$

HERA Dispersion Orbit



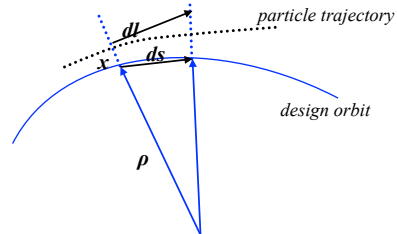
14.) Momentum Compaction Factor: α_p

The **dispersion** function relates the **momentum error** of a particle to the **horizontal orbit coordinate** and so **it changes the length of the off-energy-orbit !!**

particle with a **displacement x** to the design orbit
 → **path length dl** ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s) \frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

* The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.

Definition:

$$\frac{\delta l_{\epsilon}}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\rightarrow \alpha_p = \frac{1}{L} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

For first estimates assume:

$$\frac{1}{\rho} = \text{const.}$$

$$\int_{\text{dipoles}} D(s) ds \approx I_{\Sigma(\text{dipoles})} \cdot \langle D \rangle_{\text{dipole}}$$

$$\alpha_p = \frac{1}{L} I_{\Sigma(\text{dipoles})} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \rightarrow \alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

Assume: $v \approx c$

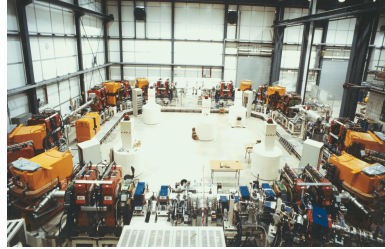
$$\rightarrow \frac{\delta T}{T} = \frac{\delta l_{\epsilon}}{L} = \alpha_p \frac{\Delta p}{p}$$

α_p combines via the dispersion function the momentum spread with the longitudinal motion of the particle.

15.) Gradient Errors

Matrix in Twiss Form

Transfer Matrix from point „0“ in the lattice to point „s“:



$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_0 \sin \psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic boundary conditions:

$$\beta(s+L) = \beta(s)$$

$$\alpha(s+L) = \alpha(s)$$

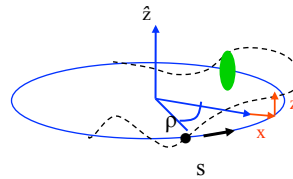
$$\gamma(s+L) = \gamma(s)$$

$$M(s) = \begin{pmatrix} \cos \psi_{turn} + \alpha_s \sin \psi_{turn} & \beta_s \sin \psi_{turn} \\ -\gamma_s \sin \psi_s & \cos \psi_{turn} - \alpha_s \sin \psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic **perturbation** described by **thin lens quadrupole**

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k ds & 1 \end{pmatrix}}_{\text{quad error}} \cdot \underbrace{\begin{pmatrix} \cos \psi_{turn} + \alpha \sin \psi_{turn} & \beta \sin \psi_{turn} \\ -\gamma \sin \psi_{turn} & \cos \psi_{turn} - \alpha \sin \psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos \psi_0 + \alpha \sin \psi_0 & \beta \sin \psi_0 \\ \Delta k ds (\cos \psi_0 + \alpha \sin \psi_0) - \gamma \sin \psi_0 & \Delta k ds \beta \sin \psi_0 + \cos \psi_0 - \alpha \sin \psi_0 \end{pmatrix}$$

rule for getting the tune

$$\text{Trace}(M) = 2 \cos \psi = 2 \cos \psi_0 + \Delta k ds \beta \sin \psi_0$$

Quadrupole error → Tune Shift

$$\psi = \psi_0 + \Delta\psi \quad \longrightarrow \quad \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta k ds \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\underbrace{\cos\psi_0 \cos\Delta\psi}_{\approx 1} - \underbrace{\sin\psi_0 \sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{k ds \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{k ds \beta}{2}$$

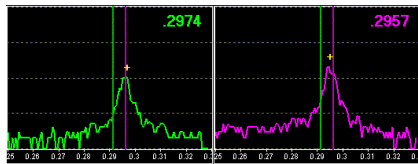
and referring to Q instead of ψ :

$$\psi = 2\pi Q$$

$$\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta k(s) \beta(s) ds}{4\pi}$$

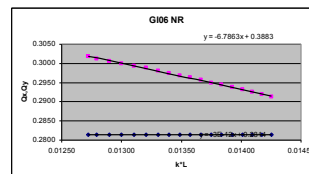
- ! the tune shift is proportional to the β -function at the quadrupole
- !! field quality, power supply tolerances etc are much tighter at places where β is large
- !!! mini beta quads: $\beta \approx 1900$ m
arc quads: $\beta \approx 80$ m
- !!!! β is a measure for the sensitivity of the beam

a quadrupole error leads to a shift of the tune:



Example: measurement of β in a storage ring: tune spectrum

$$\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta k \beta(s)}{4\pi} ds \approx \frac{\Delta k l_{quad} \bar{\beta}}{4\pi}$$



Without proof (CERN-94-01)

A quadrupole error will always lead to a tune shift, but in addition to a change of the beta-function.

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi Q)} \oint \beta(\tilde{s}) \Delta k(\tilde{s}) \cos(2|\psi(s) - \psi(\tilde{s})| - \pi Q) d\tilde{s}$$

As before the effect of the error depends on the β -function at the observation point as well as at the place of the error itself, on the error strength and there is again a resonance denominator

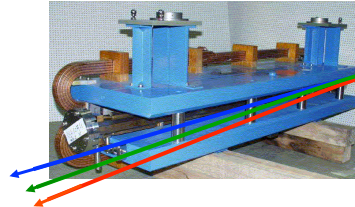
→ half integer tunes are forbidden.

16.) Chromaticity:

A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu $1/p$*

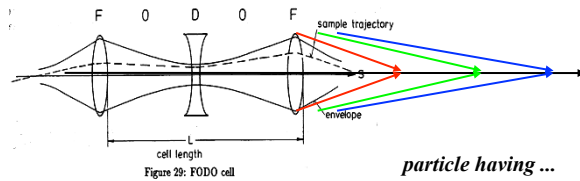
dipole magnet $\alpha = \frac{\int B dl}{p/e}$



$$x_D(s) = D(s) \frac{\Delta p}{p}$$

focusing lens

$$k = \frac{g}{p/e}$$



particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e}$$

$$p = p_0 + \Delta p$$

in case of a momentum spread:

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0} \left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

... which acts like a quadrupole error in the machine and leads to a tune spread:

$$\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds$$

definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} ; \quad Q' = -\frac{1}{4\pi} \int k(s) \beta(s) ds$$

... what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a **number** indicating the **size of the tune spot** in the working diagram,

Q' is always created if the beam is focussed

→ it is determined by the focusing strength k of all quadrupoles

$$Q' = -\frac{1}{4\pi} \oint \beta(s)k(s)ds$$

k = quadrupole strength

β = **betafunction** indicates the beam size ... and even more the **sensitivity of the beam to external fields**

Example: LHC

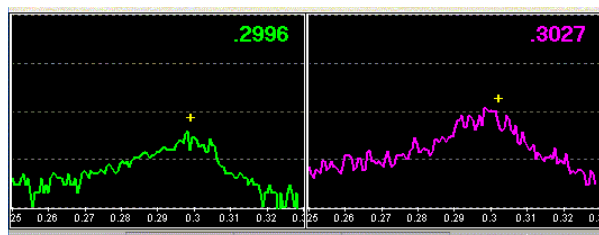
$$Q' = -250$$

$$\Delta p/p = \pm 0.2 \cdot 10^{-3}$$

$$\Delta Q = 0.256 \dots 0.36$$

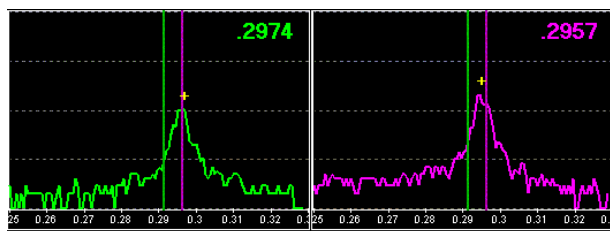
→ Some particles get very close to resonances and are lost

in other words: the tune is not a point
it is a **pancake**



Tune signal for a nearly uncompensated chromaticity ($Q' \approx 20$)

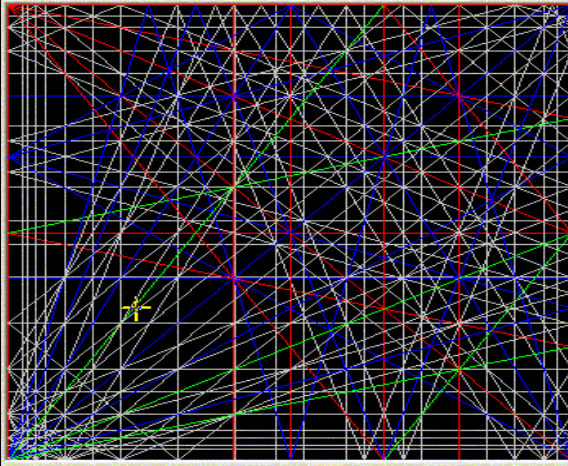
Ideal situation: chromaticity well corrected,
($Q' \approx 1$)



Tune and Resonances

$$m \cdot Q_x + n \cdot Q_y + l \cdot Q_s = \text{integer}$$

Tune diagram up to 3rd order



... and up to 7th order

Homework for the operators:
find a nice place for the tune
where against all probability
the beam will survive

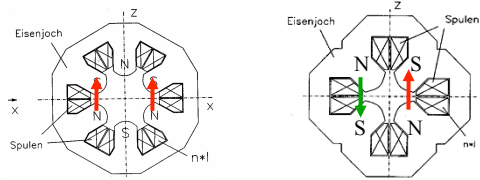
Correction of Q'

1.) sort the particles according to their momentum $x_D(s) = D(s) \frac{\Delta p}{p}$

2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left. \begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2} \tilde{g}(x^2 - z^2) \end{aligned} \right\} \quad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x \quad \text{linear rising „gradient“:}$$

Sextupole Magnets:



normalised quadrupole strength:

$$k_{\text{sext}} = \frac{\tilde{g}x}{p/e} = m_{\text{sext}} \cdot x$$

$$k_{\text{sext}} = m_{\text{sext}} \cdot D \frac{\Delta p}{p}$$

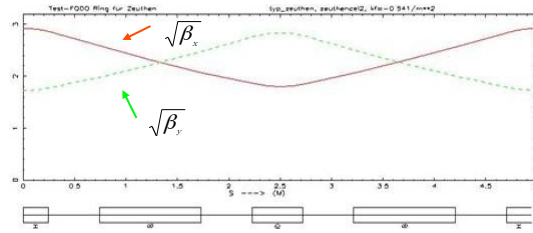
corrected chromaticity:

$$Q'_{\text{cell}_x} = -\frac{1}{4\pi} \left\{ k_{qf} \hat{\beta}_x l_{qf} - k_{qd} \tilde{\beta}_x l_{qd} \right\} + \frac{1}{4\pi} \sum_{F \text{ sext}} k_2^F l_{\text{sext}} D_x^F \beta_x^F - \frac{1}{4\pi} \sum_{D \text{ sext}} k_2^D l_{\text{sext}} D_x^D \beta_y^D$$

$$Q'_{\text{cell}_y} = -\frac{1}{4\pi} \left\{ -k_{qf} \tilde{\beta}_y l_{qf} + k_{qd} \hat{\beta}_y l_{qd} \right\} + \frac{1}{4\pi} \sum_{F \text{ sext}} k_2^F l_{\text{sext}} D_x^F \beta_x^F - \frac{1}{4\pi} \sum_{D \text{ sext}} k_2^D l_{\text{sext}} D_x^D \beta_y^D$$

Chromaticity in a FODO lattice

$$Q' = \frac{-1}{4\pi} \int k(s) \beta(s) ds$$



β -Function in a FoDo structure

$$\hat{\beta} = \frac{(1 + \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}} \quad \tilde{\beta} = \frac{(1 - \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}}$$

$$Q' = \frac{-1}{4\pi} N * \frac{\hat{\beta} - \tilde{\beta}}{f_Q}$$

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \left\{ \frac{L(1 + \sin \frac{\psi_{cell}}{2}) - L(1 - \sin \frac{\psi_{cell}}{2})}{\sin \mu} \right\}$$

using some *TLC transformations* ... ξ can be expressed in a very simple form:

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \frac{2L \sin \frac{\psi_{cell}}{2}}{\sin \psi_{cell}}$$

$$Q' = \frac{-1}{4\pi} N * \frac{1}{f_Q} * \frac{L \sin \frac{\psi_{cell}}{2}}{\sin \frac{\psi_{cell}}{2} \cos \frac{\psi_{cell}}{2}}$$

$$Q'_{cell} = \frac{-1}{4\pi f_Q} * \frac{L \tan \frac{\psi_{cell}}{2}}{\sin \frac{\psi_{cell}}{2}}$$

$$Q'_{cell} = \frac{-1}{\pi} * \tan \frac{\psi_{cell}}{2}$$

remember ...

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

putting ...

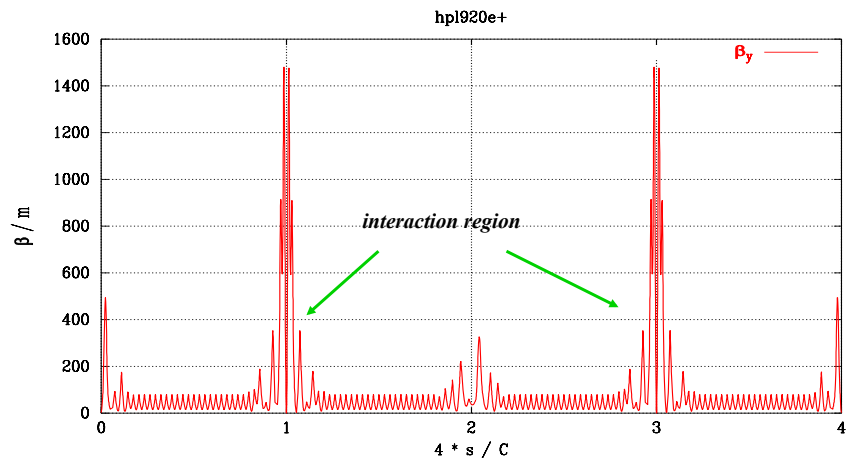
$$\sin \frac{\psi_{cell}}{2} = \frac{L}{4f_Q}$$

contribution of one FoDo Cell to the chromaticity of the ring:

Chromaticity

$$Q' = -\frac{1}{4\pi} \oint K(s) \beta(s) ds$$

question: main contribution to ξ in a lattice ... ?



Dipole Errors / Quadrupole Misalignment

The **Design Orbit** is defined by the strength and arrangement of the dipoles.

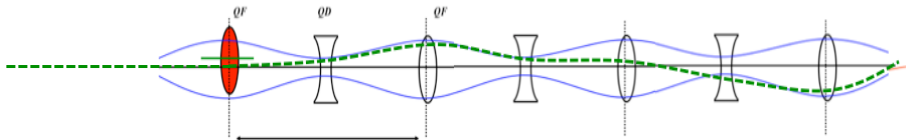
Under the influence of **dipole imperfections** and **quadrupole misalignments** we obtain a “**Closed Orbit**” which is hopefully still closed and not too far away from the design.

Dipole field error: $\theta = \frac{dl}{\rho} = \frac{\int B dl}{B\rho}$

Quadrupole offset: $g = \frac{dB}{dx} \rightarrow \Delta x \cdot g = \Delta x \frac{dB}{dx} = \Delta B$

misaligned quadrupoles (or orbit offsets in quadrupoles) create dipole effects that lead to a distorted “closed orbit”

normalised to p/e: $\Delta x \cdot k = \Delta x \cdot \frac{g}{B\rho} = \frac{1}{\rho} \quad \begin{pmatrix} x \\ x' \end{pmatrix}_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ x' \end{pmatrix}_f = \begin{pmatrix} 0 \\ \frac{l}{\rho} \end{pmatrix}$



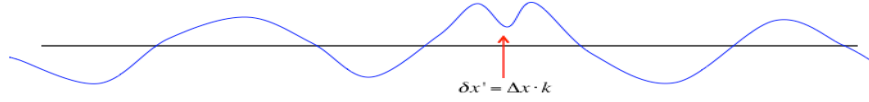
In a Linac – starting with a perfect orbit – the misaligned quadrupole creates an oscillation that is transformed from now on downstream via

$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = M \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

... and in a circular machine ??

we have to obey the periodicity condition.

The orbit is closed !! ... even under the influence of a orbit kick.



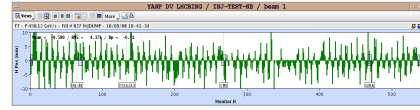
Calculation of the new closed orbit:

the general orbit will always be a solution of Hill, so ...

$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) + \varphi)$$

We set at the location of the error $s=0$, $\Psi(s)=0$

and require as 1st boundary condition:
periodic amplitude



$$x(s+L) = x(s)$$

$$\cancel{a \cdot \sqrt{\beta(s+L)} \cdot \cos(\psi(s) + 2\pi Q - \varphi)} = \cancel{a \cdot \sqrt{\beta(s)} \cdot \cos(\psi(s) - \varphi)}$$

$$\cos(2\pi Q - \varphi) = \cos(-\varphi) = \cos(\varphi)$$

$$\rightarrow \varphi = \pi Q$$

$$\beta(s+L) = \beta(s)$$

$$\psi(s+0) = 0$$

$$\psi(s+L) = 2\pi Q$$

Misalignment error in a circular machine

2nd boundary condition: $x'(s+L) + \delta x' = x'(s)$

we have to close the orbit



$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) - \varphi)$$

$$x'(s) = a \cdot \sqrt{\beta} \left(-\sin(\psi(s) - \varphi) \psi' + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s) - \varphi) \right)$$

$$x'(s) = -a \cdot \frac{1}{\sqrt{\beta}} \left(\sin(\psi(s) - \varphi) + \frac{\beta'(s)}{2\sqrt{\beta}} a \cdot \cos(\psi(s) - \varphi) \right)$$

$$\psi(s) = \int \frac{1}{\beta(s)} ds$$

$$\psi'(s) = \frac{1}{\beta(s)}$$

boundary condition: $x'(s+L) + \delta x' = x'(s)$

$$\begin{aligned} -a \cdot \frac{1}{\sqrt{\beta(\tilde{s}+L)}} \left(\sin(2\pi Q - \varphi) + \frac{\beta'(\tilde{s}+L)}{2\beta(\tilde{s}+L)} \sqrt{\beta(\tilde{s}+L)} a \cdot \cos(2\pi Q - \varphi) + \frac{\Delta \tilde{s}}{\rho} \right) = \\ = -a \cdot \frac{1}{\sqrt{\beta(\tilde{s})}} \left(\sin(-\varphi) + \frac{\beta'(\tilde{s})}{2\beta(\tilde{s})} \sqrt{\beta(\tilde{s})} a \cdot \cos(-\varphi) \right) \end{aligned}$$

Nota bene: \tilde{s} refers to the location of the kick

Misalignment error in a circular machine

Now we use: $\beta(s+L) = \beta(s)$, $\varphi = \pi Q$

$$\frac{-a}{\sqrt{\beta(\tilde{s})}} \left(\sin(\pi Q) + \frac{\beta'(\tilde{s})}{2\beta(\tilde{s})} \sqrt{\beta(\tilde{s})} a \cdot \cos(\pi Q) + \frac{\Delta\tilde{s}}{\rho} \right) = \frac{a}{\sqrt{\beta(\tilde{s})}} \left(\sin(\pi Q) + \frac{\beta'(\tilde{s})}{2\beta(\tilde{s})} \sqrt{\beta(\tilde{s})} a \cdot \cos(\pi Q) \right)$$

$$\Rightarrow 2a \cdot \frac{\sin(\pi Q)}{\sqrt{\beta(\tilde{s})}} = \frac{\Delta\tilde{s}}{\rho} \Rightarrow a = \frac{\Delta\tilde{s}}{\rho} \cdot \sqrt{\beta(\tilde{s})} \frac{1}{2\sin(\pi Q)} \quad ! \text{ this is the amplitude of the orbit oscillation resulting from a single kick}$$

inserting in the equation of motion

$$x(s) = a \cdot \sqrt{\beta} \cos(\psi(s) + \varphi)$$

$$x(s) = \frac{\Delta\tilde{s}}{\rho} \cdot \frac{\sqrt{\beta(\tilde{s})} \sqrt{\beta(s)} \cos(\psi(s) - \varphi)}{2\sin(\pi Q)}$$

! the distorted orbit depends on the kick strength,
! the local β function
! the β function at the observation point

!!! there is a resonance denominator
→ watch your tune !!!

Misalignment error in a circular machine

For completeness:

if we do not set $\psi(s=0) = 0$ we have to write a bit more but finally we get:

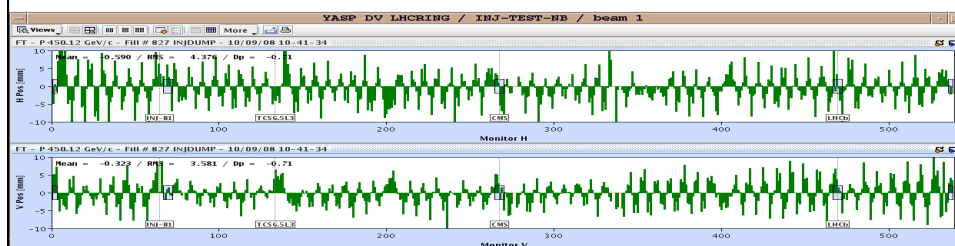
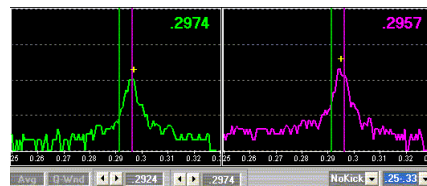
$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} * \int \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\psi(\tilde{s}) - \psi(s)| - \pi Q) d\tilde{s}$$

Reminder: LHC

Tune: $Q_x = 64.31$, $Q_y = 59.32$

Relevant for beam stability:

non integer part
avoid integer tunes



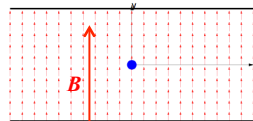
Quadrupole Rotation Errors

Short Resume until now:

- * Dipole field errors lead to closed orbit distortions
- * Quadrupole misalignments as well
- * Quadrupole gradient errors lead to tune shifts ΔQ and beta beats $\Delta\beta/\beta$

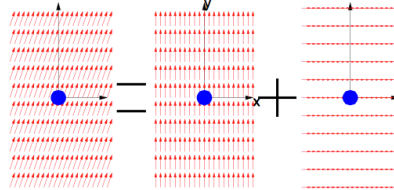
... and what does a roll angle error do ???

Dipole strength error:



An error in the strength of a main dipole causes a perturbation on the horizontal closed orbit.

Dipole roll angle error



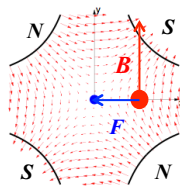
A tilt error in a main dipole causes a perturbation on the vertical closed orbit.

(court. pictures from R. Tomas)

Quadrupole Rotation Errors

quadrupole tilt errors lead to coupling of the transverse motions

Standard quadrupole:

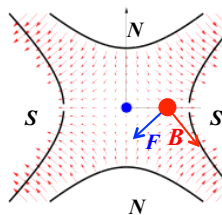


Lorents Force:

$F_x = -kx$ and $F_y = ky$ making horizontal dynamics totally decoupled from vertical.

$$F = q(\vec{v} \times \vec{B})$$

Skew Quadrupole:



Lorents Force:

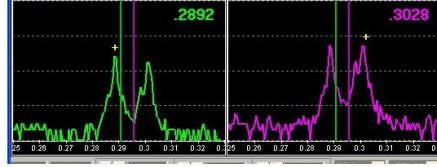
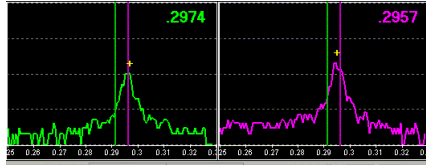
A horizontal offset leads to a horizontal and vertical component of the Lorentz force
-> to coupling between x and y plane

Quadrupole Rotation Errors

Observations on Beam:

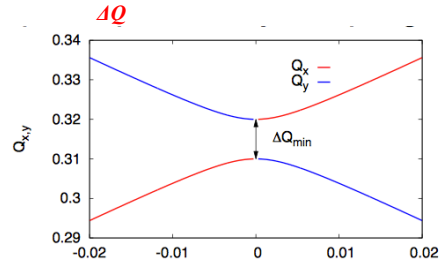
Coupling makes it impossible to approach tunes below a certain

ΔQ_{\min} that depends on the tune and the coupling strength



observed tune as a function
of the quadrupole strength
“closest tune approach”

Correction via dedicated skew quadrupoles
in the machine



Resume':

beam emittance:

$$\varepsilon \propto \frac{1}{\beta\gamma}$$

beta function in a drift:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

... and for $\alpha = 0$

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

particle trajectory for $\Delta p/p \neq 0$
inhomogeneous equation:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} \frac{1}{\rho}$$

... and its solution:

$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

momentum compaction:

$$\frac{\delta l_e}{L} = \alpha_p \frac{\Delta p}{p} \quad \alpha_p \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}$$

quadrupole error:

$$\Delta Q = \int_{s_0}^{s_0+L} \frac{\Delta K(s) \beta(s) ds}{4\pi}$$

chromaticity:

$$Q' = -\frac{1}{4\pi} \int K(s) \beta(s) ds$$