EXCEPTIONALITY , SUPERSYMMETRY AND NON-ASSOCIATIVITY IN PHYSICS

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- I first met Bruno during the summer of 1977 at CERN. I had the privilege of interacting with him scientifically during my year at CERN in 1981/82 and later at Lawrence Berkeley Lab during the period 1984-86. I co-authored two papers with Bruno.
- J. R. Ellis, M. K. Gaillard, M. Gunaydin and B. Zumino,
"Supersymmetry and Noncompact Groups in Supergravity,"
Nucl. Phys. B **224**, 427 (1983).
- M. Gunaydin and B. Zumino,
"Magnetic Charge and Non-Associative Algebras,"
Proceedings of "Old and New Problems in Fundamental Physics : Meeting in Honour of G.C. Wick" , 43-53 and Lawrence Berkeley Lab. - LBL-19200 (85,MAR.) 17p, eds. R.L. Cool, M. Jacob , E. Picasso and L.A.Radicati
- During our first meeting in 1977 Bruno wanted to talk to me about octonions. Later in LBL when we were working on non-associativity and magnetic charge we discussed how non-associativity might be integrated into the framework of quantum mechanics. Two years ago I wrote a paper with D. Minic, which we dedicated to Bruno on the occasion of his 90th birthday, in which we pointed out that a non-associative algebra that appeared recently in closed string theory is isomorphic to the non-associative magnetic algebra we studied in our paper. Bruno's last two papers in the archives involve the most famous intrinsically non-associative structure , namely the exceptional Jordan algebra and its associated Freudenthal triple system. Those are the reasons I decided to talk about "Exceptionality, supersymmetry and non-associativity in Physics".

Broad outline of the topics I will try to cover in my talk:

- Jordan formulation of quantum mechanics.
- Octonionic quantum mechanics and the exceptional Jordan algebra.
- Connections between exceptional Lie algebras and the exceptional Jordan algebra. Magic square of Freudenthal, Rozenfeld and Tits.
- Appearance of exceptional groups as global symmetry groups in supergravity.
- Orbits of extremal black hole solutions under the action of their U-duaality groups and Jordan algebras and Freudenthal triple systems
- Non-associative "magnetic algebra" studied by my paper with Bruno and its appearance in closed string theory in recent years.
- Generalization of magnetic algebra and Stuckelberg's generalization classical Poisson brackets that do not satisfy the Jacobi identity.
- Superspaces defined by Jordan superalgebras and the exceptional superspace that has no realization in terms of associative (super)matrices.
- superextensions of the magic square and existence of novel simple superalgebras over finite fields that have no counterpart in characteristic zero. Fascinating surprises!

JORDAN FORMULATION OF QUANTUM MECHANICS

- In the years 1932-33 Pascual Jordan proposed a novel formulation of the quantum mechanics that came to be called Jordan formulation. His main motivation was to generalize the quantum mechanical formalism such that the then observed beta decay phenomena can be explained within this generalized framework!
  He argued that the commutator of Hermitian operators corresponding to observables that act on an Hilbert space does not preserve their hermiticity:

$$H_1^\dagger = H_1 \quad , \quad H_2^\dagger = H_2 \longrightarrow \quad [H_1, H_2]^\dagger \neq [H_1, H_2]$$

  Jordan proposed using the symmetric anti-commutator product among Hermitian operators under which they remain Hermitian. Under the symmetric product $H_1 \circ H_2 = 1/2(H_1 H_2 + H_2 H_1)$ Hermitian operators acting on the physical Hilbert space satisfy the identities:

$$H_1 \circ H_2 = H_2 \circ H_1$$

$$H_1 \circ (H_2 \circ H_1^2) = (H_1 \circ H_2) \circ H_1^2$$

  which are taken as defining identities of Jordan algebras. Jordan hoped that there would be a rich family of algebras satisfying the above identities and which can not be realized in terms of linear operators acting on a vector (Hilbert) space with the product taken as 1/2 the anticommutator.

Dirac formulation of quantum mechanics over an Hilbert space

Pure states $\iff$ rays $|\psi\rangle q$ with $\bar{q}q = 1$

Jordan formulation

Pure states $\iff$ $|\psi\rangle\langle\psi| = P_\psi = P_\psi^2$

Propositional calculus $\iff$ Projective geometry

Axioms of quantum mechanics $\iff$ Axioms of projective geometry

Pure states: $Tr(P_\psi) = 1$ $\iff$ Points in projective geometry

Superpositions of $|\psi\rangle$ and $|\xi\rangle$ $\iff$ Line connecting the points P_ψ and P_ξ

- Jordan , von Neumann and Wigner (JvNW) (1934) gave a complete classification of all finite dimensional simple Jordan algebras and showed that with one exception all finite dimensional simple Jordan algebras are special i.e. they can be realized as linear operators acting on a vector space with the product being 1/2 the anticommutator.
- The complete list simple Euclidean Jordan algebras are as follows:
  i) Dirac gamma matrices $\Gamma(d)$ in d Euclidean dimensions.
  ii) Jordan algebras $J_n^{\mathbb{R}}$ $J_n^{\mathbb{C}}$ $J_n^{\mathbb{H}}$ generated by $n \times n$ " Hermitian" matrices over reals $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$
  iii) Exceptional Jordan algebra $J_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over the division algebra of octonions $\mathbb{O}$, which can not be realized in terms of linear operators acting on some vector space.
- The proposal of Pascual Jordan to generalize the algebraic framework of quantum mechanics led to a single novel algebraic structure, namely $J_3^{\mathbb{O}}$ in the finite dimensional case. Let alone explaining the beta decay phenomena it was not at all obvious at the time if all the axioms of quantum mechanics could be satisfied in the exceptional case due to its intrinsic non-associativity.
- During the subsequent decades the exceptional Jordan algebra $J_3^{\mathbb{O}}$ had a distinguished career in mathematics. After the work of JvNW mathematicians established deep connections between the exceptional Jordan algebra and the other exceptional groups F_4, E_6, E_7, E_8.

Freudenthal-Rozenfeld-Tits Magic square:

	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(6)$	F_4
$\mathbb{C}$	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$\mathbb{H}$	$USp(6)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

MG and Gürsey (1971):
The compact magic square contains basically all the internal symmetry groups of hadronic world that were uncovered by the physicists in the 1950s and 1960s. It begged the question whether the exceptional groups could also be relevant for the physics of elementary particles . In particular could one understand the observed internal symmetries by extending the underlying field of quantum mechanics from complex numbers to octonions!?
Our work led to the so-called algebraic confinement scheme in which quarks are represented by transverse octonionic fields with $SU(3)$ automorphisms identified as the color $SU(3)_C$. The states in color singlet sector are described by an ordinary complex Hilbert space. This proposal gave a nice mathematical model of the original suggestion of Gell-Mann that color quarks operate in a **fictitious** Hilbert space and only the color singlet sector is observable (Gell-Mann 1972). However this scheme did not incorporate dynamics. Shortly afterwards quantum chromodynamics was established as the correct theory of strong interactions in which the quarks are **confined**.

- In the 1970s grand unified theories (GUT) based on exceptional groups were formulated.
  E_6 GUT (Gürsey, Ramond & Sikivie (1975) ;....) for a single family of quarks and leptons
  E_8 GUT (Bars & MG (1980) ; ....) with family unification....
- $E_8 \times E_8$ appeared as gauge symmetry of the heterotic string (Gross, Harvey, Martinec and Rohm 1984)
- **Formulation of Octonionic Quantum Mechanics over the exceptional Jordan algebra**

  MG, Piron , Ruegg (1978)

  Quantum Mechanics with projection operators belonging to the exceptional Jordan algebra $J_3^{\mathbb{O}}$. Corresponding projective geometry is the octonionic Moufang plane. The result of successive, compatible experiments do not depend on the order in which they are performed, in spite of the non-associativity of underlying octonions. The corresponding QM is referred to as the octonionic QM and has no known Hilbert Space formulation.
  The quantum mechanics described by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ describes two octonionic degrees of freedom and its projective geometry is non-Desarguian. It was hoped that there might exist infinite dimensional exceptional Jordan algebras that describe an extension of the octonionic quantum mechanics. However these hopes were dashed by the remarkable results of Zelmanov who showed that there are no infinite dimensional exceptional Jordan algebras (Zelmanov 1979-1983). (These results were referred to as "Russian revolution in Jordan algebras" by McCrimmon.) This means that in the infinite dimensional case Jordan algebraic formulation of quantum mechanics is equivalent to the Hilbert space formulation.

Space-time Supersymmetry

Early days of spacetime supersymmetry following the seminal work of Wess and Zumino (1974) two of the most important problems were:
1. How to formulate a local gauge theory of spacetime supersymmetry that necessarily requires gravity ?
2. Is there an exceptional superalgebra whose local gauge theory would lead to a unified theory of all interactions including gravity ?

My first paper on supersymmetry was an attempt to answer the second question in which the concept of generalized spacetimes coordinatized by Jordan algebras was introduced. MG (1975)

Minkowski spacetime can be coordinatized by Hermitian 2×2 matrices:
$x = \sigma_\mu x^\mu$ where $\sigma_\nu = (1_2, \vec{\sigma})$ can be considered as elements of the Jordan algebra $J_2^{\mathbb{C}}$ with the Jordan product taken as $1/2$ the anticommutator.

Automorphism group of $J_2^{\mathbb{C}} = SU(2) \rightarrow$ rotation group

Invariance group of the norm form of $J_2^{\mathbb{C}}$:
$N(x) = Det(x) = \eta_{\mu\nu} x^\mu x^\nu = SL(2, C) \rightarrow$ Lorentz group =reduced structure group

Linear fractional group of $J_2^{\mathbb{C}} = SU(2,2) \rightarrow$ Conformal group

Twistor theory is based on such a coordinatization of Minkowski space-time in $d = 4$.

Symmetry groups of generalized space-times coordinatized by the simple Jordan algebras of degree three

J	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
$Rot(J)$	$SO(3)$	$SU(3)$	$USp(6)$	F_4
$Lor(J)$	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$
$Conf(J)$	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$

Table: Simple Euclidean Jordan algebras of degree 3 and their rotation (automorphism), "Lorentz" (reduced structure) and "Conformal" (linear fractional) groups. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. $J_3^{\mathbb{A}}$ denotes a Jordan algebra of 3×3 hermitian matrices over $\mathbb{A}$.

J	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}_s}$	$J_3^{\mathbb{H}_s}$	$J_3^{\mathbb{O}_s}$
$Rot(J)$	$SO(3)$	$SL(3,R)$	$Sp(6,R)$	$F_{4(4)}$
$Lor(J)$	$SL(3,\mathbb{R})$	$SL(3,R) \times SL(3,R)$	$SL(6,R)$	$E_{6(6)}$
$Conf(J)$	$Sp(6,\mathbb{R})$	$SL(6,R)$	$SO(6,6)$	$E_{7(7)}$

Table: Simple split Jordan algebras of degree 3 and their rotation (automorphism), "Lorentz" (reduced structure) and "Conformal" (linear fractional) groups. The symbols $\mathbb{R}$, $\mathbb{C}_s$, $\mathbb{H}_s$, $\mathbb{O}_s$ represent the split forms of composition algebras.

Tits Construction of the Magic Square

	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(6)$	$F(4)$
$\mathbb{C}$	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$\mathbb{H}$	$USp(6)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	$F(4)$	E_6	E_7	E_8

Magic Square

$$L = Aut(\mathbb{A}) \oplus \mathbb{A}_0 \otimes J_{3_0}^{\mathbb{A}'} \oplus Aut(J_3^{\mathbb{A}'})$$

$$E_8 = G_2 \oplus (7 \otimes 26) \oplus F_4$$

Goal was to construct Lie superalgebras by replacing the real parameters multiplying the spinorial components of the Jordan algebra by Grassmann parameters in the Tits construction. MG 1975

UNIFIED CONSTRUCTION OF LIE AND LIE-SUPER ALGEBRAS OVER TRIPLE SYSTEMS

Bars , MG (1978)

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$OSp(1/2)$	$SU(2/1)$	$OSp(4/2)$	$F(4)$
$\mathbb{C}$	$SU(1/2)$	$SU(1/2) \times SU(2/1)$	$SU(4/2)$	?
$\mathbb{H}$	$OSp(4/2)$	$SU(4/2)$	$OSp(4/8)$	?
$\mathbb{O}$	$F(4)$	?	?	?

Super Magic Square

Extension of the Kantor's construction of Lie algebras over triple systems to unified construction of Lie and Lie superalgebras. Taking the triple system over Grassmann variables leads to Lie superalgebras. (triple systems $\leftrightarrow$ three algebras)

$G(3) \supset G_2 \times SU(2)$

$F(4) \supset SO(7) \times SU(2)$

Construction of $G(3)$ and $F(4)$ using octonions Sudbery (1983)

NO SUPERALGEBRAS CORRESPONDING TO THE EXCEPTIONAL LIE ALGEBRAS OF THE E-SERIES!!

Classification of simple Lie superalgebras Kac (1977)

Appearance of exceptional groups in maximal supergravity:

Cremmer and Julia (1978)

d	$\#vector\ fields$	$U\ duality\ group$	$scalar\ manifold$
$d=5$	27	$E_{6(6)}$	$\frac{E_{6(6)}}{USp(8)}$
$d=4$	28	$E_{7(7)}$	$\frac{E_{7(7)}}{SU(8)}$
$d=3$	$-$	$E_{8(8)}$	$\frac{E_{8(8)}}{SO(16)}$

Table: Global symmetries of maximal supergravity in $5,4$ and 3 dimensions. Note that $E_{6(6)}$ and $E_{7(7)}$ are the Lorentz and conformal groups of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$.

Zumino's paper titled " Supersymmetry and Kähler Manifolds" (1979) established a connection between $N=1$ supersymmetric non-linear sigma models in $d=4$ and Kähler manifolds. In this paper Bruno remarks :" ..... supersymmetry implies a metric of the Kähler type, a fact that we find most remarkable."

This result turned out to be the tip of an iceberg of deep connections between geometry and supersymmetry that were uncovered later.

$N = 2$ MAXWELL-EINSTEIN SUPERGRAVITY THEORIES (MESGT) IN FIVE DIMENSIONS

MG, Sierra and Townsend (1983)

- Bosonic part of the $5D$ $N = 2$ MESGT Lagrangian

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F^{I}_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu\varphi^x)(\partial^\mu\varphi^y) + \\ + \frac{e^{-1}}{6\sqrt{6}} C_{IJK}\,\varepsilon^{\mu\nu\rho\sigma\lambda}F^{I}_{\mu\nu}F^{J}_{\rho\sigma}A^{K}_{\lambda}$$

  coupling of $(n_V - 1)$ vector multiplets $(A^a_\mu, \lambda^{ai}, \varphi^a)$ to $N = 2$ supergravity $(g_{\mu\nu}, \psi^i_\mu, A_\mu)$ ($I, J, K = 1, \ldots, n_V$, i=1,2 , $x, a = 1, \ldots, (n_V - 1)$)

- $5D$, $N = 2$ MESGT is uniquely determined by the constant symmetric tensor C_{IJK}.

- $5D$ MESGTs with symmetric scalar manifolds G/H such that G is a symmetry of the Lagrangian $\Longleftrightarrow$ C_{IJK} is given by the norm (determinant) $\mathcal{N}_3$ of a Euclidean Jordan algebra J of degree 3.

$$\mathcal{N}_3(J) = C_{IJK}h^I h^J h^K$$

  Euclidean $J :\Longleftrightarrow X^2 + Y^2 = 0 \Longrightarrow X = Y = 0\ \forall X,\ Y \in J$

- Scalar manifold in $d = 5$ is $\mathcal{M}_5 = \frac{Lorentz(J)}{Rotation(J)}$

- Unified $N = 2$ Maxwell-Einstein Supergravity theories in $5d \Leftrightarrow$ all the vectors fields including the graviphoton transform in an irreducible representation of a simple U-duality group of the action.
- There exist only four unified MESGTs in $d = 5$ with symmetric target spaces . They are defined by the four simple Euclidean Jordan algebras $J_3^{\mathbb{A}}$ of 3×3 Hermitian matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ and describe the coupling of 5, 8, 14 and 26 vector multiplets to supergravity. Their symmetries in 5,4 and 3 dimensions give the groups of the Magic Square of Freudenthal, Rozenfeld and Tits $\Longrightarrow$ *MAGICAL SUPERGRAVITY THEORIES* (GST 1983)
- Scalar manifolds of 5d magical sugras are the symmetric spaces $\mathcal{M}_5 = \frac{Lor(J_3^{\mathbb{A}})}{Rot(J_3^{\mathbb{A}})}$:

$$
\begin{array}{ccccccc}
J & = & J_3^{\mathbb{R}} & J_3^{\mathbb{C}} & J_3^{\mathbb{H}} & J_3^{\mathbb{O}} \\
\mathcal{M}_5 & = & SL(3,\mathbb{R})/SO(3) & SL(3,\mathbb{C})/SU(3) & SU^*(6)/USp(6) & E_{6(-26)}/F_4
\end{array}
$$

- Under dimensional reduction of 5d magical MESGTs to four dimensions :
$$\mathcal{M}_5 = \frac{Lor(J)}{Rot(J)} \Rightarrow \mathcal{M}_4 = \frac{Conf(J)}{\widetilde{Lor}(J) \times U(1)}$$
where $\widetilde{Lor}(J)$ is the compact real form of the Lorentz group of J.
- In 5D: Vector fields $A_I^\mu \Leftrightarrow$ Elements of Jordan algebra J
- In 4D: $F^A_{\mu\nu} \oplus \tilde{F}^A_{\mu\nu} \Leftrightarrow$ Freudenthal triple system (FTS) $\mathcal{F}(J)$:

$$\mathcal{F}(J) \ni X = \begin{vmatrix} \mathbb{R} & \mathbf{J} \\ \tilde{\mathbf{J}} & \mathbb{R} \end{vmatrix} \Leftrightarrow \begin{vmatrix} F^0_{\mu\nu} & F^I_{\mu\nu} \\ \tilde{F}^I_{\mu\nu} & \tilde{F}^0_{\mu\nu} \end{vmatrix}$$

- Scalar manifolds of 4d magical sugras are the symmetric spaces :

$$\begin{array}{cccccc} J & = & J_3^{\mathbb{R}} & J_3^{\mathbb{C}} & J_3^{\mathbb{H}} & J_3^{\mathbb{O}} \\ \mathcal{M}_4 & = & \frac{Sp(6,\mathbb{R})}{U(3)} & \frac{SU(3,3)}{S(U(3)\times U(3))} & \frac{SO^*(12)}{U(6)} & \frac{E_{7(-25)}}{E_6 \times U(1)} \end{array}$$

- $N=2$ MESGTs reduce to $N=4$ supersymmetric sigma models coupled to gravity in $d=3$. The target manifolds of magical supergravity theories in $d=3$ are the exceptional quaternionic symmetric spaces:

$$\frac{F_{4(4)}}{Usp(6)\times USp(2)}, \frac{E_{6(2)}}{SU(6)\times SU(2)}, \frac{E_{7(5)}}{SO(12)\times SU(2)}, \frac{E_{8(-24)}}{E_7 \times SU(2)}$$

U-duality Orbits and Jordan Algebras:

- Study of U-duality Orbits of Extremal , Spherically Symmetric Stationary Black Hole Solutions of Supergravity Theories with Symmetric Target Spaces in $d = 5$ and $d = 4$ using the underlying Jordan algebras and Freudenthal triple systems
  MG and Ferrara, 1997
- This work led to the proposal that $4d$ U-duality groups act as spectrum generating conformal groups of the underlying Jordan algebras that define the corresponding $5d$ supergravity theories. $Conf[J]$ leaves invariant light-like separations with respect to a cubic distance function $\mathcal{N}_3(J_1 - J_2)$ and admits a 3-grading with respect to their Lorentz subgroups

$$Conf[J] = K_J \oplus Lor(J) \times \mathcal{D} \oplus T_J$$

  $Lor(J)$ is the $5D$ U-duality group that leaves the cubic norm invariant.
- Question: Can the $3D$ U-duality groups act as spectrum generating conformal symmetries of corresponding $4D$ supergravity theories ? (MG, Koepsell, Nicolai 1997)
  Problem: $E_{8(8)}$ and $E_{8(-24)}$ appear as $3d$ U-duality groups. No conformal realization for any real forms of E_8, G_2 and F_4 $\Leftrightarrow$ No 3-grading with respect to a subgroup of maximal rank.
- However, all simple Lie algebras admit a 5-grading with respect to a subalgebra of maximal rank

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$$

  such that the grade ± 2 subspaces are one-dimensional.

$$\mathfrak{g} = \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K$$

  $A, B, C ... = 1, ...2N$ and $(K, \Delta, \tilde{K})$ form an $sl(2)$ subalgebra.
  $U_A \in \mathfrak{g}^{+1}$ and $A \in \mathcal{F}$ where $\mathcal{F}$ is a Freudenthal triple system.

QUASICONFORMAL REALIZATION OF $E_{8(8)}$ MG, Koepsell, Nicolai, 2000

$$\begin{aligned} E_{8(8)} &= \mathbf{1}_{-2} \oplus \mathbf{56}_{-1} \oplus E_{7(7)} + SO(1,1) \oplus \mathbf{56}_{+1} \oplus \mathbf{1}_{+2} \\ \mathfrak{g} &= \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K \end{aligned}$$

over a space $\mathcal{T}$ coordinatized by the elements X of the exceptional FTS $\mathcal{F}(J_3^{\mathbb{O}_S})$
plus an extra singlet variable x: $\mathbf{56}_{+1} \oplus \mathbf{1}_{+2} \Leftrightarrow (X, x) \in \mathcal{T}$:

$$\begin{aligned} K(X) = 0, \quad & U_A(X) = A, \qquad S_{AB}(X) = (A, B, X), \\ K(x) = 2, \quad & U_A(x) = \langle A, X \rangle, \quad S_{AB}(x) = 2 \langle A, B \rangle x, \end{aligned}$$

$$\tilde{U}_A(X) = \frac{1}{2}(X, A, X) - Ax$$

$$\tilde{U}_A(x) = -\frac{1}{6}\langle (X, X, X), A \rangle + \langle X, A \rangle x$$

$$\tilde{K}(X) = -\frac{1}{6}(X, X, X) + Xx$$

$$\tilde{K}(x) = \frac{1}{6}\langle (X, X, X), X \rangle + 2x^2$$

Freudenthal triple product $\Leftrightarrow (X, Y, Z)$
Skew-symmetric invariant form $\Leftrightarrow \langle X, Y \rangle = -\langle Y, X \rangle$
Quartic invariant of $E_{7(7)} \Leftrightarrow \langle (X, X, X), X \rangle$
$A, B, .. \in \mathcal{F}(J_3^{\mathbb{O}_S})$

- Geometric meaning of the quasiconformal action of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{T}$?
- Define a quartic norm of $\mathcal{X} = (X, x) \in \mathcal{T}$ as $\mathcal{N}_4(\mathcal{X}) := \mathcal{Q}_4(X) - x^2$
  $\mathcal{Q}_4(X)$ is the quartic norm of the underlying Freudenthal system and $X \in \mathcal{F}$.
- Define a quartic "distance" function between any two points $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$ in $\mathcal{T}$ as

  $$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y})$$

  $\delta(\mathcal{X}, \mathcal{Y})$ is the "symplectic" difference of $\mathcal{X}$ and $\mathcal{Y}$:

  $$\delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + \langle X, Y \rangle) = -\delta(\mathcal{Y}, \mathcal{X})$$

- Light-like separations $d(\mathcal{X}, \mathcal{Y}) = 0$ are left invariant under quasiconformal group action.
  $\longrightarrow$ *Quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function.*
- $E_{8(8)}$ is the invariance group of a quartic light-cone in 57 dimensions!
- Quasiconformal realizations extend to all Lie algebras ad Lie superalgebras.

- The space-times defined by simple Jordan algebras of degree 3 $J_3^{\mathbb{A}}$ correspond to extensions of Minkowski space-times in the critical dimensions $d = 3, 4, 6, 10$ by a dilatonic (ρ) *and* commuting spinorial coordinates (ξ^a).

$$J_3^{\mathbb{R}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, 1, 2 \quad \alpha = 1, 2$$
$$J_3^{\mathbb{C}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, 1, 2, 3 \quad \alpha = 1, 2, 3, 4$$
$$J_3^{\mathbb{H}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, \ldots, 5 \quad \alpha = 1, \ldots, 8$$
$$J_3^{\mathbb{C}} \Longleftrightarrow (\rho, x_m, \xi^\alpha) \quad m = 0, \ldots, 9 \quad \alpha = 1, \ldots, 16$$

- The commuting spinors ξ are represented by a 2×1 matrix over $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The cubic norm of a "vector" with coordinates $X^I = (\rho, x_m, \xi^\alpha)$ is given by

$$\mathcal{V}(\rho, x_m, \xi^\alpha) = C_{IJK} X^I X^J X^K = \sqrt{2} \rho x_m x_n \eta^{mn} + x^m \bar{\xi} \gamma_m \xi$$

Their Lorentz groups have the Lorentz groups Minkowskian spacetimes in $d = 3, 4, 6, 10$ as subgroups:

$$J = \mathbb{R} \oplus J_2^{\mathbb{R}} : \mathrm{SO}(1,1) \times \mathrm{SO}(2,1) \subset \mathrm{SL}(3, \mathbb{R})$$
$$J = \mathbb{R} \oplus J_2^{\mathbb{C}} : \mathrm{SO}(1,1) \times \mathrm{SO}(3,1) \subset \mathrm{SL}(3, \mathbb{C})$$
$$J = \mathbb{R} \oplus J_2^{\mathbb{H}} : \mathrm{SO}(1,1) \times \mathrm{SO}(5,1) \subset \mathrm{SU}^*(6)$$
$$J = \mathbb{R} \oplus J_2^{\mathbb{O}} : \mathrm{SO}(1,1) \times \mathrm{SO}(9,1) \subset \mathrm{E}_{6(-26)}$$

- Remarkable fact: the adjoint identity satisfied by C_{IJK} tensor $\Longleftrightarrow$ Fierz identities required for the existence of supersymmetric Yang-Mills theory in the critical dimensions $d = 3, 4, 6, 10$ Sierra (1987).

EXCEPTIONAL $N = 2$ versus MAXIMAL $N = 8$ SUPERGRAVITY:

- The exceptional $N = 2$ supergravity is defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over real octonions $\mathbb{O}$. Its global invariance group in $5D$ is $E_{6(-26)}$ with maximal compact subgroup F_4.
- The C-tensor C_{IJK} of $N = 8$ supergravity in five dimensions can be identified with the symmetric tensor given by the cubic norm of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$ defined over split octonions $\mathbb{O}_s$. Its global invariance group in $5D$ is $E_{6(6)}$ with maximal compact subgroup $USp(8)$.
- In $D = 4$ and $D = 3$ the exceptional supergravity has $E_{7(-25)}$ and $E_{8(-24)}$ as its U-duality group while the maximal $N = 8$ supergravity has $E_{7(7)}$ and $E_{8(8)}$, respectively.
- One can couple 28 hypermultiplets to exceptional $N = 2$ supergravity in $d = 4$ parametrizing the coset space $\frac{E_{8(-24)}}{E_7 \times SU(2)}$ which has the moduli space of FHSV model as a subspace $SO(10,2) \times SU(1,1) \times SO(12,4) \subset E_{7(-25)} \times E_{8(-24)}$ In three dimensions this exceptional theory with exceptional matter has the moduli space $\frac{E_{8(-24)}}{E_7 \times SU(2)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)}$ and descends from an anomaly free theory in $d = 6$.
  MG (Paris , 2006)
- How to obtain this theory from M/Supertstring theory ?
  Bianchi and Ferrara argued that octonionic magic supergravity theory admits a string interpretation closely related to the Enriques model derivation of FHSV model. (2007).
- The magical supergravity theories including the exceptional one coupled to hypermultiplets in $d = 6$ was constructed by MG, Samtleben and Sezgin (2010).

Magnetic charge and non-associative algebras

The commutators of the velocities of a non-relativistic electron in the field of a magnetic monopole violate the Jacobi identity at the position where the monopole is located Lipkin, Weisberger and Peskin (1969).
Appearance of 3-cocycles in the magnetic monopole problem. Translations become non-associative $\Rightarrow$ Dirac quantization condition. Grossmann; Jackiw ; Wu & Zee (1985)
Boulware, Deser and Zumino (1985): in any formulation of the quantum mechanical problem in which the coordinates and velocities of the electron are described by operators in a Hilbert space the Jacobi identity can not be violated since all operators acting on a Hilbert space are associative. If the Jacobi identity is violated then the operators must belong to a non-associative algebra.
MG, Zumino (October 1985):
The basic commutation relations for a non-relativistic electron moving in a magnetic field $B(x)$:

$$[x_a, x_b] = 0 \quad [x_a, v_b] = i\delta_{ab}$$

$$[v_a, v_b] = i\epsilon_{abc} B_c(x)$$

Consider the algebra of all functions of x_a and v_b. Consistency of this algebra imply certain restrictions.

$$[[v_1, v_2], v_3] + [[v_3, v_1], v_2] + [[v_2, v_3], v_1] = -\vec{\nabla} \cdot \vec{B}(x)$$

Malcev algebras correspond to generalizations of Lie algebras with an anti-symmetric product

$$a \star b = -b \star a$$

that satisfy the Malcev identity

$$(a \star b) \star (a \star c) = ((a \star b) \star c) \star a + ((b \star c) \star a) \star a + ((c \star a) \star a) \star b$$

which can be written as

$$J(a, b, a \star c) = J(a, b, c) \star a$$

where $J(a, b, c)$ is the Jacobian (Jacobiator) :

$$J(a, b, c) \equiv ((a \star b) \star c) + ((c \star a) \star b) + ((b \star c) \star a)$$

For the problem of the electron moving in the field of some magnetic charge distribution the Malcev condition requires

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}(x)) = 0$$

The case $(\vec{\nabla} \cdot \vec{B}(x)) = 0$ corresponds to a Lie algebra and the operators can be represented in terms of Hilbert space operators by taking

$$v_a = p_a - A_a \qquad B_{ab} = \partial_a A_b - \partial_b A_a$$

- The case $B_a \propto x_a$ lead to a bona fide Malcev algebra. By redefining the coordinates and the velocities one can choose $B_a(x) = x_a$ resulting in the "magnetic algebra":

$$[x_a, x_b] = 0 \quad , [x_a, v_b] = i\delta_{ab}$$

$$[v_a, v_b] = i\epsilon_{abc} x_c$$

  which corresponds to constant magnetic charge distribution. The corresponding quantum mechanical operators can not be represented by operators acting on an Hilbert space
- Non-associativity also arises in closed string theory in the presence of non-vanishing three-form H-flux. Lüst (2010); Blumenhagen (2011) , Plauschinn (2011) ,....
  Earlier related work on non-associativity in string theory Cornalba and Schiappa (2001); Ho (2001)....
- MG, Minic (2013):
  The non-associative algebra of coordinates and momenta that arises in closed string theory in the presence of constant H-flux is isomorphic to the magnetic algebra of GZ above (with roles of coordinates and momenta interchanged).
- One can generalize the magnetic algebra of GZ in a way that puts the coordinates and momenta on an equal footing:

$$[x_a, x_b] = -i\epsilon_{abc} E_c \quad , [x_a, v_b] = i\delta_{ab} \quad , [v_a, v_b] = i\epsilon_{abc} B_c(x)$$

$$[[v_1, v_2], v_3] + [[v_3, v_1], v_2] + [[v_2, v_3], v_1] = -\vec{\nabla} \cdot \vec{B}(x)$$

$$[[x_1, x_2], x_3] + [[x_3, x_1], x_2] + [[x_2, x_3], x_1] = \vec{\nabla} \cdot \vec{E}(x)$$

- Stückelberg's generalization of Poisson brackets (PB) in classical statistical mechanics while preserving Liouville's theorem such that the PB's no longer satisfy the Jacobi identity (1960). Stückelberg thought that in the corresponding quantum theory the operators must be nonlinear and there must be a fundamental length.
- Non-associativity is a particular form of nonlinearity. Therefore if one replaces non-linearity with non-associativity then string theory with a fundamental length corresponds to the kind of quantum theory Stückelberg was envisioning.
- Stueckelberg considered quantum theories with a "critical length" λ_0 such that uncertainties in the measurements of coordinates satisfy

$$(\Delta X)^2 \geqslant (\lambda_0)^2$$

and proposed modifying the minimum uncertainty relation in $d = 1$:

$$(\Delta X)^2(\Delta P)^2 = \frac{\hbar^2}{4}\left(1 - \frac{(\lambda_0)^2}{(\Delta X)^2}\right)^{-1}$$

which requires the modification of canonical commutation relations as

$$i[P, X] = \hbar\left(1 - \frac{(\lambda_0)^2}{(\Delta X)^2}\right)^{-1/2}.$$

If we formally expand the inverse $\left(1 - \frac{(\lambda_0)^2}{(\Delta X)^2}\right)^{-1}$ and use to first order $\Delta X \sim \hbar\,\Delta P^{-1}$ we get the stringy uncertainty relation which usually reads as

$$\Delta X \Delta P \sim (1 + \alpha' \Delta P^2)\hbar.$$

(This formal procedure relates λ_0 with the string scale l_s, or equivalently with $\alpha' \sim l_s^2$.)

Superspaces coordinatized by Jordan superalgebras

Jordan superalgebras are Z_2 graded algebras with a supersymmetric product (Kac , 1977). Their even subspaces are ordinary Jordan algebras. One can define generalized superspaces coordinatized by Jordan superalgebras such that their rotation, Lorentz and conformal supergroups are identified with their automorphism, reduced structure and Möbius (linear fractional) supergroups. MG (1978)

JX	SRG	SLG	SCG
$JA(m^2+n^2/2mn)$	$SU(m/n)$	$SU(m/n)\times SU(m/n)$	$SU(2m/2n)$
$JBC(r/s)$	$OSp(m/2n)$	$SU(m/2n)$	$OSp(4n/2m)$
$JD(m/2n)$	$OSp(m-1/2n)$	$OSp(m/2n)$	$OSp(m+2/2n)$
$JP(n^2/n^2)$	$P(n-1)$	$SU(n/n)$	$P(2n-1)$
$JQ(n^2/n^2)$	$Q(n-1)\times U(1)_F$	$Q(n-1)\times Q(n-1)\times U(1)_F$	$Q(2n-1)$
$JD(2/2)_\alpha$	$OSp(1/2)$	$SU(1/2)$	$D(2,1;\alpha)$
$JF(6/4)$	$OSp(1/2)\times OSp(1/2)$	$OSp(2/4)$	$F(4)$
$JK(1/2)$	$OSp(1/2)$	$SU(1/2)$	$SU(2/2)$

where $r=\frac{1}{2}m(m+1)+n(2n-1)$ and $s=2mn$.

In the list of simple Jordan superalgebras one is truly unique, namely the exceptional Jordan superalgebra $JF(6/4)$. It is the only simple Jordan superalgebra which has no realization in terms of Z_2 graded associative supermatrices. Zelmanov and Racine gave an octonionic realization of the unique exceptional Jordan superalgebra recently.

The even elements of $JF(6/4)$ belonging to grade zero subspace are denoted as S, B_0 and B_μ, $(\mu = 1,2,3,4)$. The odd elements belonging to the grade one subspace are denoted as Q_α, $(\alpha = 1,2,3,4)$. Their super-commutative Jordan products are:

$$
\begin{aligned}
B_\mu \cdot B_\nu = -\delta_{\mu\nu} B_0 \qquad B_0 \cdot B_\mu &= B_\mu \\
B_0 \cdot B_0 = B_0 \qquad B_0 \cdot S = 0 = B_\mu \cdot S \qquad S \cdot S &= S \\
Q_\alpha \cdot Q_\beta = (i\gamma_5 \gamma_\mu C)_{\alpha\beta} B^\mu + (\gamma_5 C)_{\alpha\beta} (B_0 - 3S) & \\
B_\mu \cdot Q_\alpha = \frac{i}{2} (\gamma_\mu)_{\alpha\beta} Q_\beta \quad B_0 \cdot Q_\alpha &= \frac{1}{2} Q_\alpha \\
S \cdot Q_\alpha &= \frac{1}{2} Q_\alpha \\
\mu, \nu, \ldots = 1,2,3,4 \quad ; \ \alpha, \beta, \ldots &= 1,2,3,4
\end{aligned}
$$

The B_0 and S are the two idempotents and $I = B_0 + S$ is the identity element of $JF(6/4)$. The matrices γ_μ are the four-dimensional (Euclidean) Dirac gamma matrices and C is the charge conjugation matrix:

$$
\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \quad , \quad \gamma_\mu C = -C\gamma_\mu^T
$$

Hilbert Space Formulation	Jordan Formulation	Quadratic Jordan Formulation
$\mid \alpha >$	$\mid \alpha >< \alpha \mid \;=\; P_\alpha$	P_α
$H \mid \alpha >$	$H \circ P_\alpha$	$\Pi_H P_\alpha = \{H P_\alpha H\}$
$< \alpha \mid H \mid \beta >$	?	?
$< \alpha \mid H \mid \alpha >$	$Tr\; H \circ P_\alpha$	$Tr\; \Pi_{P_\alpha} H \;=\; Tr\; \{P_\alpha H P_\alpha\}$
$\mid< \alpha \mid H \mid \beta >\mid^2$	$Tr\; P_\alpha \circ \{H P_\beta H\}$	$Tr\; \Pi_{P_\alpha} \Pi_H P_\beta$
	$=\; Tr\; \{H P_\alpha H\} \circ P_\beta$	$=\; Tr\; \Pi_{P_\beta} \Pi_H P_\alpha$
$[H_1, H_2]$	?	$4(\Pi_{H_1} \circ \Pi_{H_2} - \Pi_{H_1 \circ H_2})$

Quadratic Jordan formulation of quantum mechanics involves only the Jordan triple product: $\Longrightarrow \{ABC\} \equiv (A \circ B) \circ C + A \circ (B \circ C) - (A \circ C) \circ B$
Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$
Quadratic Jordan formulation extends to the octonionic quantum mechanics as well as to formulation of quantum mechanics over finite fields and to Jordan superalgebras, including the exceptional one.

(MG 1978,1991)

THE EXCEPTIONAL SUPERSPACE:

- Rotation Lie superalgebra the exceptional superspace coordinatized by $JF(6/4)$ is
  $OSp(1/2) \times OSp(1/2) \supset SO(4) = SU(2) \times SU(2)$.
- Lorentz Lie superalgebra of $JF(6/4)$ is
  $OSp(2/4) \supset SO(2) \times Sp(4)$
- Superconformal Lie algebra of $JF(6/4)$ is
  $F(4) \supset SO(5,2) \times SU(2)$
  Non-linear action of $F(4)$ on the exceptional superspace can be obtained using the quadratic Jordan formulation.
- The exceptional $N = 2$ superconformal algebra $F(4)$ in five dimensions can not be embedded in any six dimensional super conformal algebra
  $OSp(8^*|2N) \supset SO(6,2) \times USp(2N)$ as expected from the exceptionality of the superspace defined by $JF(6/4)$.
- Minimal unitary realization of $F(4)$ was obtained via quasiconformal techniques recently. The enveloping algebra of the minimal unitary representation of $F(4)$ is the unique higher spin superconformal algebra in five dimensions.
  (Fernando and MG , 2014).
- According to Nahm's classification $d = 6$ is the maximal dimensions for the existence of superconformal field theories! However,

Super Magic Rectangle that extends Tits Construction

	$J_3^{\mathbb{F}}$	$J_3^{\mathbb{F}\times\mathbb{F}}$	$J_3^{\mathbb{M}(\mathbb{F})_2}$	$J_3^{\mathbb{O}(\mathbb{F})}$	$J^{0\|2}$	D_t	$JF(6\|4)$
$\mathbb{F}$	$SO(3)$	$SU(3)$	$USp(6)$	$F(4)$	$Sp(2)$	$OSp(1\|2)$	$OSp(1\|2)^2$
$\mathbb{F}\times\mathbb{F}$	$SU(3)$	$SU(3)\times SU(3)$	$SU(6)$	E_6	$OSp(1\|2)$	$SU(2\|1)$	$OSp(2\|4)$
$\mathbb{M}(\mathbb{F})_2$	$USp(6)$	$SU(6)$	$SO(12)$	E_7	$SU(2\|2)$	$D(2,1;\alpha)$	$F(4)$
$\mathbb{O}(\mathbb{F})$	$F(4)$	E_6	E_7	E_8	$G(3)$	$F(4)_{t=2}$	$\mathcal{T}(55\|32)_5$

$\mathcal{T}(55|32)_5$ stands for a simple Lie superalgebra whose even subalgebra is $SO(11)$ and whose odd elements are in the spinor 32 representation of $SO(11)$ in characteristic five. Elduque 2007

Remarkable fact: Simple $AdS/Conformal$ superalgebra in $d = 10/9$ dimensions whose even subalgebra is $SO(9,2)$ and odd odd generators are in the spinor representation 32 in characteristic five.

It corresponds to the quasiconformal algebra associated with the exceptional Jordan superalgebra $JF(6|4)$ just as E_8 is the quasiconformal algebra associated with the exceptional Jordan algebra $J_3^{\mathbb{O}}$.

Benkart, Cunha, Elduque, Shestakov, Zelmanov,....on super-extensions of Tits' construction.

Super Magic Square that extends Kantor Construction in characteristic 3

	$\mathbb{F}$	$\mathbb{F}\times\mathbb{F}$	$\mathbb{M}(\mathbb{F})_2$	$\mathbb{O}(\mathbb{F})$	$B(1,2)$	$B(4,2)$
$\mathbb{F}$	$SO(3)$	$SU(3)$	$USp(6)$	$F(4)$	$psl_{2,2}$	$sp_6 \oplus (14)$
$\mathbb{F}\times\mathbb{F}$	$SU(3)$	$SU(3)\otimes SU(3)$	$SU(6)$	E_6	$(pgl_3 \oplus sl_2)\oplus(psl_3\otimes(2))$	$pgl_6 \oplus (20)$
$\mathbb{M}(\mathbb{F})_2$	$USp(6)$	$SU(6)$	$SO(12)$	E_7	$(sp_6\oplus sl_2)\oplus((13)\otimes(2))$	$so(12)\oplus(spin_{12})$
$\mathbb{O}(\mathbb{F})$	$F(4)$	E_6	E_7	E_8	$(f_4\oplus sl_2)\oplus((25)\otimes(2))$	$e_7\oplus(56)$
$B(1,2)$					$so(7)\oplus 2(spin_7)$	$sp_8\oplus(40)$
$B(4,2)$						$so(13)\oplus spin_{13}$

- $B(1,2)$ and $B(4,2)$ are composition superalgebras in characteristic three.
- Simple AdS/Conformal superalgebras 11/10 dimensions : $SO(10,2)\oplus(32)$ and 12/11 dimensions $SO(11,2)\oplus(64)$ and an exceptional simple conformal superalgebra : $E_7\oplus(56)$
- $SO(10,2)\oplus(32)\subset E_{7(-25)}\oplus 56$
  $SO(6,6)\oplus(32)\subset E_{7(7)}\oplus 56$

Elduque, Okubo, Shestakov,Cunha, Leites, ...

- How to generalize quantum mechanics so that the octonionic quantum mechanics can be embedded in a higher quantum theory?
- The deep mathematical connections between supersymmetry, exceptional groups and non-associative algebras might be a hint that the relevance of exceptional groups and related algebraic structures in describing Nature at a fundamental level is intimately tied to the relevance of supersymmetry in describing Nature.
- We may not know the answer to these questions for a long time to come. However, intellectually it has been most rewarding to work on supersymmetric theories and contribute to the remarkable connections that were established between geometry, exceptional groups, representation theory and supersymmetry.

GRAZIE MILLE BRUNO !