Bruno Zumino and the physics of phenomenological lagrangians

Curtis G. Callan, Jr.
Princeton University

Abstract: Bruno made profound contributions to the problem of incorporating spontaneously broken symmetry and anomalies into phenomenological lagrangians. These contributions may have looked rather abstract at the time, but they turned out to be essential for understanding some striking physical effects. I ran into this on several occasions in my own work and I will describe a few of them as my tribute to Bruno and the broad impact on physics of his deep physical insight and mathematical subtlety.

Overview

Phenomenological Lagrangians are meant to teach us the consequences of symmetries that don't have a simple linear realization on physical fields. At a minimum they provide a compact algorithm for deriving symmetry constraints on specific processes (think $\pi^o \rightarrow \gamma + \gamma$). At a deeper level, they reveal unsuspected non-perturbative properties of quantum field theories.

Bruno made several decisive contributions to this story. I had the good luck to be a collab-orator (by courtesy and generosity) on one of them. Later, I ran into a several problems in particle physics where Bruno's insights were essential to getting the physics right. I thought that walking you through this history would be a good way for me to honor Bruno's memory.

Here are the topics that I will revisit and attempt to tie together for you

- Chiral Lagrangians and spontaneously broken global symmetry
- The Wess-Zumino-Witten action and anomalous gauge invariance
- The Skyrmion: baryon number as topology, quarks as pseudoscalar mesons
- Monopole catalysis of proton decay: how it really happens
- Anomalies and chiral fermions on strings and domain walls.

I will summarize a few papers of mine, plus foundational papers of Bruno and others. This will necessarily be very "broad brush", with analytic details merely sketched.

"Structure of Phenomenological Lagrangians I&II"

Coleman, Wess and Zumino (plus a bit of Callan) (1968)

"The general method for constructing invariant phenomenological Lagrangians is described. The fields are assumed to transform according to nonlinear representations of an internal symmetry group. The construction proceeds through the introduction of covariant derivatives .. standard forms for field gradients. "

Represent spontaneously broken global symmetry G (with subgroup H unbroken) nonlinearly:

$$g = e^{\xi\cdot A} e^{u\cdot V} \qquad (\xi,\psi) \to (\xi',\psi') = g(\xi,\psi) \qquad g e^{\xi\cdot A} = e^{\xi'\cdot A} e^{u'\cdot V} \qquad \psi' = D(e^{u'\cdot V})\psi$$

$$e^{-\xi\cdot A}(\xi,\psi,\partial_\mu\xi,\partial_\mu\psi) = (0,\psi,D_\mu\xi,D_\mu\psi) \qquad (D_\mu\xi)' = D^{(b)}(e^{u'\cdot V})(D_\mu\xi) \qquad (D_\mu\psi)' = D(e^{u'\cdot V})(D_\mu\psi)$$

A G-invariant Lagrangian is built by making an H-invariant function out of $\psi,\ D_\mu\xi,$ and $D_\mu\psi$

$$L(\xi,\psi,\partial_\mu\xi,\partial_\mu\psi) = L(0,\psi,D_\mu\xi,D_\mu\psi)$$

For chiral SU(2)xSU(2) -> SU(2)$_V$ the method neatly packages PCAC and soft pion physics

$$D_\mu\xi = \partial_\mu\xi + \frac{1}{3!}(\partial_\mu\xi\times\xi)\times\xi + \frac{1}{5!}\cdots, \qquad D_\mu N = \partial_\mu N - \tfrac{1}{2} i\tau\cdot\left(\frac{1}{2!}\partial_\mu\xi\times\xi + \frac{1}{4!}[(\partial_\mu\xi\times\xi)\times\xi]\times\xi + \cdots\right)N$$

$$L = \frac{1}{2} D_\mu\xi\cdot D_\mu\xi + \bar{N}\gamma^\mu D_\mu N + \ldots$$

“Consequences of Anomalous Ward Identities”

Wess and Zumino (1971)

“The anomalies of Ward identities are shown to satisfy consistency conditions that restrict their possible form. We display a local functional satisfying all such anomalies. It is an effective action describing all anomaly modifications of the low-energy theorem.”

L_{eff} = W is a functional of external SU(3)xSU(3) currents V^i_μ A^i_μ and ξ are the p.s. mesons that nonlinearly represent the spontaneously broken symmetry. Anomalous contributions to the divergences of currents A^i_μ were obtained for one-loop diagrams with external V, A currents. But you can’t derive anomalies from an action unless you include the p.s. meson fields. Result :

$$W[\xi, V_\mu, A_\mu] = \frac{1-\exp(\xi\cdot U)}{\xi\cdot U}\,\xi\cdot G[V_\mu, A_\mu] = \int_0^1 dt\, \exp(-t\,\xi\cdot U)\,\xi\cdot G$$

$$-G_i = \frac{1}{4\pi^2}\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}\left[\frac{\lambda_i}{2}\left\{\frac{1}{4}V_{\mu\nu}V_{\sigma\tau} + \frac{1}{12}A_{\mu\nu}A_{\sigma\tau} + \frac{2}{3}i(A_\mu A_\nu V_{\sigma\tau} + A_\mu V_{\nu\sigma}A_\tau + V_{\mu\nu}A_\sigma A_\tau) - \frac{8}{3}A_\mu A_\nu A_\sigma A_\tau\right\}\right]$$

U^i is a complicated operation on the V, A arguments of G.

The resulting expression for W looks pretty complicated, but it can be expanded in mesons and currents to get the explicit “anomaly” contributions to any process:

2 meson -> 3 meson : $$\frac{1}{6\pi^2 F_\pi^5}\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}(\Pi\partial_\mu\Pi\partial_\nu\Pi\partial_\sigma\Pi\partial_\tau\Pi), \quad \Pi = \tfrac{1}{2}\Pi_i\lambda_i$$

γ + meson -> γ + 2 meson : $$\frac{e}{i\,24\pi^2 F_\pi^3}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu} : [(\partial_\sigma\Pi^+\partial_\tau\Pi^- + \partial_\sigma K^+\partial_\tau K^-)(\Pi^0 + \tfrac{1}{\sqrt{3}}\eta) + \partial_\sigma K^0\partial_\tau\bar{K}^0(\Pi^0 - \sqrt{3}\,\eta)]$$

"Global Aspects of Current Algebra"

Witten (1983)

"A new mathematical framework for the Wess-Zumino chiral effective action is described. This action obeys an a priori quantization law, analogous to Dirac's quantization of magnetic charge. It incorporates perturbative and non-perturbative anomalies."

SU(3): $\bar{\psi}^{\mathbf{i}}_{\mathbf{L}}\psi^{\mathbf{j}}_{\mathbf{R}} \simeq U = 1 + \frac{2i}{F_\pi}\sum_{a=1}^{8}\lambda^a\pi^a + \cdots$, $U \to AUB^{-1}$ realizes $SU(3)_L \times SU(3)_R$ --> SSB

Simplest choice of action: $\mathcal{L} = \frac{1}{16}F_\pi^2\int d^4x\,\mathrm{Tr}\,\partial_\mu U\,\partial_\mu U^{-1}$ Wrong! Conserves N_{meson} modulo 2!

We fix this by adding an unusual term to the EOM: $\partial_\mu\left(\frac{1}{8}F_\pi^2 U^{-1}\partial_\mu U\right) + \lambda\varepsilon^{\mu\nu\alpha\beta}U^{-1}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}(\partial_\alpha U)U^{-1}(\partial_\beta U) = 0$

But, the new EOM term cannot be obtained by varying a 4d Lagrange density! It does arise as the variation of an integral of a topological density over a 5d ball whose boundary is 4d spacetime.

$$I = \frac{1}{16}F_\pi^2\int d^4x\,\mathrm{Tr}\,\partial_\mu U\,\partial_\mu U^{-1} + n\Gamma$$

Γ is the WZ action! It incorporates anomaly physics

$$\Gamma = \int_Q \omega_{ijklm}\,d\Sigma^{ijklm} \qquad d\Sigma^{ijklm}\,\omega_{ijklm} = -\frac{i}{240\pi^2}d\Sigma^{ijklm}\left[\mathrm{Tr}\,U^{-1}\frac{\partial U}{\partial y^i}U^{-1}\frac{\partial U}{\partial y^j}U^{-1}\frac{\partial U}{\partial y^k}U^{-1}\frac{\partial U}{\partial y^l}U^{-1}\frac{\partial U}{\partial y^m}\right].$$

Since $\Pi_5(SU(3)) = Z$, Γ integral over all S_5 is $2\pi m$. Hence Γ as defined above is ambiguous by $2\pi m$. Path integral is well-defined only if n is integer. Match to QCD anomaly results if $n=N_c$!

The Wess-Zumino-Witten Action (*bis*)

This is a totally new view of the Wess-Zumino anomaly term. It works as an action (with quantized cc's) because the variation of a topological density is a total div.

For reference, we collect formulas for electromagnetic extension of the WZW action.

$$Q = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}$$

$$U \to U + i\varepsilon(x)[Q, U],$$

Noether Current:

$$J^\mu = \frac{1}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left[Q(\partial_\nu U\, U^{-1})(\partial_\alpha U\, U^{-1})(\partial_\beta U\, U^{-1}) + Q(U^{-1}\partial_\nu U)(U^{-1}\partial_\alpha U)(U^{-1}\partial_\beta U)\right]$$

J^μ isn't gauge invariant, but adding a further term gives a gauge invariant total action:

$$\tilde{\Gamma}(U, A_\mu) = \Gamma(U) - e\int \mathrm{d}^4x\, A_\mu J^\mu + \frac{ie^2}{24\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)A_\alpha \times \mathrm{Tr}\left[Q^2(\partial_\beta U)U^{-1} + Q^2U^{-1}(\partial_\beta U) + QUQU^{-1}(\partial_\beta U)U^{-1}\right]$$

Set Q=B to find the baryon current: $J^\mu = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\, U^{-1}\partial_\nu U\, U^{-1}\partial_\alpha U\, U^{-1}\partial_\beta U.$

N.B. If $U(\vec{x}) \in SU(2)$ $B = \int_{S^3} d^3x J_0 = n$ because $\Pi_3(SU_2) = Z$

Skyrmion: The Nucleon *ex nihilo* (well, pions)

If we take *U* in SU(2) and map R^3 once into the S^3 group manifold, we get B=1 (nucleon)!

Ansatz: $U_0(x) = \exp[iF(r)\boldsymbol{\tau} \cdot \hat{\mathbf{x}}]$, where $F(r) = \pi$ at $r = 0$ and $F(r) \to 0$ as $r \to \infty$

Finding the energy-minimizing *F*(*r*) is a standard soliton problem. A higher derivative term is needed to prevent soliton collapse. The simplest option (one parameter) is:

$$L = \tfrac{1}{16} F_\pi^2 \, \mathrm{Tr}\,(\partial_\mu U \partial_\mu U^\dagger) + \frac{1}{32 e^2} \mathrm{Tr}\,[(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger]^2$$

Since *U* is in SU(2), WZW term vanishes (but we needed it to identify J_μ^B). Minimizing the energy over *F*(r), choosing *e* to match M_N, gives a pion soliton baryon!!

This static soliton can *rotate* to create a spectrum of (*I*,*J*) excited states. Invariance of soliton under ***I + J*** rotation, means only *I = J* are allowed. The WZW term again comes in (subtle argument) to fix the quantization rule: odd N_c -> half-integral *I,J.*

Bottom line: the properly quantized Skyrme soliton “rotator” gives the expected quark model sequence of non-strange excited states of the nucleon:

$(\mathrm{I,J}):\quad (\frac{1}{2}, \frac{1}{2})\ \mathrm{N} \quad (\frac{3}{2}, \frac{3}{2})\ \Sigma\ \ldots$

Mass splitting, magnetic moments, charge radii reasonably well predicted.

"Bound State Approach to Strangeness in the Skyrme Model" Callan and Klebanov (1985)

"We show that baryons carrying heavy flavors can be described as bound states of heavy mesons in the background field of the basic SU(2) Skyrmion".

Back to SU(3) Skyrme, with added kaon mass term. Fluctuations away from the SU(2) Skyrme soliton are strongly damped. Study the 2[nd] order fluctuation eqn for the K field.

$$U=\sqrt{U_\pi}\,U_K\sqrt{U_\pi}\,,\quad U_\pi=\exp\left(i\frac{2}{F_\pi}\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right)\quad U_K=\exp\left(i\frac{2}{F_\pi}\lambda_a K_a\right)$$

Expand $L(U)$ to 2[nd] order in kaon field

$$\mathscr{L}=\mathscr{L}_{Sk}(U_\pi)+(D_\mu K)^\dagger D^\mu K-m_K^2K^\dagger K+\dots\,,$$

$$K=\frac{1}{\sqrt{2}}\begin{pmatrix}K_4-iK_5\\K_6-iK_7\end{pmatrix}=\begin{pmatrix}K^+\\K^0\end{pmatrix}$$

Rest of kaon action built out of covariant bits e.g.

$$\begin{cases}A_\mu=\frac{1}{2}(\sqrt{U_\pi^\dagger}\,\partial_\mu\sqrt{U_\pi}-\sqrt{U_\pi}\,\partial_\mu\sqrt{U_\pi^\dagger})\\ D_\mu K=\partial_\mu K+\frac{1}{2}(\sqrt{U_\pi^\dagger}\,\partial_\mu\sqrt{U_\pi}+\sqrt{U_\pi}\,\partial_\mu\sqrt{U_\pi^\dagger})K.\end{cases}$$

This is nonlinearly realized $SU(2)_L x SU(2)_R$ on a linear K field (as per CCWZ). The various L_{eff} terms have known coeffs inherited from the unbroken SU(3) of the underlying Skyrme model. Plus the Wess-Zumino term! Expand the 5d WZW action using $U=\sqrt{U_\pi}U_K\sqrt{U_\pi}$

$$-\frac{i}{240\pi^2}\int d^5x\,\varepsilon^{ijklm}\,\mathrm{tr}(U^+\partial_iUU^+\partial_jUU^+\partial_kUU^+\partial_lUU^+\partial_mU)\longrightarrow$$

$$\mathscr{L}_{WZ}=(iN_c/F_\pi^2)\,B^\mu[K^\dagger D_\mu K-(D_\mu K)^\dagger K]$$

For the pion rotator, the WZW term determines the angular momentum quantization. For the kaon field, the same term will split S=-1 from S=+1 states (important!).

Bound State Approach to Strangeness

Lagrangian for the kaon field in the static Skyrme nucleon ($U_0(\boldsymbol{r})=\exp[\mathrm{i}\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}F(r)]$) can be expanded in partial waves: $K(\boldsymbol{r},t)=k(r,t)\;Y_{TLT_z}$. ($T=I+L$). With some gymnastics, we can see that this field has isospin zero and transforms under spatial rotations with $J = T$!

$$L=4\pi\int \mathrm{d}r\, r^2\Big(f(r)\dot{k}^\dagger\dot{k}+\mathrm{i}\lambda(r)\,(k^\dagger\dot{k}-\dot{k}^\dagger k)-h(r)\frac{\mathrm{d}}{\mathrm{d}r}k^\dagger\frac{\mathrm{d}}{\mathrm{d}r}k-k^\dagger k[m_K^2+V_{\mathrm{eff}}(r;T,L)]\Big)$$

This is a "free" theory, and we can solve for its creation/annihilation operators:

$$k(r,t)=\sum_{n>0}[\tilde{k}_n(r)\exp(\mathrm{i}\tilde{\omega}_n t)\,b_n^\dagger+k_n(r)\exp(-\mathrm{i}\omega_n t)\,a_n]$$

$$H=\sum_{n>0}(\omega_n a_n^\dagger a_n+\tilde{\omega}_n b_n^\dagger b_n)\quad S=\sum_{n>0}(b_n^\dagger b_n-a_n^\dagger a_n)$$

Positive and negative frequencies are <u>not</u> equal. They correspond to modes with opposite strangeness.

Degeneracy between S = -1 and S = + 1 is broken by the WZW term (~ λ) . The lowest p.w in the S=-1 channel has (T,L)=(½,1) P = + and has one bound mode ($\omega<m_K$). Occupying this mode contributes (I,J) = (0,½), just like an s quark in an s wave! In the S=+1 channel, there is one barely bound mode with (T,L)=(½,0) P = - : just like an s quark in a p wave (exotic).

Couple the P = + bound state to the (I,J) states of the rotator to get the S = -1 excitations of the nucleon. Now the allowed (I,J) states of the rotator are (I,I) with I integer (WZW again!):

$(0,0) + (0,½) \to (0,½)_+$ Λ　　　　$(1,1) + (0,½) \to (1,½)_+$ Σ or (1,3/2) Σ^*

Couple the P = - , near threshold, S=-1 bound state to the (0,0) rotator state to get a known (weird) state, the $(0,½)_-$ Λ(1405), an exotic P = - excitation of the Λ!

Quantum number transmutation due to WZW term recreates quark model physics!

Monopole Catalysis of Proton Decay

Electrons in a Dirac monopole have a special J=0 (!?) partial wave where ψ_L (ψ_R) have flux only toward (away from) r=0. Must fix unitarity with a chirality-breaking b.c. at r=0

U(1): $A_\phi = \frac{g(1+\cos\theta)}{r\sin\theta}$ $\quad F_{\theta\phi} = \frac{g}{r^2}$ $\quad \vec{J} = \vec{L} + \vec{S} + q\hat{r}$ $\quad \psi_{J=0} = \frac{1}{r}\begin{pmatrix} u(r)\eta_0 \\ v(r)\eta_0 \end{pmatrix}$

q=eg=½ $\quad \psi_R(0) = ie^{i\theta}\psi_L(0)$

SU(5) grand unified theories also have magnetic monopoles whose mass and spatial size are set by M_{GUT}. They also have special J=0 partial waves for the chiral leptons of each generation that organize into ingoing/outgoing pairs like the U(1) case:

SU(5): Ingoing -> / Outgoing -> $\begin{pmatrix} e^+ \\ d_3 \end{pmatrix}, \begin{pmatrix} \bar{d}_3 \\ e^- \end{pmatrix}, \begin{pmatrix} u_1 \\ \bar{u}_2 \end{pmatrix}, \begin{pmatrix} u_2 \\ \bar{u}_1 \end{pmatrix}.$

Linear boundary conditions at r=0 on these fields don't even conserve charge! Analysis shows that unitary boundary conditions that conserve charge don't conserve B!

$\Delta B \neq 0$ reactions are now allowed: $\quad p + M \to$ " π^0 " $+ e^+ + M$

Rates of these $\Delta B \neq 0$ processes are unsuppressed by M_{GUT} ! The J=0 partial wave has no centrifugal barrier … cross sections saturate the unitarity bound. GUT scale is now visible to the QCD scale! Or would be, if there were any monopoles in our world …..

“Monopole Catalysis of Skyrmion Decay”

Callan and Witten (1983)

“Recently it has been found that in the presence of a magnetic monopole, reactions that are ordinarily very unlikely can have a greatly increased cross section. We describe a model that can be used to make realistic estimates of nucleon-monopole cross sections. The model helps clarify the basic physics of monopole catalysis”

We ask what happens when a magnetic monopole, with its Dirac string, moves through a Skyrme nucleon. We will need the full gauge-invariant $L_{Skyrme(}(U,A_\mu)$:

$$\mathscr{L} = \tfrac{1}{16} F_\pi^2 \,\mathrm{Tr}\, D_\mu U D^\mu U^{-1} + \frac{1}{32a^2}\mathrm{Tr}\,[U^{-1}D_\mu U, U^{-1}D_\nu U]^2 + \frac{e}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta} A_\mu \,\mathrm{Tr}\, Q(\partial_\nu U U^{-1}\partial_\alpha U U^{-1}\partial_\beta U U^{-1} + U^{-1}\partial_\nu U U^{-1}\partial_\alpha U U^{-1}\partial_\beta U) + \frac{ie^2}{8\pi^2}\varepsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)A_\alpha \,\mathrm{Tr}\,(Q^2\partial_\beta U U^{-1} + Q^2 U^{-1}\partial_\beta U + \tfrac{1}{2}Q\partial_\beta U Q U^{-1} - \tfrac{1}{2}QUQ\partial_\beta U^{-1}) .$$

We also need the expression for baryon number in the presence of monopole A_μ

$$B_\mu = \frac{\varepsilon_{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{Tr}\, U^{-1}\partial^\nu U U^{-1}\partial^\alpha U U^{-1}\partial^\beta U + \frac{\varepsilon_{\mu\nu\alpha\beta}}{24\pi^2}\partial_\nu[3ieA_\alpha \,\mathrm{Tr}\, Q(U^{-1}\partial_\beta U + \partial_\beta U U^{-1})] .$$

Because of the Dirac string in A_μ, the total derivative part may not vanish: that will be the key to monopole catalysis of baryon decay in this way of looking at the problem.

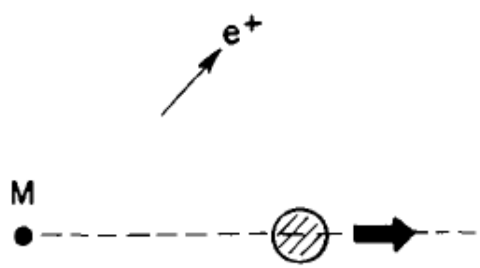
Monopole Catalysis of Skyrmion Decay (bis)

Monopole field is singular at r=0. Solve KG eqn to show that charged scalar fields must vanish as r-> 0. So, near the origin $U = \exp \imath\phi T_3 \quad (\phi = 2\pi^0/F_\pi)$ and $\phi(r=0,t) = \phi(t)$ is a boundary value that could vary with time. What would happen if it did? Baryon current for eg=½ is

$$J_r^B = -\frac{ie}{8\pi^2}\,\varepsilon^{r\theta\phi t}(\partial_\theta A_\phi)\,\mathrm{Tr}\,(-QU\partial_t U^{-1} + QU^{-1}\partial_t U) = \frac{\dot{\phi}(t)}{8\pi^2 r^2}$$

Baryon number disappears into r=0 at the rate $\dot{B} = -\dot{\phi}(t)/2\pi$. (electric charge also). We satisfy ΔB = ΔL by coupling the electron field to the monopole via the r=0 b.c. for the special partial wave: $\psi_L(0,t) = e^{\imath\phi(t)}\psi_R(0,t)$

Let's take an overall look at how a whole proton might disappear in a monopole collision. After collision, Skyrmion is drilled by the Dirac string. We can "flip" the string away from the proton with a gauge transformation Λ. What happens to the Skyrmion configuration?



$$U(x) = -\exp iF(r)\,\bar{T}\cdot\hat{x}$$
$$= -\exp\{i[F(r)(\cos(\theta)T_3 + \sin\theta\cos(\phi)T_1 + \sin\theta\sin(\phi)T_2]\}$$

$$\tilde{U} = \Lambda U \Lambda^{-1} \qquad \Lambda = \exp(\imath\phi T_3/2)$$

$$= -\exp\{iF(r)(T_3\cos\theta + T_1\sin\theta)\}$$

Final Skyrmion configuration clearly does not map R^3 *to* the full S^3 group manifold: its baryon number is zero (i.e. it is a collection of pions that will fly apart. Exactly what the *B*=1 has turned into depends on the b.c.'s imposed at r=0 on the SU(5) quarks/leptons.

Summing Up

I thank the organizers for asking me to make a contribution to this meeting convened to remember and honor Bruno Zumino's many contributions to modern theoretical physics.

What I told you in my talk barely scratches the surface of the "anomaly" story and its influence on the development of both quantum field theory and string theory. It was simply meant as a very personal reminiscence of how Bruno's injection of mathematical clarity and generality into the anomaly story led me to some surprisingly "physical" implications and applications of his work.

I will leave it to others who knew him much better than I did to talk in depth about Bruno the man. My consistent impression of him was of brilliant theoretical physicist, who was also a charming, cultivated, and generous human being. A wonderful research colleague, and an ornament to our profession.

"Anomalies and Fermion Zero Modes on Strings and Domain Walls" Callan and Harvey (1984)

"We show that the mathematical relation between non-abelian anomalies in 2n dim'ns, the parity anomaly in 2n + 1 dim'ns, and the Dirac index density in 2n + 2 dim'ns can be understood in terms of the physics of fermion zero modes on strings and domain walls."

Let there be a spontaneously broken U(1) due to complex scalar ϕ getting a v.e.v. ϕ_0 and let a fermion Ψ get a (large) mass from coupling to this v.e.v. The scalar field theory will have a string soliton lying along the z axis (cylindrical coords) with $\phi(z, r, \phi) = \phi_0 \exp(\imath\phi) f(r)$. The fermion will have a chiral massless fermion bound to the string (with massive fermions off the string)

Now couple all of this to electromagnetism, letting the fermion have q=1. The massless mode on the string is of course anomalous … its charge is not conserved if $E_z \neq 0$. Where does its charge go (or come from). Is the overall charge of the total 3+1-d system conserved? If not, why not?

The answer is "anomaly inflow" and the anomaly in question is that of $U(1)_A$ in the presence of $F_{\mu\nu}^2$.