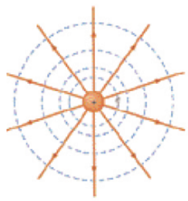
SPACE CHARGE DOMINATED BEAMS

Massimo Ferrario

INFN-LNF



THE CERN ACCELERATOR SCHOOL

EQUATION OF MOTION

The motion of charged particles is governed by the Lorentz force :

$$\frac{d(m\gamma \boldsymbol{v})}{dt} = \boldsymbol{F}_{e.m.}^{ext} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Where $\boldsymbol{m}$ is the rest mass, $\boldsymbol{\gamma}$ the relativistic factor and $\boldsymbol{v}$ the particle velocity

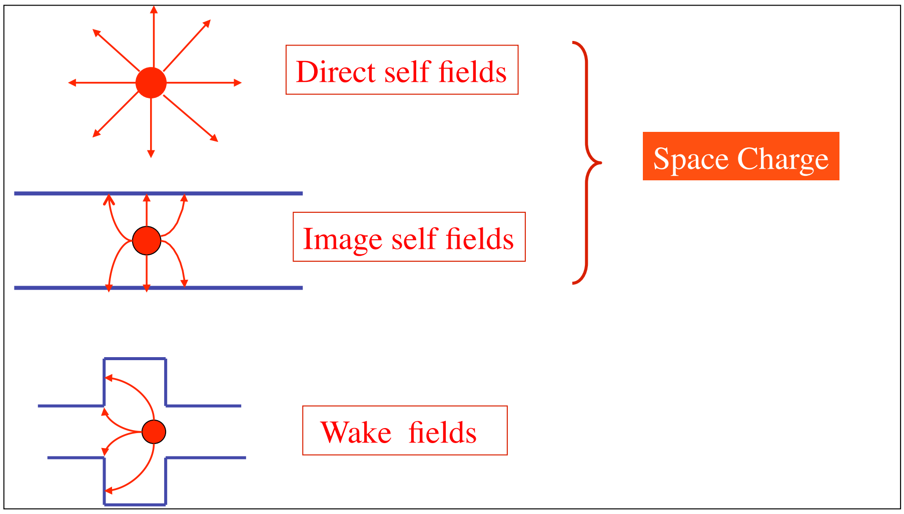
Charged particles are accelerated, guided and confined by external electromagnetic fields.

Acceleration is provided by the electric field of the RF cavity

Magnetic fields are produced in the bending magnets for guiding the charges on the reference trajectory (orbit), in the quadrupoles for the transverse confinement, in the sextupoles for the chromaticity correction.

SELF FIELDS AND WAKE FIELDS

There is another important source of e.m. fields : **the beam itself**

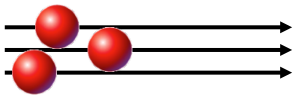


These fields depend on the current and on the charges velocity.

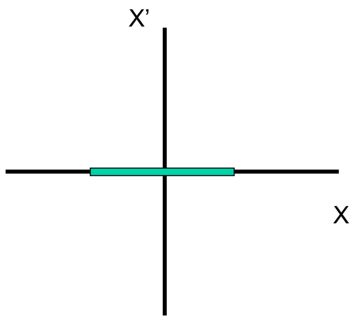
They are responsible of many phenomena of beam dynamics:

- energy loss *(wake-fields)*
- energy spread and emittance degradation
- shift of the synchronous phase and frequency (tune)
- shift of the betatron frequencies (tunes)
- instabilities.

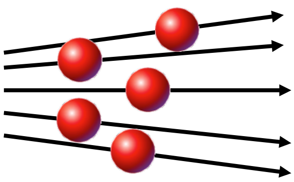
Trace space of an ideal laminar beam

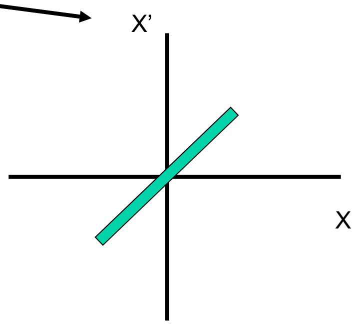


$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

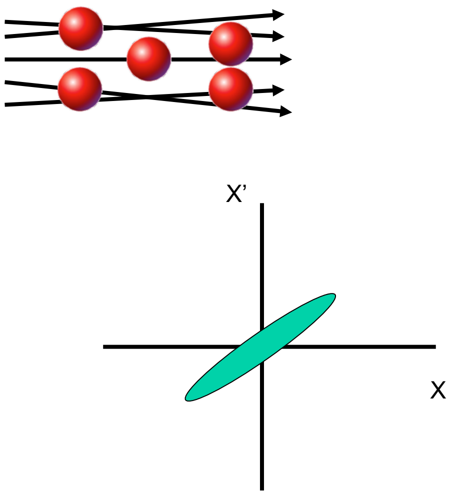


Trace space laminar beam





Trace space of non laminar beam

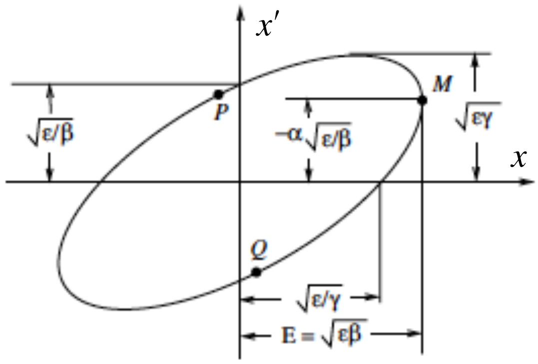


Geometric emittance (Liouville): $\boxed{\varepsilon_g}$

Ellipse equation: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters: $\beta\gamma - \alpha^2 = 1 \qquad \beta' = -2\alpha$

Ellipse area: $A = \pi\varepsilon_g$



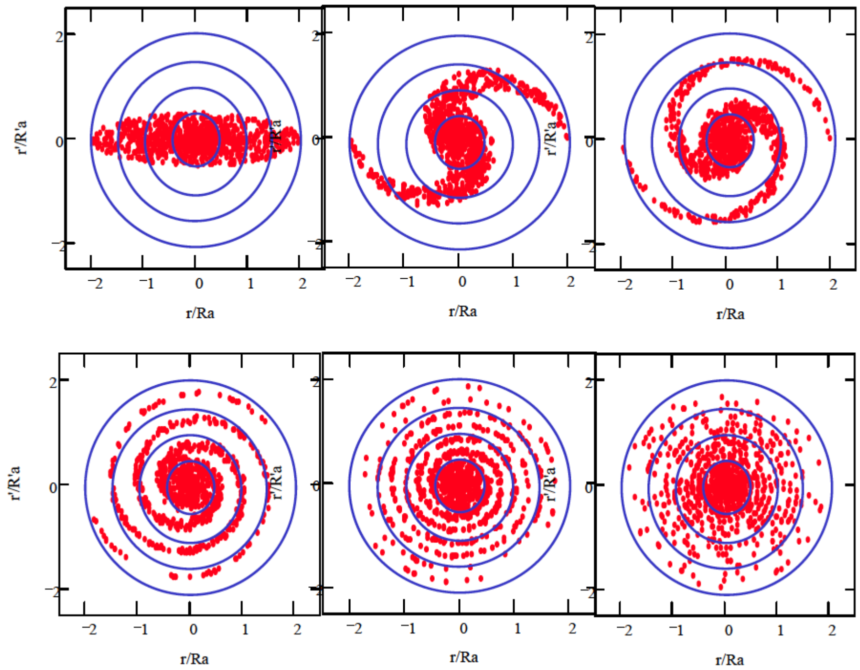
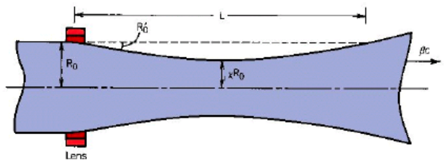
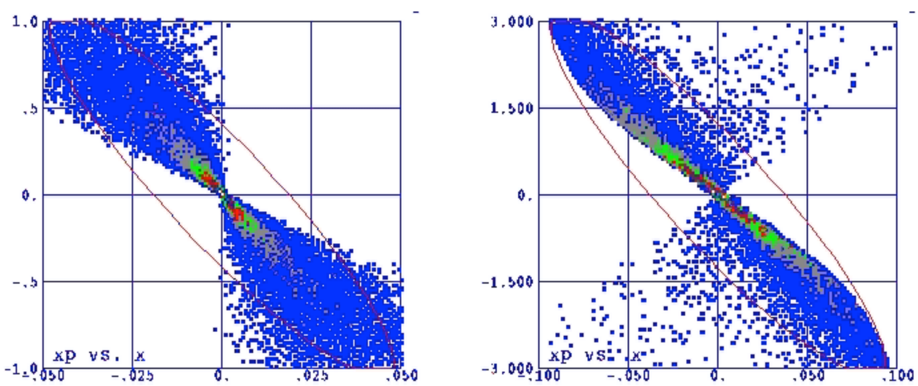


Fig. 17: Filamentation of mismatched beam in non-linear force

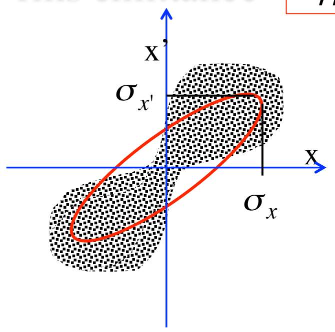
Phase space evolution



No space charge => cross over   With space charge => no cross over



rms emittance ε_{rms}



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1 \qquad \boxed{f'(x,x') = 0}$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}}$$

such that:
$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$
$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

Since: $\alpha = -\dfrac{\beta'}{2}$

it follows: $\alpha = -\dfrac{1}{2\varepsilon_{rms}} \dfrac{d}{dz}\langle x^2 \rangle = -\dfrac{\langle x x' \rangle}{\varepsilon_{rms}} = -\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}\\ \sigma_x' &= \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}\\ \sigma_{xx'} &= \langle x x' \rangle = -\alpha \varepsilon_{rms}\end{aligned}}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

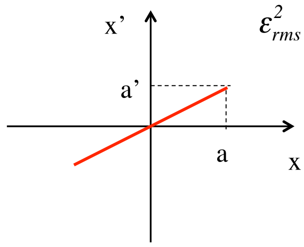
Substituting α, β, γ we get $\dfrac{\sigma_{x'}^2}{\varepsilon_{rms}} \dfrac{\sigma_x^2}{\varepsilon_{rms}} - \left(\dfrac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

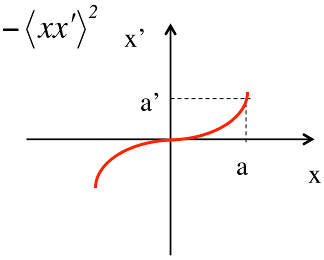
We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2\right)}} \qquad x' = \frac{p_x}{p_z}$$

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?

$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$$





Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When $n = 1$ ==> $\varepsilon_{rms} = 0$

When $n \not\leq 1$ ==> $\varepsilon_{rms} \not\leq 0$

Constant under linear transformation only

$$\frac{\mathrm{d}}{\mathrm{d}z} \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 = 2\langle xx' \rangle \langle x'^2 \rangle + 2\langle x^2 \rangle \langle x' \rangle \langle x'' \rangle - 2\langle xx'' \rangle \langle xx' \rangle = 0$$

For linear transformations, $x'' = -k_x^2 x$, and the right-hand side of the equation is

$$2k_x^2 \langle x^2 \rangle \langle xx' \rangle - 2\langle x^2 \rangle \langle xx' \rangle k_x^2 = 0,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}z} \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 = 0$$

And without acceleration: $\boxed{x' = \frac{p_x}{p_z}}$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c\beta\gamma x'$ $\quad p_z \approx p$

$$\boxed{\varepsilon_{n,rms} = \sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2\right)} \approx \langle\beta\gamma\rangle\varepsilon_{rms}}$$

Liouville theorem: the density of particles n, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant under linear tranformations.

$$\boxed{\frac{dn}{dt} = 0}$$

It hold also in the projected phase spaces (x,p_x),(y,p_y)(,z,p_z) provided that there are no couplings

Envelope Equation without Acceleration

Now take the derivatives:

$$\frac{d\sigma_x}{dz} = \frac{d}{dz}\sqrt{\langle x^2\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\langle x^2\rangle = \frac{1}{2\sigma_x}2\langle xx'\rangle = \frac{\sigma_{xx'}}{\sigma_x}$$

$$\frac{d^2\sigma_x}{dz^2} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\langle x'^2\rangle - \langle xx''\rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{\sigma_{x'}^2 + \langle xx''\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}$$

And simplify: $\boxed{\sigma_x'' = \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} - \frac{\langle xx''\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\langle xx''\rangle}{\sigma_x}}$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\boxed{\sigma_x'' - \frac{\langle xx''\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

Lets now consider for example the simple case with $\langle xx'' \rangle = 0$ describing a beam drifting in the free space.

The envelope equation reduces to:

$$\boxed{\sigma_x^3 \sigma_x'' = \varepsilon_{rms}^2}$$

With initial conditions σ_o, σ_o' at z_o, depending on the upstream transport channel, the equation has a hyperbolic solution:

$$\sigma(z) = \sqrt{\left(\sigma_o + \sigma_o'(z - z_o)\right)^2 + \frac{\varepsilon_{rms}^2}{\sigma_o^2}(z - z_o)^2}$$

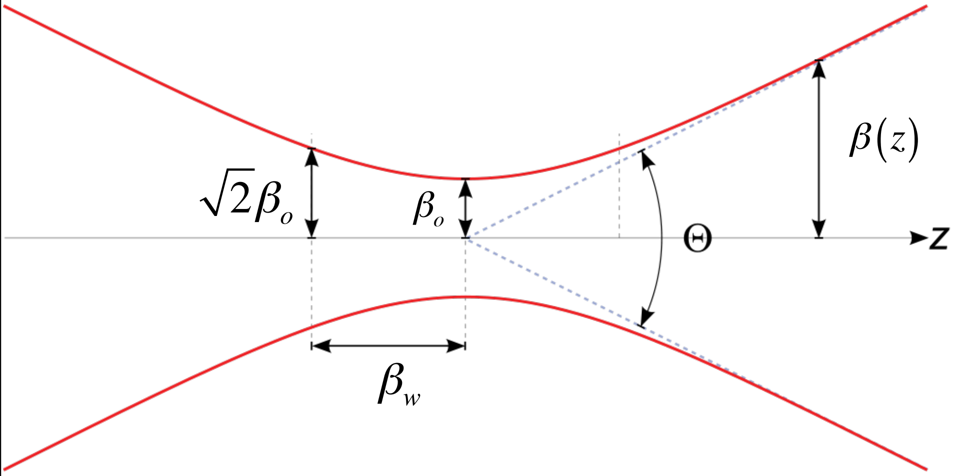
Considering the case $\sigma_o' = 0$ (beam at waist)

and using the definition $\sigma_x = \sqrt{\beta \varepsilon_{rms}}$

the solution is often written in terms of the β function as:

$$\sigma(z) = \sigma_o \sqrt{1 + \left(\frac{z - z_o}{\beta_w}\right)^2}$$

This relation indicates that without any external focusing element the beam envelope increases from the beam waist by a factor $\sqrt{2}$ with a characteristic length $\beta_w = \dfrac{\sigma_o^2}{\varepsilon_{rms}}$



For an effective transport of a beam with finite emittance is mandatory to make use of some external force providing beam confinement in the transport or accelerating line.

$$\boxed{\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle xx'' \rangle = -k_x^2 \langle x^2 \rangle$

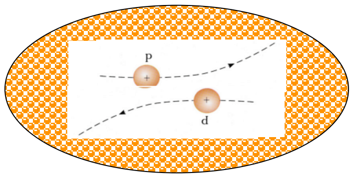
$$\boxed{\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

We obtain the rms envelope equation with a linear focusing force in which the rms emittance enters as defocusing pressure like term.

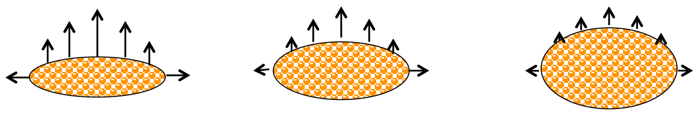
Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

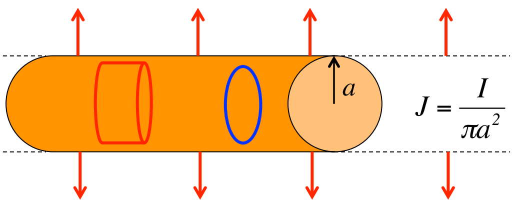
1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



Continuous Uniform Cylindrical Beam Model



$$\rho = \frac{I}{\pi a^2 v} \qquad J = \frac{I}{\pi a^2}$$

Gauss' s law

$$\int \varepsilon_o E \cdot dS = \int \rho dV$$

$$E_r = \frac{I}{2\pi\varepsilon_o a^2 v} r \quad for \quad r \le a$$

$$E_r = \frac{I}{2\pi\varepsilon_o v} \frac{1}{r} \quad for \quad r > a$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

Ampere' s law

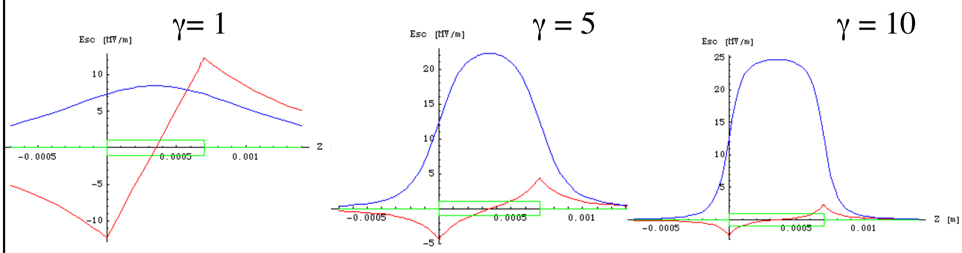
$$\int B \cdot dl = \mu_o \int J \cdot dS$$

$$B_\vartheta = \mu_o \frac{Ir}{2\pi a^2} \quad for \quad r \le a$$

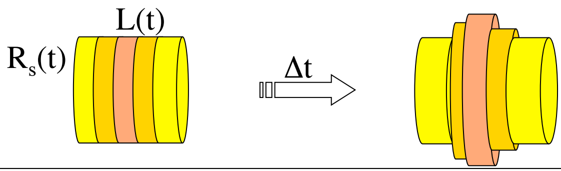
$$B_\vartheta = \mu_o \frac{I}{2\pi r} \quad for \quad r > a$$

Bunched Uniform Cylindrical Beam Model

$$E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma) \qquad E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$



$$F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)$$



$$B_\vartheta = \frac{\beta}{c} E_r$$

Lorentz Force

$$E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right)E_r = \frac{eE_r}{\gamma^2}$$

is a **linear** function of the transverse coordinate

$$\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect.

$$F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g(s,\gamma)$$

Envelope Equation with Space Charge

Single particle transverse motion: $\frac{dp_x}{dt} = F_x$ $\qquad p_x = p\,x' = \beta\gamma m_o c x'$

$$\frac{d}{dt}(px') = \beta c \frac{d}{dz}(p\,x') = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$\boxed{x'' = \frac{k_{sc}(s,\gamma)}{\sigma_x^2} x}$$

Space Charge de-focusing force

Generalized perveance

$$k_{sc}(s,\gamma) = \frac{2I}{I_A(\beta\gamma)^3} g(s,\gamma) \qquad I_A = \frac{4\pi\varepsilon_o m_o c^3}{e} = 17kA$$

Now we can calculate the term $\langle xx'' \rangle$ that enters in the envelope equation

$$\boxed{\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx'' \rangle}{\sigma_x}} \qquad \langle xx'' \rangle = \frac{k_{sc}}{\sigma_x^2} \langle x^2 \rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + k^2 \sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2 \sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\boxed{\rho = \frac{(\beta\gamma)^2 k_{sc} \sigma_x^2}{\varepsilon_n^2}}$

The beam undergoes two regimes along the accelerator

$$\sigma_x'' + k^2\sigma_x = \cancel{\frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3}} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1 Laminar Beam

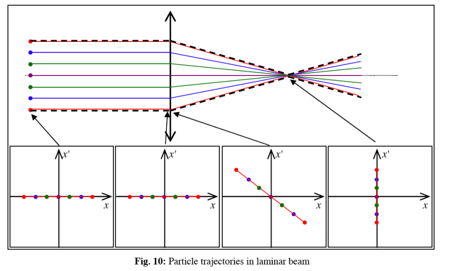


Fig. 10: Particle trajectories in laminar beam

$$\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \cancel{\frac{k_{sc}}{\sigma_x}}$$

ρ<<1 Thermal Beam

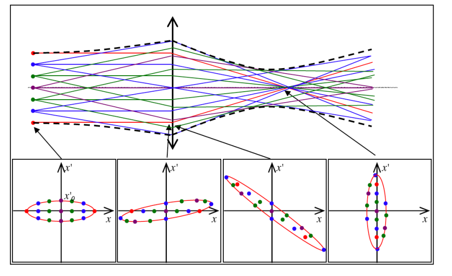


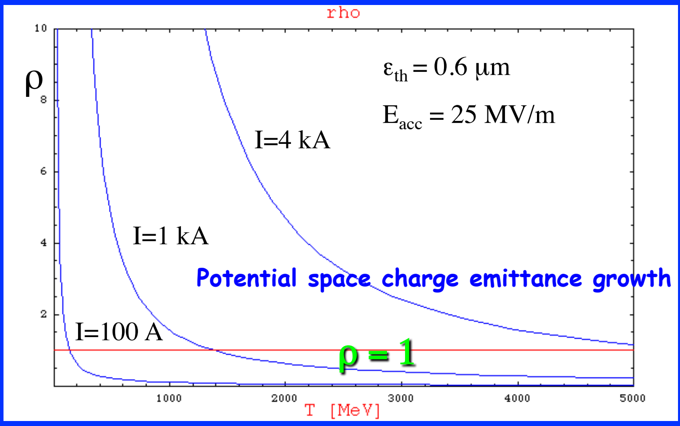
Fig. 11: Particle trajectories in non-zero emittance beam

Laminarity parameter

$$\rho = \frac{2I\sigma^2}{\gamma I_A \varepsilon_n^2} \equiv \frac{2I\sigma_q^2}{\gamma I_A \varepsilon_n^2} = \frac{4I^2}{\gamma'^2 I_A^2 \varepsilon_n^2 \gamma^2}$$

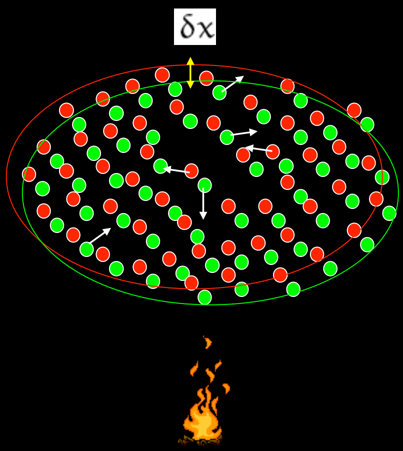
Transition Energy (ρ=1)

$$\gamma_{tr} = \frac{2I}{\gamma' I_A \varepsilon_n}$$



Surface charge density

$$\sigma = e\, n\, \delta x$$



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\, n\, \delta x/\epsilon_0$$

Restoring force

$$m \frac{d^2 \delta x}{dt^2} = e\, E_x = -m\, {\omega_p}^2\, \delta x$$

Plasma frequency

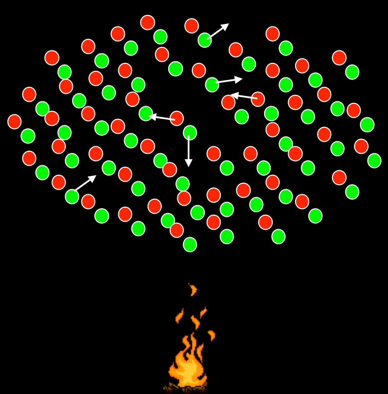
$$ {\omega_p}^2 = \frac{n\, e^2}{\epsilon_0\, m}$$

Plasma oscillations

$$\delta x = (\delta x)_0 \cos(\omega_p\, t)$$

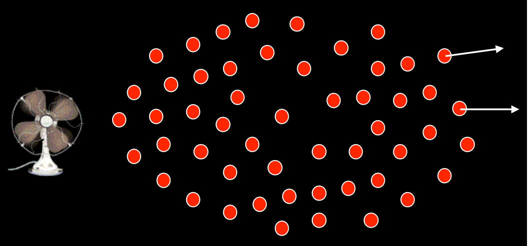
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation



Single Component Cold Relativistic Plasma

Magnetic focusing



Magnetic focusing

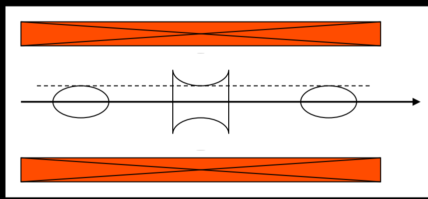
Single Component Relativistic Plasma

$$\sigma'' + k_s^2 \sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

$$\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)$$

$$\delta\sigma''(s) + 2k_s^2 \delta\sigma(s) = 0$$

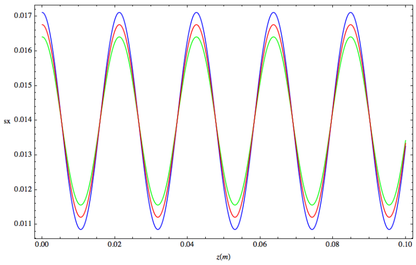
$$\delta\sigma(s) = \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

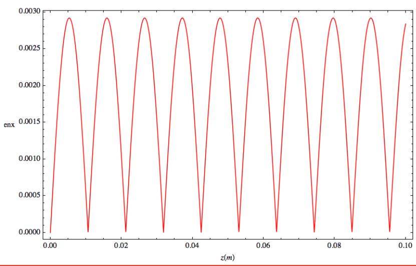
$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\cos\left(\sqrt{2}k_s z\right)$$

Envelope oscillations drive Emittance oscillations

σ(z)

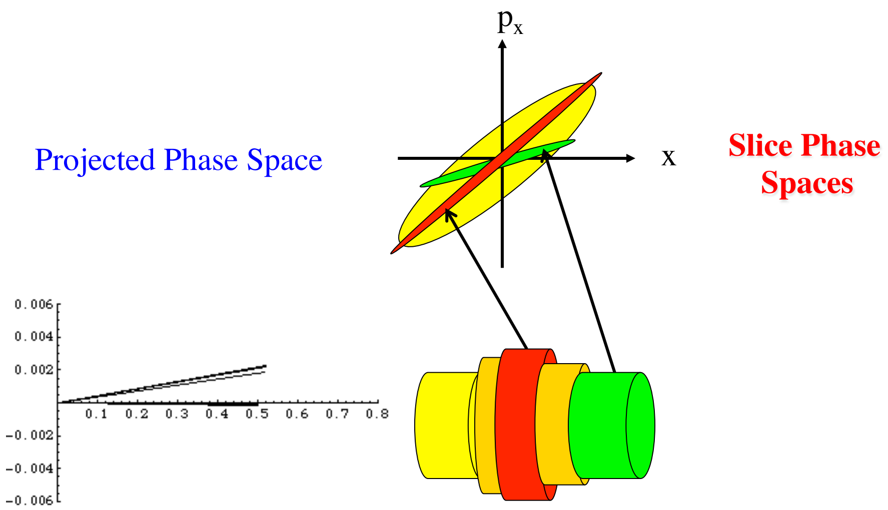


ε(z)

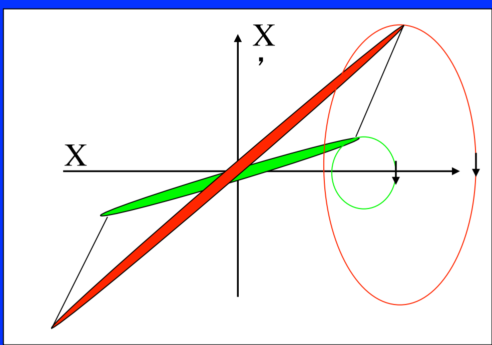


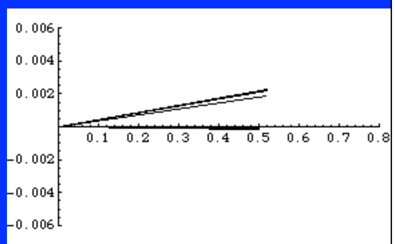
$$\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^{\ 2}} = \sqrt{\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)} \approx \left|\sin\left(\sqrt{2}k_s z\right)\right|$$

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam



Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes





energy spread induces decoherence

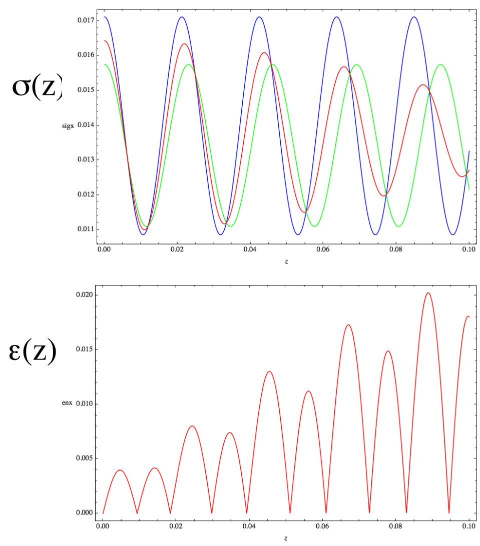
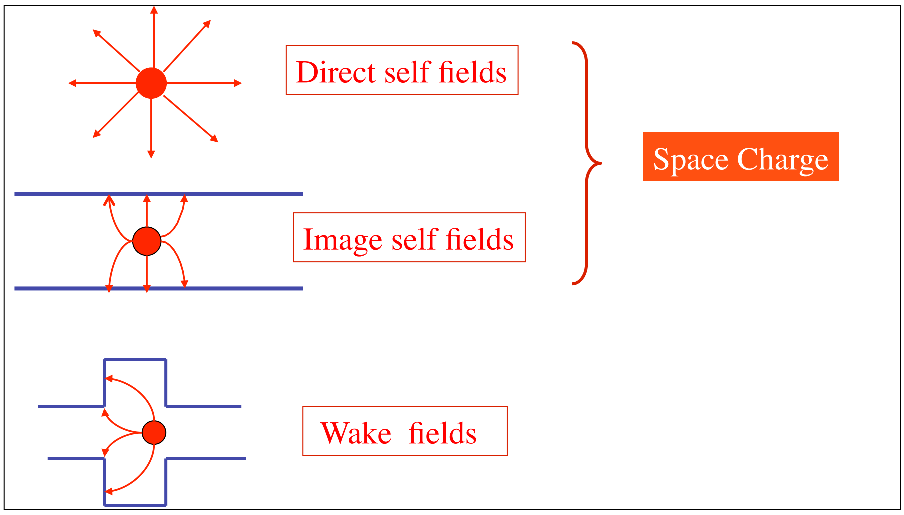


IMAGE SELF FIELDS

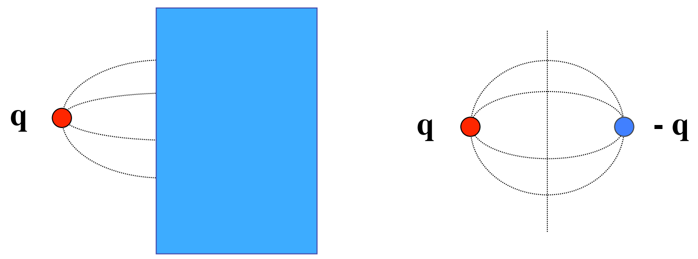
There is another important source of e.m. fields : **the beam itself**



Static Fields: conducting or magnetic screens

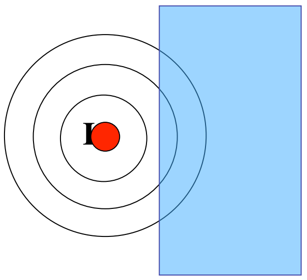
Let us consider a point charge q close to a conducting screen.

The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen

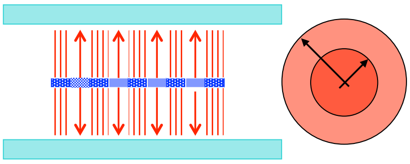


A constant current in the free space produces circular magnetic field.

If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.



Circular Perfectly Conducting Pipe (Beam at Center)



In the case of charge distribution, and $\gamma \to \infty$, the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed like in the static case, applying the Gauss and Ampere laws.

$$\lambda_o = \rho \pi a^2$$

$$\lambda(r) = \lambda_o (r/a)^2$$

$$J = \beta c \rho$$

$$I = J \pi a^2 = \beta c \lambda_o (A)$$

for $r < a$

$$E_r(r) = \frac{\lambda_o\, r}{2\pi\varepsilon_o a^2}$$

$$B_\theta(r) = \frac{\beta}{c} E_r(r) = \frac{\lambda_o \beta}{2\pi\varepsilon_o c} \frac{r}{a^2}$$

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1-\beta^2\right) E_r = \frac{eE_r}{\gamma^2} = \frac{e\lambda_o\, r}{2\pi\varepsilon_o \gamma^2 a^2}$$

there is a cancellation of the electric and magnetic forces

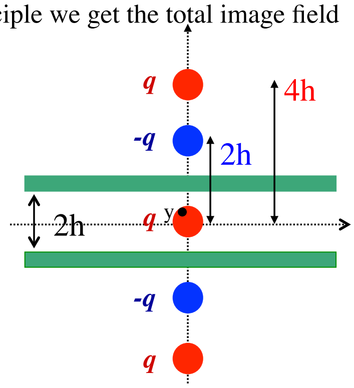
Parallel Plates (beam at center)

In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius *a* between two conducting plates *2h* apart. By applying the superposition principle we get the total image field at a position *y* inside the beam.

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_o} \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{2nh+y} - \frac{1}{2nh-y}\right]$$

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_o} \sum_{n=1}^{\infty} (-1)^n \frac{-2y}{(2nh)^2 - y^2} \cong \frac{\lambda(z)}{4\pi\,\varepsilon_o h^2} \frac{\pi^2}{12} y$$

Where we have assumed: $h >> a > y$.



For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field. We do not need “image currents “As a consequence there is no cancellation effect for the fields produced by the "image" charges.

From the divergence equation we derive also the other transverse component, notice the opposite sign:

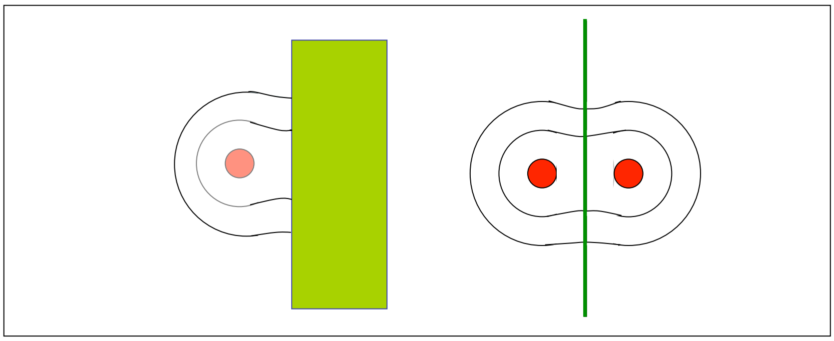
$$\frac{\partial}{\partial x}E_x^{im} = -\frac{\partial}{\partial y}E_y^{im} \Rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\,\varepsilon_o h^2}\frac{\pi^2}{12}x$$

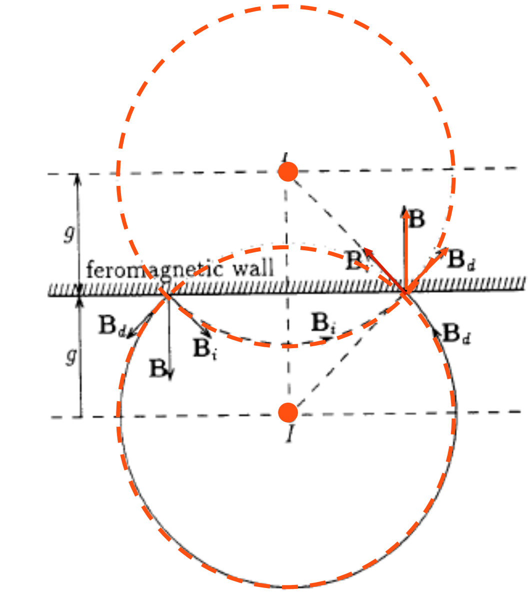
Including also the direct space charge force, we get:

$$\begin{cases} F_x(z,x) = \dfrac{e\lambda(z)x}{\pi\ \varepsilon_o}\left(\dfrac{1}{2a^2\gamma^2} - \dfrac{\pi^2}{48h^2}\right) \\ F_y(z,x) = \dfrac{e\lambda(z)y}{\pi\ \varepsilon_o}\left(\dfrac{1}{2a^2\gamma^2} + \dfrac{\pi^2}{48h^2}\right) \end{cases}$$

Therefore, for γ>>1, and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.

For ferromagnetic type, with μ_r>>1, the very high magnetic permeability makes the tangential magnetic field zero at the boundary so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.

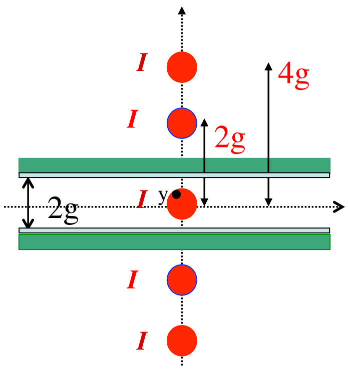




Satisfying a magnetic boundary condition by an image current

A. Hofmann

In analogy with the image method we get the magnetic field, in the region outside the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction.



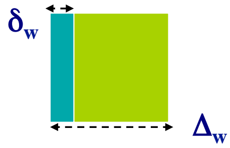
$$B_x^{im}(z,y) = \frac{\mu_o \beta c \lambda(z)}{2\pi} \sum_{n=1}^{\infty} \left[\frac{1}{2ng - y} - \frac{1}{2ng + y} \right]$$

$$B_x^{im}(z,y) \cong \frac{\mu_o \beta c \lambda(z) y}{4\pi g^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\mu_o \beta c \lambda(z) \pi^2 y}{24\pi g^2}$$

$$F_x^{im}(z,x) \cong \frac{\beta^2 \lambda(z) \pi^2}{24\pi \varepsilon_o g^2} x$$

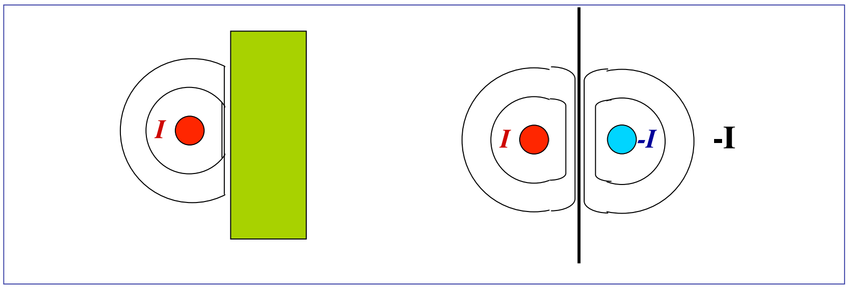
Time-varying fields

It is necessary to compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor.

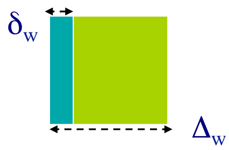


$$\delta_w \cong \sqrt{\frac{2}{\omega\sigma\mu}}$$

If the fields penetrate and pass through the material, we are practically in the static boundary conditions case. Conversely, if the skin depth is very small, fields do not penetrate, the electric filed lines are perpendicular to the wall, as in the static case, while the magnetic field line are tangent to the surface.



Parallel Plates (Beam at Center) a.c. currents

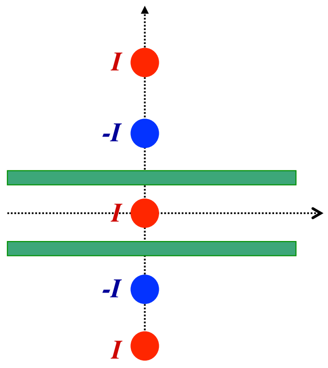


Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams.

It is convenient to decompose the current into a d.c. component, I, for which $\delta_w >> \Delta_w$, and an a.c. component, $\hat{I}$, for which $\delta_w << \Delta_w$.

While the d.c. component of the magnetic field does not perceives the presence of the material, its a.c. component is obliged to be tangent at the wall. For a charge density λ we have $I=\lambda v$.

We can see that this current produces a magnetic field able to cancel the effect of the electrostatic force.



$$\begin{cases} \tilde{E}_y(z,x) = \dfrac{\tilde{\lambda}(z)y}{\pi\ \varepsilon_o}\dfrac{\pi^2}{48h^2} \\ \tilde{B}_x(z,x) = \dfrac{\beta}{c}\tilde{E}_y(z,x) \end{cases}$$

$$\tilde{F}_y(z,x) = e\left(1-\beta^2\right)E_y = \frac{1}{\gamma^2}\frac{e\tilde{\lambda}(z)y}{\pi\ \varepsilon_o}\frac{\pi^2}{48h^2}$$

$$\begin{pmatrix} \tilde{F}_x(z,x) = \dfrac{e\tilde{\lambda}(z)x}{2\pi\ \varepsilon_o\gamma^2}\left(\dfrac{1}{a^2} - \dfrac{\pi^2}{24h^2}\right) \\ \tilde{F}_y(z,x) = \dfrac{e\tilde{\lambda}(z)y}{2\pi\ \varepsilon_o\gamma^2}\left(\dfrac{1}{a^2} + \dfrac{\pi^2}{24h^2}\right) \end{pmatrix}$$

There is cancellation of the electric and magnetic forces !!

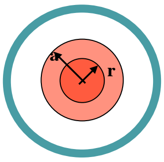
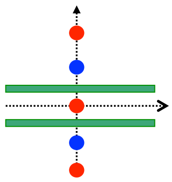
Parallel Plates - General expression of the force

Taking into account all the boundary conditions for d.c. and a.c. currents, we can write the expression of the force as:

$$F_u = \frac{e}{2\pi\ \varepsilon_o}\left[\frac{1}{\gamma^2}\left(\frac{1}{a^2} \mp \frac{\pi^2}{24h^2}\right)\lambda \mp \beta^2\left(\frac{\pi^2}{24h^2} + \frac{\pi^2}{12g^2}\right)\bar{\lambda}\right]u$$

where λ is the total current, and $\bar{\lambda}$ its d.c. part. We take the sign (+) if u=y, and the sign (–) if u=x.

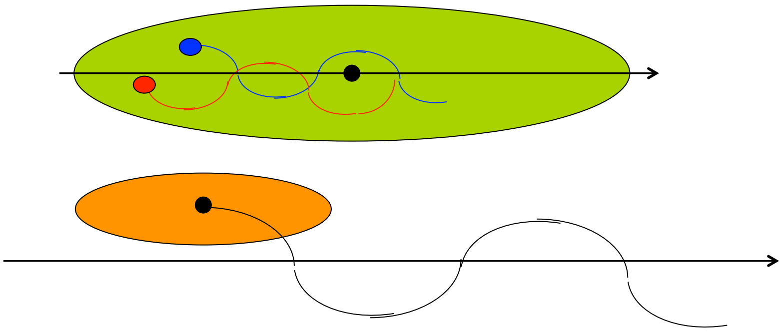
-L. J. Laslett, LBL Document PUB-6161, 1987, vol III

$\lambda(z) = \lambda_o + \tilde{\lambda} \cos(k_z z) \quad ; \quad k_z = 2\pi / l_w$		
	D.C.	**A.C.** $(\delta_w << \Delta_w)$
	$F_\perp(r) = \dfrac{e}{\gamma^2} \dfrac{\lambda(z)}{2\pi \varepsilon_0} \dfrac{r}{a^2}$	
	$F_x(z,x) = \dfrac{e\lambda_0 x}{\pi \varepsilon_0}\left(\dfrac{1}{2a^2\gamma^2} - \dfrac{\pi^2}{48h^2}\right)$ $F_y(z,x) = \dfrac{e\lambda_0 y}{\pi \varepsilon_0}\left(\dfrac{1}{2a^2\gamma^2} + \dfrac{\pi^2}{48h^2}\right)$	$\tilde{F}_x(z,x) = \dfrac{e\tilde{\lambda}(z)x}{\pi \varepsilon_0 \gamma^2}\left(\dfrac{1}{2a^2} - \dfrac{\pi^2}{48h^2}\right)$ $\tilde{F}_y(z,x) = \dfrac{e\tilde{\lambda}(z)y}{\pi \varepsilon_0 \gamma^2}\left(\dfrac{1}{2a^2} + \dfrac{\pi^2}{48h^2}\right)$

Space charge effects in storage rings

Incoherent and Coherent Transverse Effects

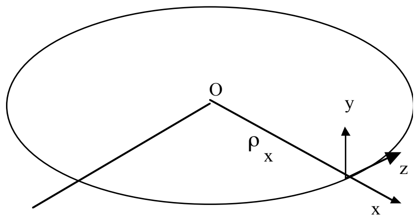
When the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass **(coherent),** but change the trajectory of individual charges in the beam **(incoherent).**



These force may have a complicate dependence on the charge position. A simple analysis is done considering only the linear expansion of the self-fields forces around the equilibrium trajectory.

Self Fields and betatron motion

Consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have



$$\frac{d(m\gamma \boldsymbol{v})}{dt} = \boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r}) \qquad \frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}^{ext}(\vec{r}) + \boldsymbol{F}^{self}(\vec{r})}{m\gamma}$$

In the analysis of the motion of the particles in presence of the self field, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.
This is the case where the focussing term is constant. Although this condition in never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

Q_x Betatron tune: n. of betatron oscillations per turn $\quad K_x = \left(\frac{Q_x}{\rho_x}\right)^2$

$$\boxed{x''(s) + \left(\frac{Q_x}{\rho_x}\right)^2 x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x,s)}$$

Transverse Incoherent Effects

We take the linear term of the transverse force in the betatron equation:

$$\left.\begin{aligned} &F_x^{s.c.}(x,z) \cong \left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x \\ &x'' + \left(\frac{Q_x}{\rho_x}\right)^2 x = \frac{1}{\beta^2 E_o}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x \end{aligned}\right\}$$

$$\boxed{x'' + \left(\left(\frac{Q_x}{\rho_x}\right)^2 - \frac{1}{\beta^2 E_o}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0}\right) x = 0}$$

$$(Q_x + \Delta Q_x)^2 \cong Q_x^2 + 2Q_x \Delta Q_x \Rightarrow \Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_o Q_x}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)$$

The betatron shift is negative since the space charge forces are defocusing on both planes. Notice that the tune shift is in general function of “z”, therefore there is a tune spread inside the beam.

Example: Incoherent betatron tune shift for an uniform electron beam of radius *a*, length l_o, inside circular perfectly conducting pipe

$$\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = \frac{\partial}{\partial x}\frac{e\lambda_o x}{2\pi\varepsilon_o\gamma^2 a^2} = \frac{e\lambda_o}{2\pi\varepsilon_o\gamma^2 a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 N e^2}{4\pi\varepsilon_o a^2\beta^2\gamma^2 E_o Q_{xo} l_o}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_o m_o c^2} \quad (electrons: 2.82\ 10^{-15} m,\ protons: 1.53\ 10^{-18} m)$$

$$\Delta Q_x = -\frac{\rho_x^2 N r_{e,p}}{a^2\beta^2\gamma^3 Q_{xo} l_o}$$

For a real bunched beams the space charge forces, and the tune shift depend on the longitudinal and radial position of the charge.

Consequences of the space charge tune shifts

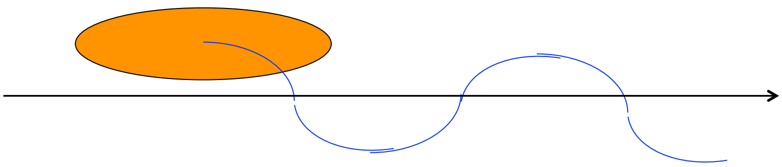
In circular accelerators the values of the betatron tunes should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable.

The tune spread induced by the space charge force can make hard to satisfy this basic requirement. Typically, in order to avoid major resonances the stability requires

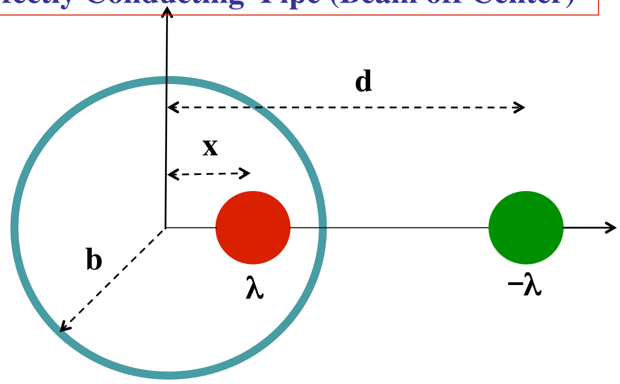
$$|\Delta Q_u| < 0.3$$

Transverse Coherent Effects

If the beam experiences a transverse deflection kick, it starts to perform betatron oscillations as a whole. The beam, source of the space charge fields moves transversely inside the pipe, while individual particles still continue their incoherent motion around the common coherent trajectory.



Circular Perfectly Conducting Pipe (Beam off Center)



$$\boxed{d = \frac{b^2}{x}}$$

The image charge is at a distance “d” such that the pipe surface is at constant voltage, and pulls the beam away from the center of the pipe.

The effect is defocusing: the horizontal electric image field E and the horizontal force F are:

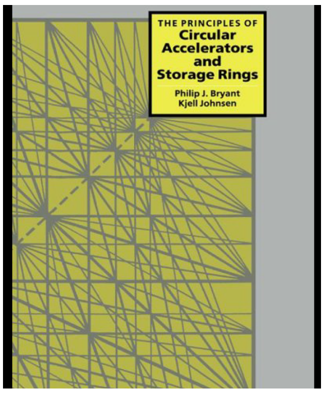
$$E_{xc}(\boldsymbol{x}) = \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{1}{d-x} \approx \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{1}{d} = \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{x}{b^2}$$

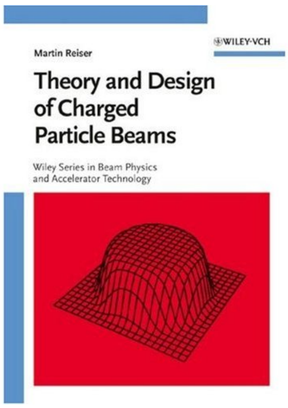
$$F_{xc}(r) \approx \frac{e\lambda(z)}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$\Delta Q_{xc} = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\left(\frac{\partial F_{xc}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\frac{e\lambda(z)}{2\pi\,\varepsilon_o b^2}$$

$$\Delta Q_{xc} = -\frac{r_e\rho_x^2}{\beta^2\gamma Q_{x0}}\frac{N}{b^2 l_0}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$





THE END