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Noncommutative Spaces and Quantum symmetries

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I started my scientific career by studying Bruno's papers on Noncommutative Geometry and then I had the priviledge of many interactions on the subject (Bruno was referee of my Ph.D. thesis). I shall recall some of his influential results on the subject, and later present recent developments.

(1989-1994) Quantum groups and spaces

(1992-1995) Differential calculus on quantum groups,
quantum Lie algebras of vector fields

(1998-1998) T-duality in Noncommutative Yang-Mills theories

(2001-2002) Nonabelian Seiberg-Witten map

(1989-1994) Quantum groups and spaces.

Quantum groups arose in solving with algebraic methods quantum integrable systems (e.g. spin chains). [Leningrad school]

They independently were studied in the noncommutative geometry context of C^*-algebras [Woronowicz].

They found applications in CFT and in the classifications of knots and 3-manifolds.

Quantum groups are symmetry groups of noncommutative spaces.

Bruno Zumino initially studies examples of q-groups and planes [Vokos, Wess, Zumino]. This lead to the classification of deformations of $GL(2)$ and its associated quantum planes. [Schirrmacher, Wess, Zumino].

Method: **Generators and relations** for quantum spaces and quantum groups.

Algebra of functions on commutative space is approximated by polynomial algebra in the coordinates x^i. They are mutually commuting: $x^i x^j = x^j x^i$. This algebra is that of words in the letters x^i modulo the ideal generated by the relations $x^i x^j = x^j x^i$. Deform these relations into the quadratic relations

$$x^i x^j = q^{-1} R^{ji}_{lm} x^l x^m \tag{1}$$

Examples of possible deformations:

$x^i x^j - x^j x^i = i\theta^{ij}$ — *canonical*

$y x^j - x^j y = i a x^j \quad , \qquad x^j x^i - x^i x^j = 0$ — *Lie algebra type*

$x^i x^j - q x^j x^i = 0$ — *quantum plane*
(ex. of quadratic rel.)

The **Poincaré-Birkhoff-Witt property** (i.e. ordering $(x^1)^{n_1}(x^2)^{n_2}(x^3)^{n_3}$.... without introducing further constraints) holds for the quadratic relations (1) if R satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\ . \tag{2}$$

The quantum group is a symmetry group of (1) i.e.

$$x^i \to x'^i = T^i_j x^j$$

with x'^i that satisfy the same relations (1). The algebra generated by the matrix entries T^i_j is noncommutative, it satisfies the RTT relations:

$$R^{ab}{}_{ef}T^e{}_c T^f{}_d = T^b{}_f T^a{}_e R^{ef}{}_{cd}$$

i.e.

$$RT_1T_2 = T_2T_1R \tag{3}$$

In particular the noncommutative two dimensional quantum plane is

$$xy = qyx \qquad\qquad \text{quantum plane}$$

it has a two parameter $GL_{q,p}(2)$ quantum symmetry group [Schirrmacher, Wess, Zumino].

Differential calculus on quantum planes, and their quantum groups symmetry

[Wess Zumino CERN-TH 5697/90]

Leibnitz rule for exterior derivative, $d(fg) = (df)g + f dg$,

Bimodule structure of 1-forms.

$d^2 = 0$.

Consistency conditions

$$d(x^i x^j - q^{-1} R^{ji}_{lm} x^l x^m) = 0$$

$$(xx - q^{-1} Rxx)dx^l = 0$$

use **associativity** and move dx to the left

Similarly we have relations and consistency conditions between partial derivatives ∂_i. Partial derivatives are related to the exterior derivative via $d = dx^i \partial_i$.

The x^i, ∂_j, dx^k algebra:

$$dx^i x^j = q^{-1} R^{-1}{}^{ij}_{ef} x^f dx^e$$

$$\partial_j x^i = \delta^i_j + q^{-1} R^{-1}{}^{ki}_{jl} x^l \partial_k$$

$$\partial_i \partial_j = q^{-1} R^{ef}_{ji} \partial_e \partial_f$$

It implies the Yang-Baxter equation for the R matrix.
The quantum group $GL_q(n)$ **emerges** as symmetry group of the quantum plane *and* its differential calculus.

$$x^i \to x'^i = T^i_j x^j \ , \quad \partial_i \to \partial'_i = T^j_i \partial_j$$

$$R T_1 T_2 = T_2 T_1 R$$

Partial derivatives are finite difference operators. Discretized Geometry.

$$\partial_y f(x,y) = \frac{f(x,y) - f(x,q^2 y)}{y(1-q^2)}$$

$$\partial_x f(x,y) = \frac{f(x,qy) - f(q^2 x,qy)}{x(1-q^2)}$$

NC spaces with these features could provide a natural ultraviolet cutoff.

q-deformed Minkowski space ($q \geq 1$)

q-Minkowski coordinates X^μ constructed as $SL_q(2)$ spinors bilinears.

$$[X^0, X^C] = 0$$

$$[X^B, X^C] = (q^2 - 1)\epsilon^{BC}_{\ \ A} X^0 X^A$$

The q-Lorentz group and the $SU_q(2)$ angular momentum subgroup act on this Minkowski space-time. The differential calculus on quantum Minkowski space gives the algebra of partial derivatives, i.e. momenta.

Momenta + Lorentz $\Rightarrow$ **Quantum Poincaré algebra** [Ogievetsky, Schmidke, Wess and Zumino]

Coordinates + Momenta $\Rightarrow$ **Quantum Phase Space** [Zumino]

Note: An extra generator, a dilatation, is needed to close the Poincaré algebra under complex conjugation. It is a unitary operator, usually denoted U (or Λ). $\overline{\partial}$ is a function of ∂ and of U. This is a typical phenomenon in q-groups.

(1992-1995) Differential calculus on quantum groups, quantum Lie algebras of vector fields

An approach that complements the previous one: more abstract techniques based on representation theory and universal R-matrices. More in the spirit of deformation quantization ($\star$-products); generators and relations then follow choosing a basis. First the differential geometry on quantum groups [Woronowicz] is further investigated. Then the geometry on quantum spaces is induced.

Differential calculus on noncommutative deformations of group manifolds, the quantum Lie algebra of left invariant vector fields (and its relation to the universal enveloping algebra).

Cartan calculus of Lie derivatives, contraction operators and exterior derivative. Then this geometry is induced on quantum homogeneous spaces: quantum planes and also projective spaces $CP_q(N)$.

[collaborations with Schlieker, Jurco, Schupp, Watts, Chryssomalakos, Chong-Sun Chu, Pei-Ming Ho].

(1998-1998) T-duality in Noncommutative Yang-Mills theories

T-duality acts within NCSYM theories on torus. [Connes, Douglas, Schwartz]

NC plane $[x^i, x^j] = \theta^{ij} \quad \Rightarrow \quad$ NC torus coordinates $U^i = e^{ix^i}$.

NCSYM: $U(n)$, θ^{ij}, G_{ij}, g_{SYM}, M first Chern number $\frac{1}{2\pi}\int TrF$.

NCSYM′: $U(n'), \theta'^{ij}$, G'_{ij}, g'_{SYM}, M'.

[Brace, Morariu, Zumino] give an explicit proof of the equality of the NCSYM and NCSYM′ actions in the case of T^2 and then of T^3, respectively with T-duality groups $SO(2,2,Z)$ and $SO(3,3,Z)$.

Let $\Lambda = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$, then

$$\theta' = (\mathcal{A}\theta + \mathcal{B})(\mathcal{C}\theta + \mathcal{D})^{-1},$$
$$G'^{ij} = (\mathcal{C}\theta + \mathcal{D})^i_{\ k}(\mathcal{C}\theta + \mathcal{D})^j_{\ l}\, G^{kl},$$
$$g'^2_{SYM} = \sqrt{|\det(\mathcal{C}\theta + \mathcal{D})|}\; g^2_{SYM},$$
$$\begin{pmatrix} n' \\ M' \end{pmatrix} = S(\Lambda)\begin{pmatrix} n \\ M \end{pmatrix} \qquad \text{spinor representation } S(\Lambda)$$

The rank and the bundle topology of the NCSYM theory determine the D0 and D2 brane charges in IIA string theory. Noncommutativity $\theta^{ij} = \int_{\gamma_{ij}} B$ captures the presence of nontrivial NS B field. The action of these $SO(d, d, Z)$ rotations matches the T-duality transformation on metric, string coupling constant, D-brane chages, NS field of type IIA string theory.

(2001-2002) Nonabelian Seiberg-Witten map A cohomological method in order to obtain the SW map. [Brace, Cerchiai, Pasqua, Varadarajan, Zumino]

Seiberg Witten Map between noncommutative and commutative gauge theories

SW map determines

$$\widehat{A} = \widehat{A}[A, \theta] \ , \quad \widehat{\epsilon} = \widehat{\epsilon}[\epsilon, A, \theta]$$

such that

$$\widehat{A}[A, \theta] + \delta_{\widehat{\epsilon}} \widehat{A}[A, \theta] = \widehat{A}[A + \delta_\epsilon A, \theta]$$

Noncommutative gauge transformations are one-to-one with commutative gauge transformations. If such a map exists then the physical degrees of freedom (gauge equivalence classes) of the commutative and noncommutative theory are the same.

(2010) Deformation of the wedge product of exterior forms covariant under general coordinates transformations. Covariant $\wedge_\star$-products [McCurdy, Zumino]

I briefly illustrate some developments of the NC geomerty thus far presented. I will use the language of $\star$-product deformation (deformation quantization), rather than generators and relations and restrict to the wide class of deformations obtained via Drinfeld twist.

Example Moyal-Weyl $\star$-product on $\mathbf{R}^4$

$$(f \star h)(x) = \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}} f(x)h(y)\Big|_{x=y}$$

We extract the bidifferential operator

$$\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}}$$

$\mathcal{F}$ is an example of Drinfeld twist on the manifold $M = \mathbf{R}^4$.

$\mathcal{F}$ deforms the geometry of M into a noncommutative geometry.

$\star$-Algebra of functions

$$f \star g = \mu \circ \mathcal{F}(f \otimes g)$$

$\star$-Algebra of functions

$$f \star g = \mu \circ \mathcal{F}(f \otimes g)$$

$\wedge_\star$-Algebra of Forms

$$\vartheta \wedge_\star \vartheta' = \wedge \circ \mathcal{F}(\vartheta \otimes \vartheta') \ .$$

Exterior forms are twisted-antisymmetric.

Similarly **$\star$-Tensorfields** $\tau \otimes_\star \tau'$

It is possible to deform via $\mathcal{F}$ also the Lie derivative along a vector field

$$\mathcal{L}_v \longrightarrow \mathcal{L}_v^\star$$

$\star$-Lie derivatives leads to a quantum Lie algebra of vector fields of the kind previously discussed.
See [P.A., Dimitrijevic, Meyer, Wess, '06] where a NC gravity theory is constructed based on NC diffeomorphisms invariance.

More in general vector bundles over M can be deformed (e.g. tangent and cotangent bundle) and connections [Aschieri, Schenkel 2014]

Twist and Cartan calculus on quantum groups and quantum spaces methods

Deformation of Principal bundles

(e.g. the bundle of ortonormal frames on a Riemannian manifold).
[Aschieri, Biliavsky, Pagani, Schenkel to appear]

Construct noncommutative principal bundles deforming commutative principal bundles with a Drinfeld twist. If the twist is related to the structure group then we have a deformation of the fiber, that becomes noncommutative. We can also have NC deformations of the base space.

We provide a general theory, construct new examples and recover in particular the instanton bundle on the noncommutative Connes-Landi sphere S^4_θ.

Motivations: The notion of **gauge group in NC geometry** can be considered from different viewpoints:

- In gauge theories on NC spaces gauge groups are mainly $U(N)$ or $GL(N)$ groups.

- A way to consider NC gauge transformations based on more general groups (e.g. $SU(N)$ or $SO(N)$) is via the *Seiberg-Witten map* between commutative and noncommutative gauge transformations.

- In geometry the gauge group is the group of authomorphisms of a Principal bundle (that are the identity on the base space). Then it is interesting to study NC gauge groups (e.g. NC SO(N) gauge transformations) as authomorphisms of NC principal bundles.

Further development. NC/NA geometry.

More general fluxes lead to nonassociative structures. These also can be described by a relaxed twist $\mathcal{F}$ (a 2-cocycle twist).

Noncommutative and nonassociative geometry emerges also in the compactification on 3-tori of *closed strings* in the presence of nonvanishing H-flux. T-dualizing along all cicles leads to a non geometric flux compactification that has a description in terms of Noncommutative and Nonassociative geometry. [Luest], [Blumenhagen, Plauschinn], [Blumenhagen, Fuchs,Hassler,Luest, Sun]

Phase space algebra:

$$[x^i, x^j] = \frac{i\ell_s^4}{3\hbar} R^{ijk} p_k \, , \qquad [x^i, p_j] = i\hbar \, \delta^i{}_j \qquad \text{and} \qquad [p_i, p_j] = 0 \, ,$$

which has a non-trivial Jacobiator

$$[x^i, x^j, x^k] = \ell_s^4 \, R^{ijk} \, .$$

This nonassociative geometry is described by a 2-cochain twist $\mathcal{F}$. It induces the expected off shell triproducts of fields on configuration spaces [Mylonas, Schupp, Szabo].

It also induces the full structure of ennary products related to the nonvanishing R-flux. (n-tripoducts) [Aschieri, Szabo].

The nonassociative differential geometry can be studied with 2-cochain twist deformation methods. The quantum Lie algebra of infinitesimal diffeomorphisms has been obtained. this is a first step toward a nonassociative theory of gravity related to the non-geometric flux compactifications of the closed string sector.

The COST research network: **Quantum Structure of Spacetime** starts on thursday! It is a network for developing these research field, where many of Bruno's students and collaborators have key responsibilities.