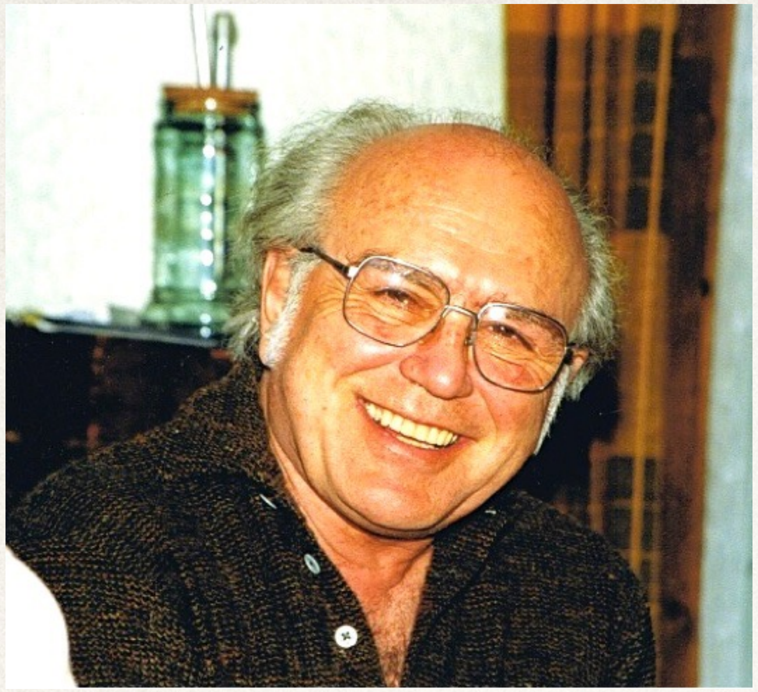


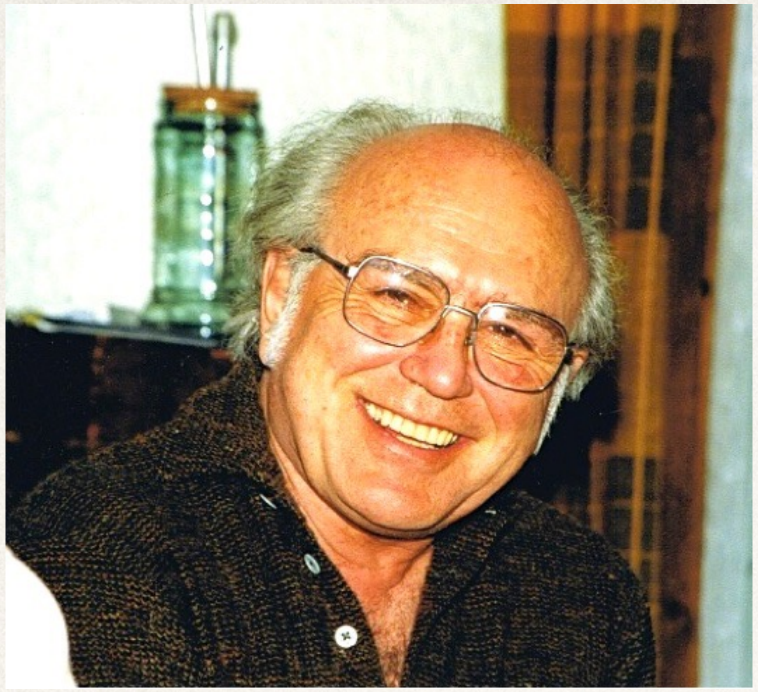
UNIVERSITY
OF MIAMI

Effective Actions, Lovelock Lagrangians and Bruno Zumino

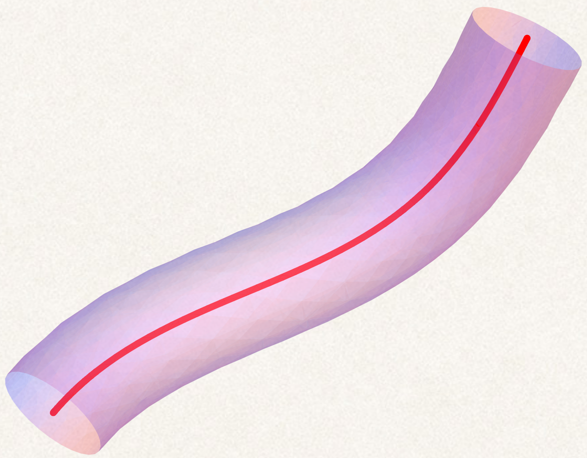
Orlando Alvarez

28 April 2015

Bruno Zumino Memorial Meeting at CERN



UNIVERSITY
OF MIAMI



Effective Actions, Lovelock Lagrangians and Bruno Zumino

Orlando Alvarez

28 April 2015

Bruno Zumino Memorial Meeting at CERN

My connection to Bruno

UNIVERSITY
OF MIAMI

My connection to Bruno

UNIVERSITY OF MIAMI

- Both of us officially started at Berkeley in the Fall of 1982 (Bruno actually arrived in 1981 as a visiting Miller Professor).

My connection to Bruno

UNIVERSITY OF MIAMI

- Both of us officially started at Berkeley in the Fall of 1982 (Bruno actually arrived in 1981 as a visiting Miller Professor).

- Our offices on campus were next door for about five years. These were tiny offices 108 ft^2 or approximately 3 m × 4 m. Afterwards Bruno moved to an office about thrice the size around the corner.

My connection to Bruno

UNIVERSITY OF MIAMI

- Both of us officially started at Berkeley in the Fall of 1982 (Bruno actually arrived in 1981 as a visiting Miller Professor).

- Our offices on campus were next door for about five years. These were tiny offices 108 ft^2 or approximately 3 m × 4 m. Afterwards Bruno moved to an office about thrice the size around the corner.

- We wrote one paper together but discussed physics and mathematics countless times.

My connection to Bruno

UNIVERSITY OF MIAMI

- Both of us officially started at Berkeley in the Fall of 1982 (Bruno actually arrived in 1981 as a visiting Miller Professor).

- Our offices on campus were next door for about five years. These were tiny offices 108 ft^2 or approximately 3 m × 4 m. Afterwards Bruno moved to an office about thrice the size around the corner.

- We wrote one paper together but discussed physics and mathematics countless times.

- We were colleagues for 12 years until I moved to Miami.

My paper with Bruno

UNIVERSITY
OF MIAMI

Commun. Math. Phys. 96, 409–417 (1984)

Communications in Mathematical Physics

Gravitational Anomalies and the Family's Index Theorem*

Orlando Alvarez[1] **, I. M. Singer[2], and Bruno Zumino[1]

1 Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, CA 94720, USA

2 Department of Mathematics and Department of Physics, University of California, Berkeley, CA 94720, USA

Abstract. We discuss the use of the family's index theorem in the study of gravitational anomalies. The geometrical framework required to apply the family's index theorem is presented and the relation to gravitational anomalies is discussed. We show how physics necessitates the introduction of the notion of *local cohomology* which is distinct from the ordinary topological cohomology. The recent results of Alvarez-Gaumé and Witten are derived by using the family's index theorem.

I. Introduction

Alvarez-Gaumé and Witten [1] have calculated the gravitational anomalies of certain parity violating theories in $4k-2$ dimensions. Their most striking result is that there is a unique minimal ten dimensionl theory where the gravitational anomalies cancel. In this communication we reproduce their results in a different way by using the family's index theorem [2] instead of Feynman diagram methods.

The relation of the family's index theorem to anomalies has been discussed by Atiyah and one of the present authors in reference [3]. In that paper, the geometric setting for the family's index theorem was presented and the relation to anomalies was discussed. The authors showed that the first characteristic class of the index bundle for the Dirac operator was related to anomalies. A number of papers have addressed the relationships among chiral anomalies, the geometry of the space of vector potentials, and the families of Dirac operators. We recommend the papers of Alvarez-Gaumé and Ginsparg [4], Lott [5], and Stora [6] to the reader. The first investigation of the behavior of the Dirac operator as a function of the metric is due to Hitchin [7].

* This work was supported in part by the National Science Foundation under Contracts PHY81-18547 and MCS80-23356; and by the Director, Office of High Energy and Nuclear Physics of the US Department of Energy under Contracts DE-AC03-76SF00098 and AT0380-ER10617

** Alfred P. Sloan Foundation Fellow

Commun. Math. Phys. 96, 409–417 (1984)

Communications in Mathematical Physics

Gravitational Anomalies and the Family's Index Theorem*

Orlando Alvarez[1] **, I. M. Singer[2], and Bruno Zumino[1]

1 Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, CA 94720, USA

2 Department of Mathematics and Department of Physics, University of California, Berkeley, CA 94720, USA

Abstract. We discuss the use of the family's index theorem in the study of gravitational anomalies. The geometrical framework required to apply the family's index theorem is presented and the relation to gravitational anomalies is discussed. We show how physics necessitates the introduction of the notion of *local cohomology* which is distinct from the ordinary topological cohomology. The recent results of Alvarez-Gaumé and Witten are derived by using the family's index theorem.

I. Introduction

Alvarez-Gaumé and Witten [1] have calculated the gravitational anomalies of certain parity violating theories in $4k-2$ dimensions. Their most striking result is that there is a unique minimal ten dimensionl theory where the gravitational anomalies cancel. In this communication we reproduce their results in a different way by using the family's index theorem [2] instead of Feynman diagram methods.

The relation of the family's index theorem to anomalies has been discussed by Atiyah and one of the present authors in reference [3]. In that paper, the geometric setting for the family's index theorem was presented and the relation to anomalies was discussed. The authors showed that the first characteristic class of the index bundle for the Dirac operator was related to anomalies. A number of papers have addressed the relationships among chiral anomalies, the geometry of the space of vector potentials, and the families of Dirac operators. We recommend the papers of Alvarez-Gaumé and Ginsparg [4], Lott [5], and Stora [6] to the reader. The first investigation of the behavior of the Dirac operator as a function of the metric is due to Hitchin [7].

* This work was supported in part by the National Science Foundation under Contracts PHY81-18547 and MCS80-23356; and by the Director, Office of High Energy and Nuclear Physics of the US Department of Energy under Contracts DE-AC03-76SF00098 and AT0380-ER10617

** Alfred P. Sloan Foundation Fellow

UNIVERSITY
OF MIAMI

Is Singer's Birthday is 3 May.
He will be 91.

Zumino & Lovelock Lagrangians

UNIVERSITY
OF MIAMI

Zumino & Lovelock L… gians

UNIVERSITY OF MIAMI

PHYSICS REPORTS (Review Section of Physics Letters) 137, No. 1 (1986) 109–114. North-Holland, Amsterdam

Gravity Theories in more than Four Dimensions[1]

Bruno ZUMINO

Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, CA 94720, U.S.A.

Abstract:
String theories suggest particular forms for gravity interactions in higher dimensions. We consider an interesting class of gravity theories in more than four dimensions, clarify their geometric meaning and discuss their special properties.

1. Introduction

The study of gravity in more than 4 dimensions is motivated by the approach of Kaluza [1] and Klein [2] to the problem of unification of gravity with electromagnetism and the other elementary interactions. The Kaluza–Klein point of view has been revived recently [3] by the study of supergravity. The most promising approach to unification seems to be that based on string theories [4–6].

The 10 dimensional gravity which emerges from supersymmetric string theories in the low energy limit contains in its action terms quadratic in the Riemann curvature tensor. Warren Siegel has emphasized that these terms give rise to ghosts and violate unitarity, while the string theory is unitary. This puzzling contradiction has been resolved by Barton Zwiebach [7] who has pointed out that the n-dimensional action

$$\int \sqrt{g}\, d^n x\, (\mathcal{R}_{ab,cd}\mathcal{R}^{ab,cd} - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}^2)\,, \tag{1.1}$$

leads to ghost-free nontrivial gravitational interactions for $n > 4$. By explicit computation Zwiebach has shown that, if one expands (1.1) about Minkowski space, the terms quadratic in the gravitational field combine to a total derivative and integrate to zero, so that (1.1), added to the usual Einstein–Hilbert action, introduces no propagator corrections. In 4 dimensions the entire expression (1.1) is a total derivative and is recognized as proportional to the Euler topological invariant.

Halpern and Zwiebach have observed that, similarly, the Einstein action in 4 dimensions, $\int \sqrt{g}\, d^4 x\, \mathcal{R}$, has exactly the form of the Euler invariant in 2 dimensions except that the indices run over 4 values instead of 2. This has led them to believe that dimensionally continued Euler densities may play a role in the low energy limit of string theories. They have also conjectured that, in each case, the leading term in an expansion around Minkowski space integrates to zero.

We shall see that this conjecture is rather easy to prove, once the geometric meaning of the dimensionally continued Euler densities is understood. As we show below, they form a particular class

[1] This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under contract DE-AC03-76-SF-00098 and in part by the National Science Foundation under research grant PHY-85-40535.

0 370-1573/86/$3.50

$$L_{0,6} = e_a e_b e_c e_d e_f e_g \epsilon^{abcdfg} , \tag{3.7}$$

$$L_{1,4} = R_{ab} e_c e_d e_f e_g \epsilon^{abcdfg} , \tag{3.8}$$

$$L_{2,2} = R_{ab} R_{cd} e_f e_g \epsilon^{abcdfg} , \tag{3.9}$$

and

$$L_{3,0} = R_{ab} R_{cd} R_{fg} \epsilon^{abcdfg} . \tag{3.10}$$

Again, the first is a cosmological term, the second is proportional to the Einstein–Hilbert action and the last to the Euler invariant. Now we have the new possibility (3.9). Similarly for higher dimensions. Odd numbers of dimensions can be considered as well, but in this case the Euler invariant is absent, of course.

To be concrete, let us stay with 6 dimensions. Can one really have a term like (3.9) in the Lagrangian? At first sight one may think that such a term, which is quadratic in the Riemann tensor, will contribute to the bilinear part of the Lagrangian for the field h which describes the deviation from Minkowski space

$$e^a{}_m = \delta^a{}_m + h^a{}_m \tag{3.11}$$

and thus spoil the particle interpretation by introducing ghosts [9]. However, one can see that this is not the case.

Let us consider an infinitesimal variation of the connection and vielbein forms. The corresponding variation of $L_{2,2}$ *is*

$$\delta L_{2,2} = 2\,\delta R_{ab} R_{cd} e_f e_g \epsilon^{abcdfg} + 2 R_{ab} R_{cd} e_f\, \delta e_g \epsilon^{abcdfg} . \tag{3.12}$$

Using (2.9), the first term on the right hand side is

$$2(\mathrm{D}\,\delta\omega_{ab}) R_{cd} e_f e_g \epsilon^{abcdfg} . \tag{3.13}$$

On the other hand, using the Bianchi identity (2.7) and the definition (2.4) of the torsion we have

$$2\mathrm{d}(\delta\omega_{ab} R_{cd} e_f e_g)\epsilon^{abcdfg} = 2(\mathrm{D}\,\delta\omega_{ab}) R_{cd} e_f e_g \epsilon^{abcdfg} + 4\,\delta\omega_{ab} R_{cd} e_f T_g \epsilon^{abcdfg} . \tag{3.14}$$

Therefore, if the torsion vanishes, (3.12) can be written

$$\delta L_{2,2} = 2\mathrm{d}(\delta\omega_{ab} R_{cd} e_f e_g \epsilon^{abcdfg}) + 2 R_{ab} R_{cd} e_f\, \delta e_g \epsilon^{abcdfg} . \tag{3.15}$$

This equation tells us that, if we consider a power series expansion in h starting from flat space, the terms in $L_{2,2}$ which are quadratic in h appear under a derivative sign (first term on the right hand side in (3.15)); for a compact manifold or with suitable conditions at infinity, they drop out after integration. The first non-trivial term in the integrated action is cubic; it comes from the second term on the right hand side of (3.15) and can be immediately obtained from it. Clearly, the same result is true for $L_{2,r}$ in

Lovelock Lagrangians in 6D

$$L_{0,6} = e_a e_b e_c e_d e_f e_g \epsilon^{abcdfg} , \tag{3.7}$$

$$L_{1,4} = R_{ab} e_c e_d e_f e_g \epsilon^{abcdfg} , \tag{3.8}$$

$$L_{2,2} = R_{ab} R_{cd} e_f e_g \epsilon^{abcdfg} , \tag{3.9}$$

and

$$L_{3,0} = R_{ab} R_{cd} R_{fg} \epsilon^{abcdfg} . \tag{3.10}$$

Again, the first is a cosmological term, the second is proportional to the Einstein–Hilbert action and the last to the Euler invariant. Now we have the new possibility (3.9). Similarly for higher dimensions. Odd numbers of dimensions can be considered as well, but in this case the Euler invariant is absent, of course.

To be concrete, let us stay with 6 dimensions. Can one really have a term like (3.9) in the Lagrangian? At first sight one may think that such a term, which is quadratic in the Riemann tensor, will contribute to the bilinear part of the Lagrangian for the field h which describes the deviation from Minkowski space

$$e^a{}_m = \delta^a{}_m + h^a{}_m \tag{3.11}$$

and thus spoil the particle interpretation by introducing ghosts [9]. However, one can see that this is not the case.

Let us consider an infinitesimal variation of the connection and vielbein forms. The corresponding variation of $L_{2,2}$ *is*

$$\delta L_{2,2} = 2\,\delta R_{ab} R_{cd} e_f e_g \epsilon^{abcdfg} + 2 R_{ab} R_{cd} e_f\, \delta e_g \epsilon^{abcdfg} . \tag{3.12}$$

Using (2.9), the first term on the right hand side is

$$2(\mathrm{D}\,\delta\omega_{ab}) R_{cd} e_f e_g \epsilon^{abcdfg} . \tag{3.13}$$

On the other hand, using the Bianchi identity (2.7) and the definition (2.4) of the torsion we have

$$2\mathrm{d}(\delta\omega_{ab} R_{cd} e_f e_g)\epsilon^{abcdfg} = 2(\mathrm{D}\,\delta\omega_{ab}) R_{cd} e_f e_g \epsilon^{abcdfg} + 4\,\delta\omega_{ab} R_{cd} e_f T_g \epsilon^{abcdfg} . \tag{3.14}$$

Therefore, if the torsion vanishes, (3.12) can be written

$$\delta L_{2,2} = 2\mathrm{d}(\delta\omega_{ab} R_{cd} e_f e_g \epsilon^{abcdfg}) + 2 R_{ab} R_{cd} e_f\, \delta e_g \epsilon^{abcdfg} . \tag{3.15}$$

This equation tells us that, if we consider a power series expansion in h starting from flat space, the terms in $L_{2,2}$ which are quadratic in h appear under a derivative sign (first term on the right hand side in (3.15)); for a compact manifold or with suitable conditions at infinity, they drop out after integration. The first non-trivial term in the integrated action is cubic; it comes from the second term on the right hand side of (3.15) and can be immediately obtained from it. Clearly, the same result is true for $L_{2,r}$ in

Lovelock Lagrangians in 6D

$$L_{0,6} = e_a e_b e_c e_d e_f e_g \epsilon^{abcdfg} , \tag{3.7}$$

$$L_{1,4} = R_{ab} e_c e_d e_f e_g \epsilon^{abcdfg} , \tag{3.8}$$

$$L_{2,2} = R_{ab} R_{cd} e_f e_g \epsilon^{abcdfg} , \tag{3.9}$$

and

$$L_{3,0} = R_{ab} R_{cd} R_{fg} \epsilon^{abcdfg} . \tag{3.10}$$

Again, the first is a cosmological term, the second is proportional to the Einstein–Hilbert action and the last to the Euler invariant. Now we have the new possibility (3.9). Similarly for higher dimensions. Odd numbers of dimensions can be considered as well, but in this case the Euler invariant is absent, of course.

To be concrete, let us stay with 6 dimensions. Can one really have a term like (3.9) in the Lagrangian? At first sight one may think that such a term, which is quadratic in the Riemann tensor, will contribute to the bilinear part of the Lagrangian for the field h which describes the deviation from Minkowski space

$$e^a{}_m = \delta^a{}_m + h^a{}_m \tag{3.11}$$

and thus spoil the particle interpretation by introducing ghosts [9]. However, one can see that this is not the case.

Let us consider an infinitesimal variation of the connection and vielbein forms. The corresponding variation of $L_{2,2}$ *is*

$$\delta L_{2,2} = 2\,\delta R_{ab} R_{cd} e_f e_g \epsilon^{abcdfg} + 2 R_{ab} R_{cd} e_f \,\delta e_g \epsilon^{abcdfg} . \tag{3.12}$$

Using (2.9), the first term on the right hand side is

$$2(\mathrm{D}\,\delta\omega_{ab}) R_{cd} e_f e_g \epsilon^{abcdfg} . \tag{3.13}$$

On the other hand, using the Bianchi identity (2.7) and the definition (2.4) of the torsion we have

$$2\mathrm{d}(\delta\omega_{ab} R_{cd} e_f e_g)\epsilon^{abcdfg} = 2(\mathrm{D}\,\delta\omega_{ab}) R_{cd} e_f e_g \epsilon^{abcdfg} + 4\,\delta\omega_{ab} R_{cd} e_f T_g \epsilon^{abcdfg} . \tag{3.14}$$

Therefore, if the torsion vanishes, (3.12) can be written

$$\delta L_{2,2} = 2\mathrm{d}(\delta\omega_{ab} R_{cd} e_f e_g \epsilon^{abcdfg}) + 2 R_{ab} R_{cd} e_f \,\delta e_g \epsilon^{abcdfg} . \tag{3.15}$$

This equation tells us that, if we consider a power series expansion in h starting from flat space, the terms in $L_{2,2}$ which are quadratic in h appear under a derivative sign (first term on the right hand side in (3.15)); for a compact manifold or with suitable conditions at infinity, they drop out after integration. The first non-trivial term in the integrated action is cubic; it comes from the second term on the right hand side of (3.15) and can be immediately obtained from it. Clearly, the same result is true for $L_{2,r}$ in

Bruno's Favorite Technology

UNIVERSITY
OF MIAMI

Bruno’s Favorite Technology

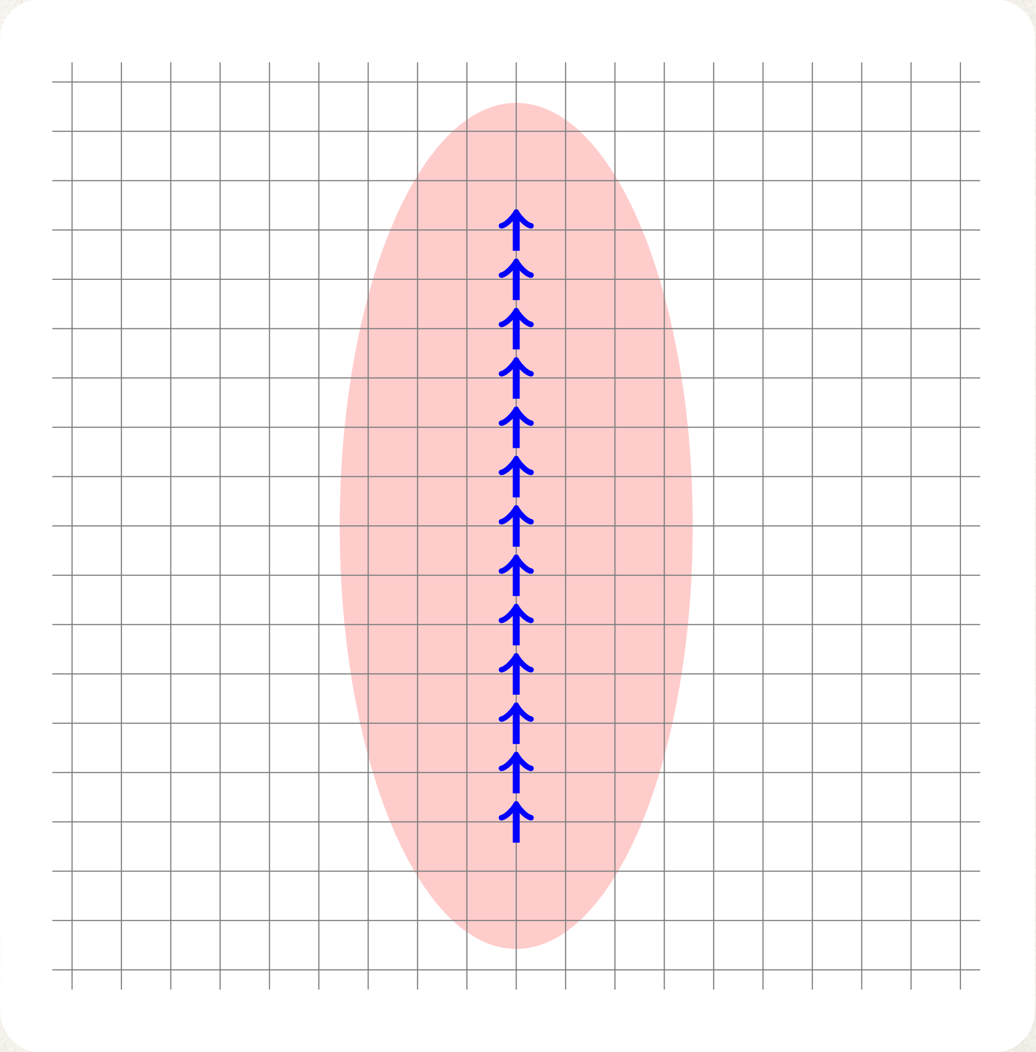
UNIVERSITY
OF MIAMI

Effective Lagrangians and Lovelock Actions

Energy Tubes

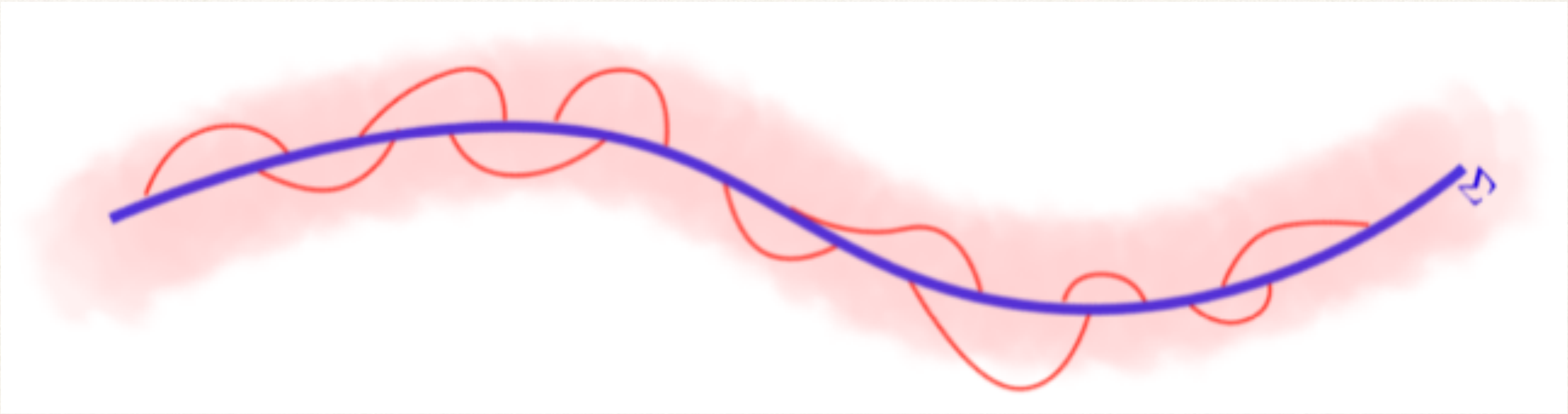
UNIVERSITY
OF MIAMI

An energy tube is a region of space where some mechanism changes the local energy relative to the vacuum. Typically you will need a field theory with a finite correlation length, and some type of boundary conditions. I do not include the traditional Casimir effect of here because you need a long range (massless) field.



Brane Neighborhood

UNIVERSITY
OF MIAMI



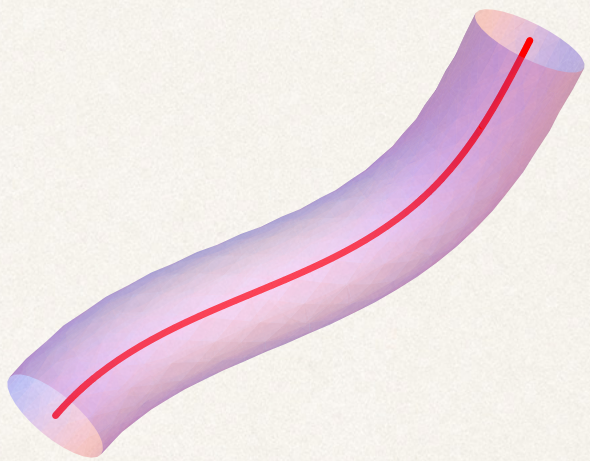
Original Motivation – Defects

Nuclear Physics B81 (1974) 84–92. North-Holland Publishing Company

DYNAMICS OF RELATIVISTIC VORTEX LINES AND THEIR RELATION TO DUAL THEORY

D. FÖRSTER *
The Niels Bohr Institute, University of Copenhagen,
DK-2100 Copenhagen Ø, Denmark

Received 24 April 1974



Original Motivation – Defects

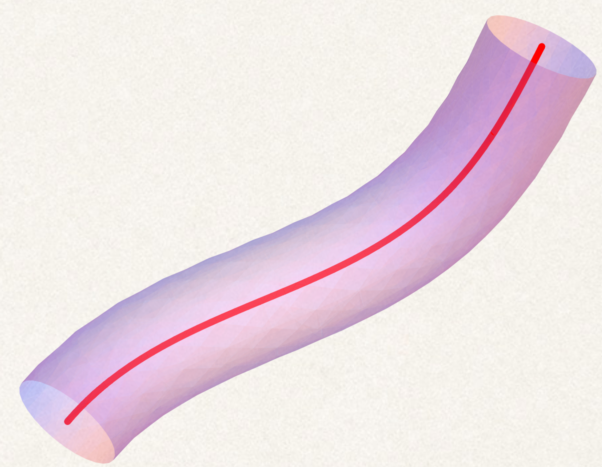
UNIVERSITY
OF MIAMI

Nuclear Physics B81 (1974) 84–92. North-Holland Publishing Company

DYNAMICS OF RELATIVISTIC VORTEX LINES AND THEIR RELATION TO DUAL THEORY

D. FÖRSTER *
The Niels Bohr Institute, University of Copenhagen,
DK-2100 Copenhagen Ø, Denmark

Received 24 April 1974



What governs the dynamics of the Nielsen-Olsen vortex?

Original Motivation – Defects

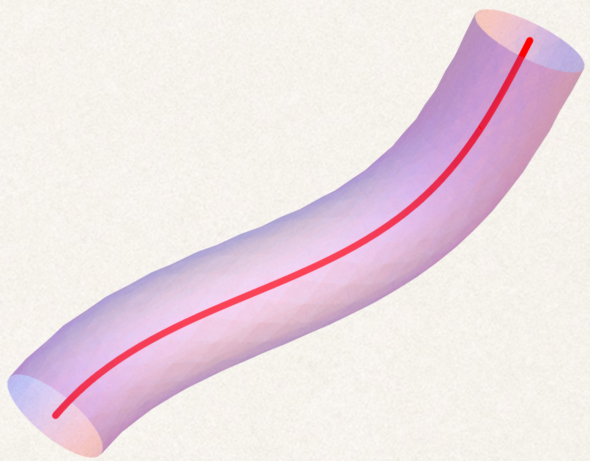
UNIVERSITY
OF MIAMI

Nuclear Physics B81 (1974) 84–92. North-Holland Publishing Company

DYNAMICS OF RELATIVISTIC VORTEX LINES AND THEIR RELATION TO DUAL THEORY

D. FÖRSTER *
The Niels Bohr Institute, University of Copenhagen,
DK-2100 Copenhagen Ø, Denmark

Received 24 April 1974



What governs the dynamics of the Nielsen-Olsen vortex?

$$S_{\text{dual}} = -\frac{1}{2\pi\alpha'}\int d^2\tau\,\sqrt{-g},$$

Nambu-Goto String Action for the dynamics of the core

14 Years Later

UNIVERSITY OF MIAMI

Volume 202, number 3 PHYSICS LETTERS B 10 March 1988

FINITE-WIDTH CORRECTIONS TO THE NAMBU ACTION FOR THE NIELSEN–OLESEN STRING

Kĕi-ichi MAEDA
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, IL 60510, USA and Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan [1]

and

Neil TUROK
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, IL 60510, USA

Received 18 November 1987

Volume 206, number 2 PHYSICS LETTERS B 19 May 1988

EFFECTIVE ACTION FOR A COSMIC STRING

R. GREGORY
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, CB3 9EW, UK

Received 10 February 1988

14 Years Later

UNIVERSITY
OF MIAMI

Volume 202, number 3 PHYSICS LETTERS B 10 March 1988

FINITE-WIDTH CORRECTIONS
TO THE NAMBU ACTION FOR THE NIELSEN–OLESEN STRING

Kêi-ichi MAEDA
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, IL 60510, USA
and Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan [1]

and

Neil TUROK
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, IL 60510, USA

Received 18 November 1987

Conclusions not totally correct

Volume 206, number 2 PHYSICS LETTERS B 19 May 1988

EFFECTIVE ACTION FOR A COSMIC STRING

R. GREGORY
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, CB3 9EW, UK

Received 10 February 1988

Better understood

UNIVERSITY
OF MIAMI

PHYSICAL REVIEW D VOLUME 43, NUMBER 2 15 JANUARY 1991

Effective actions for bosonic topological defects

Ruth Gregory
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia Illinois 60510
(Received 2 August 1990)

We consider a gauge field theory which admits p-dimensional topological defects, expanding the equations of motion in powers of the defect thickness. In this way we derive an effective action and effective equation of motion for the defect in terms of the coordinates of the p-dimensional world surface defined by the history of the core of the defect.

Better understood

UNIVERSITY
OF MIAMI

PHYSICAL REVIEW D VOLUME 43, NUMBER 2 15 JANUARY 1991

Effective actions for bosonic topological defects

Ruth Gregory
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia Illinois 60510
(Received 2 August 1990)

We consider a gauge field theory which admits p-dimensional topological defects, expanding the equations of motion in powers of the defect thickness. In this way we derive an effective action and effective equation of motion for the defect in terms of the coordinates of the p-dimensional world surface defined by the history of the core of the defect.

Clearly, upon integration, linear terms will disappear, leaving a contribution to the action of

$$S=\mu_0\int\sqrt{-\gamma}\left[1-\frac{\mu_1}{\mu_0}\epsilon^2\,{}^{(p+1)}\mathcal{R}\right]d^{p+1}\sigma\ , \tag{19}$$

where $\mu_0=\int\mathcal{L}_0 d^m\xi^i$ is the energy per unit p area of the defect, $\mu_1=\int\xi^{i2}\mathcal{L}_0 d^m\xi^i/2\epsilon^2$ is a constant of order unity, and we have used the Gauss-Codazzi relations

$$\sum_i K_i^2-K_{i\mu\nu}^2=-\,{}^{(p+1)}\mathcal{R} \tag{20}$$

to write the action in terms of the Ricci curvature of the world surface.

Better understood

UNIVERSITY OF MIAMI

PHYSICAL REVIEW D VOLUME 43, NUMBER 2 15 JANUARY 1991

Effective actions for bosonic topological defects

Ruth Gregory
NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia Illinois 60510
(Received 2 August 1990)

We consider a gauge field theory which admits p-dimensional topological defects, expanding the equations of motion in powers of the defect thickness. In this way we derive an effective action and effective equation of motion for the defect in terms of the coordinates of the p-dimensional world surface defined by the history of the core of the defect.

Clearly, upon integration, linear terms will disappear, leaving a contribution to the action of

$$S = \mu_0 \int \sqrt{-\gamma} \left[1 - \frac{\mu_1}{\mu_0} \epsilon^{2} \, {}^{(p+1)}\mathcal{R} \right] d^{p+1}\sigma \ , \qquad (19)$$

where $\mu_0 = \int \mathcal{L}_0 d^m \xi^i$ is the energy per unit p area of the defect, $\mu_1 = \int \xi^{i2} \mathcal{L}_0 d^m \xi^i / 2\epsilon^2$ is a constant of order unity, and we have used the Gauss-Codazzi relations

$$\sum_i K_i^2 - K_{i\mu\nu}^2 = - \, {}^{(p+1)}\mathcal{R} \qquad (20)$$

to write the action in terms of the Ricci curvature of the world surface.

Universal Expression

UNIVERSITY
OF MIAMI

- There is a **universal expression** that describes the “mean field energy” (mean field action) of an energy tube.
- There are corrections that I can describe to you after the talk.

What is a mathematical tube?

UNIVERSITY
OF MIAMI

What is a tube?

UNIVERSITY OF MIAMI

Let $\Sigma^q \subset \mathbb{E}^n$ be an embedded submanifold without boundary, *i.e.* a closed submanifold. The *tube* $\mathcal{T}(\Sigma, \rho)$ of radius ρ about Σ is a subset of $\mathbb{E}^n$ with the following characterization: $\boldsymbol{x}$ is in the tube if there exists a straight segment from $\boldsymbol{x}$ to Σ that intersects Σ perpendicularly and the length of the segment is less than or equal to ρ. The tube $\mathcal{T}(\Sigma, \rho)$ is a fiber bundle over Σ with fiber B^l, the l-dimensional ball (the solid $(l-1)$-sphere) where $n = q + l$.

What is a tube?

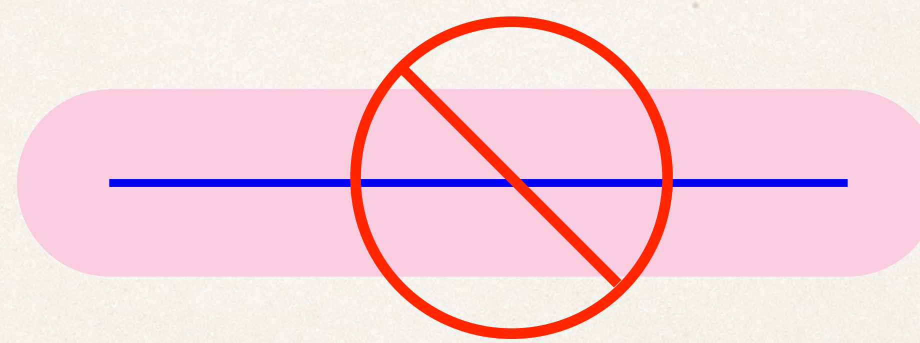
UNIVERSITY OF MIAMI

Let $\Sigma^q \subset \mathbb{E}^n$ be an embedded submanifold without boundary, *i.e.* a closed submanifold. The *tube* $\mathcal{T}(\Sigma, \rho)$ of radius ρ about Σ is a subset of $\mathbb{E}^n$ with the following characterization: $\boldsymbol{x}$ is in the tube if there exists a straight segment from $\boldsymbol{x}$ to Σ that intersects Σ perpendicularly and the length of the segment is less than or equal to ρ. The tube $\mathcal{T}(\Sigma, \rho)$ is a fiber bundle over Σ with fiber B^l, the l-dimensional ball (the solid $(l-1)$-sphere) where $n = q + l$.

What is a tube?

UNIVERSITY
OF MIAMI

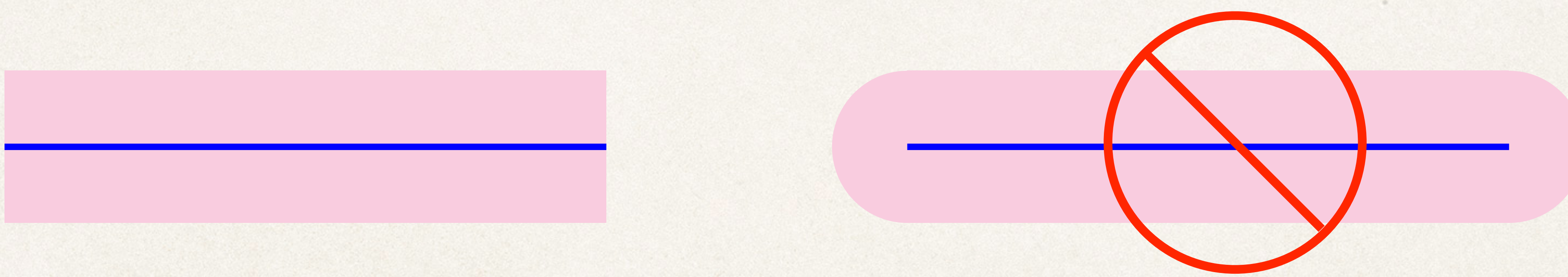
Let $\Sigma^q \subset \mathbb{E}^n$ be an embedded submanifold without boundary, *i.e.* a closed submanifold. The *tube* $\mathcal{T}(\Sigma, \rho)$ of radius ρ about Σ is a subset of $\mathbb{E}^n$ with the following characterization: $\boldsymbol{x}$ is in the tube if there exists a straight segment from $\boldsymbol{x}$ to Σ that intersects Σ perpendicularly and the length of the segment is less than or equal to ρ. The tube $\mathcal{T}(\Sigma, \rho)$ is a fiber bundle over Σ with fiber B^l, the l-dimensional ball (the solid $(l-1)$-sphere) where $n = q + l$.



What is a tube?

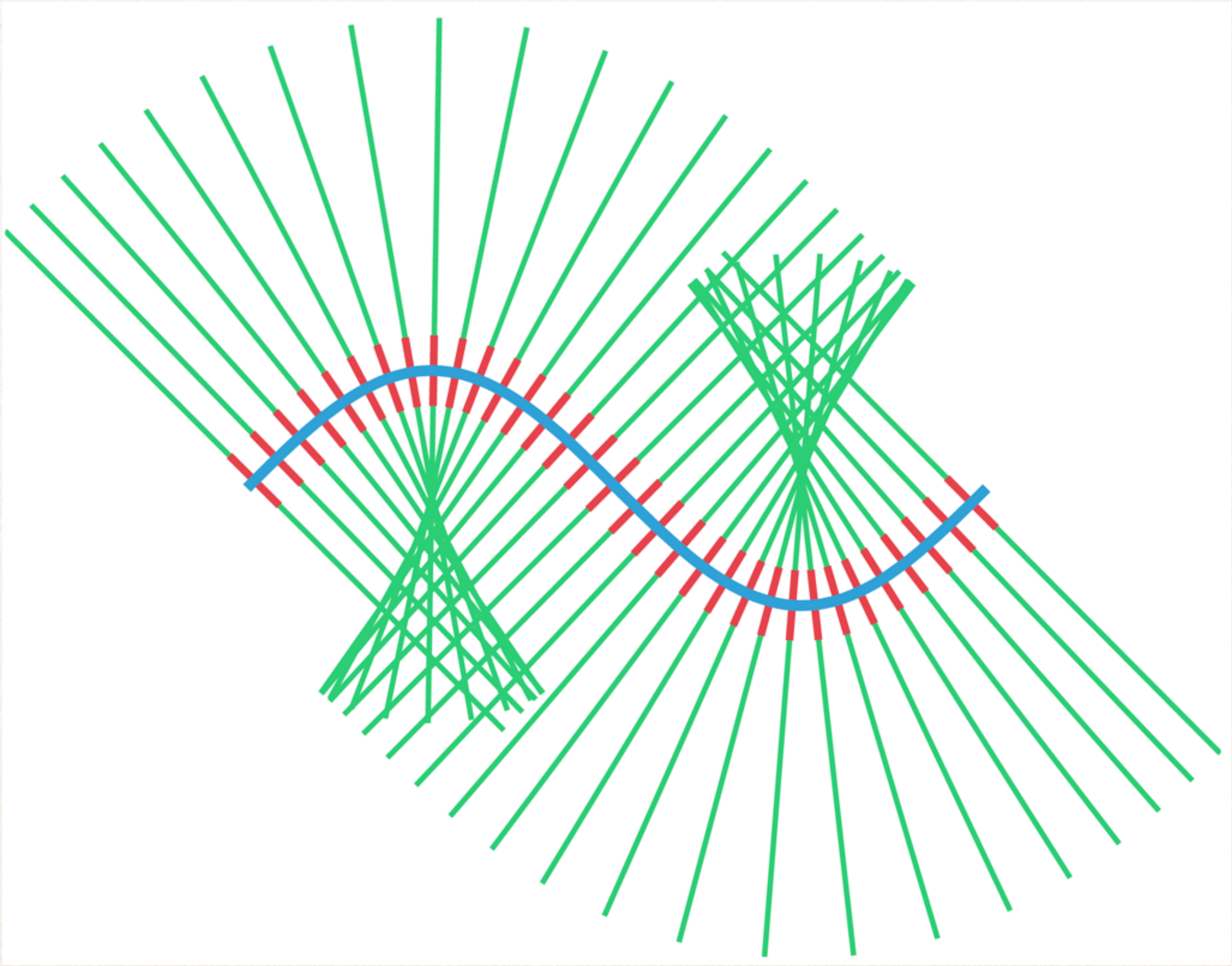
UNIVERSITY OF MIAMI

Let $\Sigma^q \subset \mathbb{E}^n$ be an embedded submanifold without boundary, *i.e.* a closed submanifold. The *tube* $\mathcal{T}(\Sigma, \rho)$ of radius ρ about Σ is a subset of $\mathbb{E}^n$ with the following characterization: $\boldsymbol{x}$ is in the tube if there exists a straight segment from $\boldsymbol{x}$ to Σ that intersects Σ perpendicularly and the length of the segment is less than or equal to ρ. The tube $\mathcal{T}(\Sigma, \rho)$ is a fiber bundle over Σ with fiber B^l, the l-dimensional ball (the solid $(l-1)$-sphere) where $n = q + l$.



$q = p + 1$

Why a short correlation length?



H. Weyl's Formula (1939)

H. Weyl's Formula (1939)

UNIVERSITY
OF MIAMI

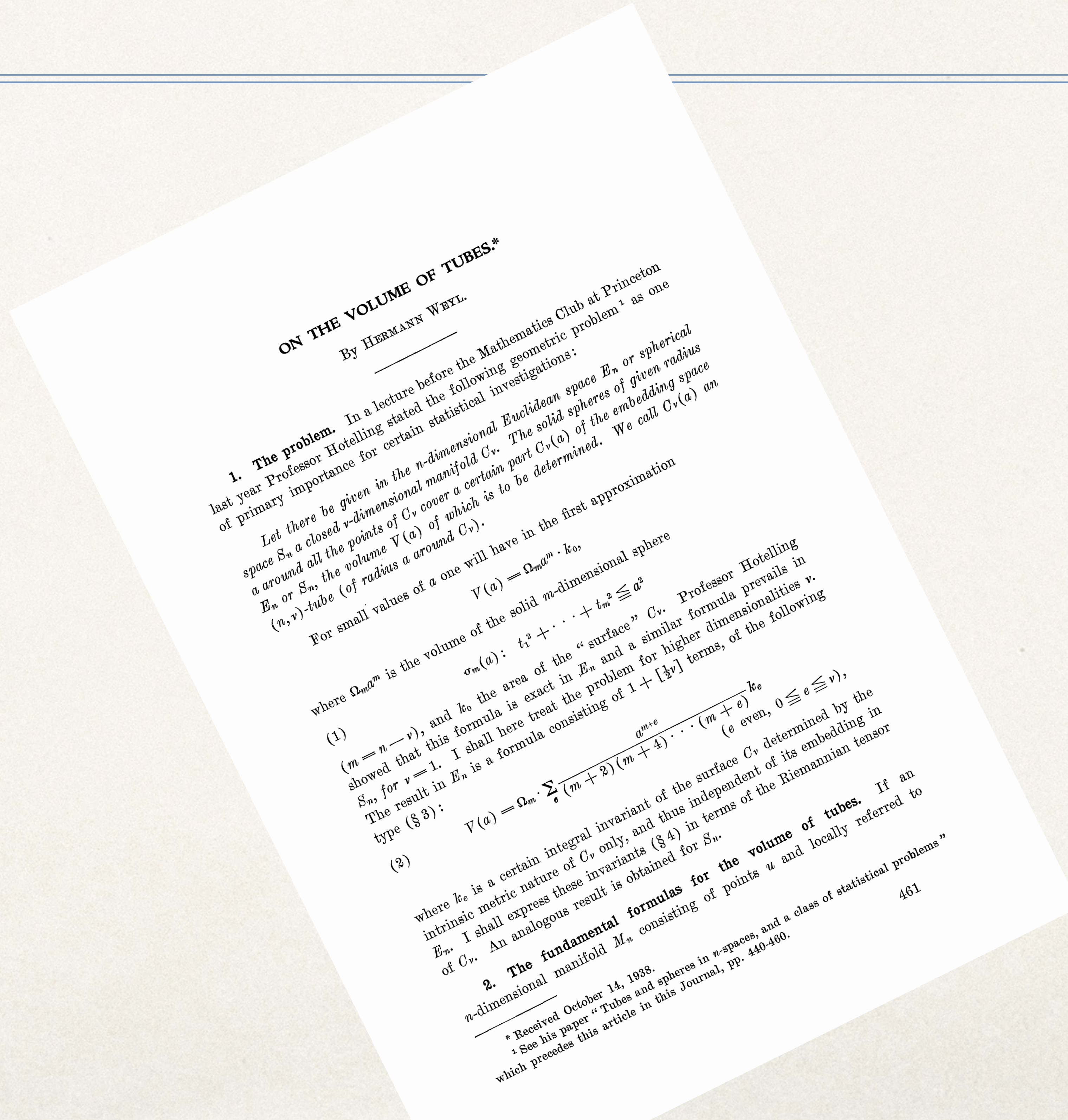
(1885-1955)

H. Weyl's Formula (1939)

UNIVERSITY
OF MIAMI

H. Weyl's Formula (1939)

UNIVERSITY OF MIAMI



ON THE VOLUME OF TUBES.*

By Hermann Weyl.

1. The problem. In a lecture before the Mathematics Club at Princeton last year Professor Hotelling stated the following geometric problem[1] as one of primary importance for certain statistical investigations:

Let there be given in the n-dimensional Euclidean space E_n or spherical space S_n a closed ν-dimensional manifold C_ν. The solid spheres of given radius a around all the points of C_ν cover a certain part $C_\nu(a)$ of the embedding space E_n or S_n, the volume $V(a)$ of which is to be determined. We call $C_\nu(a)$ an (n,ν)-tube (of radius a around C_ν).

For small values of a one will have in the first approximation

$$V(a) = \Omega_m a^m \cdot k_0,$$

where $\Omega_m a^m$ is the volume of the solid m-dimensional sphere

(1) $$\sigma_m(a):\quad t_1^2+\cdots+t_m^2 \leqq a^2$$

$(m = n - \nu)$, and k_0 the area of the "surface" C_ν. Professor Hotelling showed that this formula is exact in E_n and a similar formula prevails in S_n, *for* $\nu = 1$. I shall here treat the problem for higher dimensionalities ν. The result in E_n is a formula consisting of $1+[\frac{1}{2}\nu]$ terms, of the following type (§ 3):

(2) $$V(a)=\Omega_m\cdot\sum_e \frac{a^{m+e}}{(m+2)(m+4)\cdots(m+e)}k_e \quad (e \text{ even}, 0\leqq e\leqq \nu),$$

where k_e is a certain integral invariant of the surface C_ν determined by the intrinsic metric nature of C_ν only, and thus independent of its embedding in E_n. I shall express these invariants (§ 4) in terms of the Riemannian tensor of C_ν. An analogous result is obtained for S_n.

2. The fundamental formulas for the volume of tubes. If an n-dimensional manifold M_n consisting of points u and locally referred to

* Received October 14, 1938.

[1] See his paper "Tubes and spheres in n-spaces, and a class of statistical problems" which precedes this article in this Journal, pp. 440-460.

H. Weyl's Formula (1939)

UNIVERSITY
OF MIAMI

H. Weyl's Formula (1939)

UNIVERSITY OF MIAMI

$$
\begin{aligned}
\mathrm{vol}(\mathcal{T}(\Sigma,\rho)) = {} & V_l(B^l)\,\rho^l\ \ \mathrm{vol}(\Sigma) + V_l(B^l)\,\frac{\rho^{l+2}}{2(l+2)}\int_\Sigma R\,\eta_\Sigma \\
& + V_l(B^l)\,\frac{\rho^{l+4}}{8(l+2)(l+4)}\int_\Sigma \left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)\eta_\Sigma \\
& + O(\rho^{l+6})\,.
\end{aligned}
$$

H. Weyl's Formula (1939)

UNIVERSITY OF MIAMI

$$\mathrm{vol}(\mathcal{T}(\Sigma,\rho)) = V_l(B^l)\,\rho^l\ \mathrm{vol}(\Sigma) + V_l(B^l)\,\frac{\rho^{l+2}}{2(l+2)}\int_\Sigma R\,\eta_\Sigma$$
$$+ V_l(B^l)\,\frac{\rho^{l+4}}{8(l+2)(l+4)}\int_\Sigma \left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)\eta_\Sigma$$
$$+ O(\rho^{l+6})\,.$$

Exact formula with a finite number of terms
that only depend on the intrinsic geometry of the surface!

H. Weyl's Formula (1939)

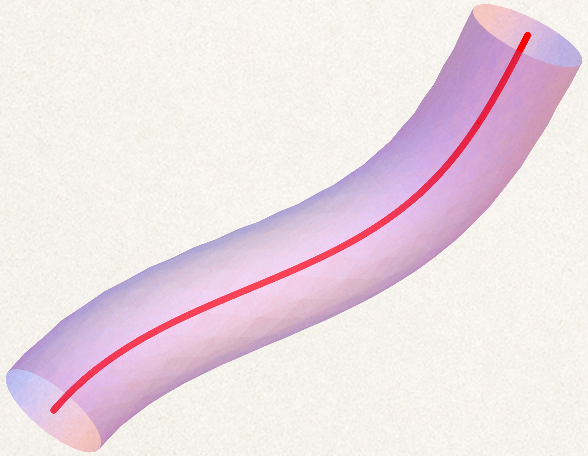
UNIVERSITY
OF MIAMI

$$\mathrm{vol}(\mathcal{T}(\Sigma,\rho)) = V_l(B^l)\,\rho^l\ \mathrm{vol}(\Sigma) + V_l(B^l)\,\frac{\rho^{l+2}}{2(l+2)}\int_\Sigma R\,\eta_\Sigma$$
$$+ V_l(B^l)\,\frac{\rho^{l+4}}{8(l+2)(l+4)}\int_\Sigma \left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)\eta_\Sigma$$
$$+ O(\rho^{l+6})\,.$$

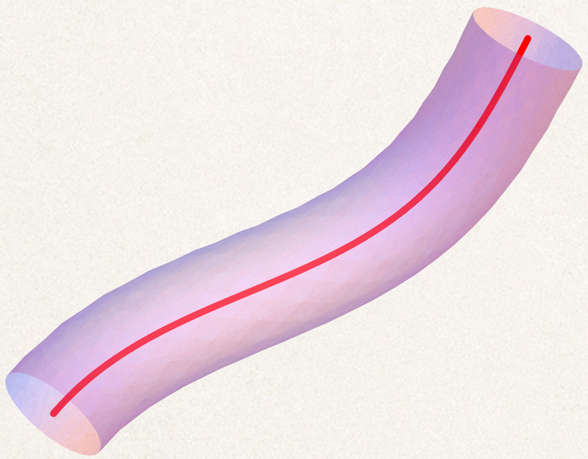
Exact formula with a finite number of terms
that only depend on the intrinsic geometry of the surface!

The terms are precisely those that appear in
Lovelock Theories of Gravity!

Thickened curve



Thickened curve



$$\text{vol}_3\left(\mathcal{T}(\Sigma,\rho)\right) = \pi r^2 \cdot \text{vol}_1(\Sigma)$$

Two dimensional surface

UNIVERSITY
OF MIAMI

Two dimensional surface

Two dimensional surface

UNIVERSITY OF MIAMI

$$\mathrm{vol}_3(\mathcal{T}(\Sigma,\rho)) = 2\rho\ \mathrm{vol}_2(\Sigma) + \frac{4\pi}{3}\,\rho^3\ \chi(\Sigma)$$

Two dimensional surface

UNIVERSITY
OF MIAMI

$$\mathrm{vol}_3(\mathcal{T}(\Sigma,\rho)) = 2\rho\ \mathrm{vol}_2(\Sigma) + \frac{4\pi}{3}\,\rho^3\,\chi(\Sigma)$$

top and bottom

Thickened S^2

UNIVERSITY
OF MIAMI

$$\mathrm{vol}_3\left(\mathcal{T}(S^2,\rho)\right) = \frac{4\pi}{3}(r+\rho)^3 - \frac{4\pi}{3}(r-\rho)^3$$
$$= 2\rho \cdot 4\pi r^2 + \frac{4\pi}{3}\rho^3 \cdot 2$$

Note that for a thickened torus the Euler characteristic term vanishes.

Thickened S^2

UNIVERSITY
OF MIAMI

$$\mathrm{vol}_3\left(\mathcal{T}(S^2,\rho)\right) = \frac{4\pi}{3}(r+\rho)^3 - \frac{4\pi}{3}(r-\rho)^3$$
$$= 2\rho \cdot 4\pi r^2 + \frac{4\pi}{3}\rho^3 \cdot 2$$

$$\mathrm{vol}_3(\mathcal{T}(\Sigma,\rho)) = 2\rho\ \mathrm{vol}_2(\Sigma) + \frac{4\pi}{3}\rho^3\ \chi(\Sigma)$$

Note that for a thickened torus the Euler characteristic term vanishes.

Lovelock Lagrangians

UNIVERSITY
OF MIAMI

$$I = \sum_{r=0}^{\lfloor q/2 \rfloor} \lambda_{2r}\, I_{2r} = \sum_{r=0}^{\lfloor q/2 \rfloor} \lambda_{2r} \int_{\Sigma} \mathcal{K}_{2r}\, \eta_{\Sigma}\ ,$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 3 MARCH 1971

The Einstein Tensor and Its Generalizations*

DAVID LOVELOCK
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada

(Received 27 August 1970)

The Einstein tensor G^{ij} is symmetric, divergence free, and a concomitant of the metric tensor g_{ab} together with its first two derivatives. In this paper all tensors of valency two with these properties are displayed explicitly. The number of independent tensors of this type depends crucially on the dimension of the space, and, in the four dimensional case, the only tensors with these properties are the metric and the Einstein tensors.

Lovelock Lagrangians

UNIVERSITY
OF MIAMI

The terms that appear to all orders in the radius in Weyl's tube volume formula are the ``dimensional continuations'' of the Euler densities. From the physics viewpoint this is astonishing. Gravitational theories defined by lagrangians containing those terms were discussed by Lovelock in the early 1970s who was interested constructing generalizations of the Einstein tensor. He required his tensors to be symmetric, rank two, divergence free and that they contained at most the first two derivatives of the metric (canonical formulation for gravity). The appearance of Lovelock lagrangians in string theory was first observed by Zwiebach (1985) who noted that compatibility of a ghost free theory with the presence of curvature squared terms in the gravitational lagrangian required a special combination that reduced to the Euler density in four dimensions. By studying the 3-graviton on shell vertex in string theory he verified that this curvature squared combination appears. *Zumino (1986) generalized Zwiebach's results and showed that gravitational theories containing higher powers of the curvature were ghost free if the additional terms in the lagrangian were ``dimensional continuations'' of Euler densities in the appropriate dimensionality, i.e., Lovelock type lagrangians.*

Lovelock Lagrangians

UNIVERSITY
OF MIAMI

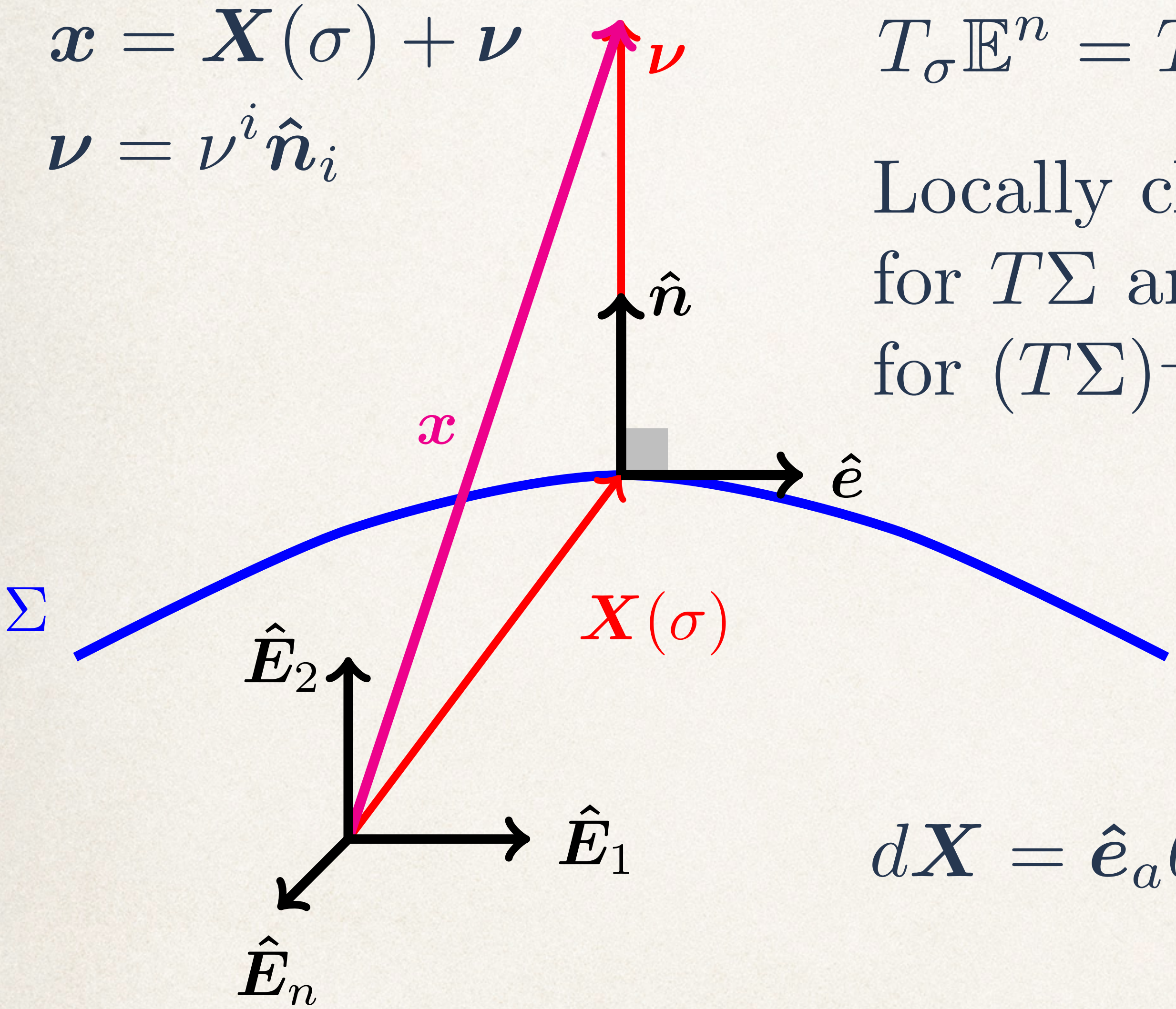
- I do not believe that any of us at Berkeley at the time were aware of Lovelock's results. They are not mentioned in any of the papers nor do I recollect any allusion to them at that time.

Weyl's Volume Element Formula

UNIVERSITY OF MIAMI

$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$\boldsymbol{\nu} = \nu^i \hat{\boldsymbol{n}}_i$$



$$T_\sigma \mathbb{E}^n = T_\sigma \Sigma \oplus (T_\sigma \Sigma)^\perp$$

Locally choose an orthonormal frame $(\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_q)$ for $T\Sigma$ and an orthonormal frame $(\hat{\boldsymbol{n}}_{q+1}, \ldots, \hat{\boldsymbol{n}}_n)$ for $(T\Sigma)^\perp$.

Let $(\sigma^1, \ldots, \sigma^q)$ be local coordinates on Σ, then $(\sigma^1, \ldots, \sigma^q, \nu^{q+1}, \ldots, \nu^n)$ are local coordinates for the tubular neighborhood.

$$d\boldsymbol{X} = \hat{\boldsymbol{e}}_a \theta^a \quad \longleftarrow$$

$$d\hat{\boldsymbol{e}}_a = \hat{\boldsymbol{e}}_b\, \omega_{ba} - \hat{\boldsymbol{n}}_j\, K_{abj}\, \theta^b$$

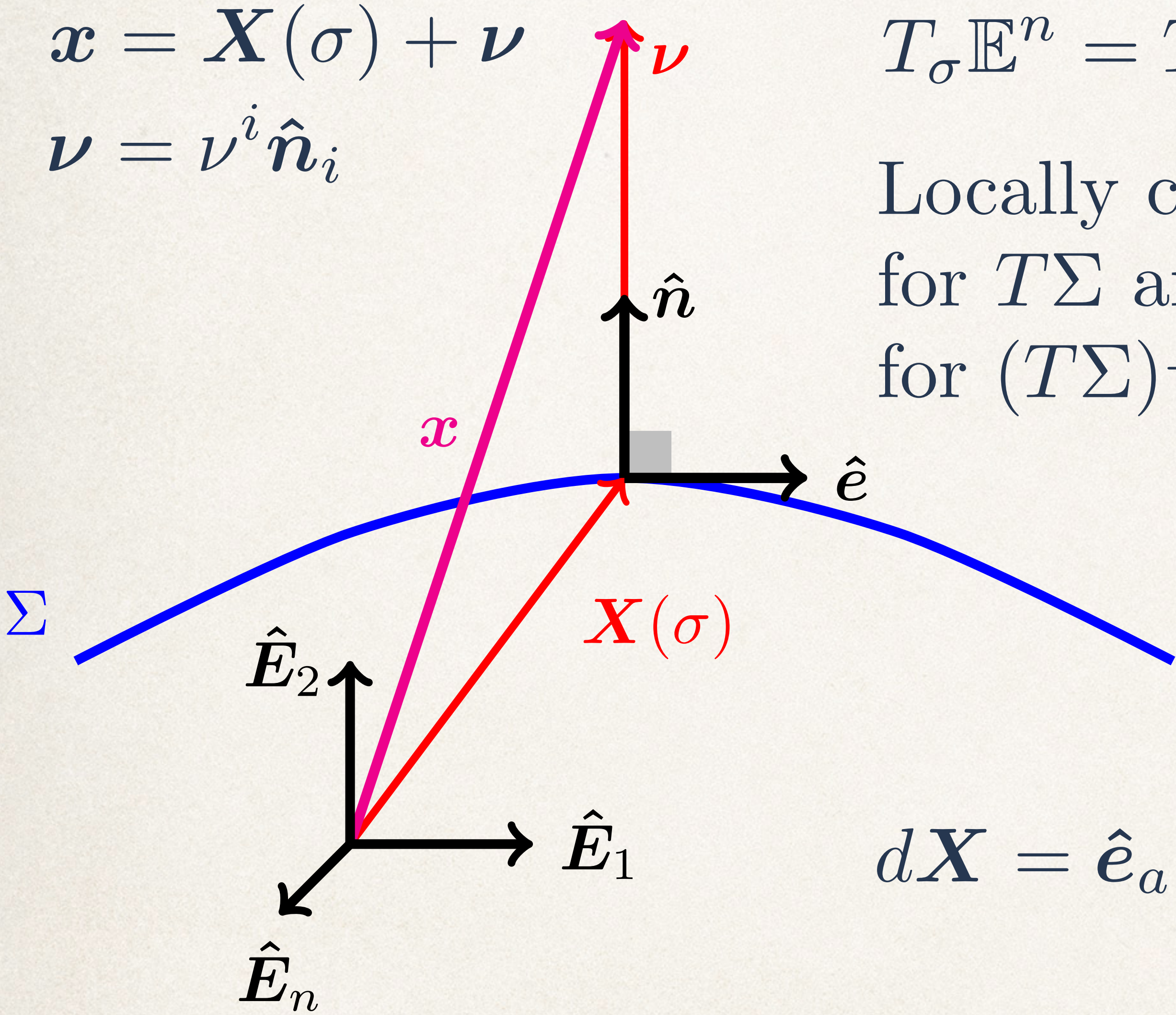
$$d\hat{\boldsymbol{n}}_i = \hat{\boldsymbol{n}}_j\, \omega_{ji} + \hat{\boldsymbol{e}}_a\, K_{abi}\, \theta^b \,,$$

Weyl's Volume Element Formula

UNIVERSITY OF MIAMI

$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$\boldsymbol{\nu} = \nu^i \hat{\boldsymbol{n}}_i$$



$$T_\sigma \mathbb{E}^n = T_\sigma \Sigma \oplus (T_\sigma \Sigma)^\perp$$

Locally choose an orthonormal frame $(\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_q)$ for $T\Sigma$ and an orthonormal frame $(\hat{\boldsymbol{n}}_{q+1}, \ldots, \hat{\boldsymbol{n}}_n)$ for $(T\Sigma)^\perp$.

Let $(\sigma^1, \ldots, \sigma^q)$ be local coordinates on Σ, then $(\sigma^1, \ldots, \sigma^q, \nu^{q+1}, \ldots, \nu^n)$ are local coordinates for the tubular neighborhood.

$$d\boldsymbol{X} = \hat{\boldsymbol{e}}_a \theta^a \longleftarrow \theta^a = M^a{}_b \, d\sigma^b$$

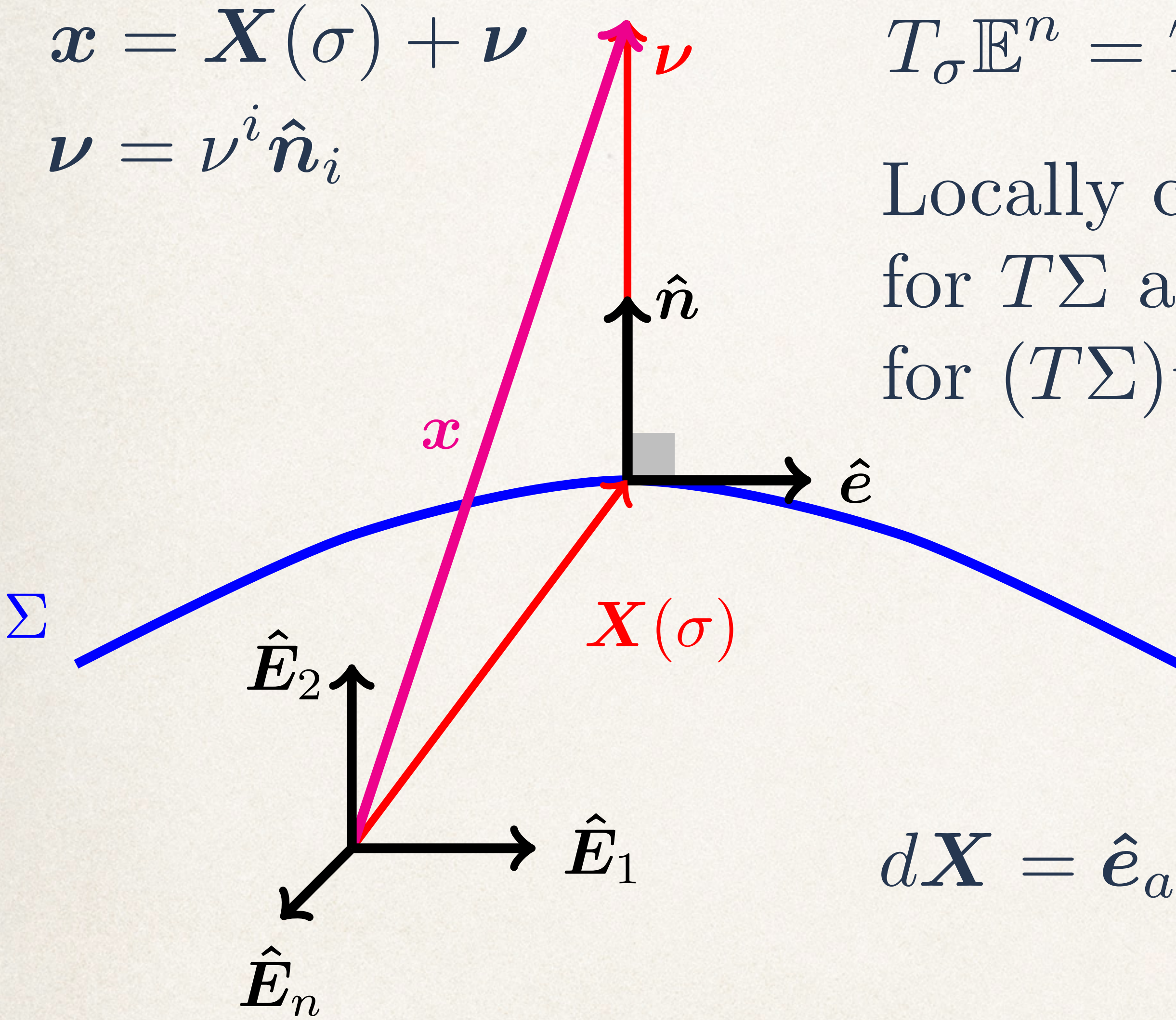
$$d\hat{\boldsymbol{e}}_a = \hat{\boldsymbol{e}}_b \, \omega_{ba} - \hat{\boldsymbol{n}}_j \, K_{abj} \, \theta^b$$

$$d\hat{\boldsymbol{n}}_i = \hat{\boldsymbol{n}}_j \, \omega_{ji} + \hat{\boldsymbol{e}}_a \, K_{abi} \, \theta^b \, ,$$

Weyl's Volume Element Formula

$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$\boldsymbol{\nu} = \nu^i \hat{\boldsymbol{n}}_i$$



$$T_\sigma \mathbb{E}^n = T_\sigma \Sigma \oplus (T_\sigma \Sigma)^\perp$$

Locally choose an orthonormal frame $(\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_q)$ for $T\Sigma$ and an orthonormal frame $(\hat{\boldsymbol{n}}_{q+1}, \ldots, \hat{\boldsymbol{n}}_n)$ for $(T\Sigma)^\perp$.

Let $(\sigma^1, \ldots, \sigma^q)$ be local coordinates on Σ, then $(\sigma^1, \ldots, \sigma^q, \nu^{q+1}, \ldots, \nu^n)$ are local coordinates for the tubular neighborhood.

$$d\boldsymbol{X} = \hat{\boldsymbol{e}}_a \theta^a$$

$$d\hat{\boldsymbol{e}}_a = \hat{\boldsymbol{e}}_b\, \omega_{ba} - \hat{\boldsymbol{n}}_j\, K_{abj}\, \theta^b$$

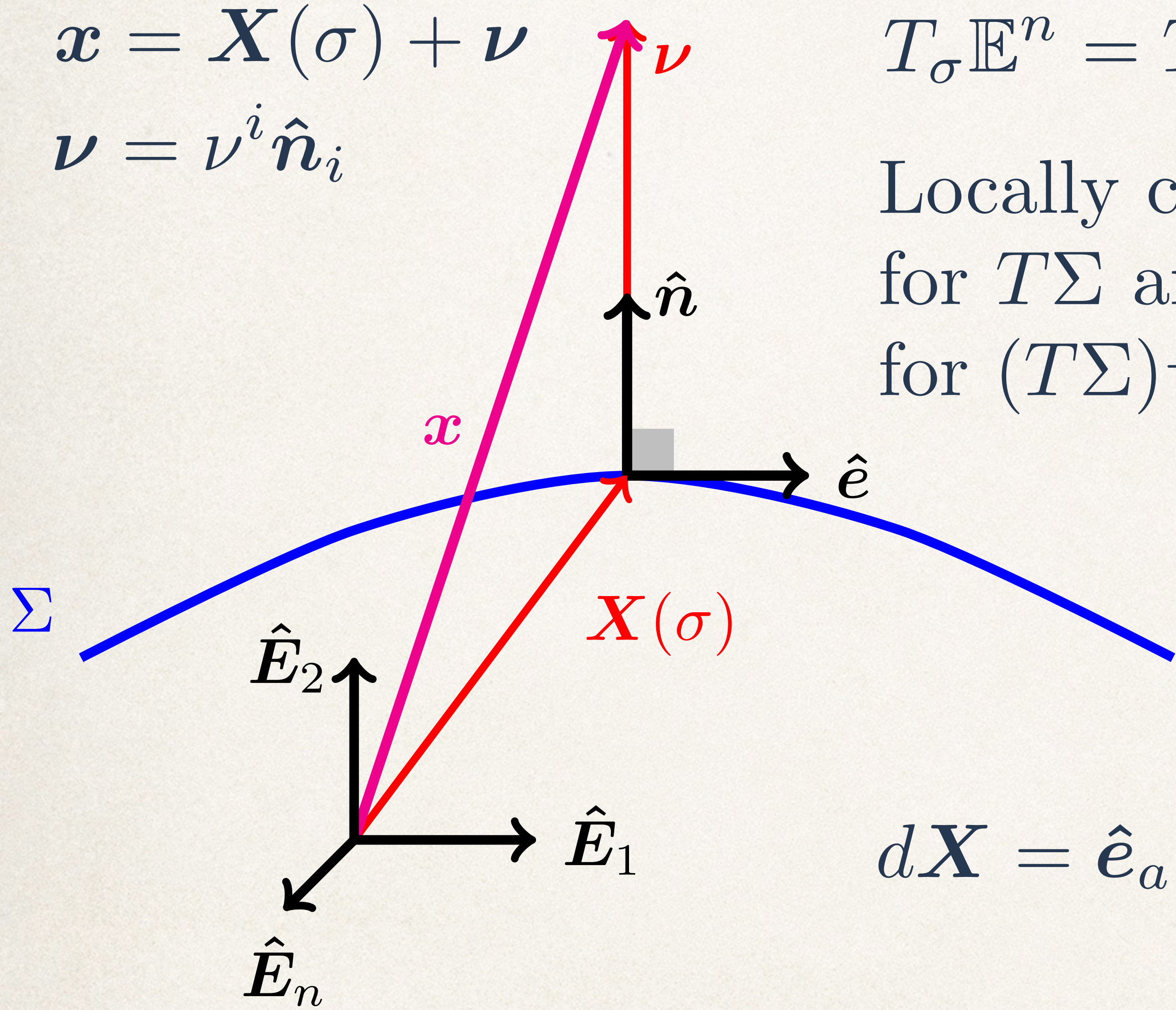
$$d\hat{\boldsymbol{n}}_i = \hat{\boldsymbol{n}}_j\, \omega_{ji} + \hat{\boldsymbol{e}}_a\, K_{abi}\, \theta^b\ ,$$

Weyl's Volume Element Formula

UNIVERSITY OF MIAMI

$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$\boldsymbol{\nu} = \nu^i \hat{\boldsymbol{n}}_i$$



$$T_\sigma \mathbb{E}^n = T_\sigma \Sigma \oplus (T_\sigma \Sigma)^\perp$$

Locally choose an orthonormal frame $(\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_q)$ for $T\Sigma$ and an orthonormal frame $(\hat{\boldsymbol{n}}_{q+1}, \ldots, \hat{\boldsymbol{n}}_n)$ for $(T\Sigma)^\perp$.

Let $(\sigma^1, \ldots, \sigma^q)$ be local coordinates on Σ, then $(\sigma^1, \ldots, \sigma^q, \nu^{q+1}, \ldots, \nu^n)$ are local coordinates for the tubular neighborhood.

$d\boldsymbol{X} = \hat{\boldsymbol{e}}_a \theta^a$ ← tangential to surface

$$d\hat{\boldsymbol{e}}_a = \hat{\boldsymbol{e}}_b \,\omega_{ba} - \hat{\boldsymbol{n}}_j \, K_{abj}\, \theta^b$$

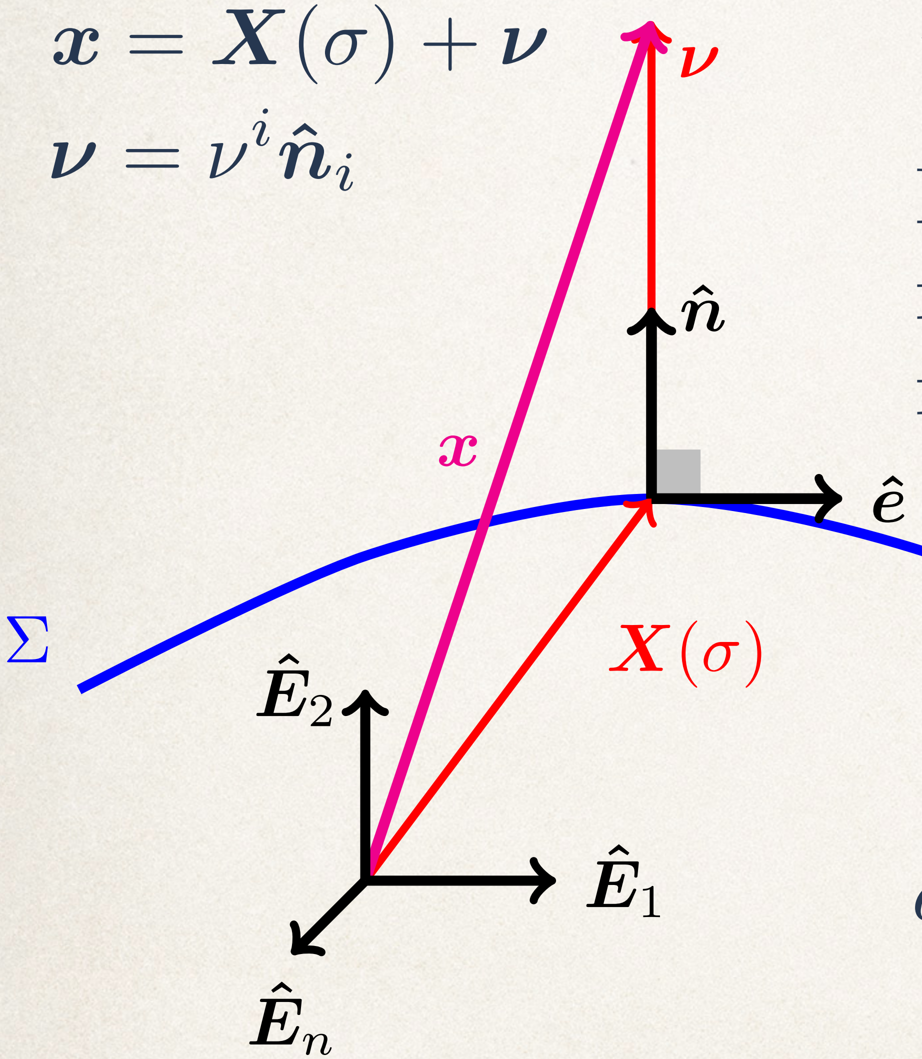
$$d\hat{\boldsymbol{n}}_i = \hat{\boldsymbol{n}}_j \,\omega_{ji} + \hat{\boldsymbol{e}}_a \, K_{abi}\, \theta^b \;,$$

Weyl's Volume Element Formula

UNIVERSITY OF MIAMI

$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$\boldsymbol{\nu} = \nu^i \hat{\boldsymbol{n}}_i$$



$$T_\sigma \mathbb{E}^n = T_\sigma \Sigma \oplus (T_\sigma \Sigma)^\perp$$

Locally choose an orthonormal frame $(\hat{\boldsymbol{e}}_1, \ldots, \hat{\boldsymbol{e}}_q)$ for $T\Sigma$ and an orthonormal frame $(\hat{\boldsymbol{n}}_{q+1}, \ldots, \hat{\boldsymbol{n}}_n)$ for $(T\Sigma)^\perp$.

Let $(\sigma^1, \ldots, \sigma^q)$ be local coordinates on Σ, then $(\sigma^1, \ldots, \sigma^q, \nu^{q+1}, \ldots, \nu^n)$ are local coordinates for the tubular neighborhood.

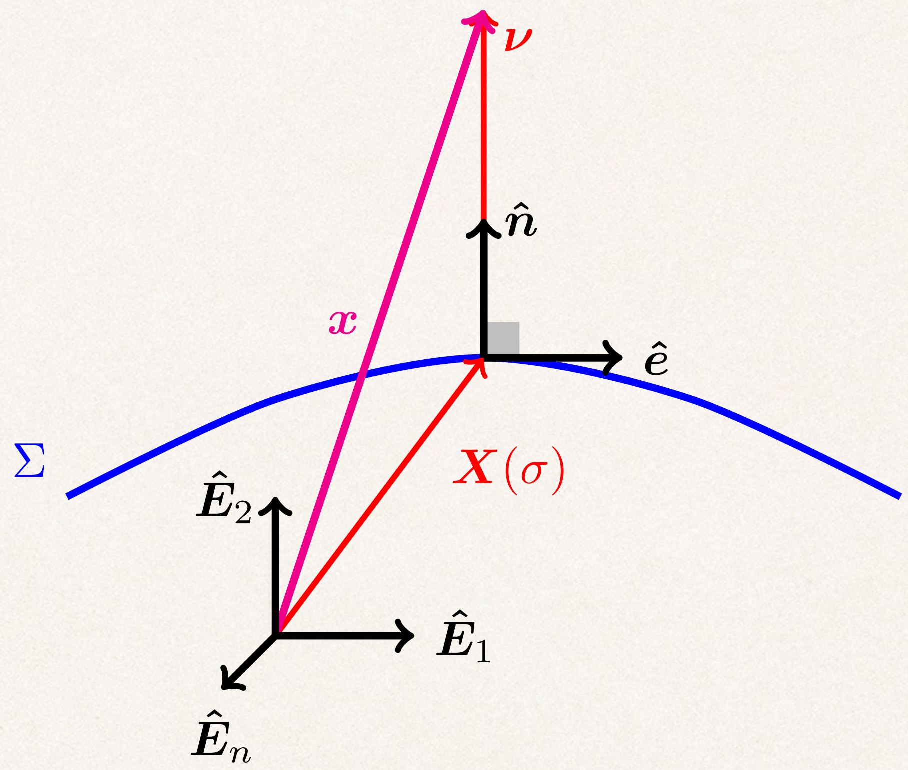
$$d\boldsymbol{X} = \hat{\boldsymbol{e}}_a \theta^a \quad \longleftarrow \quad \text{tangential to surface}$$

$$d\hat{\boldsymbol{e}}_a = \hat{\boldsymbol{e}}_b\, \omega_{ba} - \hat{\boldsymbol{n}}_j\, K_{abj}\, \theta^b$$

$$d\hat{\boldsymbol{n}}_i = \hat{\boldsymbol{n}}_j\, \omega_{ji} + \hat{\boldsymbol{e}}_a\, K_{abi}\, \theta^b \;,$$

$(\hat{\boldsymbol{e}}, \hat{\boldsymbol{n}})$ are defined on Σ and thus their differentials are only defined on Σ. This means that the objects $\theta^a, \omega_{ab}, \omega_{ij}, K^i{}_{ab}$ only depend on σ and $d\sigma$.

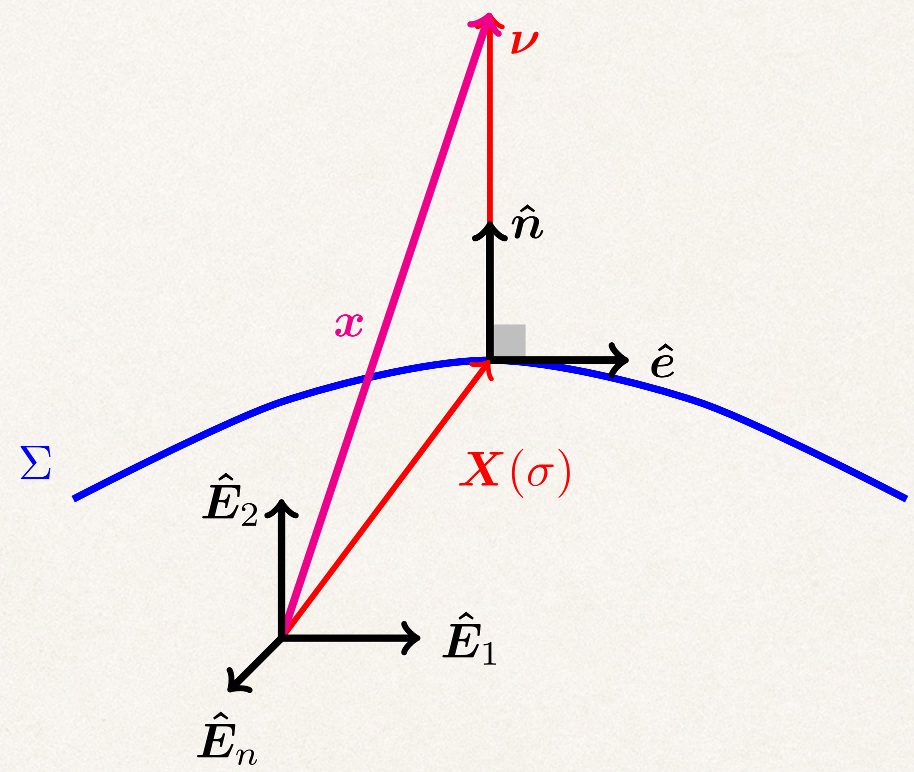
Volume Element Formula (continued)



$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

Volume Element Formula (continued)

UNIVERSITY OF MIAMI

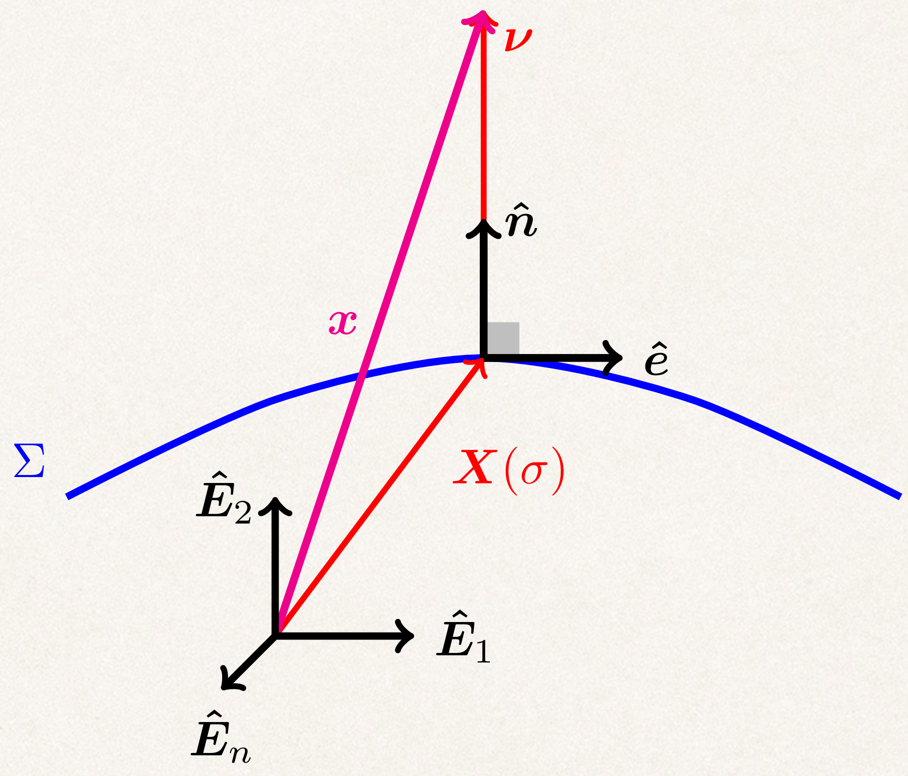


$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$d\boldsymbol{x} = \hat{\boldsymbol{E}}_\mu \, dx^\mu = \hat{\boldsymbol{e}}_a \left(\delta_{ab} + \nu^i K_{abi}\right) \theta^b + \hat{\boldsymbol{n}}_i \, D\nu^i \,, \qquad D\nu^i = d\nu^i + \omega_{ij}\nu^j$$

Volume Element Formula (continued)

UNIVERSITY
OF MIAMI

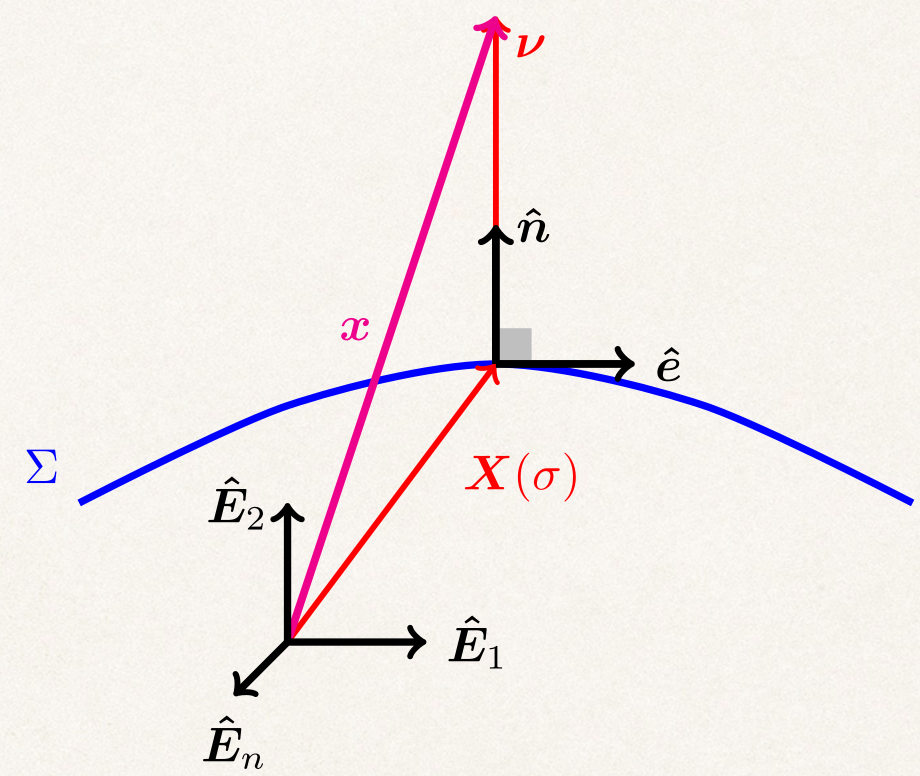


$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$d\boldsymbol{x} = \hat{\boldsymbol{E}}_\mu \, dx^\mu = \hat{\boldsymbol{e}}_a \left(\delta_{ab} + \nu^i K_{abi}\right) \theta^b + \hat{\boldsymbol{n}}_i \, D\nu^i \,, \qquad D\nu^i = d\nu^i + \omega_{ij}\nu^j$$

Volume Element Formula (continued)

UNIVERSITY OF MIAMI

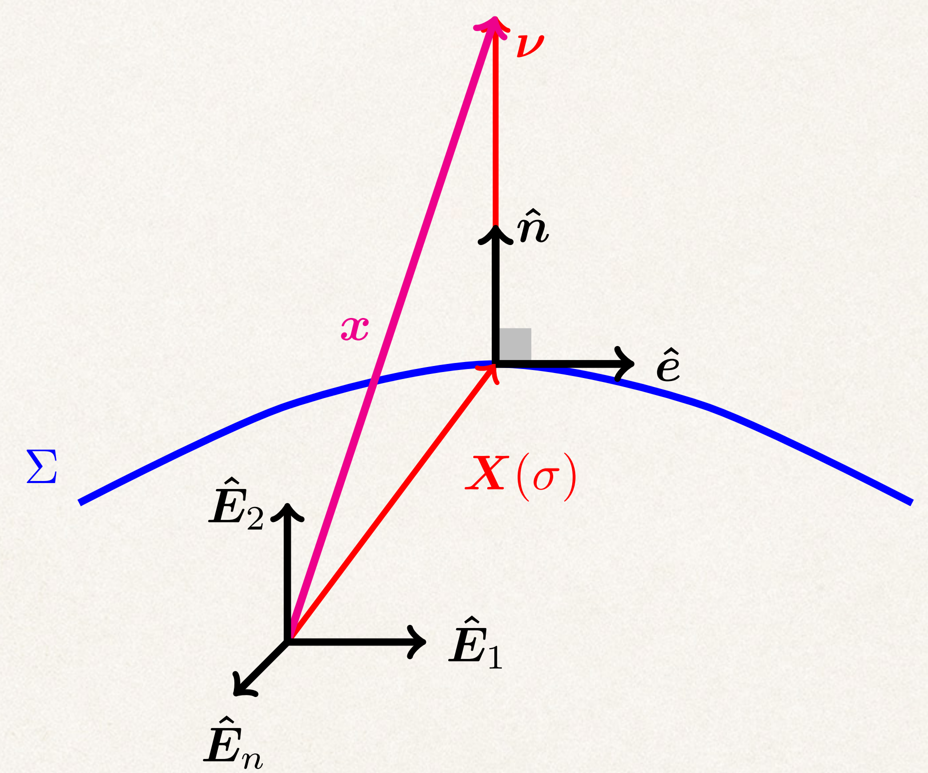


$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$d\boldsymbol{x} = \hat{\boldsymbol{E}}_\mu \, dx^\mu = \hat{\boldsymbol{e}}_a \left(\delta_{ab} + \nu^i K_{abi}\right) \theta^b + \hat{\boldsymbol{n}}_i \, D\nu^i, \qquad D\nu^i = d\nu^i + \omega_{ij}\nu^j$$

Volume Element Formula (continued)

UNIVERSITY OF MIAMI



$$\boldsymbol{x} = \boldsymbol{X}(\sigma) + \boldsymbol{\nu}$$

$$d\boldsymbol{x} = \hat{\boldsymbol{E}}_\mu \, dx^\mu = \hat{\boldsymbol{e}}_a \left(\delta_{ab} + \nu^i K_{abi}\right) \theta^b + \hat{\boldsymbol{n}}_i \, D\nu^i, \qquad D\nu^i = d\nu^i + \omega_{ij}\nu^j$$

An orthonormal coframe at $\boldsymbol{x}$ is given by $\left((\delta_{ab} + \nu^i K_{abi})\theta^b, D\nu^i\right)$.

$$ds^2 = d\boldsymbol{x} \cdot d\boldsymbol{x}$$

Weyl's volume element

UNIVERSITY OF MIAMI

$$
\begin{aligned}
d^n x &= \det(I + \boldsymbol{\nu}\cdot\boldsymbol{K})\,\theta^1\wedge\theta^2\wedge\cdots\wedge\theta^q\wedge D\nu^{q+1}\wedge\cdots\wedge D\nu^n\,,\\
&= \det(I + \boldsymbol{\nu}\cdot\boldsymbol{K})\,\theta^1\wedge\theta^2\wedge\cdots\wedge\theta^q\wedge d\nu^{q+1}\wedge d\nu^{q+2}\wedge\cdots\wedge d\nu^n\,,\\
&= \det(I + \boldsymbol{\nu}\cdot\boldsymbol{K})\,\eta_\Sigma\wedge d\nu^{q+1}\wedge d\nu^{q+2}\wedge\cdots\wedge d\nu^n\,.
\end{aligned}
$$

By linearizing the determinant: $d\det(I+\boldsymbol{\nu}\cdot\boldsymbol{K})\big|_{\boldsymbol{\nu}=0} = d\boldsymbol{\nu}\cdot\mathrm{Tr}(\boldsymbol{K})$ you see that an extremal surface has vanishing mean curvature vector $\delta^{ab}K^i{}_{ab}\,\hat{\boldsymbol{n}}_i$. The mean curvature vector points in the direction of fastest increase in local volume of the surface.

Weyl's volume element

UNIVERSITY OF MIAMI

$$
\begin{aligned}
d^n x &= \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^q \wedge D\nu^{q+1} \wedge \cdots \wedge D\nu^n\,, \\
&= \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^q \wedge d\nu^{q+1} \wedge d\nu^{q+2} \wedge \cdots \wedge d\nu^n\,, \\
&= \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\, \eta_\Sigma \wedge d\nu^{q+1} \wedge d\nu^{q+2} \wedge \cdots \wedge d\nu^n\,.
\end{aligned}
$$

By linearizing the determinant: $d \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\big|_{\boldsymbol{\nu}=0} = d\boldsymbol{\nu} \cdot \mathrm{Tr}(\boldsymbol{K})$ you see that an extremal surface has vanishing mean curvature vector $\delta^{ab} K^i{}_{ab}\, \hat{\boldsymbol{n}}_i$. The mean curvature vector points in the direction of fastest increase in local volume of the surface.

Weyl's volume element

UNIVERSITY OF MIAMI

$$
\begin{aligned}
d^n x &= \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^q \wedge D\nu^{q+1} \wedge \cdots \wedge D\nu^n\,, \\
&= \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^q \wedge d\nu^{q+1} \wedge d\nu^{q+2} \wedge \cdots \wedge d\nu^n\,, \\
&= \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\, \eta_\Sigma \wedge d\nu^{q+1} \wedge d\nu^{q+2} \wedge \cdots \wedge d\nu^n\,.
\end{aligned}
$$

By linearizing the determinant: $d\det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\big|_{\boldsymbol{\nu}=0} = d\boldsymbol{\nu} \cdot \mathrm{Tr}(\boldsymbol{K})$ you see that an extremal surface has vanishing mean curvature vector $\delta^{ab} K^i{}_{ab}\, \hat{\boldsymbol{n}}_i$. The mean curvature vector points in the direction of fastest increase in local volume of the surface.

Energy Tubes

UNIVERSITY
OF MIAMI

Explain what is an energy tube and energy formula

$$(\Delta E)_{\text{eff}}(X) = \int_{\mathbb{E}^n} u(\boldsymbol{x}, X)\, d^n x$$

$$E = \int_\Sigma \eta_\Sigma(\sigma) \left(\int_{(T_\sigma \Sigma)^\perp} u(\sigma, \boldsymbol{\nu})\ \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K})\, d^l \nu \right).$$

Spherically Symmetric Energy Density

UNIVERSITY OF MIAMI

$$E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_{\Sigma} \mathcal{K}_{2r}(\Sigma)\, \eta_{\Sigma} \int_0^{\infty} d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma, \nu)$$

$$C_0 = 1, \quad C_{2r} = \prod_{k=0}^{r-1} \frac{1}{l+2k}$$

$$\mathcal{K}_0(\Sigma) = 1\,,$$

$$\mathcal{K}_1(\Sigma) = \frac{1}{2}\, R\,,$$

$$\mathcal{K}_2(\Sigma) = \frac{1}{8}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right).$$

Spherically Symmetric Energy Density

UNIVERSITY OF MIAMI

$$E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_\Sigma \mathcal{K}_{2r}(\Sigma)\, \eta_\Sigma \int_0^\infty d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma,\nu)$$

Where did the ${K_{ab}}^i$ go?

$$C_0 = 1, \quad C_{2r} = \prod_{k=0}^{r-1} \frac{1}{l+2k}$$

$$\mathcal{K}_0(\Sigma) = 1\,,$$

$$\mathcal{K}_1(\Sigma) = \frac{1}{2}\, R\,,$$

$$\mathcal{K}_2(\Sigma) = \frac{1}{8}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right).$$

Spherically Symmetric Energy Density

UNIVERSITY OF MIAMI

$$E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_\Sigma \mathcal{K}_{2r}(\Sigma)\, \eta_\Sigma \int_0^\infty d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma,\nu)$$

Where did the ${K_{ab}}^i$ go?

$$C_0 = 1, \quad C_{2r} = \prod_{k=0}^{r-1} \frac{1}{l+2k}$$

$$\mathcal{K}_0(\Sigma) = 1\,,$$

$$\mathcal{K}_1(\Sigma) = \frac{1}{2}\, R\,,$$

$$\mathcal{K}_2(\Sigma) = \frac{1}{8}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right).$$

Spherically Symmetric Energy Density

UNIVERSITY OF MIAMI

radial moments

$$E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_\Sigma \mathcal{K}_{2r}(\Sigma)\, \eta_\Sigma \int_0^\infty d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma, \nu)$$

Where did the ${K_{ab}}^i$ go?

$$C_0 = 1, \quad C_{2r} = \prod_{k=0}^{r-1} \frac{1}{l+2k}$$

$$\mathcal{K}_0(\Sigma) = 1\,,$$

$$\mathcal{K}_1(\Sigma) = \frac{1}{2}\, R\,,$$

$$\mathcal{K}_2(\Sigma) = \frac{1}{8}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right).$$

Convert extrinsic geometry to intrinsic geometry by using the Gauss equation:

$$R_{abcd} = K_{ac}{}^{i} K_{bd}{}^{i} - K_{ad}{}^{i} K_{bc}{}^{i}$$

condition for an isometric embedding

Induced Scalar Fields

$$E^{(0)} = \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_{\Sigma} \mu_{2r}^{(0)}(\sigma)\, \mathcal{K}_{2r}(\Sigma)\, \eta_{\Sigma} \,,$$

$$\mu_{2r}^{(0)}(\sigma) = \int_{(T_\sigma \Sigma)^\perp} \|\boldsymbol{\nu}\|^{2r}\, u^{(0)}(\sigma, \|\boldsymbol{\nu}\|)\, d^l \nu = V_{l-1}(S^{l-1}) \int_0^\infty d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma, \nu) \,.$$

Induced Scalar Fields

UNIVERSITY OF MIAMI

$$E^{(0)} = \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_\Sigma \mu_{2r}^{(0)}(\sigma)\, \mathcal{K}_{2r}(\Sigma)\, \eta_\Sigma \,,$$

Induced Scalar Fields
(live on Σ)

$$\mu_{2r}^{(0)}(\sigma) = \int_{(T_\sigma\Sigma)^\perp} \|\boldsymbol{\nu}\|^{2r}\, u^{(0)}(\sigma, \|\boldsymbol{\nu}\|)\, d^l\nu = V_{l-1}(S^{l-1}) \int_0^\infty d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma, \nu) \,.$$

Induced Scalar Fields

UNIVERSITY OF MIAMI

$$E^{(0)} = \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_{\Sigma} \mu_{2r}^{(0)}(\sigma)\, \mathcal{K}_{2r}(\Sigma)\, \eta_{\Sigma}\ ,$$

$$\mu_{2r}^{(0)}(\sigma) = \int_{(T_\sigma \Sigma)^\perp} \|\boldsymbol{\nu}\|^{2r}\, u^{(0)}(\sigma, \|\boldsymbol{\nu}\|)\, d^l \nu = V_{l-1}(S^{l-1}) \int_0^\infty d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma, \nu)\ .$$

Induced Scalar Fields

UNIVERSITY OF MIAMI

$$E^{(0)} = \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_\Sigma \mu_{2r}^{(0)}(\sigma)\, \mathcal{K}_{2r}(\Sigma)\, \eta_\Sigma\,,$$

Effective Lovelock Action

$$\mu_{2r}^{(0)}(\sigma) = \int_{(T_\sigma\Sigma)^\perp} \|\boldsymbol{\nu}\|^{2r}\, u^{(0)}(\sigma, \|\boldsymbol{\nu}\|)\, d^l\nu = V_{l-1}(S^{l-1}) \int_0^\infty d\nu\, \nu^{2r+l-1}\, u^{(0)}(\sigma, \nu)\,.$$

Weyl's Volume of a Tube

UNIVERSITY OF MIAMI

For the volume of a tube you have: $u(\sigma, \boldsymbol{\nu}) = \begin{cases} 1 & \text{if } \|\boldsymbol{\nu}\| < \rho, \\ 0 & \text{if } \|\boldsymbol{\nu}\| > \rho, \end{cases}$

$$\mathrm{vol}_n(\mathcal{T}(\Sigma, \rho)) = V_l(B^l)\rho^l \ \mathrm{vol}_q(\Sigma) + V_l(B^l) \sum_{r=1}^{\lfloor q/2 \rfloor} \frac{\rho^{l+2r}}{\prod_{k=1}^{r}(l+2k)} \int_\Sigma \mathcal{K}_{2r}(\Sigma)\, \eta_\Sigma \ .$$

This formula is exact!

Multilinear Algebra

UNIVERSITY
OF MIAMI

$$\delta^{j_1 j_2 \cdots j_m}_{i_1 i_2 \cdots i_m} = \det \begin{pmatrix} \delta^{j_1}{}_{i_1} & \delta^{j_1}{}_{i_2} & \cdots & \delta^{j_1}{}_{i_m} \\ \delta^{j_2}{}_{i_1} & \delta^{j_2}{}_{i_2} & \cdots & \delta^{j_2}{}_{i_m} \\ \cdots & \cdots & \cdots & \cdots \\ \delta^{j_m}{}_{i_1} & \delta^{j_m}{}_{i_2} & \cdots & \delta^{j_m}{}_{i_m} \end{pmatrix}$$

$$= \frac{1}{(n-m)!}\, \epsilon_{i_1 i_2 \cdots i_m k_{m+1} k_{m+2} \cdots k_n} \epsilon^{j_1 j_2 \cdots j_m k_{m+1} k_{m+2} \cdots k_n} \, .$$

$$\det(I + tS) = \sum_{m=0}^{n} \frac{t^m}{m!}\, \delta^{j_1 \cdots j_m}_{i_1 \cdots i_m}\, S^{i_1}{}_{j_1} S^{i_2}{}_{j_2} \cdots S^{i_m}{}_{j_m} \text{ if } S \text{ is symmetric}$$

Multilinear Algebra

UNIVERSITY OF MIAMI

$$\eta_{i_1 i_2 \cdots i_m} = \star\left(\theta^{i_1} \wedge \cdots \wedge \theta^{i_m}\right) = \frac{1}{(n-m)!}\,\epsilon_{i_1 i_2 \cdots i_m j_{m+1} \cdots j_n}\,\theta^{j_{m+1}} \wedge \cdots \wedge \theta^{j_n}\ .$$

$$\Omega_{ab} = \frac{1}{2}\,R_{abcd}\,\theta^c \wedge \theta^d$$

$$\mathcal{K}_{2r}(\Sigma)\,\eta_\Sigma = \frac{1}{4^r\,r!}\,\delta^{b_1\cdots b_{2r}}_{a_1\cdots a_{2r}}\,R_{a_1a_2b_1b_2}\cdots R_{a_{2r-1}a_{2r}b_{2r-1}b_{2r}}\,\eta_\Sigma\,,$$
$$= \frac{1}{2^r\,r!}\,\eta^{a_1a_2\cdots a_{2r-1}a_{2r}}\wedge\Omega_{a_1a_2}\wedge\cdots\wedge\Omega_{a_{2r-1}a_{2r}}\,.$$

If $\dim\Sigma = q = 2r$ is even then the differential form above of maximal degree is

$$\mathcal{K}_q(\Sigma)\,\eta_\Sigma = \frac{1}{2^r\,r!}\,\epsilon^{a_1a_2\cdots a_{2r-1}a_{2r}}\,\Omega_{a_1a_2}\wedge\cdots\wedge\Omega_{a_{2r-1}a_{2r}}\,,$$

where $\mathcal{K}_q(\Sigma) = \mathrm{pf}(\Omega)$ is the pfaffian of the "antisymmetric matrix valued 2-form" Ω_{ab}. The Euler characteristic is $\chi(\Sigma) = (1/2\pi)^{q/2}\int_\Sigma \mathrm{pf}(\Omega)\,\eta_\Sigma$ by the generalized Gauss-Bonnet Theorem.

$$\int_\Sigma \eta_\Sigma \int_{(T_\sigma \Sigma)^\perp} u(\sigma, \boldsymbol{\nu}) \; \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K}) \; d^l \nu$$

$$= \int_\Sigma \eta_\Sigma \sum_{r=0}^{q} \int_{\nu=0}^{\infty} \int_{S^{l-1}} u^{(0)}(\sigma, \nu) \; \frac{\nu^r}{r!} \; \delta^{b_1 \cdots b_r}_{a_1 \cdots a_r} \; K^{a_1}{}_{b_1 i_1} K^{a_2}{}_{b_2 i_2} \cdots K^{a_r}{}_{b_r i_r}$$

$$\times \hat{\nu}^{i_1} \hat{\nu}^{i_2} \cdots \hat{\nu}^{i_r} \cdot \nu^{l-1} \; d\nu \; d\,\mathrm{vol}_{S^{l-1}} \; ,$$

$$= \int_\Sigma \eta_\Sigma \sum_{r=0}^{\lfloor q/2 \rfloor} \frac{1}{(2r)!} \int_0^\infty d\nu \; \nu^{2r+l-1} u^{(0)}(\sigma, \nu)$$

$$\times V_{l-1}(S^{l-1}) \; (2r-1)!! \; C_{2r} \; \delta^{b_1 \cdots b_{2r}}_{a_1 \cdots a_{2r}} \; K^{a_1}{}_{b_1 i_1} K^{a_2}{}_{b_2 i_1} \cdots K^{a_{2r-1}}{}_{b_{2r-1} i_r} K^{a_{2r}}{}_{b_{2r} i_r} \; .$$

$$R_{abcd} = K_{ac}{}^i K_{bd}{}^i - K_{ad}{}^i K_{bc}{}^i$$

$$E^{(0)} = V_{l-1}(S^{l-1}) \sum_{r=0}^{\lfloor q/2 \rfloor} C_{2r} \int_\Sigma \mathcal{K}_{2r}(\Sigma) \; \eta_\Sigma \int_0^\infty d\nu \; \nu^{2r+l-1} \; u^{(0)}(\sigma, \nu)$$

General Formula

UNIVERSITY OF MIAMI

$$E = \int_\Sigma \eta_\Sigma(\sigma) \left(\int_{(T_\sigma \Sigma)^\perp} u(\sigma, \boldsymbol{\nu}) \; \det(I + \boldsymbol{\nu} \cdot \boldsymbol{K}) \; d^l \nu \right) .$$

Spherical multipole expansion for SO(l)

$$u(\sigma, \boldsymbol{\nu}) = \sum_{j=0}^{\infty} \sum_{M=1}^{\dim W^j} u_M^{(j)}(\sigma, \|\boldsymbol{\nu}\|) \, {Y^j}_M(\hat{\boldsymbol{\nu}}) \, ,$$

Faux Cartesian Spherical Harmonics $\mathcal{Y}^{j}_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$

UNIVERSITY OF MIAMI

The faux cartesian spherical harmonics are not a basis but an over complete set for the irreducible representation.

$$\mathcal{Y}^0(\hat{\boldsymbol{\nu}}) = 1 \,,$$

$$\mathcal{Y}^1_i(\hat{\boldsymbol{\nu}}) = \hat{\nu}^i \,,$$

$$\mathcal{Y}^2_{ii'}(\hat{\boldsymbol{\nu}}) = \hat{\nu}^i \hat{\nu}^{i'} - \frac{1}{l}\, \delta^{ii'} \,,$$

$$\mathcal{Y}^3_{i_1 i_2 i_3}(\hat{\boldsymbol{\nu}}) = \hat{\nu}^{i_1} \hat{\nu}^{i_2} \hat{\nu}^{i_3} - \frac{1}{l+2} \left[\delta^{i_1 i_2} \hat{\nu}^{i_3} + \delta^{i_2 i_3} \hat{\nu}^{i_1} + \delta^{i_3 i_1} \hat{\nu}^{i_2} \right] \,.$$

Faux Cartesian Spherical Harmonics $\mathcal{Y}^{j}_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$

UNIVERSITY
OF MIAMI

The faux cartesian spherical harmonics are not a basis but an over complete set for the irreducible representation.

$$\mathcal{Y}^2_{11} = -\mathcal{Y}^2_{22} - \mathcal{Y}^2_{33}$$

$$\mathcal{Y}^0(\hat{\boldsymbol{\nu}}) = 1\,,$$

$$\mathcal{Y}^1_i(\hat{\boldsymbol{\nu}}) = \hat{\nu}^i\,,$$

$$\mathcal{Y}^2_{ii'}(\hat{\boldsymbol{\nu}}) = \hat{\nu}^i \hat{\nu}^{i'} - \frac{1}{l}\,\delta^{ii'}\,,$$

$$\mathcal{Y}^3_{i_1 i_2 i_3}(\hat{\boldsymbol{\nu}}) = \hat{\nu}^{i_1}\hat{\nu}^{i_2}\hat{\nu}^{i_3} - \frac{1}{l+2}\left[\delta^{i_1 i_2}\hat{\nu}^{i_3} + \delta^{i_2 i_3}\hat{\nu}^{i_1} + \delta^{i_3 i_1}\hat{\nu}^{i_2}\right]\,.$$

For $l > 1$ are uniquely specified by

1. $\mathcal{Y}^{j}_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$ is totally symmetric under any permutation of $i_1, i_2, \ldots, i_j$.

2. $\mathcal{Y}^{j}_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$ is traceless with respect to contraction on any pair of indices. Because the harmonic is totally symmetric this reduces to $\mathcal{Y}^{j}_{i i i_3 \cdots i_j}(\hat{\boldsymbol{\nu}}) = 0$.

3. The parity of $\mathcal{Y}^{j}$ is $(-1)^j$.

4. $\mathcal{Y}^{j}_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$ is an inhomogeneous polynomial of degree j in the $\hat{\nu}^i$ with normalization determined by

$$\mathcal{Y}^{j}_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}}) = \hat{\nu}^{i_1} \hat{\nu}^{i_2} \cdots \hat{\nu}^{i_j} + (\text{polynomial of degree } j - 2)\,.$$

For $l > 1$ are uniquely specified by

1. $\mathcal{Y}^j_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$ is totally symmetric under any permutation of $i_1, i_2, \ldots, i_j$.

2. $\mathcal{Y}^j_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$ is traceless with respect to contraction on any pair of indices. Because the harmonic is totally symmetric this reduces to $\mathcal{Y}^j_{i i i_3 \cdots i_j}(\hat{\boldsymbol{\nu}}) = 0$. (symmetric traceless)

3. The parity of $\mathcal{Y}^j$ is $(-1)^j$.

4. $\mathcal{Y}^j_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}})$ is an inhomogeneous polynomial of degree j in the $\hat{\nu}^i$ with normalization determined by

$$\mathcal{Y}^j_{i_1 i_2 \cdots i_j}(\hat{\boldsymbol{\nu}}) = \hat{\nu}^{i_1} \hat{\nu}^{i_2} \cdots \hat{\nu}^{i_j} + (\text{polynomial of degree } j-2)\,.$$

Our Multipole Expansion

UNIVERSITY OF MIAMI

$$u(\sigma, \boldsymbol{\nu}) = \sum_{j=0}^{\infty} \sum_{i_1,\ldots,i_j} u^{(j)}_{i_1\cdots i_j}(\sigma, \|\boldsymbol{\nu}\|)\, \mathcal{Y}^j_{i_1\cdots u_j}(\hat{\boldsymbol{\nu}})\,.$$

$u^{(j)}_{i_1\cdots i_j}$ is totally symmetric and traceless in the indices $i_1 \cdots i_j$ and is the 2^j-pole

$$\mu^{(j)}_{k_1\cdots k_j, 2s+j}(\sigma) = \int_{(T_\sigma\Sigma)^\perp} \|\boldsymbol{\nu}\|^{2s+j}\, u^{(j)}_{k_1\cdots k_j}(\sigma, \|\boldsymbol{\nu}\|)\, d^l\nu\,,$$

$$= V_{l-1}(S^{l-1}) \int_0^\infty d\nu\, \nu^{2s+j+l-1}\, u^{(j)}_{k_1\cdots k_j}(\sigma, \nu)\,.$$

radial moments

General Formula

UNIVERSITY OF MIAMI

$$E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s\ s!}$$

$$\times \int_\Sigma \mu^{(j)}_{k_1\cdots k_j, 2s+j}(\sigma)\ {\kappa_{b_1}}^{k_1} \wedge \cdots \wedge {\kappa_{b_j}}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2s-1} a_{2s}} \wedge \eta^{b_1 \cdots b_j a_1 \cdots a_{2s}}\ .$$

$${\kappa_a}^k = {K_{ab}}^k\, \theta^b$$

Note that the Gauss equation may be written as $\Omega_{ab} = {\kappa_a}^k \wedge {\kappa_b}^k$ and the SO(l)-curvature 2-form of the normal bundle is $F^{ij} = {\kappa_a}^i \wedge {\kappa_a}^j$. Since the cartesian multipole moments $\mu^{(j)}_{k_1\cdots k_j}$ are traceless in the k indices we see that the κ terms above cannot be transformed into terms involving the intrinsic curvature R_{abcd} of the surface.

General Formula

UNIVERSITY OF MIAMI

$$E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s \; s!}$$

$$\times \int_\Sigma \mu^{(j)}_{k_1 \cdots k_j, 2s+j}(\sigma) \; {\kappa_{b_1}}^{k_1} \wedge \cdots \wedge {\kappa_{b_j}}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2s-1} a_{2s}} \wedge \eta^{b_1 \cdots b_j a_1 \cdots a_{2s}} \, .$$

$${\kappa_a}^k = {K_{ab}}^k \, \theta^b$$

General Formula

UNIVERSITY
OF MIAMI

$$E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s \; s!}$$

$$\times \int_{\Sigma} \mu^{(j)}_{k_1 \cdots k_j, 2s+j}(\sigma) \; \kappa_{b_1}{}^{k_1} \wedge \cdots \wedge \kappa_{b_j}{}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2s-1} a_{2s}} \wedge \eta^{b_1 \cdots b_j a_1 \cdots a_{2s}} \, .$$

$$\kappa_a{}^k = K_{ab}{}^k \, \theta^b$$

There are only finite number of terms in the expansion. There are roughly $q^4/4$.

Generalizes to Constant Curvature Spaces

UNIVERSITY OF MIAMI

$$\Omega^{\mu\nu} = k\,\theta^\mu \wedge \theta^\nu \qquad R_{\mu\nu\rho\sigma} = k\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

$$E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s\ s!}$$

$$\times \int_\Sigma \mu^{(j)}_{k_1\cdots k_j, 2s+j}(\sigma)\ \kappa_{b_1}{}^{k_1} \wedge \cdots \wedge \kappa_{b_j}{}^{k_j} \wedge \Omega_{a_1 a_2} \wedge \cdots \wedge \Omega_{a_{2s-1}a_{2s}} \wedge \eta^{b_1\cdots b_j a_1 \cdots a_{2s}}\,.$$

Generalizes to Constant Curvature Spaces

UNIVERSITY OF MIAMI

$$\Omega^{\mu\nu} = k\,\theta^{\mu} \wedge \theta^{\nu} \qquad R_{\mu\nu\rho\sigma} = k\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

$$E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s\ s!}$$
$$\times \int_{\Sigma} \mu^{(j)}_{k_1\cdots k_j,2s+j}(\sigma)\ \kappa_{b_1}{}^{k_1} \wedge \cdots \wedge \kappa_{b_j}{}^{k_j} \wedge \Omega_{a_1a_2} \wedge \cdots \wedge \Omega_{a_{2s-1}a_{2s}} \wedge \eta^{b_1\cdots b_j a_1\cdots a_{2s}}\,.$$

Case $k < 0$

$$\mu^{(j)}_{k_1\cdots k_j,2s+j}(\sigma) = V_{l-1}(S^{l-1}) \int_0^{\infty} d\nu\ \left(\cosh|k|^{1/2}\nu\right)^{q-j-2s} \left(\frac{\sinh|k|^{1/2}\nu}{|k|^{1/2}}\right)^{2s+j+l-1} u^{(j)}_{k_1\cdots k_j}(\sigma,\nu)\,.$$

Generalizes to Constant Curvature Spaces

UNIVERSITY OF MIAMI

$$\Omega^{\mu\nu} = k\,\theta^\mu \wedge \theta^\nu \qquad R_{\mu\nu\rho\sigma} = k\,(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

$$E = \sum_{j=0}^{q} \sum_{s=0}^{\lfloor (q-j)/2 \rfloor} \frac{C_{2j+2s}}{2^s\ s!}$$
$$\times \int_\Sigma \mu^{(j)}_{k_1\cdots k_j, 2s+j}(\sigma)\ \kappa_{b_1}{}^{k_1} \wedge \cdots \wedge \kappa_{b_j}{}^{k_j} \wedge \Omega_{a_1a_2} \wedge \cdots \wedge \Omega_{a_{2s-1}a_{2s}} \wedge \eta^{b_1\cdots b_j a_1 \cdots a_{2s}}\,.$$

Case $k < 0$

$$\mu^{(j)}_{k_1\cdots k_j, 2s+j}(\sigma) = V_{l-1}(S^{l-1}) \int_0^\infty d\nu\ \left(\cosh|k|^{1/2}\nu\right)^{q-j-2s} \left(\frac{\sinh|k|^{1/2}\nu}{|k|^{1/2}}\right)^{2s+j+l-1} u^{(j)}_{k_1\cdots k_j}(\sigma,\nu)\,.$$

For $k > 0$ replace the hyperbolic functions by the corresponding trigonometric functions.

Emergent Gravity?

UNIVERSITY
OF MIAMI

Emergent Gravity?

UNIVERSITY
OF MIAMI

For another talk…

Emergent Gravity?

UNIVERSITY
OF MIAMI

Emergent Gravity?

UNIVERSITY
OF MIAMI

THE END