Feynman integrals in QCD made simple

Johannes M. Henn

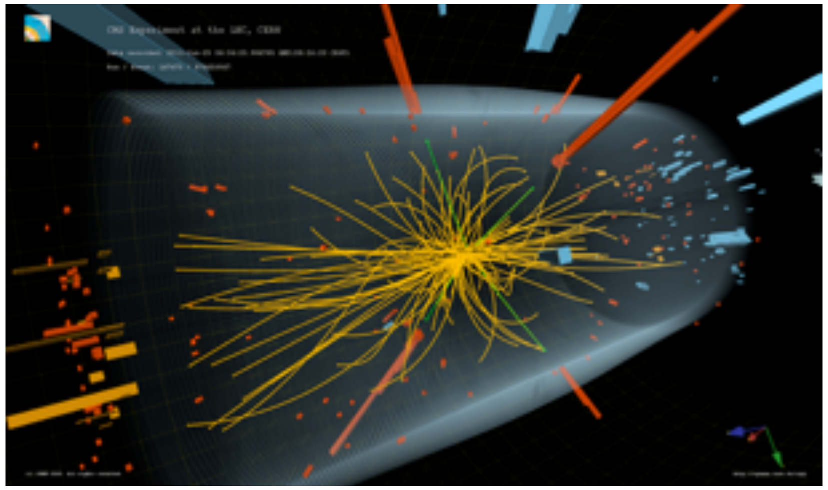
Institute for Advanced Study

supported in part by the Department of Energy grant DE-SC0009988
Marvin L. Goldberger Member

CERN Theory colloquium, May 6, 2015

Experiment and theory

- The Higgs boson has ben found at the LHC

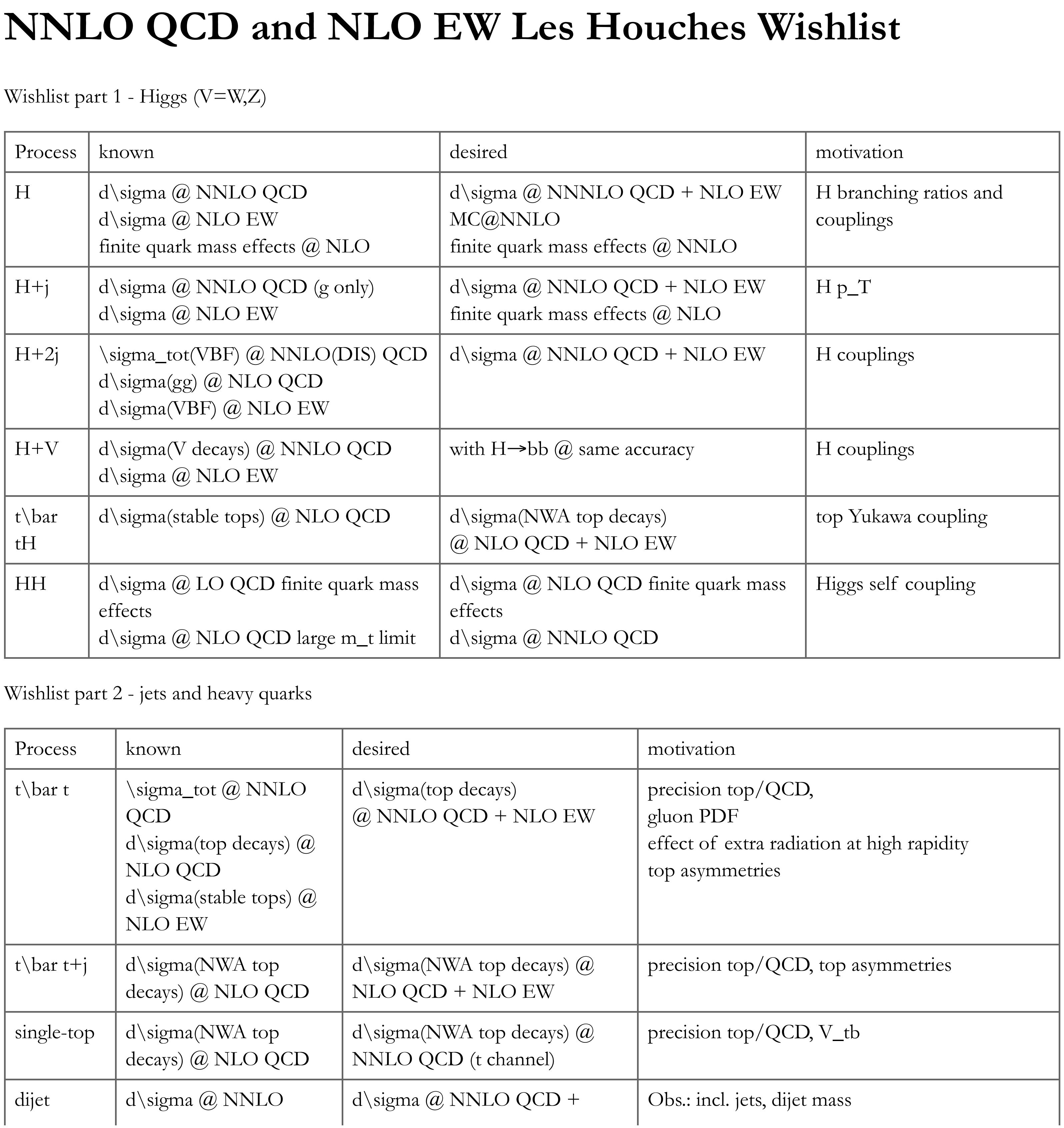


Huge success both for theory and experiment

- What's next?
  - determine properties of the new particle
  - search for deviations from the standard model

- Increasing experimental precision puts new challenges to theory community

Les Houches wishlist



NNLO QCD and NLO EW Les Houches Wishlist

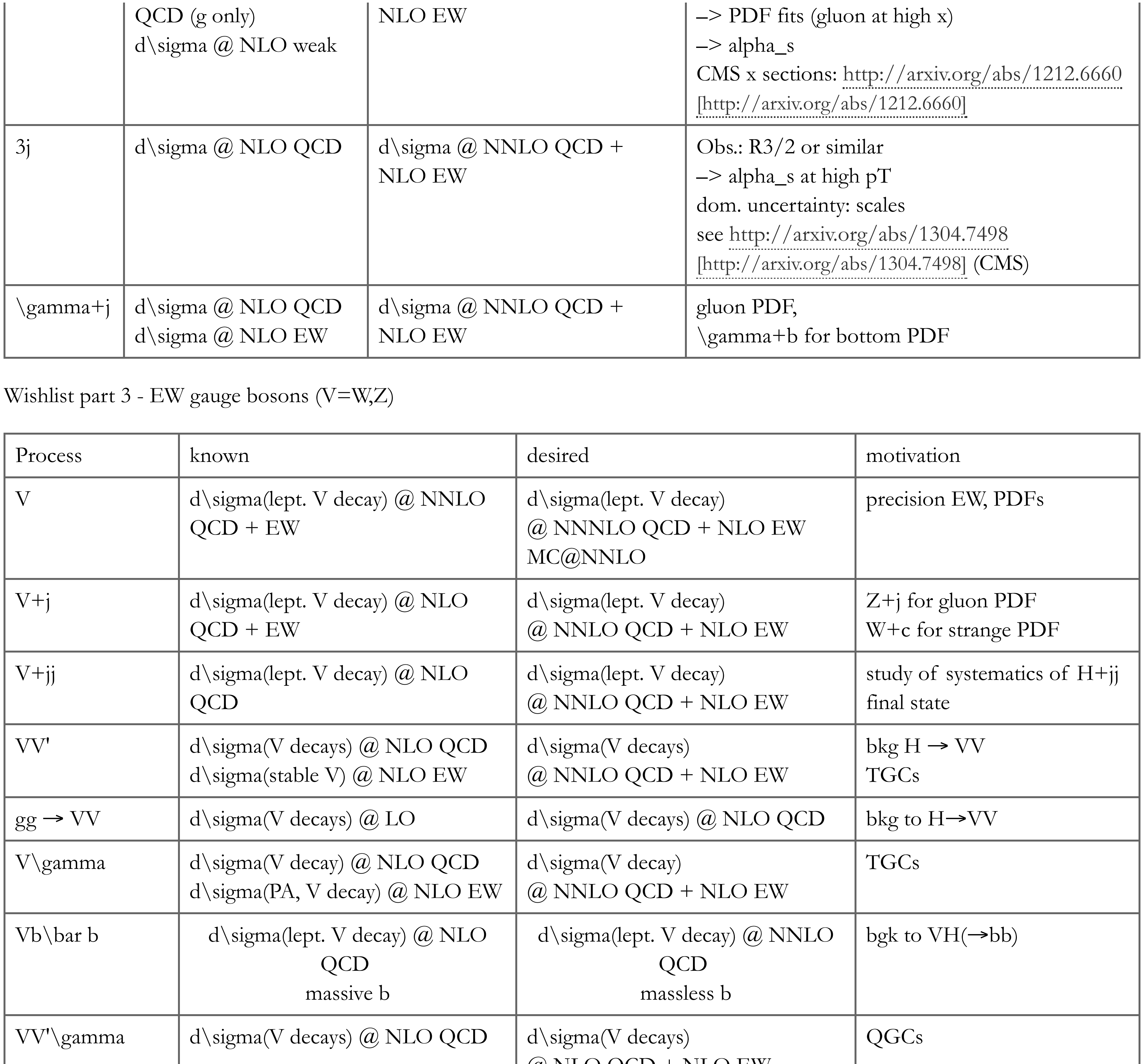
Wishlist part 1 - Higgs (V=W,Z)

Process	known	desired	motivation
H	d\sigma @ NNLO QCD d\sigma @ NLO EW finite quark mass effects @ NLO	d\sigma @ NNNLO QCD + NLO EW MC@NNLO finite quark mass effects @ NNLO	H branching ratios and couplings
H+j	d\sigma @ NNLO QCD (g only) d\sigma @ NLO EW	d\sigma @ NNLO QCD + NLO EW finite quark mass effects @ NLO	H p_T
H+2j	\sigma_tot(VBF) @ NNLO(DIS) QCD d\sigma(gg) @ NLO QCD d\sigma(VBF) @ NLO EW	d\sigma @ NNLO QCD + NLO EW	H couplings
H+V	d\sigma(V decays) @ NNLO QCD d\sigma @ NLO EW	with H→bb @ same accuracy	H couplings
t\bar tH	d\sigma(stable tops) @ NLO QCD	d\sigma(NWA top decays) @ NLO QCD + NLO EW	top Yukawa coupling
HH	d\sigma @ LO QCD finite quark mass effects d\sigma @ NLO QCD large m_t limit	d\sigma @ NLO QCD finite quark mass effects d\sigma @ NNLO QCD	Higgs self coupling

Wishlist part 2 - jets and heavy quarks

Process	known	desired	motivation
t\bar t	\sigma_tot @ NNLO QCD d\sigma(top decays) @ NLO QCD d\sigma(stable tops) @ NLO EW	d\sigma(top decays) @ NNLO QCD + NLO EW	precision top/QCD, gluon PDF effect of extra radiation at high rapidity top asymmetries
t\bar t+j	d\sigma(NWA top decays) @ NLO QCD	d\sigma(NWA top decays) @ NLO QCD + NLO EW	precision top/QCD, top asymmetries
single-top	d\sigma(NWA top decays) @ NLO QCD	d\sigma(NWA top decays) @ NNLO QCD (t channel)	precision top/QCD, V_tb
dijet	d\sigma @ NNLO	d\sigma @ NNLO QCD +	Obs.: incl. jets, dijet mass

Les Houches wishlist



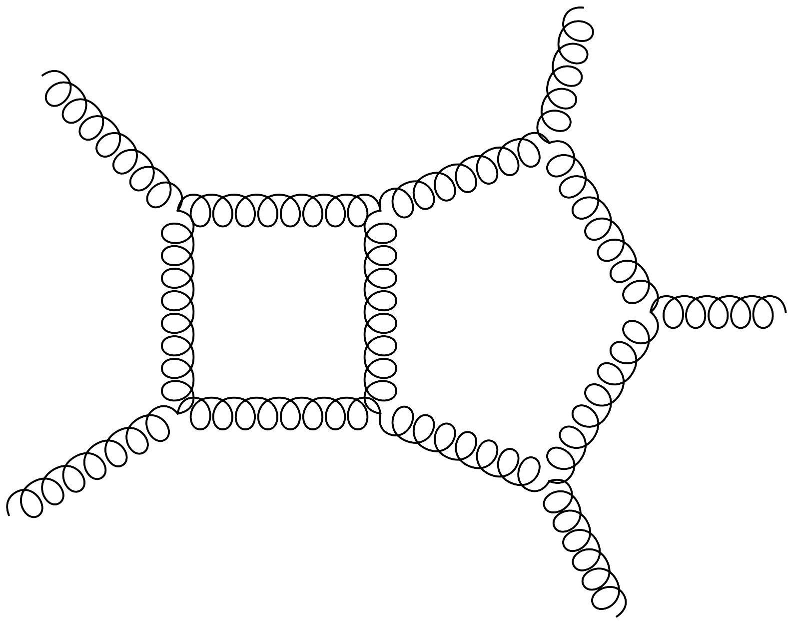
	QCD (g only) d\sigma @ NLO weak	NLO EW	–> PDF fits (gluon at high x) –> alpha_s CMS x sections: http://arxiv.org/abs/1212.6660 [http://arxiv.org/abs/1212.6660]
3j	d\sigma @ NLO QCD	d\sigma @ NNLO QCD + NLO EW	Obs.: R3/2 or similar –> alpha_s at high pT dom. uncertainty: scales see http://arxiv.org/abs/1304.7498 [http://arxiv.org/abs/1304.7498] (CMS)
\gamma+j	d\sigma @ NLO QCD d\sigma @ NLO EW	d\sigma @ NNLO QCD + NLO EW	gluon PDF, \gamma+b for bottom PDF

Wishlist part 3 - EW gauge bosons (V=W,Z)

Process	known	desired	motivation
V	d\sigma(lept. V decay) @ NNLO QCD + EW	d\sigma(lept. V decay) @ NNNLO QCD + NLO EW MC@NNLO	precision EW, PDFs
V+j	d\sigma(lept. V decay) @ NLO QCD + EW	d\sigma(lept. V decay) @ NNLO QCD + NLO EW	Z+j for gluon PDF W+c for strange PDF
V+jj	d\sigma(lept. V decay) @ NLO QCD	d\sigma(lept. V decay) @ NNLO QCD + NLO EW	study of systematics of H+jj final state
VV'	d\sigma(V decays) @ NLO QCD d\sigma(stable V) @ NLO EW	d\sigma(V decays) @ NNLO QCD + NLO EW	bkg H → VV TGCs
gg → VV	d\sigma(V decays) @ LO	d\sigma(V decays) @ NLO QCD	bkg to H→VV
V\gamma	d\sigma(V decay) @ NLO QCD d\sigma(PA, V decay) @ NLO EW	d\sigma(V decay) @ NNLO QCD + NLO EW	TGCs
Vb\bar b	d\sigma(lept. V decay) @ NLO QCD massive b	d\sigma(lept. V decay) @ NNLO QCD massless b	bgk to VH(→bb)
VV'\gamma	d\sigma(V decays) @ NLO QCD	d\sigma(V decays) @ NLO QCD + NLO EW	QGCs

Challenges for calculations in QFT

- Many processes involve several variables (masses, scattering angles), e.g. 2->3 processes
- One of the main obstacles: often, no analytic expressions for the Feynman integrals are available
- In this talk, I will focus on virtual contributions and present tools for the evaluation of the Feynman integrals



Outline

- Real versus ideal scattering amplitudes
- new ideas for integrands and integrals in quantum field theory
- differential equations for Feynman integrals
- application (new result):
  all massless planar 2->3 NNLO Feynman integrals

‘Ideal’ and ‘real’ scattering amplitudes

‘formal theory’

supersymmetric
scattering
amplitudes

QCD at the
LHC

This talk: tools for ‘real’ QCD coming from ‘ideal’ amplitudes

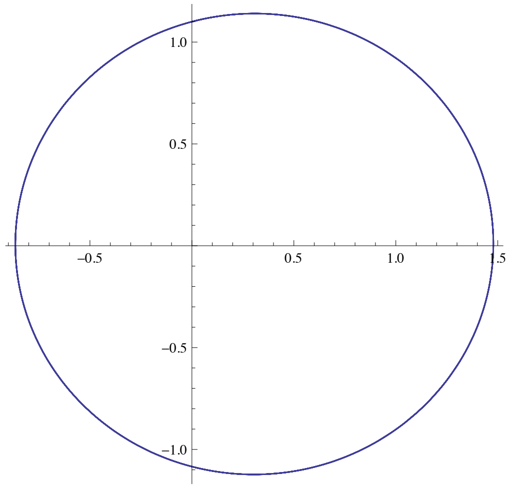
Idealized ‘toy’ theories: from Kepler to QFT

Idealized systems play an important role in physics

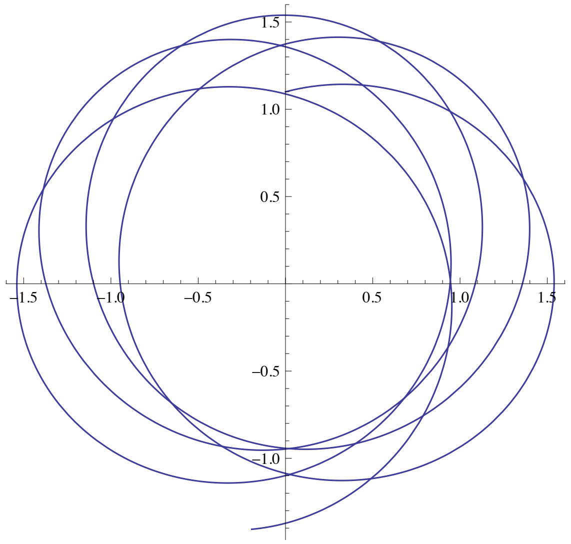
Often, (hidden) symmetries help to solve a problem

Example 1: Kepler problem

$$V = 1/r$$



$$V = 1/r^{0.9}$$



- Laplace-Runge-Lenz (LRL) vector is conserved

$$\vec{A} = \frac{1}{2}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \mu \frac{\lambda}{4\pi} \frac{\vec{x}}{|x|}$$

- consequence: orbits do not precess

Example 2: Hydrogen atom

- described by quantum mechanics
- hidden symmetry:
Laplace-Runge-Lenz-Pauli vector
- gives elegant algebraic way to find spectrum

$$E_n = -\frac{mk^2}{2\hbar^2}\frac{1}{n^2} \qquad n = 1, 2, \ldots$$

- explains why there are n^2 states of energy E_n.

Is there a quantum field theory (preferably a gauge theory) that has the same symmetry?

Example 3: N=4 super Yang-Mills theory

- generalization of massless QCD
  - gluons, plus 4 complex fermions and 6 scalars in adjoint representation
  - masses can be added via Higgs mechanism
- conformal symmetry and (extended) supersymmetry
- has a hidden dual conformal symmetry

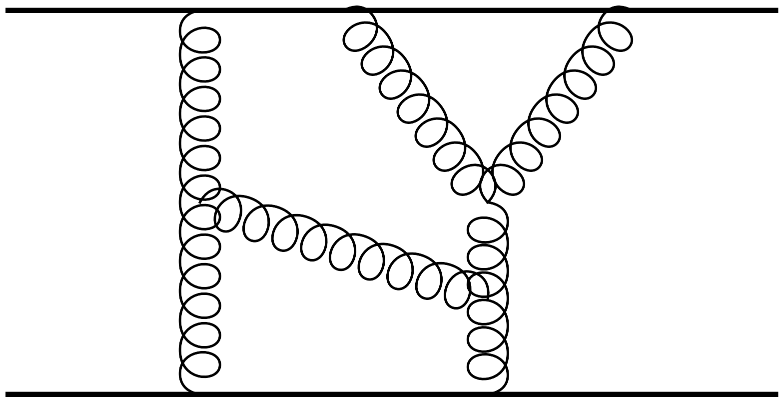
[Drummond, JMH, Korchemsky, Sokatchev, 2008]

[Yangian interpretation: Drummond, JMH, Plefka, 2009]

- this symmetry is a generalization of the LRL symmetry to a (planar) relativistic quantum field theory

[JMH and Caron-Huot, 2013]

e.g., extra symmetry governs spectrum of bound states of massive W bosons



Laplace-Runge-Lenz symmetry

classical mechanics	Kepler problem
quantum mechanics	Hydrogen atom
quantum field theory	(planar) N=4 super Yang-Mills theory

Some people call N=4 SYM the 'hydrogen atom of quantum field theory' — perhaps they are not completely wrong…

Open questions:

- Is this the unique gauge theory with this property?
- Is there a generalization to the non-planar level?

What gauge theory to study scattering amplitudes?

Our toy model will be (planar) N=4 supersymmetric theory

Many properties that allow to find spectacular results

- conformal field theory, ultraviolet finite
- conjectured string theory dual (AdS/CFT)
- dual conformal, Yangian symmetry

At the same time, very similar to QCD in perturbation theory

- Feynman diagrams, loop integrals
- infrared properties

Examples of developments and ideas

- on-shell techniques

  (generalized) unitarity techniques [Bern, Dixon, Dunbar, Kosower], …

  on-shell recursion relations at tree level, and for loop integrands [Britto, Cachazo, Feng, Witten], [Arkani-Hamed et al.]

- better understanding of the loop integrand

  singularity structure

  physical properties (e.g. infrared properties)

  [Arkani-Hamed et al.] [Caron-Huot] [JMH, Drummond] [Bourjaily et al.]

- progress in analytically computing loop integrals

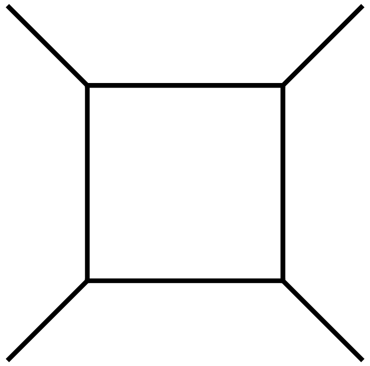
  algebra of iterated integrals ('symbol', coproduct) [Brown, Goncharov, Spradlin, Vergu, Volovich, Duhr, Gangl, …]

  progress in differential equations technique

Analyzing loop integrands: maximal cuts, leading singularities

- maximal cuts

$$D_1 = k^2 \quad D_2 = (k+p_1)^2 \quad D_3 = (k+p_1+p_2)^2 \quad D_4 = (k+p_1+p_2+p_3)^2$$



$$= \int d^4k\,\delta(D_1)\delta(D_2)\delta(D_3)\delta(D_4) \sim \frac{1}{st}$$

note: there are two solutions that localize the loop momentum (related by complex conjugation); these correspond to the leading singularities

- at higher loops, maximal cuts do not completely localize the loop momenta; leading singularities \`cut\` also Jacobian factors

Pentagon example

- one-loop pentagon integrals

$D_1 = k^2 \quad D_2 = (k+p_1)^2 \quad D_3 = (k+p_1+p_2)^2 \quad D_4 = (k+p_1+p_2+p_3)^2 \quad D_5 = (k-p_5)^2$

- now there are five different maximal cuts we can take

- leading singularities of the scalar pentagon integral cannot all be normalized to one

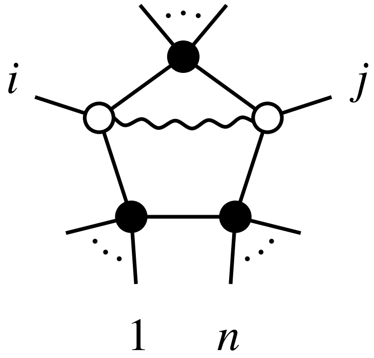
- consider a pentagon integral with numerator:

$$\int d^4k \frac{N(k)}{D_1 D_2 D_3 D_4 D_5}$$

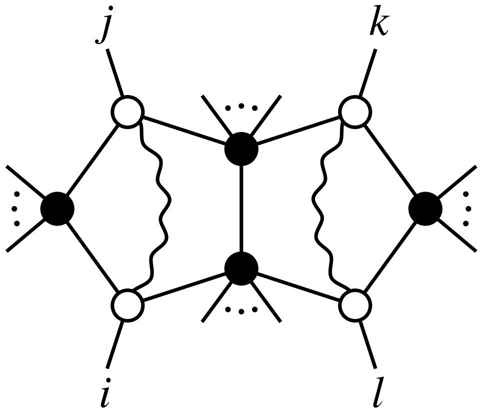
- can choose numerator such that integral has constant leading singularities

- Such integrals `naturally` appear in N=4 SYM [Arkani-Hamed et al, 2010]

$$\mathcal{A}^{1-\text{loop}}_{\text{MHV}} = \sum_{i<j}$$



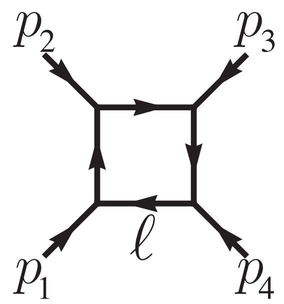
$$\mathcal{A}^{2-\text{loop}}_{\text{MHV}} = \frac{1}{2}\sum_{i<j<k<l<i}$$



`d-log forms`

- observation: sometimes, loop integrand can be rewritten in suggestive form

$\mathcal{A}_4^{\ell=0} \times$



$= \mathcal{A}_4^{\ell=0} \times \int \frac{d^4\ell \ \ (p_1+p_2)^2(p_1+p_3)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$

[Arkani-Hamed et al, 2012]
[Caron-Huot, talk at Trento, 2012]
[Lipstein and Mason, 2013-2014]

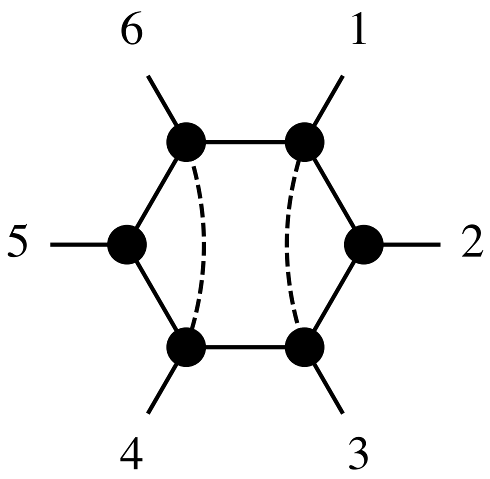
[also see recent work, on non-planar cases: Arkani-Hamed et al, 2014; Bern et al., 2015]

$$\frac{d^4\ell \ \ (p_1+p_2)^2(p_1+p_3)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$
$$= \mathrm{dlog}\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) \mathrm{dlog}\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

- `d-log forms`: make leading singularities obvious

Leading singularities, weight conjecture

- observation: these integrals have homogeneous logarithmic weight (`transcendentality`); e.g.,



$$= \text{Li}_2(1-u_1) + \text{Li}_2(1-u_2) + \text{Li}_2(1-u_3) + \log(u_3)\log(u_1) - \frac{\pi^2}{3}$$

assign log weight: $w(\log) = 1 \quad w(\text{Li}_n) = n \qquad w(\pi) = 1$

$$w(ab) = w(a) + w(b)$$

function has uniform weight 2 and kinematic-independent prefactors

- weight conjecture

[Arkani-Hamed, Bourjaily, Cachzao, Trnka, 2010]

[Arkani-Hamed et al, 2012]

integrals with constant leading singularities should have uniform weight

- as we will see, differential equations can shed more light on the weight properties

[JMH, 2013]

Differential equations (DE) technique

- idea: differentiate Feynman integral w.r.t. external variables, e.g. s, t, masses

Some general facts:

- a given Feynman integral f satisfies an n-th order DE
- equivalently described by a system of n first-order equations for $\vec{f}$

$$\partial_x \vec{f}(x,\epsilon) = A(x,\epsilon)\vec{f}(x,\epsilon)$$

since they come from Feynman integrals, they can only have regular singularities. Constrains matrix $A(x,\epsilon)$

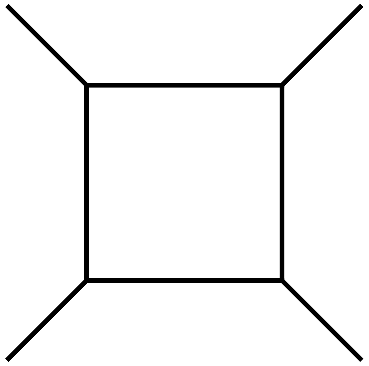
Long and successful history:

[Kotikov, 1991] [Remiddi, 1997] [Gehrmann, Remiddi, 2000] […]

New idea: use integrals with constants leading singularities as basis for DE system [JMH, 2013]

Example: one-loop four-point integral

- choose basis according to [JMH, 2013]
- differential equations $x = t/s$ $D = 4 - 2\epsilon$



$$\partial_x \vec{f}(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] \vec{f}(x,\epsilon)$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

  - make singularities manifest
  - asymptotic behavior governed by matrices a, b

- Solution: expand to any order in ϵ

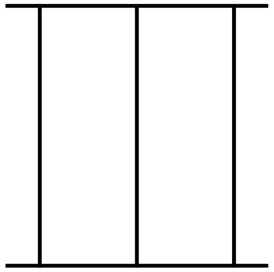
$$\vec{f} = \epsilon^{-p} \sum_{k\geq 0} \epsilon^k \vec{f}^{(k)}$$

$\vec{f}^{(k)}$ is k-fold iterated integral (uniform weight k)

Technique applies to QCD integrals

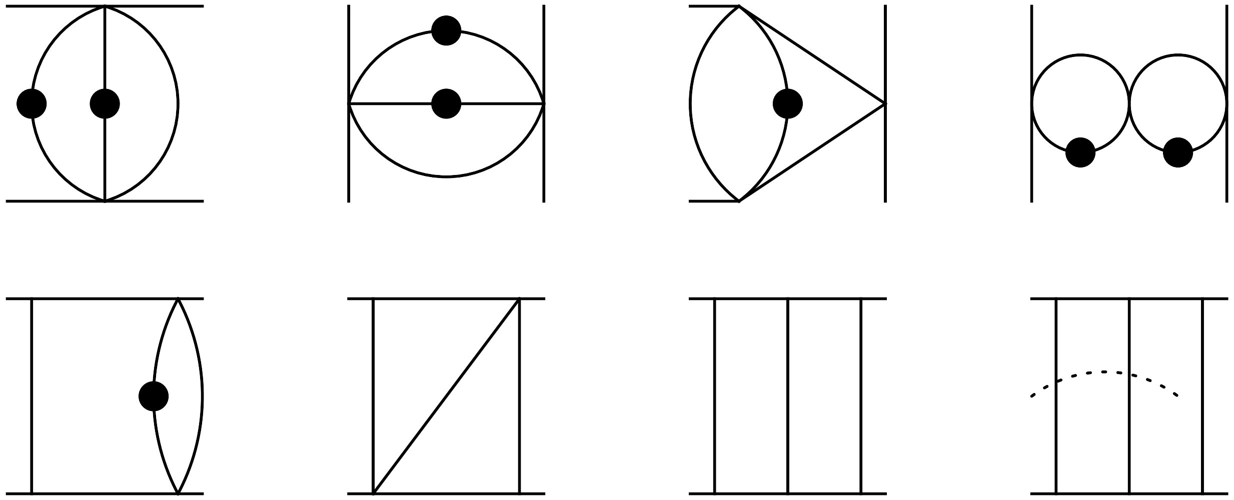
- system of DE for `N=4` integral contains QCD integrals

$$f(x,\epsilon) =$$



$$x = t/s$$
$$D = 4 - 2\epsilon$$

$$\vec{f}(x,\epsilon) =$$



$$\partial_x \vec{f}(x,\epsilon) = \epsilon\left[\frac{a}{x} + \frac{b}{1+x}\right]\vec{f}(x,\epsilon)$$

Multi-variable case and the alphabet

- Natural generalization to multi-variable case

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$

constant matrices letters (alphabet)

- Examples of alphabets:

4-point on-shell	$\alpha = \{x, 1+x\}$
two-variable example (from 1-loop Bhabha scattering):	$\alpha = \{x\,, 1\pm x\,, y\,, 1\pm y\,, x+y\,, 1+xy\}$ [J.M.H., Smirnov]
``hexagon functions`` in N=4 SYM	$\alpha = \{x, y, z, 1-x, 1-y, 1-z, 1-xy, 1-xz, 1-yz, 1-xyz\}$ [Goncharov, Spradlin, Vergu, Volovich] [Caron-Huot, He] [Dixon, Drummond, J.M.H.] [Dixon et al.]

- Matrices and letters determine solution
- Immediate to solve in terms of iterated integrals

Physics applications of new ideas for DE

[JMH, 2013]

- vector boson production

VV' planar and non-planar NNLO integrals

[Caola, JMH, Melnikov, Smirnov, Smirnov, 2014]

equal mass case: [Gehrmann, von Manteuffel, Tancredi, Weihs, 2014]

essential ingredient for ZZ and W+W- production at NNLO

[Cascioli et al, 2014] [Gehrmann et al, 2014]

- 3-loop QCD cusp anomalous dimension (determines IR structure of planar QCD scattering amplitudes)

[Grozin, JMH, Korchemsky, Marquard, 2014]

- B physics [Bell, Huber, 2014] [Huber, Kraenkl, 2015]

- integrals for H production in gluon fusion at N3LO

[Dulat, Mistlberger, 2014] [Hoeschele,Hoff,Ueda, 2014]

physics result: [Anastasiou et al, 2014]

...

Beyond iterated integrals

- Note: functions beyond iterated integrals can appear in Feynman integrals

- One such class are elliptic functions, needed e.g. in top quark physics [Czakon and Mitov, 2010]

- A generalization of the above methods is required here

New results for penta-box integrals and five-particle amplitudes at NNLO

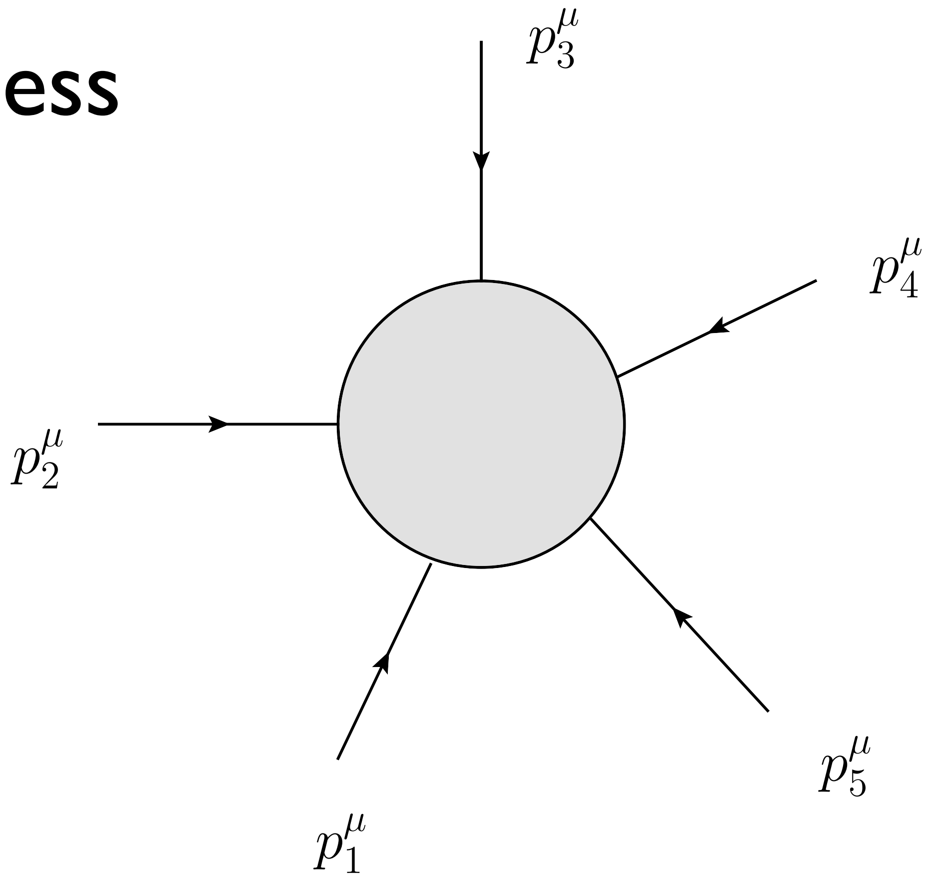
[Gehrmann, JMH, Lo Presti, to appear]

[related work with Frellesvig on one-loop pentagon integrals]

five-point kinematics

- massless 5->0 process

$$s_{ij} = (p_i + p_j)^2$$

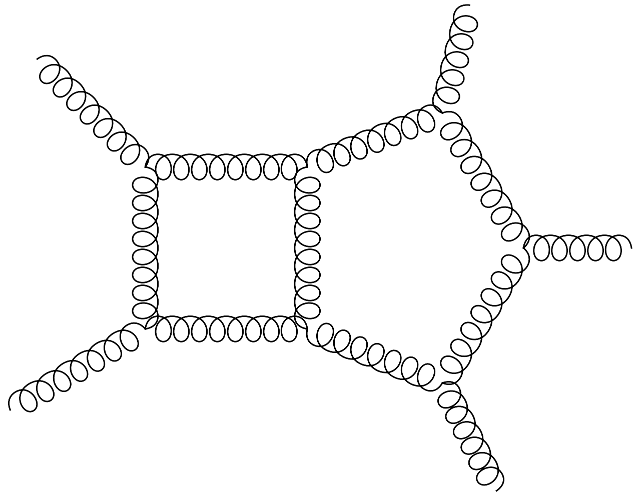


- independent variables $\vec{x} = \{s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{51}\}$

- convenient to start with non-physical region where all planar integrals are real-valued $s_{i,i+1} < 0$

- other kinematic regions can be reached by analytic continuation

differential equations for penta-box integrals



- 61 planar master integrals

$$d\vec{f}(\vec{x};\epsilon) = \epsilon\, d\left[\sum_k A_k \log \alpha_k(\vec{x})\right] \vec{f}(\vec{x};\epsilon)$$

- integral basis chosen following [JMH, 2013]

$$\vec{x} = \{s_{12}\,, s_{23}\,, s_{34}\,, s_{45}\,, s_{51}\}$$

- alphabet of 24 letters $\alpha_k(\vec{x})$ e.g.

$$s_{12} \qquad\qquad s_{12} - s_{34}$$

$$s_{12} + s_{23} \qquad\qquad s_{12} - s_{34} + s_{51}$$

$$(s_{23} - s_{51})\sqrt{\Delta} + s_{12}s_{23}^2 - s_{34}s_{23}^2 + s_{34}s_{45}s_{23} - 2s_{12}s_{51}s_{23}$$
$$+s_{34}s_{51}s_{23} + s_{45}s_{51}s_{23} + s_{12}s_{51}^2 - s_{45}s_{51}^2 + s_{34}s_{45}s_{51}$$

Gram determinant Δ

boundary conditions

- the boundary conditions can be obtained from physical conditions

- no singularities in non-physical region $s_{i,i+1} < 0$

- this means that certain singularities are spurious (on the first sheet of the multivalued functions), e.g. at

$$s_{12} = s_{34}$$
$$s_{12} + s_{51} = s_{34}$$

- similarly, no branch cuts should start at $\Delta = 0$

- these conditions fix everything except trivial single-scale integrals that are evaluated in terms of gamma functions

analytic solution

- we have

$$d\vec{f}(\vec{x},\epsilon) = \epsilon\, d\tilde{A}\, \vec{f}(\vec{x},\epsilon) \qquad\qquad \tilde{A} = \sum_k A_k \alpha_k(\vec{x})$$

- solution in terms of iterated integrals

$$\vec{f}(\vec{x},\epsilon) = \mathbb{P}\exp\left[\epsilon \int_\gamma d\,\tilde{A}\right] \vec{f}(\vec{x}_0,\epsilon)$$

$$\gamma : [0,1] \longrightarrow \mathcal{M}$$

$$\gamma(0) = \vec{x}_0 \qquad \gamma(1) = \vec{x}$$

- can be written in terms of Goncharov polylogarithms
(for a convenient choice of γ)

application to five-particle amplitudes

- five-particle scattering amplitudes were conjectured to have the following form (in modern language) [Bern, Dixon, Smirnov, 2003]

$$\log M_5 = \sum_{L\geq 1} a^L \left[-\frac{\gamma^{(L)}}{8(L\epsilon)^2} - \frac{\mathcal{G}_0^{(L)}}{4L\epsilon} + f^{(L)} \right] \sum_{i=1}^{5} \left(\frac{\mu^2}{s_{i,i+1}} \right)^{L\epsilon}$$

$$+ \frac{\gamma(a)}{4} F_n^{(1)}(s_{ij}) + C(a) + \mathcal{O}(\epsilon)$$

- This is in part due to the infrared structure of amplitudes
- The BDS conjecture fixes the finite part; it is now understood to follow from dual conformal symmetry

[Drummond, JMH, Korchemsky, Sokatchev, 2008]

- previously, this formula had been tested numerically

[Cachazo, Spradlin, Volovich, 2006] (parity-even part)
[Bern, Czakon, Kosower, Roiban, Smirnov, 2006]

- we verified the parity-even part of it using our analytic results

Summary and conclusions

- supersymmetric toy models valuable for perturbative QCD

- unitarity-based methods for determining integrands complemented with a new method for evaluating the integrals
- both rely on analyzing the integrand's singularity structure

- many recent new results obtained with DE method
- method particularly useful for problems with many scales
- presented new results for five-particle two-loop integrals
- can be used to compute QCD +++++ amplitude

[Badger, Frellsvig, Zhang, 2013]

Thank you!

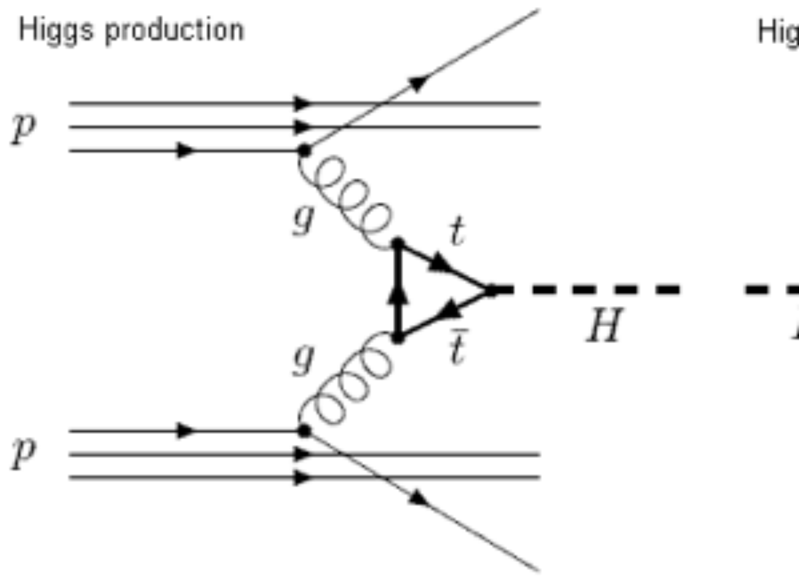
Extra slides

The alphabet and perfect bricks (1)

Can we parametrize variables such that alphabet is rational?

Not essential, but nice feature.

- Example: Higgs production



encounter $\sqrt{1-4m^2/s}$

choose $-m^2/s = x/(1-x)^2$

$\alpha = \{x, 1-x, 1+x\}$ (to two loops)

Note: this is a purely kinematical question. Independent of basis choice.

- Related to diophantine equations

e.g. find rational solutions to equations such as

$$1 + 4\,a = b^2$$

here we found a 1-parameter solution

$$a = \frac{x}{(1-x)^2} \qquad b = \frac{1+x}{1-x}$$

The alphabet and perfect bricks (2)

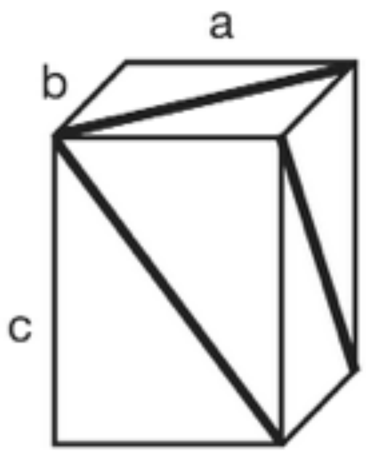
- Classic example: Euler brick problem

Find a brick with sides a, b, c and diagonals d, e, f integers

smallest solution (P. Halcke):

(a,b,c)=(44,117,240)

$$a^2 + b^2 = d^2\,, \quad a^2 + c^2 = e^2\,, \quad b^2 + c^2 = f^2\,.$$

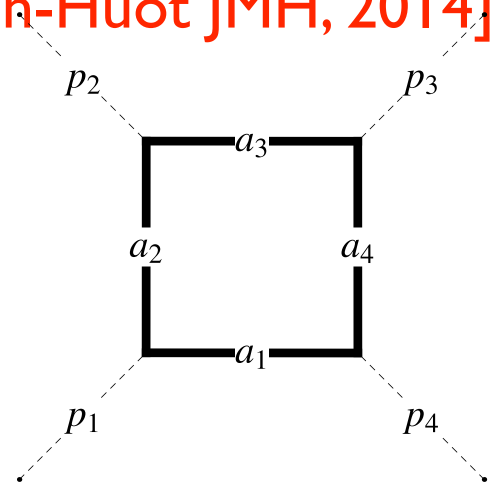


Perfect cuboid (add eq. $a^2 + b^2 + c^2 = g^2$): open problem in mathematics!

- Similar equations for particle kinematics

[Caron-Huot JMH, 2014]

e.g encountered in 4-d light-by-light scattering



$$u = -4m^2/s \qquad v = -4m^2/t$$

$$\beta_u = \sqrt{1+u},\ \beta_v = \sqrt{1+v},\ \beta_{uv} = \sqrt{1+u+v}$$

Need two-parameter solution to

$$\beta_u^2 + \beta_v^2 = \beta_{uv}^2 + 1$$

e.g. $$\beta_u = \frac{1-wz}{w-z}\,, \quad \beta_v = \frac{w+z}{w-z}\,, \quad \beta_{uv} = \frac{1+wz}{w-z}\,.$$

more roots in D-dim and at 3 loops! - in general alphabet changes with the loop order!

Find such solutions systematically? Minimal polynomial order?

Feynman integrals as iterated integrals (1)

- Logarithm and dilogarithm are first examples of iterated integrals with special ``d-log`` integration kernels

$$\frac{dt}{t} = d\log t \qquad \frac{-dt}{1-t} = d\log(1-t) \qquad \frac{dt}{1+t} = d\log(1+t)$$

- these are called harmonic polylogarithms (HPL) [Remiddi, Vermaseren]

e.g. $H_{1,-1}(x) = \int_0^x \frac{dx_1}{1-x_1} \int_0^{x_1} \frac{dx_2}{1+x_2}$

  - shuffle product algebra
  - coproduct structure
  - Mathematica implementation [Maitre]

- weight: number of integrations

- special values related to multiple zeta values (MZV)

$$\zeta_{i_1,i_2,\ldots,i_k} = \sum_{a_1>a_2>\ldots a_k\geq 1} \frac{1}{a_1^{i_1} a_2^{i_2} \ldots a_k^{i_k}}$$

cf. e.g. [Bluemlein, Broadhurst, Vermaseren]

e.g. $H_{0,1}(1) = \mathrm{Li}_2(1) = \zeta_2$

Feynman integrals as iterated integrals (2)

- Natural generalization: multiple polylogarithms

[also called hyperlogarithms; Goncharov polylogarithms]

allow kernels $w = d\log(t-a)$

$$G_{a_1,\ldots a_n}(z) = \int_0^z \frac{dt}{t-a_1} G_{a_2,\ldots,a_n}(t)$$

numerical evaluation: GINAC [Vollinga, Weinzierl]

- Chen iterated integrals

$$\int_C \omega_1 \omega_2 \ldots \omega_n \qquad C : [0,1] \longrightarrow M$$

(space of kinematical variables)

Alphabet: set of differential forms $\omega_i = d\log \alpha_i$

integrals we discuss will be monodromy invariant on $M \setminus S$

S (set of singularities)

more flexible than multiple polylogarithms!

- Uniform weight functions (pure functions):

$\mathbb{Q}$ -linear combinations of functions of the same weight