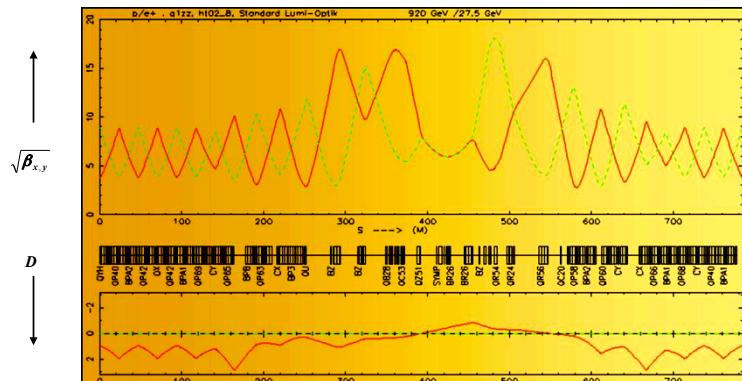


Lattice Design in Particle Accelerators

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*1952: Courant, Livingston, Snyder:
Theory of strong focusing in particle beams*

Lattice Design: „... how to build a storage ring“

High energy accelerators → **circular machines**

somewhere in the lattice we need a number of **dipole magnets**,
that are bending the design orbit to a **closed ring**

Geometry of the ring:

centrifugal force = Lorentz force

$$e * v * B = \frac{mv^2}{\rho}$$

$$\rightarrow e * B = \frac{mv}{\rho} = p / \rho$$

$$\rightarrow B * \rho = p / e$$

*p = momentum of the particle,
 ρ = curvature radius*

Bp = beam rigidity

*Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength*

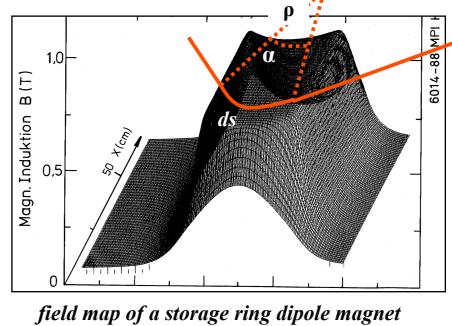


1.) Circular Orbit:

„... defining the geometry“

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$$

$$\alpha = \frac{B * dl}{B * \rho}$$



The angle swept out in one revolution must be 2π , so

$$\alpha = \frac{\int B dl}{B * \rho} = 2\pi \quad \rightarrow \quad \int B dl = 2\pi * \frac{p}{q} \quad \dots \text{for a full circle}$$

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ is usually required !!

Example LHC:



7000 GeV Proton storage ring
dipole magnets $N = 1232$
 $l = 15 \text{ m}$
 $q = +1 \text{ e}$

$$\int B dl \approx N l B = 2\pi p / e$$

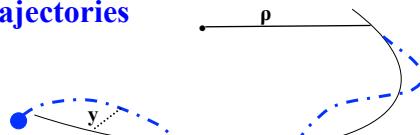
$$B \approx \frac{2\pi \cdot 7000 \cdot 10^9 \text{ eV}}{1232 \cdot 15 \text{ m} \cdot 3 \cdot 10^8 \frac{\text{m}}{\text{s}}} = 8.3 \text{ Tesla}$$

Focusing Forces: single particle trajectories

$$y'' + K * y = 0$$

$$K = -k + 1/\rho^2 \quad \text{hor. plane}$$

$$K = k \quad \text{vert. plane}$$



dipole magnet

$$\frac{1}{\rho} = \frac{B}{p/q}$$

quadrupole magnet

$$k = \frac{g}{p/q}$$

Example: HERA Ring:

Bending radius: $\rho = 580 \text{ m}$

Quadrupole Gradient: $g = 110 \text{ T/m}$

$$k = 33.64 * 10^{-3} / \text{m}^2$$

$$1/p^2 = 2.97 * 10^{-6} / \text{m}^2$$

For estimates in large accelerators the weak focusing term $1/p^2$ can in general be neglected

Solution for a focusing magnet

$$y(s) = y_0 * \cos(\sqrt{K} * s) + \frac{y'_0}{\sqrt{K}} * \sin(\sqrt{K} * s)$$

$$y'(s) = -y_0 * \sqrt{K} * \sin(\sqrt{K} * s) + y'_0 * \cos(\sqrt{K} * s)$$

Or written more convenient in matrix form:

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = M * \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

Hor. **focusing** Quadrupole Magnet

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

Hor. **defocusing** Quadrupole Magnet

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} * l) \\ \sqrt{K} \sinh(\sqrt{K} * l) & \cosh(\sqrt{K} * l) \end{pmatrix}$$

Drift space

$$M_{Drift} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$



$$M_{lattice} = M_{QF1} * M_{D1} * M_{QD} * M_{D1} * M_{QF2} \dots$$

8.) Transfer Matrix M ... yes we had the topic already

general solution
of Hill's equation

$$\left\{ \begin{array}{l} x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos \{\psi(s) + \phi\} \\ x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}} [\alpha(s) \cos \{\psi(s) + \phi\} + \sin \{\psi(s) + \phi\}] \end{array} \right.$$

remember the trigonometrical gymnastics: $\sin(a + b) = \dots$ etc

$$x(s) = \sqrt{\varepsilon} \sqrt{\beta_s} (\cos \psi_s \cos \phi - \sin \psi_s \sin \phi)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}} [\alpha_s \cos \psi_s \cos \phi - \alpha_s \sin \psi_s \sin \phi + \sin \psi_s \cos \phi + \cos \psi_s \sin \phi]$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\left. \begin{array}{l} \cos \phi = \frac{x_0}{\sqrt{\varepsilon \beta_0}}, \\ \sin \phi = -\frac{1}{\sqrt{\varepsilon}} (x'_0 \sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}) \end{array} \right\}$$

inserting above ...

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}} \{ \cos \psi_s + \alpha_0 \sin \psi_s \} x_0 + \sqrt{\beta_s \beta_0} \sin \psi_s x'_0$$

$$x'(s) = \frac{1}{\sqrt{\beta_s \beta_0}} \{ (\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s \} x_0 + \sqrt{\frac{\beta_0}{\beta_s}} \{ \cos \psi_s - \alpha_s \sin \psi_s \} x'_0$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$

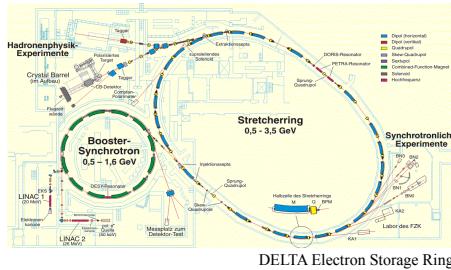
- * we can calculate the single particle trajectories between two locations in the ring,
if we know the $\alpha \beta \gamma$ at these positions.
- * and nothing but the $\alpha \beta \gamma$ at these positions.

* ... !

* Äquivalenz der Matrizen

9.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\beta_s} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ \frac{(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\beta_0} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}$$



„This rather formidable looking matrix simplifies considerably if we consider one complete revolution ...“

$$M(s) = \begin{pmatrix} \cos \psi_{turn} + \alpha_s \sin \psi_{turn} & \beta_s \sin \psi_{turn} \\ -\gamma_s \sin \psi_{turn} & \cos \psi_{turn} - \alpha_s \sin \psi_{turn} \end{pmatrix} \quad \psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)} \quad \psi_{turn} = \text{phase advance per period}$$

Tune: Phase advance per turn in units of 2π

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

Stability Criterion:

Question: what will happen, if we do not make too many mistakes and your particle performs one complete turn ?



Matrix for 1 turn:

$$M = \begin{pmatrix} \cos \psi_{turn} + \alpha_s \sin \psi_{turn} & \beta_s \sin \psi_{turn} \\ -\gamma_s \sin \psi_{turn} & \cos \psi_{turn} - \alpha_s \sin \psi_{turn} \end{pmatrix} = \underbrace{\cos \psi}_{\mathbf{1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\sin \psi}_{\mathbf{J}} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

Matrix for N turns:

$$M^N = (1 \cdot \cos \psi + J \cdot \sin \psi)^N = 1 \cdot \cos N\psi + J \cdot \sin N\psi$$

The motion for N turns remains bounded, if the elements of M^N remain bounded

$$\psi = \text{real} \quad \Leftrightarrow \quad |\cos \psi| \leq 1 \quad \Leftrightarrow \quad \text{Tr}(M) \leq 2$$

stability criterion proof for the disbelieving colleagues !!

$$\text{Matrix for 1 turn: } M = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix} = \cos\psi \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_I + \sin\psi \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_J$$

Matrix for 2 turns:

$$M^2 = (I \cos\psi_1 + J \sin\psi_1)(I \cos\psi_2 + J \sin\psi_2) \\ = I^2 \cos\psi_1 \cos\psi_2 + IJ \cos\psi_1 \sin\psi_2 + JI \sin\psi_1 \cos\psi_2 + J^2 \sin\psi_1 \sin\psi_2$$

now ...

$$I^2 = I \\ IJ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \\ JI = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad \left. \right\} IJ = JI \\ J^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} * \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$M^2 = I \cos(\psi_1 + \psi_2) + J \sin(\psi_1 + \psi_2)$$

$$M^2 = I \cos(2\psi) + J \sin(2\psi)$$

10.) Transformation of α, β, γ

consider two positions in the storage ring: s_0, s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$

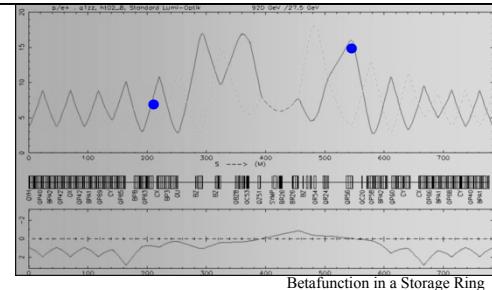
$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

since $\varepsilon = \text{const}$ (Liouville):

$$\varepsilon = \beta_s x'^2 + 2\alpha_s x x' + \gamma_s x^2 \\ \varepsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x'_0 + \gamma_0 x_0^2$$

... remember $W = CS^{-1}SC^{-1} = 1$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_0 = M^{-1} * \begin{pmatrix} x \\ x' \end{pmatrix}_s \\ M^{-1} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \quad \left. \right\} \rightarrow \begin{aligned} x_0 &= m_{22}x - m_{12}x' \\ x'_0 &= -m_{21}x + m_{11}x' \end{aligned} \quad \dots \text{inserting into } \varepsilon$$



$$\varepsilon = \beta_0(m_{11}x' - m_{21}x)^2 + 2\alpha_0(m_{22}x - m_{12}x')(m_{11}x' - m_{21}x) + \gamma_0(m_{22}x - m_{12}x')^2$$

sort via x, x' and compare the coefficients to get

The Twiss parameters α, β, γ can be transformed through the lattice via the matrix elements defined above.

$$\beta(s) = m_{11}^2 \beta_0 - 2m_{11}m_{12}\alpha_0 + m_{12}^2\gamma_0$$

$$\alpha(s) = -m_{11}m_{21}\beta_0 + (m_{12}m_{21} + m_{11}m_{22})\alpha_0 - m_{12}m_{22}\gamma_0$$

$$\gamma(s) = m_{21}^2\beta_0 - 2m_{21}m_{22}\alpha_0 + m_{22}^2\gamma_0$$

in matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s1}$$



- 1.) this expression is important
- 2.) given the twiss parameters α, β, γ at any point in the lattice we can transform them and calculate their values at any other point in the ring.
- 3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to calculate single particle trajectories.

... and here starts the **lattice design !!!**

Most simple example: drift space

$$M_{drift} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

particle coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix}_l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_0 \quad \rightarrow \quad \boxed{x(l) = x_0 + l * x'_0 \\ x'(l) = x'_0}$$

transformation of twiss parameters:

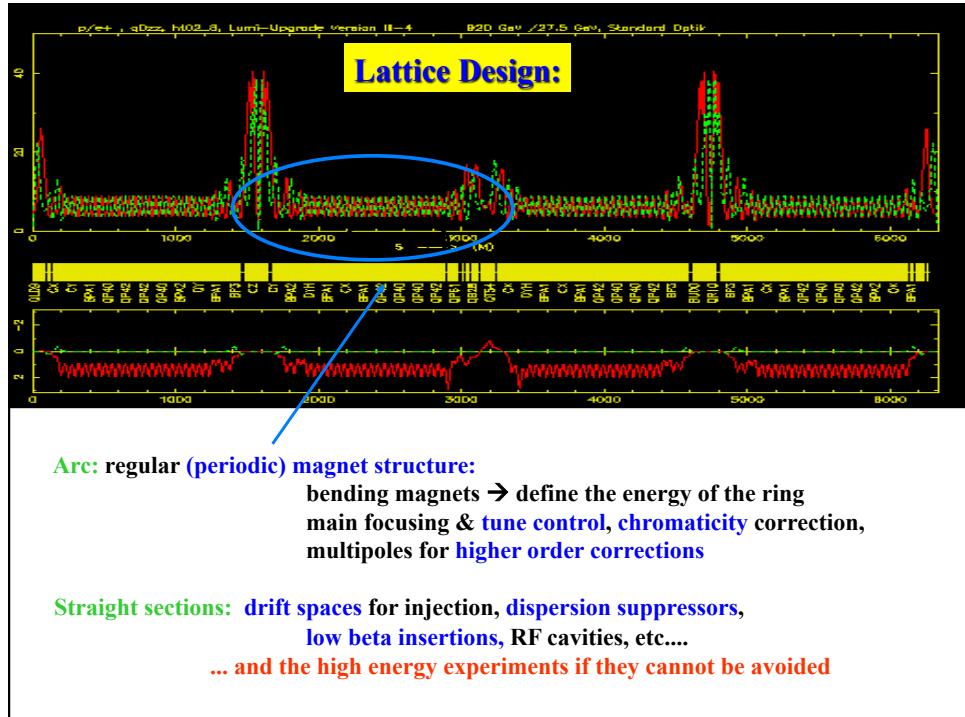
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_l = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

$$\boxed{\beta(s) = \beta_0 - 2l * \alpha_0 + l^2 * \gamma_0}$$

Stability ...?

$$\text{trace}(M) = 1 + 1 = 2$$

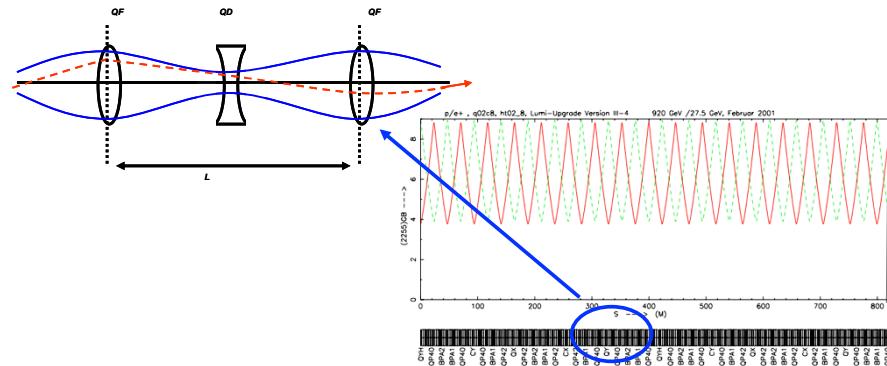
→ A periodic solution doesn't exist in a lattice built exclusively out of drift spaces.



3.) The FoDo-Lattice

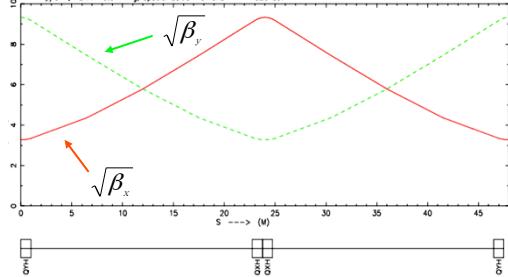
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with **nothing** in between.

(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45^\circ$,
→ calculate the twiss parameters for a periodic solution

Periodic Solution of a FoDo Cell



Output of the optics program:

Nr	Type	Length m	Strength 1/m ²	β_x m	α_x 1/2π	φ_x 1/2π	β_z m	α_z 1/2π	φ_z 1/2π
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

$QX = 0,125 \quad QZ = 0,125$

$$0.125 * 2\pi = 45^\circ$$

Can we understand what the optics code is doing ?

matrices

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l_q) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l_q) \\ -\sqrt{K} \sin(\sqrt{K} * l_q) & \cos(\sqrt{K} * l_q) \end{pmatrix}, \quad M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix}$$

strength and length of the FoDo elements

$$K = +/- 0.54102 \text{ m}^{-2}$$

$$lq = 0.5 \text{ m}$$

$$ld = 2.5 \text{ m}$$

The matrix for the **complete cell** is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and **multiplying out** ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for 1 period gives us all the information that we need !

1.) is the motion stable?

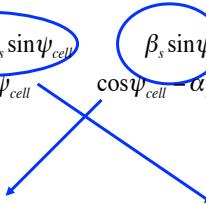
$$\text{trace}(M_{FoDo}) = 1.415 \rightarrow$$

< 2

2.) Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi_{cell} + \alpha_s \sin\psi_{cell} & \beta_s \sin\psi_{cell} \\ -\gamma_s \sin\psi_{cell} & \cos\psi_{cell} - \alpha_s \sin\psi_{cell} \end{pmatrix}$$

$\cos\psi_{cell} = \frac{1}{2} \text{trace}(M) = 0.707$
 $\psi_{cell} = \cos^{-1}\left(\frac{1}{2} \text{trace}(M)\right) = 45^\circ$



3.) hor β -function

$$\beta = \frac{m_{12}}{\sin\psi_{cell}} = 11.611 \text{ m}$$

4.) hor α -function

$$\alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = 0$$

Can we do a bit easier ?
We can ... in thin lens approximation !

Matrix of a focusing quadrupole magnet:

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K} * l) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} * l) \\ -\sqrt{K} \sin(\sqrt{K} * l) & \cos(\sqrt{K} * l) \end{pmatrix}$$

If the **focal length f** is much larger than the length of the quadrupole magnet,

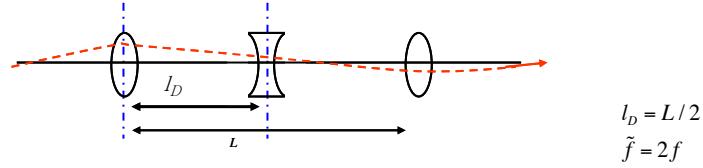
$$f = \cancel{k l_q} \gg l_q$$

the transfer matrix can be approximated using ↪

$$k l_q = \text{const}, \quad l_q \rightarrow 0$$

$$M = \begin{pmatrix} 1 & 0 \\ \cancel{1/f} & 1 \end{pmatrix}$$

4.) FoDo in thin lens approximation



Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

$$M_{halfCell} = M_{QD/2} * M_{ID} * M_{QF/2}$$

$$M_{halfCell} = \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix} * \begin{pmatrix} 1 & l_D \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ -1/\tilde{f} & 1 \end{pmatrix} \quad \text{note: } \tilde{f} \text{ denotes the focusing strength of half a quadrupole, so } \tilde{f} = 2f$$

$$M_{halfCell} = \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix} \quad \text{for the second half cell set } f \rightarrow -f$$

FoDo in thin lens approximation

Matrix for the complete FoDo cell

$$M = \begin{pmatrix} 1 + l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 - l_D/\tilde{f} \end{pmatrix} * \begin{pmatrix} 1 - l_D/\tilde{f} & l_D \\ -l_D/\tilde{f}^2 & 1 + l_D/\tilde{f} \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{2l_d^2}{\tilde{f}^2} & 2l_d(1 + \frac{l_d}{\tilde{f}}) \\ 2(\frac{l_d^2}{\tilde{f}^3} - \frac{l_d}{\tilde{f}^2}) & 1 - 2\frac{l_d^2}{\tilde{f}^2} \end{pmatrix}$$

Now we know, that the phase advance is related to the transfer matrix by

$$\cos\psi_{cell} = \frac{1}{2} \text{trace}(M) = \frac{1}{2} * (2 - \frac{4l_d^2}{\tilde{f}^2}) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

After some beer and with a little bit of trigonometric gymnastics

$$\cos(x) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) = 1 - 2\sin^2(\frac{x}{2})$$

we can calculate the phase advance as a function of the FoDo parameter ...

$$\cos\psi_{cell} = 1 - 2 \sin^2(\psi_{cell}/2) = 1 - \frac{2l_d^2}{\tilde{f}^2}$$

$$\sin(\psi_{cell}/2) = l_d / \tilde{f} = \frac{L_{cell}}{2\tilde{f}}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}$$

Example:
45-degree Cell

$$L_{cell} = l_{QF} + l_D + l_{QD} + l_D = 0.5m + 2.5m + 0.5m + 2.5m = 6m$$

$$1/f = k * l_Q = 0.5m * 0.541 m^{-2} = 0.27 m^{-1}$$

$$\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405$$

$$\rightarrow \psi_{cell} = 47.8^\circ$$

$$\rightarrow \beta = 11.4 m$$

Remember:
Exact calculation yields:

$$\rightarrow \psi_{cell} = 45^\circ$$

$$\rightarrow \beta = 11.6 m$$

Stability in a FoDo structure



SPS Lattice

$$M_{FoDo} = \begin{pmatrix} 1 - \frac{2l_D^2}{\tilde{f}^2} & 2l_D(1 + \frac{l_D}{\tilde{f}}) \\ 2(\frac{l_D^2}{\tilde{f}^3} - \frac{l_D}{\tilde{f}^2}) & 1 - 2\frac{l_D^2}{\tilde{f}^2} \end{pmatrix}$$

Stability requires:

$$|Trace(M)| < 2$$

$$|Trace(M)| = \left| 2 - \frac{4l_d^2}{\tilde{f}^2} \right| < 2$$

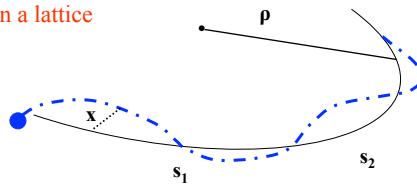
$$\rightarrow f > \frac{L_{cell}}{4}$$

For stability the focal length has to be larger than a quarter of the cell length
... don't focus too strong!

Transformation Matrix in Terms of the Twiss Parameters

Transformation of the coordinate vector (x, x') in a lattice

$$\begin{pmatrix} \mathbf{x}(s) \\ \mathbf{x}'(s) \end{pmatrix} = M_{s1,s2} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}'_0 \end{pmatrix}$$



General solution of the equation of motion

$$x(s) = \sqrt{\varepsilon * \beta(s)} * \cos(\psi(s) + \varphi)$$

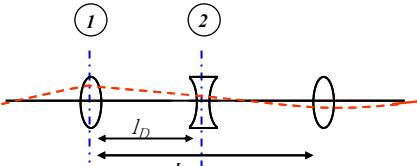
$$x'(s) = \sqrt{\varepsilon / \beta(s)} * \{ \alpha(s) \cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \}$$

Transformation of the coordinate vector (x, x') expressed as a function of the twiss parameters

$$M_{1 \rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \psi_{12} + \alpha_1 \sin \psi_{12}) & \sqrt{\beta_1 \beta_2} \sin \psi_{12} \\ \frac{(\alpha_1 - \alpha_2) \cos \psi_{12} - (1 + \alpha_1 \alpha_2) \sin \psi_{12}}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \psi_{12} - \alpha_2 \sin \psi_{12}) \end{pmatrix}$$

Transfer Matrix for half a FoDo cell:

$$M_{halfcell} = \begin{pmatrix} 1 - \frac{l_p}{f} & l_D \\ -\frac{l_D}{f^2} & 1 + \frac{l_D}{f} \end{pmatrix}$$



Compare to the twiss parameter form of M

$$M_{1 \rightarrow 2} = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \psi_{12} + \alpha_1 \sin \psi_{12}) & \sqrt{\beta_1 \beta_2} \sin \psi_{12} \\ \frac{(\alpha_1 - \alpha_2) \cos \psi_{12} - (1 + \alpha_1 \alpha_2) \sin \psi_{12}}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \psi_{12} - \alpha_2 \sin \psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we always have $\alpha = 0$, and the half cell will lead us from β_{max} to β_{min}

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta'}} \cos \frac{\psi_{cell}}{2} & \sqrt{\beta \beta'} \sin \frac{\psi_{cell}}{2} \\ \frac{-1}{\sqrt{\beta \beta'}} \sin \frac{\psi_{cell}}{2} & \sqrt{\frac{\beta}{\beta'}} \cos \frac{\psi_{cell}}{2} \end{pmatrix}$$

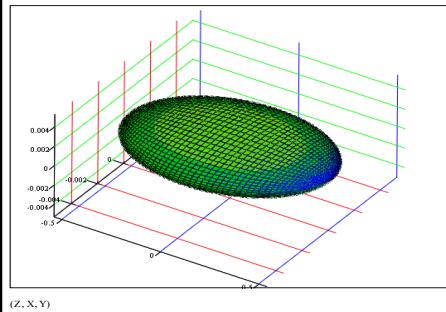
Solving for β_{\max} and β_{\min} and remembering that $\sin \frac{\psi_{cell}}{2} = \frac{l_d}{\tilde{f}} = \frac{L}{4f}$

$$\frac{m_{22}}{m_{11}} = \frac{\hat{\beta}}{\check{\beta}} = \frac{1 + l_d / \tilde{f}}{1 - l_d / \tilde{f}} = \frac{1 + \sin(\psi_{cell}/2)}{1 - \sin(\psi_{cell}/2)}$$

$$\frac{m_{12}}{m_{21}} = \hat{\beta} \check{\beta} = \tilde{f}^2 = \frac{l_d^2}{\sin^2(\psi_{cell}/2)}$$

$$\hat{\beta} = \frac{(1 + \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}}$$

$$\check{\beta} = \frac{(1 - \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}}$$



The maximum and minimum values of the β -function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger β

typical shape of a proton bunch in a FoDo Cell

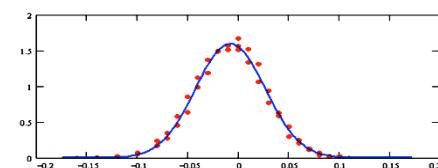
5.) Beam dimension:

Optimisation of the FoDo Phase advance

In both planes a gaussian particle distribution is assumed, given by the beam emittance ϵ and the β -function

$$\sigma = \sqrt{\epsilon \beta}$$

HERA beam size

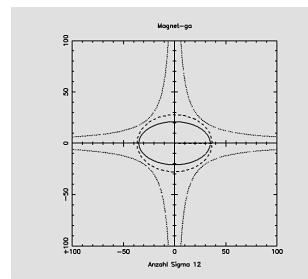


In general proton beams are „round“ in the sense that

$$\epsilon_x \approx \epsilon_y$$

So for highest aperture we have to minimise the β -function in both planes:

$$r^2 = \epsilon_x \beta_x + \epsilon_y \beta_y$$



typical beam envelope, vacuum chamber and pole shape in a foc. Quadrupole lens in HERA

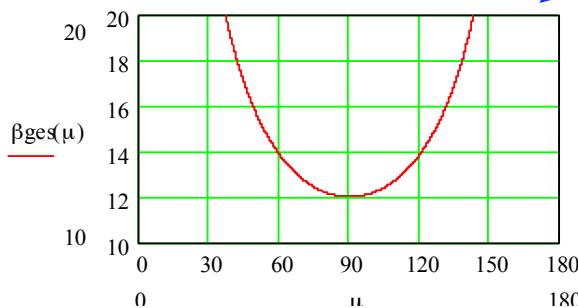
Optimisation of the FoDo Phase advance

$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

search for the phase advance μ that results in a minimum of the sum of the beta's

$$\hat{\beta} + \bar{\beta} = \frac{(1 + \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}} + \frac{(1 - \sin \frac{\psi_{cell}}{2})L}{\sin \psi_{cell}}$$

$$\hat{\beta} + \bar{\beta} = \frac{2L}{\sin \psi_{cell}} \quad \frac{d}{d\psi_{cell}} \left(\frac{2L}{\sin \psi_{cell}} \right) = 0$$



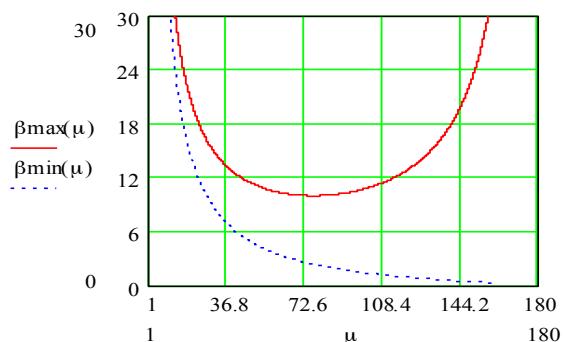
$$\frac{L}{\sin^2 \psi_{cell}} * \cos \psi_{cell} = 0 \rightarrow \psi_{cell} = 90^\circ$$

Electrons are different

electron beams are usually flat, $\varepsilon_y \approx 2 - 10\% \varepsilon_x$
 → optimise only β_{hor}

$$\frac{d}{d\psi_{cell}} (\hat{\beta}) = \frac{d}{d\psi_{cell}} \frac{L(1 + \sin \frac{\psi_{cell}}{2})}{\sin \psi_{cell}} = 0 \rightarrow \psi_{cell} = 76^\circ$$

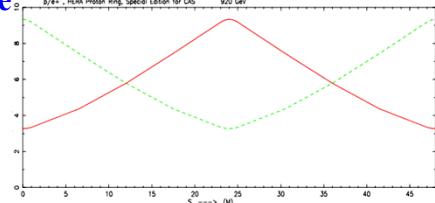
red curve: β_{max}
 blue curve: β_{min}
 as a function of the phase advance μ



Orbit distortions in a periodic lattice

field error of a dipole/distorted quadrupole

$$\rightarrow \delta(mrad) = \frac{ds}{\rho} = \frac{\int B ds}{p/e}$$



the particle will follow a new closed trajectory, the distorted orbit:

$$x(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\pi Q)} * \int \sqrt{\beta(\tilde{s})} \frac{1}{\rho(\tilde{s})} \cos(|\psi(\tilde{s}) - \psi(s)| - \pi Q) d\tilde{s}$$

* the orbit amplitude will be large if the β function at the location of the kick is large
 $\beta(\tilde{s})$ indicates the sensitivity of the beam \rightarrow here orbit correctors should be placed in the lattice

* the orbit amplitude will be large at places where in the lattice $\beta(s)$ is large \rightarrow here beam position monitors should be installed

Orbit Correctors and Beam Instrumentation in a Storage Ring



Elsa ring, Bonn

*

Résumé

1.) Dipole strength

$$\int B ds = N * B_0 * l_{\text{eff}} = 2\pi \frac{p}{q}$$

l_{eff} effective magnet length, N number of magnets

2.) Stability condition

$$\text{Trace}(M) < 2$$

for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine

3.) Transfer matrix for periodic cell

$$M(s) = \begin{pmatrix} \cos\psi_{\text{cell}} + \alpha_s \sin\psi_{\text{cell}} & \beta_s \sin\psi_{\text{cell}} \\ -\gamma_s \sin\psi_{\text{cell}} & \cos\psi_{\text{cell}} - \alpha_s \sin\psi_{\text{cell}} \end{pmatrix}$$

α, β, γ depend on the position s in the ring, μ (phase advance) is independent of s

4.) Thin lens approximation

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f_Q} & 1 \end{pmatrix} \quad f_Q = \frac{1}{k_Q l_Q}$$

focal length of the quadrupole magnet $f_Q = 1/(k_Q l_Q) \gg l_Q$

5.) Tune (rough estimate)

$$\psi_{\text{period}} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Tune = phase advance
in units of 2π

$$Q = N * \frac{\psi_{\text{period}}}{2\pi} = \frac{1}{2\pi} * \oint \frac{ds}{\beta(s)} \approx \frac{1}{2\pi} * \frac{2\pi \bar{R}}{\bar{\beta}} = \frac{\bar{R}}{\bar{\beta}}$$

$\bar{R}, \bar{\beta}$ average radius
and β -function

$$Q \approx \frac{\bar{R}}{\bar{\beta}}$$

6.) Phase advance per FoDo cell
(thin lens appro)

$$\sin \frac{\psi_{\text{cell}}}{2} = \frac{l_d}{f} = \frac{L_{\text{cell}}}{4f_Q}$$

L_{cell} length of the complete FoDo cell, f_Q focal length of the quadrupole, μ phase advance per cell

7.) Stability in a FoDo cell
(thin lens approx)

$$f_Q > \frac{L_{\text{cell}}}{4}$$

8.) Beta functions in a FoDo cell
(thin lens approx)

$$\hat{\beta} = \frac{(1 + \sin \frac{\psi_{\text{cell}}}{2}) L_{\text{cell}}}{\sin \psi_{\text{cell}}} \quad \bar{\beta} = \frac{(1 - \sin \frac{\psi_{\text{cell}}}{2}) L_{\text{cell}}}{\sin \psi_{\text{cell}}}$$

L_{cell} length of the complete FoDo cell, μ phase advance per cell