Anomalies in QFT and Hydrodynamics

Juan L. Mañes

University of the Basque Country, Bilbao, Spain

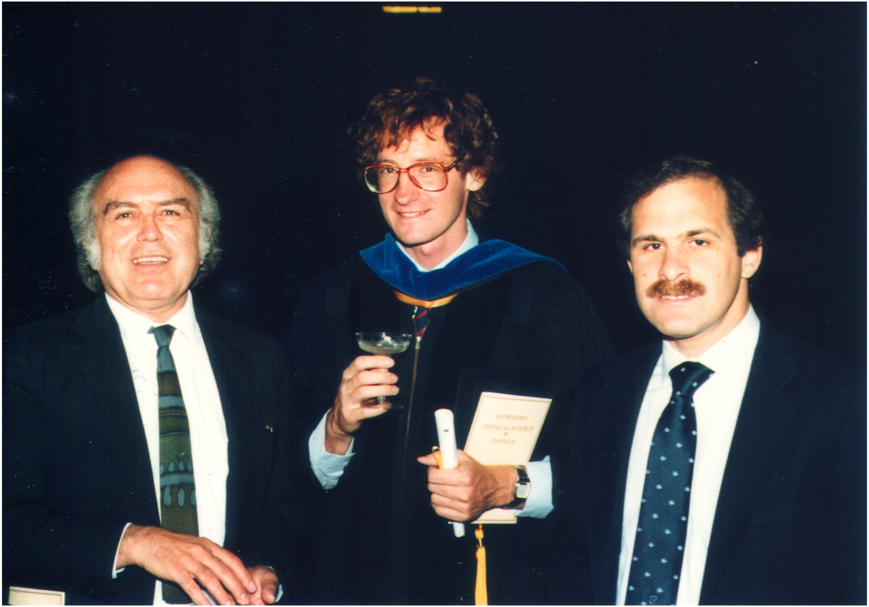
Ph. D. thesis with Bruno (Berkeley, 1983-86)

Thesis Title: Anomalies in QFT and Differential Geometry

- *Differential geometric construction of the gauged Wess-Zumino action*
- *WKB method, SUSY quantum mechanics and the index theorem* (with B. Zumino)
- *Algebraic study of chiral anomalies* (with B. Zumino and R. Stora)
- *Non-triviality of chiral anomalies* (with B. Zumino)

That was a very exciting time for QFT anomalies:

- Anomalies and differential geometry (Stora and Zumino 1983)
- Gravitational anomalies (Alvarez-Gaumé and Witten 1984)
- Consistent and covariant anomalies (Bardeen and Zumino 1984)
- Gravitational anomalies and the family index (O. Alvarez, I. Singer and Zumino 1984)
- Anomaly cancellation in Superstring Theory (Green and Schwarz 1984)
- Anomaly inflow (Callan and Harvey 1985)
- Anomalies in odd dimensions (Niemi and Semenoff 1983, Alvarez-Gaumé, Della Pietra and Moore 1985)
- Gauge anomalies and index theorems (Alvarez-Gaumé and Ginsparg 1985)
- Hamiltonian interpretation (Alvarez-Gaumé and Nelson 1985)
- ....



Graduation day (June 86)

Plan of the talk

- **Basic facts about anomalies**
- Relativistic hydrodynamics
- Anomalous hydrodynamics
- Structure of anomalous partition functions

Wess-Zumino consistency condition (1971)

- In 1969 the non-abelian gauge anomaly (ABJ) was computed

$$D_\mu J^\mu_a = c_A\, \epsilon^{\kappa\lambda\mu\nu} \mathrm{tr}\Big\{ T^a \partial_\kappa \big(A_\lambda \partial_\mu A_\nu + \frac{1}{2} A_\lambda A_\mu A_\nu \big) \Big\} \equiv -G_a[A]$$

with $c_A = \frac{1}{24\pi^2}$ and $A_\mu = A^a_\mu T^a$.

Wess-Zumino consistency condition (1971)

- In 1969 the non-abelian gauge anomaly (ABJ) was computed

$$D_\mu J_a^\mu = c_A\, \epsilon^{\kappa\lambda\mu\nu} \mathrm{tr}\Big\{ T^a \partial_\kappa \big(A_\lambda \partial_\mu A_\nu + \frac{1}{2} A_\lambda A_\mu A_\nu \big) \Big\} \equiv -G_a[A]$$

with $c_A = \frac{1}{24\pi^2}$ and $A_\mu = A_\mu^a T^a$.

- The anomaly results from the non-invariance of the vacuum functional under gauge transformations

$$\delta_\Lambda A_\mu^a = \partial_\mu \Lambda^a + [A_\mu, \Lambda]^a$$

Wess-Zumino consistency condition (1971)

- In 1969 the non-abelian gauge anomaly (ABJ) was computed

$$D_\mu J^\mu_a = c_A\,\epsilon^{\kappa\lambda\mu\nu}\mathrm{tr}\Big\{T^a\partial_\kappa\big(A_\lambda\partial_\mu A_\nu + \frac{1}{2}A_\lambda A_\mu A_\nu\big)\Big\} \equiv -G_a[A]$$

with $c_A = \frac{1}{24\pi^2}$ and $A_\mu = A^a_\mu T^a$.

- The anomaly results from the non-invariance of the vacuum functional under gauge transformations

$$\delta_\Lambda A^a_\mu = \partial_\mu \Lambda^a + [A_\mu, \Lambda]^a$$

Namely,

$$J^\mu_a = \frac{\delta W}{\delta A^a_\mu} \implies \delta_\Lambda W[A] = \int dx \Lambda^a(x) G_a[A]$$

- The commutator of two gauge transformations acting on $W[A]$

$$(\delta_\Lambda \delta_{\Lambda'} - \delta_{\Lambda'} \delta_\Lambda) W[A] = \delta_{[\Lambda,\Lambda']} W[A]$$

- The commutator of two gauge transformations acting on $W[A]$

$$(\delta_\Lambda \delta_{\Lambda'} - \delta_{\Lambda'} \delta_\Lambda) W[A] = \delta_{[\Lambda,\Lambda']} W[A]$$

implies the Wess-Zumino consistency condition for the anomaly

$$\int dx \left(\Lambda'^a \delta_\Lambda G_a - \Lambda^a \delta_{\Lambda'} G_a \right) = \int dx [\Lambda, \Lambda']^a G_a$$

- The commutator of two gauge transformations acting on $W[A]$

$$(\delta_\Lambda \delta_{\Lambda'} - \delta_{\Lambda'} \delta_\Lambda) W[A] = \delta_{[\Lambda,\Lambda']} W[A]$$

implies the Wess-Zumino consistency condition for the anomaly

$$\int dx \left(\Lambda'^a \delta_\Lambda G_a - \Lambda^a \delta_{\Lambda'} G_a \right) = \int dx [\Lambda, \Lambda']^a G_a$$

- In 1983 Stora and Zumino discovered a very systematic and elegant way to obtain non-trivial solutions to these conditions, based on the use of the 'descent equations'.

- The commutator of two gauge transformations acting on $W[A]$

$$(\delta_\Lambda \delta_{\Lambda'} - \delta_{\Lambda'} \delta_\Lambda) W[A] = \delta_{[\Lambda,\Lambda']} W[A]$$

  implies the Wess-Zumino consistency condition for the anomaly

$$\int dx \left(\Lambda'^a \delta_\Lambda G_a - \Lambda^a \delta_{\Lambda'} G_a\right) = \int dx [\Lambda, \Lambda']^a G_a$$

- In 1983 Stora and Zumino discovered a very systematic and elegant way to obtain non-trivial solutions to these conditions, based on the use of the 'descent equations'.

Note: A solution that can be obtained as the gauge variation of a *local* functional of the gauge field is trivial. A trivial anomaly can be eliminated by adding local counterterms to the action.

Descent equations

- To compute the non-abelian anomaly in any even dimension $2n$, one starts from a symmetric, invariant polynomial in $2n+2$ dimensions

$$\mathcal{P} = \operatorname{tr} F^{n+1}, \quad \text{‘Anomaly Polynomial’}$$

Descent equations

- To compute the non-abelian anomaly in any even dimension $2n$, one starts from a symmetric, invariant polynomial in $2n+2$ dimensions

$$\mathcal{P} = \mathrm{tr}\, F^{n+1}, \quad \text{`Anomaly Polynomial'}$$

We are using the notation of differential forms

$$\mathrm{tr}\, F^{n+1} = \varepsilon^{\mu_1\mu_2\cdots\mu_{2n+1}\mu_{2n+2}}\mathrm{tr}\, F_{\mu_1\mu_2}\ldots F_{\mu_{2n+1}\mu_{2n+2}}$$

and $F = dA + A^2$ is the field strength.

Descent equations

- To compute the non-abelian anomaly in any even dimension $2n$, one starts from a symmetric, invariant polynomial in $2n+2$ dimensions

$$\mathcal{P} = \mathrm{tr}\, F^{n+1}, \quad \text{`Anomaly Polynomial'}$$

We are using the notation of differential forms

$$\mathrm{tr}\, F^{n+1} = \varepsilon^{\mu_1\mu_2\ldots\mu_{2n+1}\mu_{2n+2}} \mathrm{tr}\, F_{\mu_1\mu_2} \ldots F_{\mu_{2n+1}\mu_{2n+2}}$$

and $F = dA + A^2$ is the field strength.

- From the fact that $\mathcal{P}$ is closed we have

$$d\mathcal{P} = 0 \implies \mathcal{P} = d\omega^0_{2n+1}(A)\,, \quad (d^2 = 0)$$

where $\omega^0_{2n+1}(A)$ is the Chern-Simons form.

- From the gauge invariance of $\mathcal{P}$

$$0 = \delta_\Lambda \mathcal{P} = \delta_\Lambda d\omega^0_{2n+1}(A) = d\delta_\Lambda \omega^0_{2n+1}(A)$$

- From the gauge invariance of $\mathcal{P}$

$$0=\delta_\Lambda \mathcal{P} = \delta_\Lambda d\omega^0_{2n+1}(A) = d\delta_\Lambda \omega^0_{2n+1}(A)$$

the gauge variation of $\omega^0_{2n+1}(A)$ is a total differential

$$\delta_\Lambda \omega^0_{2n+1}(A) = d\omega^1_{2n}(\Lambda, A)$$

- From the gauge invariance of $\mathcal{P}$

$$0 = \delta_\Lambda \mathcal{P} = \delta_\Lambda d\omega^0_{2n+1}(A) = d\delta_\Lambda \omega^0_{2n+1}(A)$$

the gauge variation of $\omega^0_{2n+1}(A)$ is a total differential

$$\delta_\Lambda \omega^0_{2n+1}(A) = d\omega^1_{2n}(\Lambda, A)$$

- Integrating this equation over a $\mathcal{M}_{2n+1}$ with boundary $2n$-dimensional space-time and using Stokes theorem

$$\delta_\Lambda \int_{\mathcal{M}_{2n+1}} \omega^0_{2n+1}(A) = \int_{\mathcal{M}_{2n+1}} d\omega^1_{2n}(\Lambda, A) = \int_{\partial\mathcal{M}_{2n+1}} \omega^1_{2n}(\Lambda, A)$$

- From the gauge invariance of $\mathcal{P}$

$$0 = \delta_\Lambda \mathcal{P} = \delta_\Lambda d\omega^0_{2n+1}(A) = d\delta_\Lambda \omega^0_{2n+1}(A)$$

the gauge variation of $\omega^0_{2n+1}(A)$ is a total differential

$$\delta_\Lambda \omega^0_{2n+1}(A) = d\omega^1_{2n}(\Lambda, A)$$

- Integrating this equation over a $\mathcal{M}_{2n+1}$ with boundary $2n$-dimensional space-time and using Stokes theorem

$$\delta_\Lambda \int_{\mathcal{M}_{2n+1}} \omega^0_{2n+1}(A) = \int_{\mathcal{M}_{2n+1}} d\omega^1_{2n}(\Lambda, A) = \int_{\partial\mathcal{M}_{2n+1}} \omega^1_{2n}(\Lambda, A)$$

shows that we can make the identification

$$\int_{\partial\mathcal{M}_{2n+1}} \omega^1_{2n}(\Lambda, A) = \int d^{2n}x \Lambda^a(x) G_a[A]$$

The Wess-Zumino action

- $\omega_4^1(\Lambda, A) = \Lambda^a(x) G_a[A]$ satisfies the Wess-Zumino consistency conditions in 4-dimensional space-time because it can be written as the variation of a functional

$$\int d^4x \Lambda^a(x) G_a[A] = \delta_\Lambda \int_{\mathcal{M}_5} \omega_{2n+1}^0(A)$$

The Wess-Zumino action

- $\omega_4^1(\Lambda, A) = \Lambda^a(x) G_a[A]$ satisfies the Wess-Zumino consistency conditions in 4-dimensional space-time because it can be written as the variation of a functional

$$\int d^4x \Lambda^a(x) G_a[A] = \delta_\Lambda \int_{\mathcal{M}_5} \omega^0_{2n+1}(A)$$

It is non-trivial because the RHS is not a local functional in 4 dimensional space-time.

The Wess-Zumino action

- $\omega_4^1(\Lambda, A) = \Lambda^a(x) G_a[A]$ satisfies the Wess-Zumino consistency conditions in 4-dimensional space-time because it can be written as the variation of a functional

$$\int d^4x \Lambda^a(x) G_a[A] = \delta_\Lambda \int_{\mathcal{M}_5} \omega^0_{2n+1}(A)$$

  It is non-trivial because the RHS is not a local functional in 4 dimensional space-time.

- One can make sense of the RHS if we define $A = A(x, s)$, where x is 4-dimensional and $A(x, s)$ is obtained from $A(x)$ by a finite gauge transformation

$$A(x, s) = g^{-1} dg + g^{-1} A(x) g$$

  Here $g(x, s) = \exp(s\xi(x))$ and the 'pion field' $\xi(x)$ transforms in the adjoint representation. One can show that this is equivalent to the action proposed by Wess and Zumino in 1971.

Plan of the talk

- Basic facts about anomalies
- **Relativistic hydrodynamics**
- Anomalous hydrodynamics
- Structure of anomalous partition functions

From Thermodynamics to Hydrodynamics

Assume a fluid in thermodynamic equilibrium with equation of state

$$p = p\,(T, \mu)$$

from which we may obtain the entropy, particle and energy densities

$$s = \partial p/\partial T \quad , \quad n = \partial p/\partial \mu \quad , \quad \epsilon = -p + Ts + \mu n$$

From Thermodynamics to Hydrodynamics

Assume a fluid in thermodynamic equilibrium with equation of state

$$p = p\,(T, \mu)$$

from which we may obtain the entropy, particle and energy densities

$$s = \partial p / \partial T \quad , \quad n = \partial p / \partial \mu \quad , \quad \epsilon = -p + Ts + \mu n$$

- In **Hydrodynamics** the thermodynamic variables T and μ are promoted to slowly varying functions $T(x)$ and $\mu(x)$. To these, one has to add a local fluid velocity $u^{\mu}(x)$, with $u^2 = u^{\mu} u_{\mu} = -1$. Thus, we take as hydrodynamic fields

$$\{T(x), \mu(x), u^{\mu}(x)\}$$

From Thermodynamics to Hydrodynamics

Assume a fluid in thermodynamic equilibrium with equation of state

$$p = p\,(T, \mu)$$

from which we may obtain the entropy, particle and energy densities

$$s = \partial p/\partial T \quad , \quad n = \partial p/\partial \mu \quad , \quad \epsilon = -p + Ts + \mu n$$

- In **Hydrodynamics** the thermodynamic variables T and μ are promoted to slowly varying functions $T(x)$ and $\mu(x)$. To these, one has to add a local fluid velocity $u^\mu(x)$, with $u^2 = u^\mu u_\mu = -1$. Thus, we take as hydrodynamic fields

$$\{T(x), \mu(x), u^\mu(x)\}$$

- In **Ideal Hydrodynamics** the local the entropy, particle and energy densities are obtained form the equation of state by the expressions above.

Ideal Hydrodynamics

- The energy-momentum and particle current density are given by

$$
\begin{aligned}
T^{\mu\nu} &= (\epsilon + p)\, u^{\mu} u^{\nu} + p\, \eta^{\mu\nu} \\
J^{\mu} &= n u^{\mu}
\end{aligned}
$$

Ideal Hydrodynamics

- The energy-momentum and particle current density are given by

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$
$$J^\mu = n u^\mu$$

These 'constitutive relations' can be understood by noting that in the local rest frame defined by u^μ one has

$$T^{00} = \epsilon \ , \quad T^{ij} = p\, \delta^{ij} \ , \quad J^0 = n \ , \quad T^{0i} = J^i = 0$$

Ideal Hydrodynamics

- The energy-momentum and particle current density are given by

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$
$$J^\mu = n u^\mu$$

These 'constitutive relations' can be understood by noting that in the local rest frame defined by u^μ one has

$$T^{00} = \epsilon \ , \ \ T^{ij} = p\, \delta^{ij} \ , \ \ J^0 = n \ , \ \ T^{0i} = J^i = 0$$

- The equations of motion are

$$\partial_\mu T^{\mu\nu} = 0 \quad (1) \quad , \quad \partial_\mu J^\mu = 0 \quad (2)$$

Ideal Hydrodynamics

- The energy-momentum and particle current density are given by

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$
$$J^\mu = n u^\mu$$

These 'constitutive relations' can be understood by noting that in the local rest frame defined by u^μ one has

$$T^{00} = \epsilon \ , \quad T^{ij} = p\, \delta^{ij} \ , \quad J^0 = n \ , \quad T^{0i} = J^i = 0$$

- The equations of motion are

$$\partial_\mu T^{\mu\nu} = 0 \quad (1) \quad , \quad \partial_\mu J^\mu = 0 \quad (2)$$

- Combining (1) and (2) yields

$$\partial_\mu (s\, u^\mu) = 0$$

i. e., ideal hydrodynamics is non-dissipative.

Dissipative hydrodynamics and constitutive relations

- Ideal hydrodynamics is generalized by writing the most general expression for $T^{\mu\nu}$ and J^{μ}

$$
\begin{aligned}
T^{\mu\nu} &= (\mathcal{E} + \mathcal{P})u^{\mu}u^{\nu} + \mathcal{P}\eta^{\mu\nu} + (q^{\mu}u^{\nu} + q^{\nu}u^{\mu}) + t^{\mu\nu} \\
J^{\mu} &= \mathcal{N}u^{\mu} + j^{\mu}
\end{aligned}
$$

where $u_{\mu}q^{\mu} = u_{\mu}j^{\mu} = u_{\mu}t^{\mu\nu} = 0$, and $t^{\mu\nu}$ is symmetric and traceless.

Dissipative hydrodynamics and constitutive relations

- Ideal hydrodynamics is generalized by writing the most general expression for $T^{\mu\nu}$ and J^μ

$$\begin{aligned} T^{\mu\nu} &= (\mathcal{E} + \mathcal{P})u^\mu u^\nu + \mathcal{P}\eta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu} \\ J^\mu &= \mathcal{N} u^\mu + j^\mu \end{aligned}$$

  where $u_\mu q^\mu = u_\mu j^\mu = u_\mu t^{\mu\nu} = 0$, and $t^{\mu\nu}$ is symmetric and traceless.
- $\mathcal{E}$, $\mathcal{P}$, q^μ, j^μ and $t^{\mu\nu}$ depend on the hydrodynamic fields

$$\{T(x), \mu(x), u^\mu(x)\}$$

  and their derivatives.

Dissipative hydrodynamics and constitutive relations

- Ideal hydrodynamics is generalized by writing the most general expression for $T^{\mu\nu}$ and J^μ

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P})u^\mu u^\nu + \mathcal{P}\eta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu}$$
$$J^\mu = \mathcal{N} u^\mu + j^\mu$$

  where $u_\mu q^\mu = u_\mu j^\mu = u_\mu t^{\mu\nu} = 0$, and $t^{\mu\nu}$ is symmetric and traceless.
- $\mathcal{E}$, $\mathcal{P}$, q^μ, j^μ and $t^{\mu\nu}$ depend on the hydrodynamic fields

$$\{T(x), \mu(x), u^\mu(x)\}$$

  and their derivatives.
- One may use the ambiguities in the definition of u^μ to choose a 'frame'. In the **Landau frame** one defines the fluid velocity in such a way that $q^\mu = 0$.

First order Hydrodynamics

- In the Landau frame, the constitutive relations to first derivative order are given by $\mathcal{E} = \epsilon$ and

First order Hydrodynamics

- In the Landau frame, the constitutive relations to first derivative order are given by $\mathcal{E} = \epsilon$ and

$$\begin{aligned} \mathcal{P} &= p - \zeta\, \partial_\lambda u^\lambda \\ t^{\mu\nu} &= -\eta\, \sigma^{\mu\nu} \\ j^\mu &= -\sigma T \Delta^{\mu\nu} \partial_\nu (\mu/T) + \chi_T \Delta^{\mu\nu} \partial_\nu T \end{aligned}$$

First order Hydrodynamics

- In the Landau frame, the constitutive relations to first derivative order are given by $\mathcal{E} = \epsilon$ and

$$\begin{aligned} \mathcal{P} &= p - \zeta\, \partial_\lambda u^\lambda \\ t^{\mu\nu} &= -\eta\, \sigma^{\mu\nu} \\ j^\mu &= -\sigma T \Delta^{\mu\nu} \partial_\nu (\mu/T) + \chi_T \Delta^{\mu\nu} \partial_\nu T \end{aligned}$$

where $\Delta^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu}$ is the transverse projector ($\Delta^{\mu\nu} u_\mu = 0$) and

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial_\mu u^\mu \right)$$

is the shear tensor.

First order Hydrodynamics

- In the Landau frame, the constitutive relations to first derivative order are given by $\mathcal{E} = \epsilon$ and

$$\mathcal{P} = p - \zeta\, \partial_\lambda u^\lambda$$
$$t^{\mu\nu} = -\eta\, \sigma^{\mu\nu}$$
$$j^\mu = -\sigma T \Delta^{\mu\nu} \partial_\nu (\mu/T) + \chi_T \Delta^{\mu\nu} \partial_\nu T$$

  where $\Delta^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu}$ is the transverse projector ($\Delta^{\mu\nu} u_\mu = 0$) and

$$\sigma^{\mu\nu} \equiv \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial_\mu u^\mu \right)$$

  is the shear tensor.
- Writing the second law of thermodynamics in the form $\partial_\mu S^\mu \geqslant 0$ imposes

$$\eta \geq 0 \ , \ \zeta \geq 0 \ , \ \sigma \geq 0 \ , \ \chi_T = 0$$

Plan of the talk

- Basic facts about anomalies
- Relativistic hydrodynamics
- **Anomalous hydrodynamics**
- Structure of anomalous partition functions

QFT anomalies and Hydrodynamics

- Hydrodynamics arises as the description of long wavelength, low frequency modes in QFT at finite temperature and chemical potential.

QFT anomalies and Hydrodynamics

- Hydrodynamics arises as the description of long wavelength, low frequency modes in QFT at finite temperature and chemical potential.
- This suggests that anomalies should play a role in the hydrodynamics of chiral fluids. But this was not clearly realized until the work of Son and Surówka in 2009. (See however G. Newman 2005)

QFT anomalies and Hydrodynamics

- Hydrodynamics arises as the description of long wavelength, low frequency modes in QFT at finite temperature and chemical potential.
- This suggests that anomalies should play a role in the hydrodynamics of chiral fluids. But this was not clearly realized until the work of Son and Surówka in 2009. (See however G. Newman 2005)
- In their 2009 work *Hydrodynamics with Triangle Anomalies* they considered a charged chiral fluid in the presence of an external $U(1)$ gauge field with equations of motion

$$\partial_\mu T^{\mu\nu} = F^{\mu\lambda} J_\lambda, \quad \partial_\mu J^\mu = \frac{C}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

QFT anomalies and Hydrodynamics

- Hydrodynamics arises as the description of long wavelength, low frequency modes in QFT at finite temperature and chemical potential.
- This suggests that anomalies should play a role in the hydrodynamics of chiral fluids. But this was not clearly realized until the work of Son and Surówka in 2009. (See however G. Newman 2005)
- In their 2009 work *Hydrodynamics with Triangle Anomalies* they considered a charged chiral fluid in the presence of an external $U(1)$ gauge field with equations of motion

$$\partial_\mu T^{\mu\nu} = F^{\mu\lambda} J_\lambda, \quad \partial_\mu J^\mu = \frac{C}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

They showed that the positivity of entropy production demands the modification of the constitutive relations.

- Concretely $J^\mu = nu^\mu + j^\mu$, with

$$j^\mu = -\sigma\Big(T\Delta^{\mu\nu}\partial_\nu(\mu/T) - E^\mu\Big) + \xi\omega^\mu + \xi_B B^\mu$$

- Concretely $J^\mu = nu^\mu + j^\mu$, with

$$j^\mu = -\sigma\Big(T\Delta^{\mu\nu}\partial_\nu(\mu/T) - E^\mu\Big) + \xi\omega^\mu + \xi_B B^\mu$$

Here E^μ and B^μ are the electric and magnetic fields in the local rest frame $E^\mu = F^{\mu\nu}u_\nu$,

$$E^\mu = F^{\mu\nu}u_\nu, \quad B^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$$

- Concretely $J^\mu = nu^\mu + j^\mu$, with

$$j^\mu = -\sigma\Big(T\Delta^{\mu\nu}\partial_\nu(\mu/T) - E^\mu\Big) + \xi\omega^\mu + \xi_B B^\mu$$

Here E^μ and B^μ are the electric and magnetic fields in the local rest frame $E^\mu = F^{\mu\nu}u_\nu$,

$$E^\mu = F^{\mu\nu}u_\nu, \quad B^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$$

and ω^μ is the vorticity

$$\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$$

- Concretely $J^\mu = nu^\mu + j^\mu$, with

$$j^\mu = -\sigma\Big(T\Delta^{\mu\nu}\partial_\nu(\mu/T) - E^\mu\Big) + \xi\omega^\mu + \xi_B B^\mu$$

Here E^μ and B^μ are the electric and magnetic fields in the local rest frame $E^\mu = F^{\mu\nu}u_\nu$,

$$E^\mu = F^{\mu\nu}u_\nu, \quad B^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$$

and ω^μ is the vorticity

$$\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$$

- From the entropy current condition they were able to determine the two new transport coefficients

$$\xi = C\Big(\mu^2 - \frac{2}{3}\frac{n\mu^3}{\epsilon + P}\Big), \quad \xi_B = C\Big(\mu - \frac{1}{2}\frac{n\mu^2}{\epsilon + P}\Big)$$

A QFT computation

- Son and Sorowka used only the anomalous divergence of the $U(1)$ current to obtain their results. It was later realized, throught explicit QFT computations, that they were missing crucial contributions to the anomalous transport coefficients.

A QFT computation

- Son and Sorowka used only the anomalous divergence of the $U(1)$ current to obtain their results. It was later realized, throught explicit QFT computations, that they were missing crucial contributions to the anomalous transport coefficients.
- That was part of the motivation behind our work

  J. Mañes and M. Valle, *Parity violating gravitational response and anomalous constituive relations*, JHEP 1301 (2013) 008

  where we do a QFT computation to third order in the derivative expansion.

- We place an ideal gas of Weyl fermions in a curved background at finite T and μ. The action is

$$\int d^4x\sqrt{-g}\frac{i}{2}\left[\bar{\psi}\gamma^\mu\nabla_\mu\psi-(\nabla_\mu\bar{\psi})\gamma^\mu\psi\right]$$

- We place an ideal gas of Weyl fermions in a curved background at finite T and μ. The action is

$$\int d^4x\sqrt{-g}\frac{i}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - (\nabla_{\mu}\bar{\psi})\gamma^{\mu}\psi\right]$$

where $\nabla_{\mu}\psi = \partial_{\mu}\psi - \Gamma_{\mu}\psi$ and the spin connection is related to the vierbein e_a^{ν} by

$$\Gamma_{\mu} = \frac{1}{8}[\gamma^a, \gamma^b]e_a^{\nu}e_{b\nu;\mu} = \frac{1}{8}[\gamma^a, \gamma^b]e_a^{\nu}(\partial_{\mu}e_{b\nu} - \Gamma^{\alpha}_{\mu\nu})e_{\beta\alpha}$$

- We place an ideal gas of Weyl fermions in a curved background at finite T and μ. The action is

$$\int d^4x\sqrt{-g}\frac{i}{2}\left[\bar{\psi}\gamma^\mu\nabla_\mu\psi-(\nabla_\mu\bar{\psi})\gamma^\mu\psi\right]$$

where $\nabla_\mu\psi=\partial_\mu\psi-\Gamma_\mu\psi$ and the spin connection is related to the vierbein e_a^ν by

$$\Gamma_\mu=\frac{1}{8}[\gamma^a,\gamma^b]e_a^\nu e_{b\nu;\mu}=\frac{1}{8}[\gamma^a,\gamma^b]e_a^\nu(\partial_\mu e_{b\nu}-\Gamma^\alpha_{\mu\nu})e_{\beta\alpha}$$

- In terms of the effective action Γ we may define the graviton polarization tensor

$$\Pi^{\mu\nu\,\rho\sigma}(x-y)=-4\;\frac{\delta\Gamma}{\delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y)}\bigg|_{g=\eta}=-2\;\frac{\delta}{\delta g_{\mu\nu}(x)}\left(\sqrt{-g}\langle T^{\rho\sigma}(y)\rangle\right)$$

with

$$\langle T^{\mu\nu}\rangle=\frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}}$$

- Linear response theory gives the corresponding induced change to linear order in $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$.

$$\delta\left(\sqrt{-g}\langle T^{\mu\nu}(x)\rangle\right) = -\frac{1}{2}\int d^4y\, \Pi^{\mu\nu\,\rho\sigma}(x-y)h_{\rho\sigma}(y)$$

where the retarded version of $\Pi^{\mu\nu\,\rho\sigma}(x-y)$ has to be used.

- Linear response theory gives the corresponding induced change to linear order in $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$.

$$\delta\left(\sqrt{-g}\langle T^{\mu\nu}(x)\rangle\right) = -\frac{1}{2}\int d^4y\, \Pi^{\mu\nu\,\rho\sigma}(x-y)h_{\rho\sigma}(y)$$

where the retarded version of $\Pi^{\mu\nu\,\rho\sigma}(x-y)$ has to be used.

- There are two contributions to $\Pi^{\mu\nu\,\rho\sigma}$:

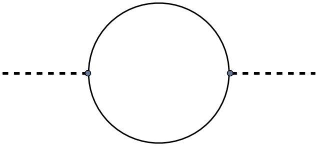
$$\begin{aligned}\Pi^{\mu\nu\,\rho\sigma}(x-y) \equiv & -i\,\theta(x^0-y^0)\left\langle\left[T^{\mu\nu}(x), T^{\rho\sigma}(y)\right]\right\rangle \\ & -2\left\langle \left.\frac{\delta\left(\sqrt{-g(x)}\,T^{\mu\nu}(x)\right)}{\delta g_{\rho\sigma}(y)}\right|_{g=\eta}\right\rangle\end{aligned}$$

For an ideal gas of left-handed Weyl fermions, the first term takes the following form in the imaginary time formalism

$$\Pi_1^{\mu\nu\,\rho\sigma}(i\nu_n, \boldsymbol{q}) = T\sum_{\omega_n}\int \frac{d^3k}{(2\pi)^3}\mathrm{tr}\big[\mathcal{P}_- \not{K} V^{\mu\nu}(K, K+Q)(\not{K}+\not{Q}) \times V^{\rho\sigma}(K+Q, K)\big]\frac{1}{K^2(K+Q)^2}, \qquad K^0 = i\omega_n + \mu,$$

where $\mathcal{P}_- = (1-\gamma_5)/2$ and the fermion-fermion-graviton three-vertex is

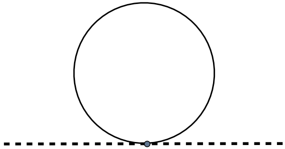
$$V^{\mu\nu}(K,P) = \frac{1}{4}\big[\gamma^\mu(K+P)^\nu + \gamma^\nu(K+P)^\mu\big] - \frac{1}{2}\eta^{\mu\nu}(\not{K}+\not{P})$$



Up to parity-even contributions, the second ('seagull') term can be written

$$\Pi_2^{\mu\nu\,\rho\sigma}(i\nu_n, \boldsymbol{q}) = \frac{1}{8}\eta^{\mu\rho} T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \mathrm{tr}\big[\{\sigma^{\nu\sigma}, \not{Q}\}\mathcal{P}_- \not{K}\big] \frac{1}{K^2}$$
$$+ \frac{1}{8}\eta^{\nu\rho} T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \mathrm{tr}\big[\{\sigma^{\mu\sigma}, \not{Q}\}\mathcal{P}_- \not{K}\big] \frac{1}{K^2} + (\rho \leftrightarrow \sigma),$$

where $\sigma^{\nu\sigma} \equiv \frac{1}{4}[\gamma^\nu, \gamma^\sigma]$.



Energy-momentum tensor and metric perturbations

- Once the parity-odd response function is computed, the parity violating part of the energy-momentum tensor is given by

$$\delta\langle T^{\mu\nu}\rangle = -\frac{1}{2}\Pi^{\mu\nu\,\rho\sigma}(q^0, \boldsymbol{q})h_{\rho\sigma}$$

Energy-momentum tensor and metric perturbations

- Once the parity-odd response function is computed, the parity violating part of the energy-momentum tensor is given by

$$\delta\langle T^{\mu\nu}\rangle = -\frac{1}{2}\Pi^{\mu\nu\,\rho\sigma}(q^0, \boldsymbol{q})h_{\rho\sigma}$$

- It is convenient to decompose a general perturbation of the metric into $SO(3)$ components, where $a_i^{(S)}$ and $a_i^{(L)} = \partial_i b$ are the solenoidal and irrotational parts of $h_{0i} = -a_i(t, \boldsymbol{x})$.

	Scalar	Vector	Tensor
h_{00}	-2σ	–	–
h_{0i}	$-\partial_i b$	$-a_i^{(S)}$	–
h_{ij}	$c\delta_{ij} + \partial_i\partial_j d$	$\partial_i F_j + \partial_j F_i$	$\tilde{h}_{ij}$

- Direct substitution shows that scalar perturbations do not produce any parity-violating effect on $T^{\mu\nu}$.

- Direct substitution shows that scalar perturbations do not produce any parity-violating effect on $T^{\mu\nu}$.
- For vector perturbations, one finds

$$\delta\langle T^{0i}\rangle = c_{\mathbb{V}}(q^0, q)\, i\epsilon^{ijk} q^j(-a_k + iq^0 F_k),$$
$$\delta\langle T^{ij}\rangle = c_{\mathbb{V}}(q^0, q)\, iq^0 \left(\epsilon^{imn}\hat{q}^m\hat{q}^j + \epsilon^{jmn}\hat{q}^m\hat{q}^i\right)(-a_n + iq^0 F_n),$$

where $\hat{q}^j = q^j/q$. $c_{\mathbb{V}}(q^0, q)$ parametrizes the response to vector perturbations of the metric.

- Direct substitution shows that scalar perturbations do not produce any parity-violating effect on $T^{\mu\nu}$.
- For vector perturbations, one finds

$$\delta\langle T^{0i}\rangle = c_{\mathbb{V}}(q^0,q)\, i\epsilon^{ijk} q^j(-a_k + iq^0 F_k),$$
$$\delta\langle T^{ij}\rangle = c_{\mathbb{V}}(q^0,q)\, iq^0 \left(\epsilon^{imn}\hat{q}^m\hat{q}^j + \epsilon^{jmn}\hat{q}^m\hat{q}^i\right)(-a_n + iq^0 F_n),$$

where $\hat{q}^j = q^j/q$. $c_{\mathbb{V}}(q^0,q)$ parametrizes the response to vector perturbations of the metric.

- Similarly, $c_{\mathbb{T}}(q^0,q)$ parametrizes the response to tensor perturbations $\tilde{h}_{ij}$

$$\delta\langle T^{ij}\rangle = -c_{\mathbb{T}}(q^0,q)\epsilon^{ilm}\delta^{jn}\, iq^l \tilde{h}_{mn} + (i \leftrightarrow j)$$

The functions $c_{\mathbb{L},\mathbb{T}}$ are explicitly given by

$$
\begin{aligned}
c_{\mathbb{V}}(q^0,q) &= \frac{1}{24\pi^2}\left(\mu^3+\pi^2\mu T^2\right)\left(1+\frac{3Q^2}{q^2}L(q^0,q)\right)\\
&\quad+\frac{\mu\, q^2}{192\pi^2}\left[-\frac{2Q^2}{q^2}+\frac{3Q^2(q^2-2Q^2)}{q^4}L(q^0,q)\right]\\
c_{\mathbb{T}}(q^0,q) &= -\frac{1}{96\pi^2}\left(\mu^3+\pi^2\mu T^2\right)\left(2+\frac{Q^2}{q^2}+\frac{3Q^4}{q^4}L(q^0,q)\right)\\
&\quad+\frac{\mu\, q^2}{192\pi^2}\left[\frac{Q^4}{2q^4}+\frac{3Q^6}{2q^6}L(q^0,q)\right]
\end{aligned}
$$

where

$$L(q^0,q) = -1+\frac{q^0}{2q}\ln\left|\frac{q^0+q}{q^0-q}\right| - \frac{i\pi}{2}\frac{q^0}{q}\theta\Big(1-\frac{(q^0)^2}{q^2}\Big)$$

This gives the response function to third derivative order.

Static limit and anomalous constitutive relations

- The complete expressions to linear order in Q have been used in

  M. Valle, *Kinetic theory and evolution of cosmological fluctuations with neutrino number asymmetry*, PRD88 (2013)041304

  to derive the Botzmann equation that governs the evolution of the μ-dependent part of the chiral fermion distribution.
- In the static limit $q^0 \to 0$

$$c_{\mathbb{V}}(0,q) = -\frac{1}{12\pi^2}\left(\mu^3 + \pi^2\mu\, T^2\right) + \frac{\mu}{192\pi^2}q^2$$
$$c_{\mathbb{T}}(0,q) = -\frac{\mu}{192\pi^2}q^2$$

Static limit and anomalous constitutive relations

- To first derivative order, this implies the following the parity-odd contributions to the energy-momentum tensor

$$\delta\langle T^{0i}\rangle = -c_{\mathbb{V}}^{(0)}(0,q)\epsilon^{ijk} i q^j a_k \ , \quad \delta\langle T^{ij}\rangle = 0$$

Static limit and anomalous constitutive relations

- To first derivative order, this implies the following the parity-odd contributions to the energy-momentum tensor

$$\delta\langle T^{0i}\rangle = -c_{\mathbb{V}}^{(0)}(0,q)\epsilon^{ijk} iq^j a_k \ , \quad \delta\langle T^{ij}\rangle = 0$$

Identifying $h_{0i} = -a_i$ with the fluid velocity v^i and T^{0i} with the induced momentum density g_i gives

$$\delta \boldsymbol{g} = \chi_{\mathbb{V}} \boldsymbol{\nabla} \times \boldsymbol{v}$$

Static limit and anomalous constitutive relations

- To first derivative order, this implies the following the parity-odd contributions to the energy-momentum tensor

$$\delta\langle T^{0i}\rangle = -c_{\mathbb{V}}^{(0)}(0,q)\epsilon^{ijk}iq^j a_k \ , \quad \delta\langle T^{ij}\rangle = 0$$

Identifying $h_{0i} = -a_i$ with the fluid velocity v^i and T^{0i} with the induced momentum density g_i gives

$$\delta \boldsymbol{g} = \chi_{\mathbb{V}} \boldsymbol{\nabla} \times \boldsymbol{v}$$

and implies the following value for the anomalous susceptibility

$$\chi_{\mathbb{V}} = c_{\mathbb{V}}^{(0)}(0,q) = -\frac{1}{12\pi^2}\left(\mu^3 + \pi^2 \mu T^2\right)$$

Static limit and anomalous constitutive relations

- To first derivative order, this implies the following the parity-odd contributions to the energy-momentum tensor

$$\delta\langle T^{0i}\rangle = -c_{\mathbb{V}}^{(0)}(0,q)\epsilon^{ijk} i q^j a_k \ , \quad \delta\langle T^{ij}\rangle = 0$$

Identifying $h_{0i} = -a_i$ with the fluid velocity v^i and T^{0i} with the induced momentum density g_i gives

$$\delta \boldsymbol{g} = \chi_{\mathbb{V}} \boldsymbol{\nabla} \times \boldsymbol{v}$$

and implies the following value for the anomalous susceptibility

$$\chi_{\mathbb{V}} = c_{\mathbb{V}}^{(0)}(0,q) = -\frac{1}{12\pi^2}\left(\mu^3 + \pi^2 \mu T^2\right)$$

- The term $\propto T^2$ was missing in the work of Son and Surowka.

Plan of the talk

- Basic facts about anomalies
- Relativistic hydrodynamics
- Anomalous hydrodynamics
- **Structure of anomalous partition functions**

Around 2012 it was realized that the consequences of QFT anomalies in hydrodynamics could be conveniently obtained from the analysis of an equilibrium partition function:

Around 2012 it was realized that the consequences of QFT anomalies in hydrodynamics could be conveniently obtained from the analysis of an equilibrium partition function:

- K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, *Towards hydrodynamics without an entropy current*, PRL109 (2012) 101601
- K. Jensen, *Triangle Anomalies, Thermodynamics, and Hydrodynamics*, PRD85 (2012) 125017
- M. Valle, *Hydrodynamics in* $1+1$ *dimensions with gravitational anomalies*, JHEP 1208 (2012) 113
- N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on Fluid Dynamics from Equilibrium Partition Functions*, JHEP 1209 (2012) 046

Around 2012 it was realized that the consequences of QFT anomalies in hydrodynamics could be conveniently obtained from the analysis of an equilibrium partition function:

- K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, *Towards hydrodynamics without an entropy current*, PRL109 (2012) 101601
- K. Jensen, *Triangle Anomalies, Thermodynamics, and Hydrodynamics*, PRD85 (2012) 125017
- M. Valle, *Hydrodynamics in* $1+1$ *dimensions with gravitational anomalies*, JHEP 1208 (2012) 113
- N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on Fluid Dynamics from Equilibrium Partition Functions*, JHEP 1209 (2012) 046

This is possible because the new anomaly induced transport coefficients, which violate parity, are non-dissipative.

- The anomalous partition function is closely related to the Wess-Zumino effective action, but has two distinctive features

- The anomalous partition function is closely related to the Wess-Zumino effective action, but has two distinctive features

  (1) It can be written as the 3d integral of a local polynomial in the background fields.

- The anomalous partition function is closely related to the Wess-Zumino effective action, but has two distinctive features

  (1) It can be written as the 3d integral of a local polynomial in the background fields.
  (2) It can be decomposed into two different pieces

$$W[A] = W_{anom}[A] + W_{inv}[A]$$

- The anomalous partition function is closely related to the Wess-Zumino effective action, but has two distinctive features

  (1) It can be written as the 3d integral of a local polynomial in the background fields.
  (2) It can be decomposed into two different pieces

$$W[A] = W_{anom}[A] + W_{inv}[A]$$

- W_{anom} is essentially given by the dimensional reduction of ω^0_{2n+1} on the thermal cycle.

$$\omega^0_{2n+1} \to n\,\mathrm{tr}\left(A_0 F^{n-1}\right) + d\Gamma[A]$$

The non-invariance of $W_{anom}[A] \sim \Gamma[A]$ under gauge transformations gives rise to the anomaly.

- The anomalous partition function is closely related to the Wess-Zumino effective action, but has two distinctive features

  (1) It can be written as the 3d integral of a local polynomial in the background fields.
  (2) It can be decomposed into two different pieces

$$W[A] = W_{anom}[A] + W_{inv}[A]$$

- W_{anom} is essentially given by the dimensional reduction of ω^0_{2n+1} on the thermal cycle.

$$\omega^0_{2n+1} \to n\,\mathrm{tr}\,(A_0 F^{n-1}) + d\Gamma[A]$$

  The non-invariance of $W_{anom}[A] \sim \Gamma[A]$ under gauge transformations gives rise to the anomaly.

- W_{inv} is the integral of a Chern-Simons form. It is parity violating but gauge invariant.

$U(1)$ anomalous partition function in $3+1$ dimensions

Consider a gas of Weyl fermions in the presence of a time independent gravitational background and $U(1)$ gauge field

$$
\begin{aligned}
ds^2 &= -e^{2\sigma(\boldsymbol{x})}(dt + a_i(\boldsymbol{x})dx^i)^2 + g_{ij}(\boldsymbol{x})dx^i dx^j \\
A_\mu &= (A_0(\boldsymbol{x}), \boldsymbol{A}(\boldsymbol{x}))
\end{aligned}
$$

where $i, j = 1, 2, 3$.

$U(1)$ anomalous partition function in $3+1$ dimensions

Consider a gas of Weyl fermions in the presence of a time independent gravitational background and $U(1)$ gauge field

$$ds^2 = -e^{2\sigma(\boldsymbol{x})}(dt + a_i(\boldsymbol{x})dx^i)^2 + g_{ij}(\boldsymbol{x})dx^i dx^j$$
$$A_\mu = (A_0(\boldsymbol{x}), \boldsymbol{A}(\boldsymbol{x}))$$

where $i, j = 1, 2, 3$. We have gauge and mixed (covariant) anomalies

$$\nabla_\mu J^\mu_{\rm cov} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\left(3c_A F_{\mu\nu}F_{\rho\sigma} + c_m R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\rho\sigma}\right)$$
$$\nabla_\nu T^{\mu\nu}_{\rm cov} = F^\mu{}_\nu J^\nu_{\rm cov} + \frac{1}{2}c_m\nabla_\nu\left(\epsilon^{\rho\sigma\alpha\beta}F_{\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}\right)$$

For a left-handed spinor in $(3+1)$ dimensions,

$$c_A = 8c_m = \frac{1}{24\pi^2}$$

- To first order in the derivatives of the background fields the anomalous partition function is given by

$$W[A,a] = \frac{c_A}{T_0}\int\left(2A_0\tilde{A}d\tilde{A} + A_0^2\tilde{A}da\right) - \tilde{c}_{4d}\,T_0\int\tilde{A}da$$

where a_i is the Kaluza-Klein field and $\tilde{A}_i = A_i - A_0 a_i$.

- To first order in the derivatives of the background fields the anomalous partition function is given by

$$W[A,a] = \frac{c_A}{T_0}\int\left(2A_0\tilde{A}d\tilde{A} + A_0^2\tilde{A}da\right) - \tilde{c}_{4d}T_0\int \tilde{A}da$$

  where a_i is the Kaluza-Klein field and $\tilde{A}_i = A_i - A_0 a_i$.

- The integral proportional to c_A is W_{anom}. The last integral, which is the gauge invariant W_{inv}, has the structure of a (mixed) Chern-Simons term.

- To first order in the derivatives of the background fields the anomalous partition function is given by

$$W[A,a] = \frac{c_A}{T_0}\int \Big(2A_0\tilde{A}d\tilde{A} + A_0^2\tilde{A}da\Big) - \tilde{c}_{4d}T_0\int \tilde{A}da$$

where a_i is the Kaluza-Klein field and $\tilde{A}_i = A_i - A_0 a_i$.

- The integral proportional to c_A is W_{anom}. The last integral, which is the gauge invariant W_{inv}, has the structure of a (mixed) Chern-Simons term.
- The coefficient $\tilde{c}_{4d}$ is in principle undetermined. However, it has been argued (K. Jensen, R. Loganayagam and A. Yarom, JHEP02(2013) 088) that $\tilde{c}_{4d} = -8\pi^2 c_m$.

- This can be checked against explicit QFT computations by noting that the partition function makes the following prediction for the susceptibility considered earlier ($\delta \boldsymbol{g} = \chi_{\mathbb{V}} \boldsymbol{\nabla} \times \boldsymbol{v}$)

$$\chi_{\mathbb{V}} = 2(\tilde{c}_{4d}\, \mu\, T^2 - c_A \mu^3)$$

- This can be checked against explicit QFT computations by noting that the partition function makes the following prediction for the susceptibility considered earlier ($\delta \boldsymbol{g} = \chi_{\mathbb{V}} \boldsymbol{\nabla} \times \boldsymbol{v}$)

$$\chi_{\mathbb{V}} = 2(\tilde{c}_{4d}\, \mu\, T^2 - c_A \mu^3)$$

- Our QFT computation gave

$$\chi_{\mathbb{V}} = -\frac{1}{12\pi^2}(\mu^3 + \pi^2 \mu T^2)$$

- This can be checked against explicit QFT computations by noting that the partition function makes the following prediction for the susceptibility considered earlier ($\delta \boldsymbol{g} = \chi_{\mathbb{V}} \boldsymbol{\nabla} \times \boldsymbol{v}$)

$$\chi_{\mathbb{V}} = 2(\tilde{c}_{4d}\, \mu\, T^2 - c_A \mu^3)$$

- Our QFT computation gave

$$\chi_{\mathbb{V}} = -\frac{1}{12\pi^2}(\mu^3 + \pi^2 \mu T^2)$$

which confirms that

$$\tilde{c}_{4d} = -8\pi^2 c_m = -\frac{1}{24}$$

Anomalous hydrodynamics in $2+1$ dimensions

- There are no gauge anomalies in odd space-time dimensions. As a consequence, $W_{anom} = 0$ and

$$W[A, a] = W_{inv}[A, a] = \int \Big(\alpha(\sigma, A_0) d\tilde{A} + T_0 \beta(\sigma, A_0) da\Big)$$

where $\tilde{A}_i = A_i - A_0 a_i$ and the coefficient *functions* of the $U(1)$ and KK field strengths are in principle arbitrary.

Anomalous hydrodynamics in $2 + 1$ dimensions

- There are no gauge anomalies in odd space-time dimensions. As a consequence, $W_{anom} = 0$ and

$$W[A, a] = W_{inv}[A, a] = \int \Big(\alpha(\sigma, A_0) d\tilde{A} + T_0 \beta(\sigma, A_0) da\Big)$$

  where $\tilde{A}_i = A_i - A_0 a_i$ and the coefficient *functions* of the $U(1)$ and KK field strengths are in principle arbitrary.

- In J. Mañes and M. Valle, *Parity odd equilibrium partition function in 2+1 dimensions*, JHEP 1311 (2013) 178

  we used Schwinger's proper time method to compute the unknown functions in the partition function for an ideal gas of fermions

They are given by

$$\alpha(\sigma, A_0) = \frac{1}{4\pi}\log\left[\cosh[\frac{\beta}{2}(A_0 - e^{\sigma} m)]\right] - \frac{1}{4\pi}\log\left[\cosh[\frac{\beta}{2}(A_0 + e^{\sigma} m)]\right] + \frac{A_0\beta}{4\pi}\mathrm{sgn}(m)$$

$$\begin{aligned}\beta(\sigma, A_0) = &-\frac{\beta^2}{4\pi}A_0 e^{\sigma} m + \frac{\beta}{8\pi}e^{\sigma} m \log[2\cosh(A_0\beta) + 2\cosh(e^{\sigma}\beta m)] \\ &-\frac{\beta}{4\pi}(A_0 + e^{\sigma} m)\log\left[1 + e^{-\beta(A_0 + e^{\sigma} m)}\right] \\ &+\frac{\beta}{4\pi}(A_0 - e^{\sigma} m)\log\left[1 + e^{-\beta(A_0 - e^{\sigma} m)}\right] \\ &+\frac{1}{4\pi}\mathrm{Li}_2\left[-e^{-\beta(A_0 + e^{\sigma} m)}\right] - \frac{1}{4\pi}\mathrm{Li}_2\left[-e^{-\beta(A_0 - e^{\sigma} m)}\right] \\ &+\frac{A_0^2\beta^2}{8\pi}\mathrm{sgn}(m)\end{aligned}$$

Hall viscosity

- In $2+1$ dimensions the stress tensor gets an anomalous contribution

$$t^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu}$$

where $\tilde{\eta}$ is the Hall viscosity and

$$\tilde{\sigma}^{\mu\nu} = \frac{1}{2}\left(\varepsilon^{\mu\alpha\beta}u_\alpha\sigma_\rho^\nu + \varepsilon^{\nu\alpha\rho}u_\alpha\sigma_\rho^\mu\right)$$

For the background considered above $\tilde{\sigma}^{\mu\nu}$ vanishes in equilibrium, and $\tilde{\eta}$ can not be computed from the partition function.

Hall viscosity

- In $2+1$ dimensions the stress tensor gets an anomalous contribution

$$t^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu}$$

  where $\tilde{\eta}$ is the Hall viscosity and

$$\tilde{\sigma}^{\mu\nu} = \frac{1}{2}\left(\varepsilon^{\mu\alpha\beta}u_\alpha\sigma_\rho^\nu + \varepsilon^{\nu\alpha\rho}u_\alpha\sigma_\rho^\mu\right)$$

  For the background considered above $\tilde{\sigma}^{\mu\nu}$ vanishes in equilibrium, and $\tilde{\eta}$ can not be computed from the partition function.
- M. Valle has recently shown (arXiv:1503.04020) that in the presence of torsion $\tilde{\sigma}^{\mu\nu} \neq 0$, and the equilibrium partition function can be used to obtain

$$\begin{aligned}\tilde{\eta} &= -\frac{\langle\bar{\Psi}\Psi\rangle}{4} \\ &= \frac{mT}{8\pi}\left[\ln\left(1+\exp\left(\frac{\mu-|m|}{T}\right)\right) + \ln\left(1+\exp\left(\frac{-\mu-|m|}{T}\right)\right)\right]\end{aligned}$$

State of the art

For spacetimes of even dimension $2n$:

- There is a well established method to obtain the non-invariant part W_{anom} of the anomalous partition function by dimensional reduction of the Chern-Simons form ω^0_{2n+1} on the thermal cycle. The only input is the anomaly polynomial.

State of the art

For spacetimes of even dimension $2n$:

- There is a well established method to obtain the non-invariant part W_{anom} of the anomalous partition function by dimensional reduction of the Chern-Simons form ω^0_{2n+1} on the thermal cycle. The only input is the anomaly polynomial.
- Regarding the invariant piece W_{inv}, the authors of

  K. Jensen, R. Loganayagam and A. Yarom, *Chern-Simons terms from thermal circles and anomalies*, JHEP05(2014)110

  propose the use of a 'thermal anomaly polynomial'. They derive the form of this polynomial from the condition of 'consistency with the thermal vacuum'.

THANK YOU