Low-Energy Behavior of Massless Particles from Gauge Invariance

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Foreword

- This talk is based on the work done together with

  Zvi Bern, Scott Davies and Josh Nohle
  "Low-Energy Behavior of Gluons and Gravitons:
  from Gauge Invariance", arXiv:1406:6987 [hep-th]
  Phys. Rev. D **90** (2014) 084035

  and

  Raffaele Marotta and Matin Mojaza
  "Soft theorem for the graviton, dilaton and the Kalb-Ramond field
  in the bosonic string", arXiv:1502.05258.

Plan of the talk

Introduction

- In particle physics we deal with three kinds of symmetries.
- They all leave the action invariant, but have different physical consequences.
- GLOBAL symmetries as isotopic spin (if $m_u = m_d$) in 2-flavor QCD.
- Unique vacuum annihilated by the symmetry gener.: $Q_a|0\rangle = 0$
- Particles are classified according to multiplets of this symmetry and all particles of a multiplet have the same mass.
- If $m_u = m_d$, QCD is invariant under an $SU(2)_V$ flavor symmetry.
- and the proton and the neutron would have the same mass.
- This is not the case because the quark mass matrix breaks explicitly $SU(2)$ $(m_u \neq m_d)$.

- Then, we have the GLOBAL spontaneously broken symmetries.
- For zero quark mass, QCD with two flavors is invariant under $SU(2)_L \times SU(2)_R$.
- Degenerate vacua: $Q_a|0\rangle = |0'\rangle$.
- Not realized in the spectrum, but presence in the spectrum of massless particles, called Goldstone bosons.
- They are the pions in QCD with 2 flavors.
- This is one physical consequence of the spontaneous breaking.
- Another one is the existence of low-energy theorems.
- The $\pi\pi$ scattering amplitude is fixed at low energy.
- Actually the scattering amplitude for massless pions is zero at low energy because Goldstone bosons interact with derivative coupling (shift symmetry $\leftrightarrow$ Adler zeroes).
- If one introduces a mass term, breaking explicitly chiral symmetry and giving a small mass to the pion, one gets the two Weinberg scattering lengths:

$$a_0 = \frac{7m_\pi}{32\pi F_\pi^2} \quad ; \quad a_2 = -\frac{m_\pi}{16\pi F_\pi^2}$$

- Finally, we have the LOCAL gauge symmetries for massless spin 1 and spin 2 particles.
- Local gauge invariance is necessary to reconcile the theory of relativity with quantum mechanics.
- It allows a fully relativistic description, eliminating, at the same time, the presence of negative norm states in the spectrum of physical states.
- Although described by A_μ and $G_{\mu\nu}$, both photons and gravitons have only two physical degrees of freedom in d=4.
- Another consequence of gauge invariance is charge conservation that, however, follows from the global part of the gauge group.
- Yet another physical consequence of local gauge invariance is the existence of low-energy theorems for photons and gravitons: [F. Low, 1958; S. Weinberg, 1964]

- Let us consider Compton scattering on spinless particles.
- The scattering amplitude $M_{\mu\nu}$ is gauge invariant:

$$k_1^\mu M_{\mu\nu} = k_2^\nu M_{\mu\nu} = 0$$

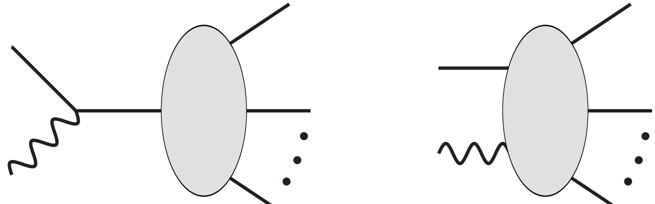
- The previous conditions determine the scattering amplitude for zero frequency photons and one gets the Thompson cross-section:

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = \frac{8\pi}{3} r_{cl}^2$$

where r_{cl} is the classical radius of a point particle of mass m and charge e.

- The interest on the soft theorems was recently revived by [Cachazo and Strominger, arXiv:1404.1491[hep-th]].
- They study the behavior of the n-graviton amplitude when the momentum q of one graviton becomes soft ($q \sim 0$).
- The leading term $O(q^{-1})$ was shown to be universal by Weinberg in the sixties.
- They suggest a universal formula for the subleading term $O(q^{0})$.
- They speculate that, as the leading term, it may be a consequence of BMS symmetry of asymptotically flat space-times.
- In this seminar we show that the first three leading terms of order q^{-1}, q^{0}, q are a direct consequence of gauge invariance.
- This result is valid for an arbitrary space-time dimension d.
- In the second part study soft theorems in string theory, in particular for dilaton and $B_{\mu\nu}$.

One photon and n scalar particles



(a) (b)

- The scattering amplitude $M_\mu(q; k_1 \dots k_n)$, involving one photon and n scalar particles, consists of two pieces:

$$A_n^\mu(q; k_1, \dots, k_n) = \sum_{i=1}^n e_i \frac{k_i^\mu}{k_i \cdot q} T_n(k_1, \dots, k_i + q, \dots, k_n) + N_n^\mu(q; k_1, \dots, k_n).$$

- and must be gauge invariant for any value of q:

$$q_\mu A_n^\mu = \sum_{i=1}^n e_i T_n(k_1, \dots, k_i + q, \dots, k_n) + q_\mu N_n^\mu(q; k_1, \dots, k_n) = 0$$

- Expanding around $q = 0$, we have

$$0 = \sum_{i=1}^{n} e_i \left[T_n(k_1, \ldots, k_i, \ldots, k_n) + q_\mu \frac{\partial}{\partial k_{i_\mu}} T_n(k_1, \ldots, k_i, \ldots, k_n) \right]$$

$$+ q_\mu N_n^\mu(q = 0; k_1, \ldots, k_n) + \mathcal{O}(q^2) \,.$$

- At leading order, this equation is

$$\sum_{i=1}^{n} e_i = 0 \,,$$

which is simply a statement of charge conservation [Weinberg, 1964]

- At the next order, we have

$$q_\mu N_n^\mu(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i q_\mu \frac{\partial}{\partial k_{i_\mu}} T_n(k_1, \ldots, k_n) \,.$$

- This equation tells us that $N_n^\mu(0; k_1, \ldots, k_n)$ is entirely determined in terms of T_n up to potential pieces that are separately gauge invariant.
- It can be shown that they are of higher order in q.
- We can therefore remove the q_μ leaving

$$N_n^\mu(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) ,$$

thereby determining $N_n^\mu(0; k_1, \ldots, k_n)$ as a function of the amplitude without the photon.

- Inserting this into the original expression yields

$$A_n^\mu(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q) \, ,$$

  where

$$J_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right)$$

  is the orbital angular-momentum operator.
- The amplitude with a soft photon with momentum q is entirely determined, up to $\mathcal{O}(q^0)$, in terms of the amplitude $T_n(k_1, \ldots, k_n)$, involving n scalar particles and no photon.
- This goes under the name of F. Low's low-energy theorem.

- Low's theorem for photons is unchanged at loop level.
- Can we get any further information at higher orders in the soft expansion?
- One order further in the expansion, we find the extra condition,

$$\frac{1}{2}\sum_{i=1}^{n} e_i q_\mu q_\nu \frac{\partial^2}{\partial k_{i\mu}\partial k_{i\nu}} T_n(k_1,\ldots,k_n) + q_\mu q_\nu \frac{\partial N_n^\mu}{\partial q_\nu}(0;k_1,\ldots,k_n) = 0\,.$$

- This implies

$$\sum_{i=1}^{n} e_i \frac{\partial^2}{\partial k_{i\mu}\partial k_{i\nu}} T_n(k_1,\ldots,k_n) + \left[\frac{\partial N_n^\mu}{\partial q_\nu} + \frac{\partial N_n^\nu}{\partial q_\mu}\right](0;k_1,\ldots,k_n) = 0\,,$$

- Gauge invariance determines only the symmetric part of the quantity $\frac{\partial N_n^\nu}{\partial q_\mu}(0; k_1, \ldots, k_n)$.
- The antisymmetric part is not fixed by gauge invariance.
- Then, up to this order, we have

$$
\begin{aligned}
& A_n^\mu(q; k_1, \ldots, k_n) \\
& = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu} \left(1 + \frac{1}{2} q_\rho \frac{\partial}{\partial k_{i\rho}} \right) \right] T_n(k_1, \ldots, k_n) \\
& + \frac{1}{2} q_\nu \left[\frac{\partial N_n^\mu}{\partial q_\nu} - \frac{\partial N_n^\nu}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) + O(q^2) \, .
\end{aligned}
$$

- Get an amplitude by contracting $A_n^\mu(q; k_1, \ldots, k_n)$ with the photon polarization $\varepsilon_{q\mu}$.
- Soft-photon limit:

$$A_n(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)}\right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q)\,,$$

where

$$S^{(0)} \equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q}\,,$$

where $J_i^{\mu\nu}$ is the angular momentum.

One graviton and n scalar particles

- In the case of a graviton scattering on n scalar particles, one can write

$$M_n^{\mu\nu}(q;k_1,\ldots,k_n) = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i\cdot q} T_n(k_1,\ldots,k_i+q,\ldots,k_n) + N_n^{\mu\nu}(q;k_1,\ldots,k_n)\,,$$

- $N_n^{\mu\nu}(q;k_1,\ldots,k_n)$ is symmetric under the exchange of μ and ν.
- On-shell gauge invariance implies

$$0 = q_\mu M_n^{\mu\nu}(q;k_1,\ldots,k_n) = \sum_{i=1}^{n} k_i^\nu T_n(k_1,\ldots,k_i+q,\ldots,k_n) + q_\mu N_n^{\mu\nu}(q;k_1,\ldots,k_n)\,.$$

- More precisely, gauge invariance imposes:

$$q_\mu M_n^{\mu\nu}(q,k_i) = f(q,k_i)q^\nu \Longrightarrow q_\mu\left(M_n^{\mu\nu} - f(k_i)\eta^{\mu\nu}\right) = 0$$

but the extra term is irrelevant for gravitons: $\epsilon_G^{\mu\nu}\eta_{\mu\nu} = 0$.

- At leading order in q, we then have

$$\sum_{i=1}^{n} k_i^{\mu} = 0\,,$$

- It is satisfied due to momentum conservation.
- Different couplings to different particles would have prevented the leading term to vanish: Gravitons have universal coupling [Weinberg, 1964].
- At first order in q, one gets

$$\sum_{i=1}^{n} k_i^{\nu} \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) + N_n^{\mu\nu}(0; k_1, \ldots, k_n) = 0\,,$$

- while at second order in q, it gives

$$\sum_{i=1}^{n} k_i^{\nu} \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\rho}} T_n(k_1, \ldots, k_n) + \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} + \frac{\partial N_n^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) = 0\,.$$

- As for the photon, this is true up to gauge-invariant contributions to $N_n^{\mu\nu}$.
- However, the requirement of locality prevents us from writing any expression that is local in q and not sufficiently suppressed in q.
- Using the previous equations, we write the expression for a soft graviton as

$$M_n^{\mu\nu}(q; k_1 \dots k_n) = \sum_{i=1}^{n} \frac{k_i^\nu}{k_i \cdot q} \left[k_i^\mu - i q_\rho J_i^{\mu\rho} \left(1 + \frac{1}{2} q_\sigma \frac{\partial}{\partial k_{i\sigma}} \right) \right] T_n(k_1, \dots, k_n) + \frac{1}{2} q_\rho \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} - \frac{\partial N_n^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \dots, k_n) + \mathcal{O}(q^2) \, .$$

- This is essentially the same as for the photon except that there is a second Lorentz index in the graviton case.
- Unlike the case of the photon, the antisymmetric quantity in the last line of the previous equation can also be determined from the amplitude $T_n(k_1, \dots, k_n)$ without the graviton.

- From the equation above (implied by gauge invariance) + remembering that $N_n^{\mu\nu}$ is a symmetric matrix, one gets the following relation:

$$-i \sum_{i=1}^{n} J_i^{\mu\rho} \frac{\partial}{\partial k_{i\nu}} T_n(k_1, \ldots, k_n) = \left[\frac{\partial N_n^{\rho\nu}}{\partial q_\mu} - \frac{\partial N_n^{\mu\nu}}{\partial q_\rho} \right] (0; k_1, \ldots, k_n) \, ,$$

which fixes the antisymmetric part of the derivative of $N_n^{\mu\nu}$ in terms of the amplitude $T_n(k_1, \ldots, k_n)$ without the graviton.

- Using the previous equation, we can then rewrite the terms of $\mathcal{O}(q)$ as follows:

$$M_n^{\mu\nu}(q; k_1, \ldots, k_n)\big|_{\mathcal{O}(q)}$$

$$= -\frac{i}{2}\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{k_i \cdot q}\left[k_i^\nu J_i^{\mu\rho}\frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma J_i^{\mu\rho}\frac{\partial}{\partial k_{i\nu}}\right] T_n(k_1, \ldots, k_n)$$

$$= -\frac{i}{2}\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{k_i \cdot q}\left[J_i^{\mu\rho}k_i^\nu\frac{\partial}{\partial k_{i\sigma}} - \left(J_i^{\mu\rho}k_{i\nu}\right)\frac{\partial}{\partial k_{i\sigma}}\right.$$

$$\left.-J_i^{\mu\rho}k_i^\sigma\frac{\partial}{\partial k_{i\nu}} + \left(J_i^{\mu\rho}k_i^\sigma\right)\frac{\partial}{\partial k_{i\nu}}\right] T_n(k_1, \ldots, k_n)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\frac{1}{k_i \cdot q}\left[\left((k_i \cdot q)(\eta^{\mu\nu}q^\sigma - q^\mu\eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma\right)\frac{\partial}{\partial k_i^\sigma}\right.$$

$$\left.- q_\rho J_i^{\mu\rho}q_\sigma J_i^{\nu\sigma}\right] T_n(k_1, \ldots, k_n)\,.$$

- Finally, we contract with the physical polarization tensor of the soft graviton, $\varepsilon_{q\mu\nu}$.

- We see that the physical-state conditions

$$q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = 0 \quad ; \quad \eta^{\mu\nu}\epsilon_{\mu\nu} = 0$$

set to zero the terms that are proportional to $\eta^{\mu\nu}$, q^μ and q^ν.
- We are then left with the following expression for the graviton soft limit of a single-graviton, n-scalar amplitude:

$$M_n(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)} + S^{(2)} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q^2)\,,$$

- where

$$S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q}\,,$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}\,.$$

- These soft factors follow entirely from gauge invariance.

Soft limit of $(n+1)$-gluon amplitude

- We consider a tree-level color-ordered amplitude where gluon $(n+1)$ becomes soft with $q \equiv k_{n+1}$.
- Being the amplitude color-ordered, we have to consider only the two poles with the soft particle attached to the two adjacent legs.
- We proceed as before.

- Contract with external polarization vectors:

$$A_{n+1}(q;k_1,\ldots,k_n) \to \left[S^{(0)} + S^{(1)}\right] A_n(k_1,\ldots,k_n) + \mathcal{O}(q)\,,$$

where

$$S^{(0)} \equiv \frac{k_1\cdot\varepsilon_q}{\sqrt{2}\,(k_1\cdot q)} - \frac{k_n\cdot\varepsilon_q}{\sqrt{2}\,(k_n\cdot q)}\,,$$

$$S^{(1)} \equiv -i\varepsilon_{q\mu}q_\sigma\left(\frac{J_1^{\mu\sigma}}{\sqrt{2}\,(k_1\cdot q)} - \frac{J_n^{\mu\sigma}}{\sqrt{2}\,(k_n\cdot q)}\right).$$

- Here

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + S_i^{\mu\sigma}\,,$$

where

$$L_i^{\mu\nu} \equiv i\left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}}\right) \quad , \quad S_i^{\mu\sigma} \equiv i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right).$$

Soft limit of $(n+1)$-graviton amplitude

$$M_{n+1}(q; k_1, \ldots, k_n) = \left[S^{(0)} + S^{(1)} + S^{(2)}\right] M_n(k_1, \ldots, k_n) + \mathcal{O}(q^2)\,,$$

$$S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q}\,,$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q}\,,$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}\,.$$

where $J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + S_i^{\mu\sigma}$ and $(\epsilon_i^{\mu\nu} \equiv \epsilon_i^\mu \epsilon_i^\nu)$

$$L_i^{\mu\sigma} \equiv i\left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}}\right)\,, \qquad S_i^{\mu\sigma} \equiv i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right)\,.$$

These soft factors follow from gauge invariance and agree with those computed by Cachazo and Strominger.

What about soft theorems in string theory?

- In superstring the soft theorems have been investigated by B.U.W. Schwab, arXiv:1406.4172 and arXiv:1411.6661 M. Bianchi, Song He, Yu-tin Huang and Congkao Wen, arXiv:1406.5155.
- Soft theorems for gluons and gravitons are of course satisfied, as one can check computing explicitly the amplitude.
- Study the soft theorems for other massless particles as the dilaton and the $B_{\mu\nu}$.

Soft theorem for dilaton and $B_{\mu\nu}$

- The field theory action for the dilaton and $B_{\mu\nu}$:

$$S_{\text{string}} = \frac{1}{2\hat{\kappa}_d^2} \int d^d x \sqrt{-G}\ \mathrm{e}^{-2\phi} \left[R + 4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2\cdot 3!} H^2_{\mu\nu\rho} \right]$$

- There is no gauge invariance for the dilaton as for the graviton.
- Therefore, we don't expect low-energy theorems for the dilaton.
- No long range force associated with the $B_{\mu\nu}$ (no term of $\mathcal{O}(q^{-1})$).
- We cannot use its gauge invariance as for gravitons.
- On the other hand, the soft dilaton behavior in string theory goes back to the 70s [Ademollo et al , 1975] and [Shapiro, 1975].

- In string theory the scattering amplitudes involving a graviton or a dilaton or a Kalb-Ramond field are all obtained from the same two-index tensor $M^{\mu\nu}(q; k_i)$ by saturating it with a polarization tensor satisfying respectively the following conditions:

$$\begin{aligned} \text{Graviton } (g_{\mu\nu}) &\implies \epsilon_g^{\mu\nu} = \epsilon_g^{\nu\mu} \;\; ; \;\; \eta_{\mu\nu}\epsilon_g^{\mu\nu} = 0 \\ \text{Dilaton } (\phi) &\implies \epsilon_d^{\mu\nu} = \eta^{\mu\nu} - q^\mu \bar{q}^\nu - q^\nu \bar{q}^\mu \\ \text{Kalb-Ramond } (B_{\mu\nu}) &\implies \epsilon_B^{\mu\nu} = -\epsilon_B^{\nu\mu} \end{aligned}$$

  where $\bar{q}$ is, similarly to q, a lightlike vector such that $q \cdot \bar{q} = 1$.
- The soft theorem for a dilaton can, in principle, be computed starting from the expression that we obtained for the graviton.
- But now we cannot neglect extra terms proportional to $\eta^{\mu\nu}$ as we did in the case of a graviton.

- Imposing $q^\mu M_{\mu\nu} = q^\nu M_{\mu\nu} = 0$, for the graviton we got:

$$M_n^{\mu\nu}(q; k_1 \dots k_n)$$
$$= \sum_{i=1}^{n} \frac{k_i^\nu}{k_i \cdot q} \left[k_i^\mu - i q_\rho J_i^{\mu\rho} \right] T_n(k_1, \dots, k_n)$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \frac{1}{k_i \cdot q} \left[\left((k_i \cdot q)(\eta^{\mu\nu} q^\sigma - q^\mu \eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma \right) \frac{\partial}{\partial k_i^\sigma} \right.$$
$$\left. - q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma} \right] T_n(k_1, \dots, k_n) \, .$$

  and we have neglected the terms in the third line because the graviton polarization satisfies the identities:

$$q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = \eta^{\mu\nu} \epsilon_{\mu\nu} = 0$$

- More precisely, gauge invariance imposes:

$$q_\mu M_n^{\mu\nu}(q, k_i) = f(q, k_i) q^\nu \Longrightarrow q_\mu \left(M_n^{\mu\nu} - f(k_i) \eta^{\mu\nu} \right) = 0$$

- The extra term with $\eta^{\mu\nu}$ is irrelevant for the graviton, but not for the dilaton.

- Because of this we cannot in general get low-energy theorems for the dilaton.
- But, let us forget for a moment this problem, and compute the amplitude with a massless closed string state and n closed string tachyons:

$$M_n^{\mu\nu} \sim \int \frac{\prod_{i=1}^{n} d^2 z_i}{dV_{abc}} \prod_{i<j} |z_i - z_j|^{\alpha' k_i k_j} \int d^2 z \prod_{i=1}^{n} |z - z_i|^{\alpha' k_i q}$$
$$\times \alpha' \sum_{i=1}^{n} \frac{k_i^\mu}{z - z_i} \sum_{i=1}^{n} \frac{k_i^\nu}{\bar{z} - \bar{z}_i}$$

- We have explicitly computed the first three terms of order q^{-1}, q^0 and q^1.

- The calculation is rather long and at the end we get the following expression:

$$M_n^{\mu\nu}(q;k_1\dots k_n) = \kappa_d \left\{ \sum_{i=1}^n \frac{k_{i\mu}k_{i\nu}}{k_i q} + \sum_{i=1}^n \frac{k_{i\nu}q^\rho}{k_i q}\left(k_{i\mu}\frac{\partial}{\partial k_{i\rho}} - k_{i\rho}\frac{\partial}{\partial k_{i\mu}}\right) \right.$$

$$+ \frac{1}{2}\sum_{i=1}^n \frac{q^\rho q^\sigma}{k_i q}\left[k_{i\nu}\left(k_{i\mu}\frac{\partial}{\partial k_{i\rho}} - k_{i\rho}\frac{\partial}{\partial k_{i\mu}}\right)\frac{\partial}{\partial k_{i\sigma}} \right.$$

$$\left.\left. - k_{i\sigma}\left(k_{i\mu}\frac{\partial}{\partial k_{i\rho}} - k_{i\rho}\frac{\partial}{\partial k_{i\mu}}\right)\frac{\partial}{\partial k_{i\nu}}\right]\right\} T_n$$

where

$$T_n = \frac{8\pi}{\alpha'}\left(\frac{\kappa_d}{2\pi}\right)^{n-2}\int \frac{\prod_{i=1}^4 d^2 z_i}{dV_{abc}} \prod_{i\neq j} |z_i - z_j|^{\frac{\alpha'}{2}k_i k_j}$$

is the correctly normalized n-tachyon amplitude.

- This is precisely the expression obtained from gauge invariance with the general argument imposing the conditions: $q_\mu M^{\mu\nu} = q_\nu M^{\mu\nu} = 0$.

- By saturating it with the graviton polarization one gets, of course, the previous general expression.
- By saturating it with the dilaton "polarization"

$$\epsilon_{\mu\nu} = (\eta_{\mu\nu} - q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu) \quad ; \quad q^2 = \bar{q}^2 = 0 \;\; ; \;\; q\bar{q} = 1$$

- one gets ($m_i^2 = -\frac{4}{\alpha'}$)

$$\begin{aligned} S^{(0)} + S^{(1)} + S^{(2)} &= -\sum_{i=1}^{n} \frac{m_i^2 \left(1 + q^\rho \frac{\partial}{\partial k_{i\rho}} + \frac{1}{2} q_\rho q^\sigma \frac{\partial^2}{\partial k_{i\rho} \partial k_{i\sigma}}\right)}{k_i q} \\ &- \sum_{i=1}^{n} k_{i\mu} \frac{d}{dk_{i\mu}} + 2 \\ &+ \sum_{i=1}^{n} \left(-k_{i\mu} q_\sigma \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\sigma}} + \frac{1}{2}(k_i q) \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\mu}}\right) \\ &+ \mathcal{O}(q^2) \end{aligned}$$

- $B_{\mu\nu}$ not coupled to n tachyons (invariance under w.s. parity Ω).

- Soft behavior of a massless closed string in an amplitude involving an arbitrary number of other massless closed strings (bosonic+superstring) .
- In this case we have performed the calculation up to the $\mathcal{O}(q^0)$.
- For the symmetric part of $M_{\mu\nu}$ we get:

$$M_S^{\mu\nu}(q; k_i, \epsilon_i) = \kappa_d \sum_{i=1}^{n} \left(\frac{k_i^\mu k_i^\nu - \frac{i}{2} k_i^\nu q_\rho J_i^{\mu\rho} - \frac{i}{2} k_i^\mu q_\rho J_i^{\nu\rho}}{qk_i} \right) M_n(k_i, \epsilon_i)$$

where $M_n(k_i, \epsilon_i)$ is the amplitude with n massless states,

$$J_i^{\mu\nu} = L_i^{\mu\nu} + S_i^{\mu\nu} + \bar{S}_i^{\mu\nu} \ ,$$

$$L_i^{\mu\nu} = i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right) \ , \ S_i^{\mu\nu} = i \left(\epsilon_i^\mu \frac{\partial}{\partial \epsilon_{i\nu}} - \epsilon_i^\nu \frac{\partial}{\partial \epsilon_{i\mu}} \right) \ ,$$

$$\bar{S}_i^{\mu\nu} = i \left(\bar{\epsilon}_i^\mu \frac{\partial}{\partial \bar{\epsilon}_{i\nu}} - \bar{\epsilon}_i^\nu \frac{\partial}{\partial \bar{\epsilon}_{i\mu}} \right) \quad ; \quad \epsilon_i^{\mu\nu} \equiv \epsilon_i^\mu \bar{\epsilon}_i^\nu$$

- By saturating with the polarization of the graviton, one gets (of course) the soft behavior obtained from gauge invariance.
- If we instead saturate it with the polarization of the dilaton we get:

$$M_{n+1} = \kappa_d \left[2 - \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} \right] M_n + \mathcal{O}(q) \ ,$$

- It can be written in a more suggestive way by observing that, in general, M_n has the following form:

$$M_n = \frac{8\pi}{\alpha'} \left(\frac{\kappa_d}{2\pi} \right)^{n-2} F_n(\sqrt{\alpha'} k_i) \, , \ \kappa_d = \frac{1}{2^{\frac{d-10}{4}}} \frac{g_s}{\sqrt{2}} (2\pi)^{\frac{d-3}{2}} (\sqrt{\alpha'})^{\frac{d-2}{2}} \ ,$$

where F_n is dimensionless and obviously satisfies the equation:

$$\sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} F_n = \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} F_n \ .$$

- One gets:

$$M_{n+1} = \kappa_d \left[-\sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + \frac{d-2}{2} g_s \frac{\partial}{\partial g_s} \right] M_n + \mathcal{O}(q) \ .$$

- Same result if we include massless open strings (on a Dp-brane).
- No extra term proportional to $\eta^{\mu\nu}$ is needed to reproduce the previous amplitude.
- The amplitude of a soft dilaton is obtained from the amplitude without a dilaton by a simultaneous rescaling of the Regge slope α' and the string coupling constant g_s.
- Same rescaling that leaves Newton's constant invariant:

$$\left[-\sqrt{\alpha'}\frac{\partial}{\partial\sqrt{\alpha'}} + \frac{d-2}{2} g_s \frac{\partial}{\partial g_s}\right] \kappa_d = 0$$

- No fundamental dimensionless constant in string theory.
- From it we can rewrite the soft dilaton theorem:

$$M_{n+1} = \kappa_d \frac{d-2}{2} \frac{d}{d\phi_0} M_n + \mathcal{O}(q) \;\; ; \;\; g_s \equiv \mathrm{e}^{\phi_0}$$

- Apply to the case $n = 5$ with 5 dilatons:

$$M_5 = \kappa_d \left(2 - \sum_{i=1}^{n} k_{i\mu} \frac{\partial}{\partial k_{i\mu}} \right) M_4 + \mathcal{O}(q)$$

where

$$M_4 = 2\kappa_d^2 \left(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u} \right) \frac{\Gamma(1 - \frac{\alpha' s}{4})\Gamma(1 - \frac{\alpha' u}{4})\Gamma(1 - \frac{\alpha' t}{4})}{\Gamma(1 + \frac{\alpha' s}{4})\Gamma(1 + \frac{\alpha' u}{4})\Gamma(1 + \frac{\alpha' t}{4})}$$

- In the field theory limit ($\alpha' \to 0$), one gets zero because one has a homogenous function of degree 2.
- In string theory one gets a non-trivial right-hand-side.

Soft theorem for $B_{\mu\nu}$

- In order to formulate a soft theorem for the antisymmetric tensor we have to make a distinction between the momentum of the holomorphic part, which we call k_i, from that of the anti-holomorphic part, which we call $\bar{k}_i$.
- This means that the amplitude $M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)$, on which the soft operator acts, is a function of both k_i and $\bar{k}_i$.
- Together with the operators L_i, S_i and $\bar{S}_i$, we then also introduce:

$$\bar{L}_i^{\mu\nu} = i \left(\bar{k}_i^{\mu} \frac{\partial}{\partial \bar{k}_{i\nu}} - \bar{k}_i^{\nu} \frac{\partial}{\partial \bar{k}_{i\mu}} \right) .$$

- In terms of these operators, the soft behavior for $B_{\mu\nu}$ reads:

$$M_{n+1} = -i\epsilon^B_{q\,\mu\nu}\kappa_d \sum_{i=1}^{n} \left[\frac{k_i^{\nu} q_{\rho}(L_i + S_i)^{\mu\rho}}{qk_i} - \frac{k_i^{\nu} q_{\rho}(\bar{L}_i + \bar{S}_i)^{\mu\rho}}{qk_i} \right]$$

$$M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)\Big|_{k=\bar{k}} + \mathcal{O}(q)$$

- It is equal to

$$M_{n+1} = -i\epsilon^{B}_{q\,\mu\nu}\kappa_d \sum_{i=1}^{n} \left[\frac{1}{2}(L_i - \bar{L}_i)^{\mu\nu} + \frac{k_i^{\nu} q_{\rho}}{k_i q}(S_i - \bar{S}_i)^{\mu\rho}\right]$$
$$\times M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)\Big|_{k=\bar{k}} + \mathcal{O}(q)\,.$$

- As expected from Weinberg's general argument, we do not get any term of $\mathcal{O}(q^{-1})$, corresponding to a long range force, but there are several terms of $\mathcal{O}(q^0)$.
- It is not clear how to get the soft operator of the antisymmetric field by directly using its own gauge symmetry, as it has been done for the graviton.
- It is not really a soft theorem because the amplitude $M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)$ is not a physical amplitude before we act with the soft operators.
- It is nevertheless easy to show that it is gauge invariant.

- Under a gauge transformation for the Kalb-Ramond field, $\epsilon^{B}_{q\,\mu\nu} \to \epsilon^{B}_{q\,\mu\nu} + q^{\mu}\chi_{\nu} - q^{\nu}\chi_{\mu}$, the amplitude changes as follows

$$\hat{S}^{(1)} M_n \to \hat{S}^{(1)} M_n + i q_{\rho} \chi_{\mu} \sum_{i=1}^{n} \left[(L_i + S_i)^{\mu\rho} - (\bar{L}_i + \bar{S}_i)^{\mu\rho} \right] \times M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)\Big|_{k=\bar{k}} .$$

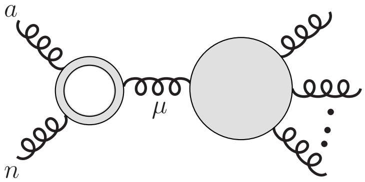
- The extra term vanishes as a consequence of the identity

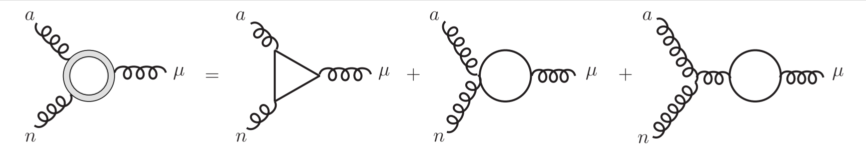
$$\sum_{i=1}^{n} (L_i + S_i)^{\mu\rho} M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)\Big|_{k=\bar{k}} = \sum_{i=1}^{n} (\bar{L}_i + \bar{S}_i)^{\mu\rho} M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i)\Big|_{k=\bar{k}}$$

which can be proved by a direct calculation, ensuring gauge invariance of the amplitude.

Comments on loop corrections: gauge theory

- At one-loop the amplitude will have in general IR and UV divergences.
- We are not giving here a complete study of them.
- The one-loop contributions have been classified into the factorizing ones and the non-factorizing ones.
- We will concentrate here to the factorizing ones.
- They modify the vertex present in the pole term.
- For the gauge theory they are of the type shown in the figure.





- They have been computed in QCD and are given by:

$$D^{\mu,\text{fact}} = \frac{i}{\sqrt{2}} \frac{1}{3} \frac{1}{(4\pi)^2} \left(1 - \frac{n_f}{N_c} + \frac{n_s}{N_c}\right) (q - k_a)^\mu \left[(\varepsilon_n \cdot \varepsilon_a) - \frac{(q \cdot \varepsilon_a)(k_a \cdot \varepsilon_n)}{(k_a \cdot q)} \right]$$

[Z. Bern, V. Del Duca, C.R. Schmidt, 1998]
[Z. Bern, V. Del Duca, W.B. Kilgore, C.R. Schmidt, 1999]

- It is both IR and UV finite and the limit $\epsilon \to 0$ has been taken.
- It is non-local because of the pole in (qk_a).
- It is gauge invariant under the substitution $\epsilon_n \to q$.
- It does not contribute to the leading soft behavior.

- Attaching to it the rest of the amplitude

$$D_\mu^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu \,,$$

- $\mathcal{J}^\mu$ is a conserved current:

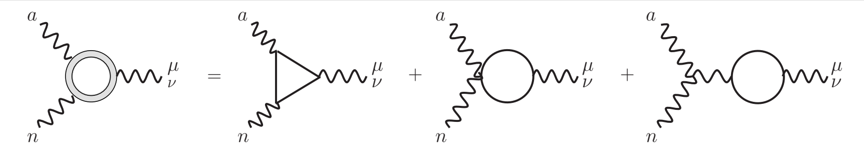
$$(q + k_a)_\mu \mathcal{J}^\mu = 0 \,,$$

  assuming that all the remaining legs are contracted with on-shell polarizations.
- We can trade k_a with q and we get immediately:

$$D_\mu^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu = \mathcal{O}(q^0) \,,$$

- No leading $\mathcal{O}(\frac{1}{q})$ correction from the factorizing contribution to the one-loop soft functions.

Comments on loop corrections: gravity



- A similar calculation can be done for the gravity case.
- We consider only the case in which scalar fields circulate in the loop.
- The result of this calculation is:

$$\mathcal{D}^{\mu\nu,\text{fact,s}} = \frac{i}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^3 \frac{1}{30}\left[(\varepsilon_n \cdot \varepsilon_a) - \frac{(q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n)}{(q\cdot k_a)}\right] \times \Big((q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n) - (\varepsilon_n\cdot\varepsilon_a)(q\cdot k_a)\Big) k_a^\mu k_a^\nu + \mathcal{O}(q^2)\,,$$

- As in the gauge-theory case, the diagrams $\mathcal{D}^{\mu\nu,\text{fact,s}}$ contract into a conserved current:

$$(k_a + q)^\mu \mathcal{J}_{\mu\nu} = f(k_i, \epsilon_i)(k_a + q)_\nu \,, \; (k_a + q)^\nu \mathcal{J}_{\mu\nu} = f(k_i, \epsilon_i)(k_a + q)_\mu \,.$$

- This means

$$\begin{aligned} k_a^\mu k_a^\nu \mathcal{J}_{\mu\nu} &= (k_a + q)^\mu (k_a + q)^\nu \mathcal{J}_{\mu\nu} + \mathcal{O}(q) \\ &= f(k_i, \epsilon_i)(k_a + q)^2 + \mathcal{O}(q) = 2f(k_i, \epsilon_i) q \cdot k_a + \mathcal{O}(q) = \mathcal{O}(q) \end{aligned}$$

- We therefore have

$$\mathcal{D}^{\mu\nu,\text{fact,s}} \frac{i}{2q \cdot k_a} \mathcal{J}_{\mu\nu} = \mathcal{O}(q) \,.$$

- No modification of the two first leading terms.
- As in QCD, we expect that the contribution of other particles circulating in the loop will not modify this result.

Conclusions

- We have extended Low's proof of the universality of sub-leading behavior of photons to non-abelian gauge theory and to gravity.
- On-shell gauge invariance fully determines the first sub-leading soft-gluon and the first two sub-leading soft-graviton behavior at tree level.
- Factorizing one-loop contributions preserve the leading behavior in gauge theories and the first two leading behaviors in gravity.
- One computes the low-energy behavior of $M_{\mu\nu}$ by imposing the Eqs. $q^{\mu} M_{\mu\nu} = q^{\nu} M_{\mu\nu} = 0$.
- Saturating $M_{\mu\nu}$ with the polarization of Graviton/Dilaton, one gets automatically their soft behavior.
- This is the result for all amplitudes we have looked at: BCJ/KLT?
- We get also a kind of soft theorem for $B_{\mu\nu}$.
- Extend our considerations to one-loop diagrams.
- Study the double-soft behavior both in field and string theory.