Adler's Zero Condition
and
A Minimally Symmetric Higgs Boson

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We know only two ways to stabilize the Higgs mass:

1) bosonic global symmetry ----> higgs as a pseudo Nambu-Goldstone boson (PNGB)!

2) fermionic global symmetry -----> supersymmetry

- supersymmetric theories are all built upon a minimal lagrangian
  -- the MSSM:

$$W_{\rm MSSM} = \overline{u}\mathbf{y_u}QH_u - \overline{d}\mathbf{y_d}QH_d - \overline{e}\mathbf{y_e}LH_d + \mu H_u H_d$$

$$\begin{aligned}
\mathcal{L}_{\rm soft}^{\rm MSSM} \;=\; & -\frac{1}{2}\left(M_3\widetilde{g}\widetilde{g} + M_2\widetilde{W}\widetilde{W} + M_1\widetilde{B}\widetilde{B} + {\rm c.c.}\right) \\
& -\left(\widetilde{\overline{u}}\,\mathbf{a_u}\,\widetilde{Q}H_u - \widetilde{\overline{d}}\,\mathbf{a_d}\,\widetilde{Q}H_d - \widetilde{\overline{e}}\,\mathbf{a_e}\,\widetilde{L}H_d + {\rm c.c.}\right) \\
& -\widetilde{Q}^\dagger\,\mathbf{m^2_Q}\,\widetilde{Q} - \widetilde{L}^\dagger\,\mathbf{m^2_L}\,\widetilde{L} - \widetilde{\overline{u}}\,\mathbf{m^2_{\overline{u}}}\,\widetilde{\overline{u}}^\dagger - \widetilde{\overline{d}}\,\mathbf{m^2_{\overline{d}}}\,\widetilde{\overline{d}}^\dagger - \widetilde{\overline{e}}\,\mathbf{m^2_{\overline{e}}}\,\widetilde{\overline{e}}^\dagger \\
& -m^2_{H_u}H_u^*H_u - m^2_{H_d}H_d^*H_d - (bH_uH_d + {\rm c.c.})\,.
\end{aligned}$$

This is the minimal lagrangian the makes standard model supersymmetric

On the other hand, the space of a PNGB Higgs looks huge:

$\mathcal{G}$	$\mathcal{H}$	C	N_G	$\mathbf{r}_{\mathcal{H}} = \mathbf{r}_{SU(2)\times SU(2)}\ (\mathbf{r}_{SU(2)\times U(1)})$	Ref.
SO(5)	SO(4)	✓	4	$\mathbf{4} = (\mathbf{2},\mathbf{2})$	[11]
SU(3) × U(1)	SU(2) × U(1)		5	$\mathbf{2}_{\pm 1/2} + \mathbf{1}_0$	[10, 35]
SU(4)	Sp(4)	✓	5	$\mathbf{5} = (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	[29, 47, 64]
SU(4)	$[SU(2)]^2 \times U(1)$	✓*	8	$(\mathbf{2},\mathbf{2})_{\pm 2} = 2 \cdot (\mathbf{2},\mathbf{2})$	[65]
SO(7)	SO(6)	✓	6	$\mathbf{6} = 2 \cdot (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	–
SO(7)	G_2	✓*	7	$\mathbf{7} = (\mathbf{1},\mathbf{3}) + (\mathbf{2},\mathbf{2})$	[66]
SO(7)	SO(5) × U(1)	✓*	10	$\mathbf{10}_0 = (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{2},\mathbf{2})$	–
SO(7)	$[SU(2)]^3$	✓*	12	$(\mathbf{2},\mathbf{2},\mathbf{3}) = 3 \cdot (\mathbf{2},\mathbf{2})$	–
Sp(6)	Sp(4) × SU(2)	✓	8	$(\mathbf{4},\mathbf{2}) = 2 \cdot (\mathbf{2},\mathbf{2})$	[65]
SU(5)	SU(4) × U(1)	✓*	8	$\mathbf{4}_{-5} + \bar{\mathbf{4}}_{+5} = 2 \cdot (\mathbf{2},\mathbf{2})$	[67]
SU(5)	SO(5)	✓*	14	$\mathbf{14} = (\mathbf{3},\mathbf{3}) + (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1})$	[9, 47, 49]
SO(8)	SO(7)	✓	7	$\mathbf{7} = 3 \cdot (\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	–
SO(9)	SO(8)	✓	8	$\mathbf{8} = 2 \cdot (\mathbf{2},\mathbf{2})$	[67]
SO(9)	SO(5) × SO(4)	✓*	20	$(\mathbf{5},\mathbf{4}) = (\mathbf{2},\mathbf{2}) + (\mathbf{1}+\mathbf{3},\mathbf{1}+\mathbf{3})$	[34]
$[SU(3)]^2$	SU(3)		8	$\mathbf{8} = \mathbf{1}_0 + \mathbf{2}_{\pm 1/2} + \mathbf{3}_0$	[8]
$[SO(5)]^2$	SO(5)	✓*	10	$\mathbf{10} = (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1}) + (\mathbf{2},\mathbf{2})$	[32]
SU(4) × U(1)	SU(3) × U(1)		7	$\mathbf{3}_{-1/3} + \bar{\mathbf{3}}_{+1/3} + \mathbf{1}_0 = 3 \cdot \mathbf{1}_0 + \mathbf{2}_{\pm 1/2}$	[35, 41]
SU(6)	Sp(6)	✓*	14	$\mathbf{14} = 2 \cdot (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{3}) + 3 \cdot (\mathbf{1},\mathbf{1})$	[30, 47]
$[SO(6)]^2$	SO(6)	✓*	15	$\mathbf{15} = (\mathbf{1},\mathbf{1}) + 2 \cdot (\mathbf{2},\mathbf{2}) + (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$	[36]

Table 1: Symmetry breaking patterns $\mathcal{G} \to \mathcal{H}$ for Lie groups. The third column denotes whether the breaking pattern incorporates custodial symmetry. The fourth column gives the dimension N_G of the coset, while the fifth contains the representations of the GB's under $\mathcal{H}$ and $SO(4) \cong SU(2)_L \times SU(2)_R$ (or simply $SU(2)_L \times U(1)_Y$ if there is no custodial symmetry). In case of more than two SU(2)'s in $\mathcal{H}$ and several different possible decompositions we quote the one with largest number of bi-doublets.

Bellazzni, Csaki and Serre:1401.2457

This is because a different symmetry breaking pattern gives a seemingly different effective lagrangian.

any young hot shot can come up with his/her own symmetry breaking pattern G/H, and becomes famous!

Conventional Wisdom:

Each G/H is a different nonlinear sigma model.

Nonlinear sigma models (nlsm) were introduced by Gell-Mann and Levy in 1960 as a "toy model" for pion-nucleon interactions

The Axial Vector Current in Beta Decay (*).

M. GELL-MANN (**)

Collège de France and Ecole Normale Supérieure - Paris (***)

M. LÉVY

Faculte des Sciences, Orsay, and Ecole Normale Supérieure - Paris (***)

(ricevuto il 19 Febbraio 1960)

$$\mathscr{L}_3 = \left\{ -\bar{N}[\gamma\partial - g_0(\sigma' + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\gamma_5)]N - \frac{(\partial\boldsymbol{\pi})^2}{2} - \frac{(\partial\sigma')^2}{2} - \frac{\mu_0^2}{2f_0}\sigma' \right\}$$

$$\sigma' = -\sqrt{\frac{1}{4f_0^2} - \pi^2} = -\frac{1}{2f_0}\sqrt{1 - 4f_0^2\pi^2}$$

Nowadays we understand the original nlsm as describing spontaneously broken $SU(2)_L$ x $SU(2)_R$ chiral symmetry. The pions are the (pseudo) Nambu-Goldstone bosons.

The symmetry perspective is very powerful, and has led to important insights in many areas of theoretical physics, including condensed matter, statistical, particle, and mathematical physics.

The symmetry viewpoint also makes it clear that nlsm is about the infrared structure of the theory in the presence of nonlinearly realized symmetries, in particular the nontrivial space of vacua.

This is the viewpoint I wish to emphasize in this talk!

In two seminal papers Coleman, Callan, Wess and Zumino (CCWZ) considered the most general approach to writing down effective lagrangians for nlsm:

Structure of Phenomenological Lagrangians. I*

S. Coleman
Harvard University, Cambridge, Massachusetts 02138

and

J. Wess† and Bruno Zumino
New York University, New York, New York 10003
(Received 13 June 1968)

Structure of Phenomenological Lagrangians. II*

Curtis G. Callan, Jr. and Sidney Coleman
Harvard University, Cambridge, Massachusetts 02138

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New York University, New York, New York 10003
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- CCWZ is a very geometrical approach:

standard forms, which are described in detail. The mathematical problem is equivalent to that of finding all (nonlinear) realizations of a (compact, connected, semisimple) Lie group which become linear when restricted to a given subgroup. The relation between linear representations and nonlinear realizations is

- To crank the machinery, one decides on a nonlinearly realized group G, and a subgroup H of G that is linearly realized.

  We say G is the broken group and H the unbroken group:

$$\xi = e^{i\Pi/f}, \quad \Pi = \pi^a X^a,$$

$$g\,\xi = \xi'\,U(g,\xi)\ , \quad U(g,\xi) \in H$$

- The "pions" are the coordinates on the coset manifold G/H, and the action of the full group G on pions is complicated and nonlinear!

CCWZ thus looked for objects that have “simple” transformation properties under the action of G.

These are contained in the Cartan-Maurer one-form:

$$\xi^\dagger \partial_\mu \xi \;=\; i\mathcal{D}_\mu^a X^a + i\mathcal{E}_\mu^i T^i \equiv i\mathcal{D}_\mu + i\mathcal{E}_\mu$$

They are the “Goldstone covariant derivative” and the “associated gauge field”,

$$\mathcal{D}_\mu \to U\mathcal{D}_\mu U^{-1} \;, \quad \mathcal{E}_\mu \to U\mathcal{E}_\mu U^{-1} - (\partial_\mu U)U^{-1}$$

upon which the complete effective lagrangian can be built (apart from the topological terms)

$$\mathcal{L}_{eff} = \frac{f^2}{2}\mathrm{Tr}\mathcal{D}_\mu\mathcal{D}^\mu + \cdots$$

For example, applying CCWZ to low-energy QCD with G=SU(3)$_L$xSU(3)$_R$ and H=SU(3)$_V$, we have

$$\pi = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{bmatrix}$$

One is led to the following effective interactions at the two-derivative level

$$\mathcal{L}_2 = \mathrm{Tr}\ \partial_\mu \pi \partial^\mu \pi + \frac{1}{3f^2}\,\mathrm{Tr}\ [\pi, \partial_\mu \pi]^2 + \ldots$$

Notice the second term describes pi-pi scatterings.

More generally, one can crank the machinery of CCWZ once the Lie algebra is identified

1) $$[T^i, T^j] = if^{ijk}T^k$$

2) $$[T^i, X^a] = if^{iab}X^b$$

3) $$[X^a, X^b] = if^{abi}T^i + if^{abc}X^c$$

1) is a statement that the unbroken generator forms a sub-algebra.

2) is a statement that the broken generators forms a linear representation of the unbroken sub-algebra.

If f^{abc}=0 in 3), this is a symmetric coset.

It’s clear that CCWZ is an extremely powerful formalism. And it adopts a “Top-down” perspective, which requires knowing ahead of time what the broken group “G” is in the UV.

On the other hand, let’s not forget that nlsm is all about the presence of non-trivial vacua:

Goldstone bosons are long-range degrees of freedom that connect different vacua!

They are our gateway to understanding the IR structure of the theory!

This IR viewpoint was pursued vigorously in the context of pions in the ‘60s by Adler, Nambu, Goldstone, Weinberg, etc.

This body of work was collectively known as “soft pion theorems,” although a significant part of them does not depend on the particular symmetry breaking pattern!

- one particularly important “soft-pion” theorem is the Adler’s zero condition:

on-shell scattering amplitudes of Goldstone bosons must vanish in the limit the momentum of one Goldstone boson is taken off-shell and soft.

- often this is over-simplified as saying “the Goldstone boson is derivatively coupled.”

it is an over-simplification because it doesn’t do justice to the full power of the Adler’s zero condition.

for now assume only one flavor of Goldstone boson and consider 4-pt scattering amplitudes, written in terms of the Mandelstam variables.

- Adler's zero condition forbids a constant term!

$$\mathcal{A}(\pi\pi \to \pi\pi) = c_1 s + c_2 t + c_3 u + \mathcal{O}(p^4)$$

- Bose symmetry implies $c_1 = c_2 = c_3$!

$$\mathcal{A}(\pi\pi \to \pi\pi) = \mathcal{O}(p^4)$$

the argument can be generalized to n-pt amplitudes to show that $O(p^2)$ term always vanishes!

$$\begin{aligned}\mathcal{A}(\pi\pi\cdots \to \pi\pi\cdots) &= c(p_1 + p_2 + \cdots + p_n)^2 + \mathcal{O}(p^4) \\ &= \mathcal{O}(p^4)\end{aligned}$$

- the simplest lagrangian satisfying these properties is the familiar one:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \mathcal{O}(\partial^4)$$

- as is well known, L_0 can be obtained by requiring that there is a constant "shift symmetry" acting on pion:

$$\pi \longrightarrow \pi + \epsilon$$

- the derivative of pion has simpler transformation under the broken symmetry:

$$\partial_\mu \pi \to \partial_\mu \pi$$

$\partial_\mu \pi$ is the building block of the effective lagrangian!

we have learned a simple yet powerful statement that is universal in nlsm:

self-interactions among Goldstones of the same flavor are fixed by Adler's zero condition and Bose symmetry, and must have the form:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \mathcal{O}(\partial^4)$$

- when there are multiple flavors of Goldstones, higher order terms appear in the shift symmetry.

- to warm up, let's consider two flavors of goldstones transforming as a complex scalar under an unbroken U(1):

$$\phi = (\pi_1 + i\pi_2)/\sqrt{2} \quad \rightarrow \quad e^{i\alpha}\ \phi$$

- nonlinear shift symmetry at NLO can be written as

$$\phi \mapsto \phi' = \phi + \epsilon - \frac{c_1}{f^2}(\phi^* \epsilon)\phi - \frac{c_2}{f^2}(\epsilon^* \phi)\phi$$

- when we turn off one of the two flavors , we must return to the single flavor case, $\pi_i \rightarrow \pi_i + \varepsilon_i$,

$$\phi \mapsto \phi' = \phi + \epsilon - \frac{c_1}{f^2}(\phi^* \epsilon - \epsilon^* \phi)\phi$$

- to construct the lagrangian, it pays to recall that the derivative of goldstones usually has simpler transformation!

- define the “covariant derivative” of the Goldstone,

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - \frac{d_1}{f^2}(\phi\, \partial_\mu \phi^* - \partial_\mu \phi\ \phi^*)\phi$$

- the form is again fixed by reducing to the single flavor case:

$$\mathcal{D}_\mu \phi\big|_{\pi_{1,2}=0} = \partial_\mu \phi$$

how does the covariant derivative transform under the broken symmetry acting on the Goldstone?

- as the name suggests, it should change by a *field-dependent* U(1) rotation under the broken symmetry:

$$\mathcal{D}_\mu\phi \mapsto \mathcal{D}_\mu\phi' = e^{i\alpha\, u(\phi,\epsilon)/f}\ \mathcal{D}_\mu\phi$$

- the field-dependent phase again should have the property:

$$u(\phi,\epsilon)\big|_{\pi_{1,2}=0} = 0$$

$$u(\phi,\epsilon) = \frac{e_1}{f}(\phi^*\epsilon - \epsilon^*\phi)$$

let's recap here. we postulate

$$\phi \mapsto \phi' = \phi + \epsilon - \frac{c_1}{f^2}(\phi^* \epsilon - \epsilon^* \phi)\phi$$

$$\mathcal{D}_\mu \phi = \partial_\mu \phi - \frac{d_1}{f^2}(\phi\, \partial_\mu \phi^* - \partial_\mu \phi\ \phi^*)\phi$$

$$u(\phi, \epsilon) = \frac{e_1}{f}(\phi^* \epsilon - \epsilon^* \phi)$$

and require

$$\mathcal{D}_\mu \phi \mapsto \mathcal{D}_\mu \phi' = e^{i\alpha\, u(\phi,\epsilon)/f}\ \mathcal{D}_\mu \phi$$

which is not difficult to solve

$$d_1 = -c_1/2$$

$$e_1 = -3i\ c_1/2$$

- the two-deriative effective lagrangian is built out of the covariant derivative:

$$\begin{aligned}\mathcal{L}^{(2)} &= \mathcal{D}_\mu \phi^* \mathcal{D}^\mu \phi \\ &= \partial_\mu \phi^* \partial^\mu \phi - \frac{c_1}{f^2} |\partial_\mu \phi^* \ \phi - \partial_\mu \phi \ \phi^*|^2 + \mathcal{O}(1/f^4)\end{aligned}$$

- this process can be continued order-by-order in 1/f:

$$\mathcal{D}_\mu \phi = \partial_\mu \phi + \phi \frac{\partial_\mu \phi^* \ \phi - \partial_\mu \phi \ \phi^*}{2|\phi|^2} \left(1 - \frac{\tilde{f}}{|\phi|} \sin \frac{|\phi|}{\tilde{f}} \right)$$

$$\mathcal{L}^{(2)} = \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi = \partial_\mu \phi^* \partial^\mu \phi - \frac{|\partial_\mu \phi^* \ \phi - \partial_\mu \phi \ \phi^*|^2}{4|\phi|^2} \left(1 - \frac{\tilde{f}^2}{|\phi|^2} \sin^2 \frac{|\phi|}{\tilde{f}} \right)$$

$$\tilde{f} = f/\sqrt{6c_1}$$

a few comments:

- we managed to derive the two-derivative lagrangian without referring to any symmetry breaking pattern.

- the only undetermined parameter, c_1, reflects the arbitrariness in the normalization of "f".

$$\mathcal{D}_\mu \phi = \partial_\mu \phi + \phi \frac{\partial_\mu \phi^* \, \phi - \partial_\mu \phi \, \phi^*}{2|\phi|^2} \left(1 - \frac{\tilde{f}}{|\phi|} \sin \frac{|\phi|}{\tilde{f}} \right)$$

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$$\tilde{f} = f / \sqrt{6 c_1}$$

a few comments:

- the sign of c_1 is not fixed:
  a positive sign implies a compact G/H (suppression), while a negative sign implies a non-compact G/H (enhancement).

- if UV completion is a concern, $c_1 > 0$ and the sign of the dim-6 operator is negative.

$$\mathcal{D}_\mu \phi = \partial_\mu \phi + \phi \frac{\partial_\mu \phi^* \, \phi - \partial_\mu \phi \, \phi^*}{2|\phi|^2} \left(1 - \frac{\tilde{f}}{|\phi|} \sin \frac{|\phi|}{\tilde{f}} \right)$$

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$$\tilde{f} = f / \sqrt{6 c_1}$$

- one could introduce another object that transforms non-homogeneously like a gauge field:

$$\mathcal{E}_\mu \mapsto e^{-iu}\mathcal{E}_\mu e^{iu} - ie^{-iu}\partial_\mu e^{iu} = \mathcal{E}_\mu + \partial_\mu u(\phi, \epsilon)$$

$$\mathcal{E}_\mu = \frac{i}{\alpha}\frac{\partial_\mu\phi^*\,\phi - \partial_\mu\phi\,\phi^*}{|\phi|^2}\sin^2\frac{|\phi|}{2\tilde{f}}$$

Let's pause for a moment and reflect on what's happened…

We derived the two-derivative lagrangian for a complex Goldstone boson charged under an unbroken U(1):

$$\mathcal{L}^{(2)} = \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi = \partial_\mu \phi^* \partial^\mu \phi - \frac{|\partial_\mu \phi^* \, \phi - \partial_\mu \phi \, \phi^*|^2}{4|\phi|^2} \left(1 - \frac{\tilde{f}^2}{|\phi|^2} \sin^2 \frac{|\phi|}{\tilde{f}} \right)$$

The only assumptions are

1. The Adler's zero condition.
2. There exists an unbroken U(1).

As such, this is the universal lagrangian among all nlsm's containing a complex Goldstone!

We can check against the universality using explicit examples.

In SU(2)/U(1):

$$\xi = e^{i\Pi/f}\ , \quad \Pi = \sum_{a=1,2} \phi^a X^a = \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix}$$

$$|\partial_\mu \phi|^2 - \frac{1}{3f^2}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2 + \frac{8}{45f^4}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2 |\phi|^2 \\ - \frac{16}{315f^6}\left|\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right|^2 |\phi|^4 + \cdots \ .$$

In SU(5)/SO(5):

$$\Pi = \begin{pmatrix} & \frac{H}{\sqrt{2}} & \\ \frac{H^\dagger}{\sqrt{2}} & & \frac{H^T}{\sqrt{2}} \\ & \frac{H^*}{\sqrt{2}} & \end{pmatrix} \qquad H = (h^+, h^0)^T$$

$$|\partial_\mu h^+|^2 - \frac{1}{48f^2}\left|h^-\partial_\mu h^+ - h^+\partial_\mu h^-\right|^2 + \frac{1}{1440f^4}\left|h^-\partial_\mu h^+ - h^+\partial_\mu h^-\right|^2 |h^+|^2 \\ - \frac{1}{161280f^6}\left|h^-\partial_\mu h^+ - h^+\partial_\mu h^-\right|^2 |h^+|^4 + \cdots \ .$$

The two lagrangians agree under f -> 4f !

After the "warm-up" exercise, let's see how to extend this approach to the general case.

We assume a set of scalars furnishing a linear representation under a simple Lie group H:

$$\pi^a(x) \to \pi^a(x) + i\alpha^i (T^i)_{ab} \pi^b(x)$$

This is another simple yet important statement:

The infinitesimal action of H on π^a is characterized by its group generators T^i.

A convenient choice of basis for group generators:

For unitary representations all generators can be chosen to be Hermitian

$$(T^i)^\dagger = T^i$$

In addition, we'd like all generators to be purely imaginary (and hence anti-symmetric!)

$$(T^i)^T = -T^i \text{ and } (T^i)^* = -T^i$$

This choice will simplify the analysis and makes the correspondence with CCWZ more transparent.

Now what does the “shift symmetry” look like in the general case?

$$\pi^a \to \pi^a + \epsilon^a + \ ??$$

Now what does the "shift symmetry" look like in the general case?

$$\pi^a \to \pi^a + \epsilon^a + \,??$$

One thing we do know: the shift symmetry cannot take π^a out of the representation of H.

So one must be able to "compensate" the shift symmetry by a field-dependent H-action:

$$|\pi\rangle \to |\pi'\rangle \approx |\pi\rangle + |\epsilon\rangle + i\,\alpha^i(\epsilon,\pi)\,|T^i\pi\rangle$$

$$|T^i\pi\rangle \equiv T^i|\pi\rangle$$

This is simply rephrasing the coset statement:

$$\xi = e^{i\Pi/f}, \quad \Pi = \pi^a X^a,$$
$$g\,\xi = \xi'\,U(g,\xi)\;, \quad U(g,\xi) \in H$$

What's the requirement on $\alpha^i(\pi, \epsilon)$?

- We are only interested in infinitesimal actions. So it should be linear in ε .
- Adler's zero condition must be fulfilled. Ie., when all but one Goldstone boson are turned off, α->0 and we recover the single Goldstone case.

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- We are only interested in infinitesimal actions. So it should be linear in ε .
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A simple "ansatz" is

$$\alpha^i = \frac{1}{f^2} \langle \pi T^i \epsilon \rangle + \mathcal{O}\left(\frac{1}{f^4}\right)$$

$$\langle \pi T^i \epsilon \rangle = \langle \pi | T^i | \epsilon \rangle = \pi^a (T^i)_{ab} \epsilon^b$$

In our basis T^i is anti-symmetric so Adler's zero condition is satisfied!

So the nonlinear shift symmetry at this order in 1/f is

$$|\pi\rangle \to |\pi'\rangle = |\pi\rangle + |\epsilon\rangle + \frac{A_1}{f^2}\,|T^i\pi\rangle\langle\pi T^i\epsilon\rangle$$

We'd like to construct the Goldstone covariant derivative such that, under the above shift symmetry,

$$(*) \qquad |\mathcal{D}\pi\rangle \to |\mathcal{D}\pi'\rangle = e^{i\,u^i(\pi,\epsilon)T^i/f}|\mathcal{D}\pi\rangle$$

where, in similar veins,

$$\begin{aligned} |\mathcal{D}\pi\rangle &= |\partial\pi\rangle + \frac{B_1}{f^2}|T^i\pi\rangle\langle\pi T^i\partial\pi\rangle \\ u^i(\pi,\epsilon) &= \frac{C_1}{f}\langle\pi T^i\epsilon\rangle \ . \end{aligned}$$

Plugging in (*) now gives the following equation:

$$(T^i)_{ab}(T^i)_{cd} - \frac{A_1 + iC_1}{B_1}(T^i)_{ac}(T^i)_{db} + \frac{A_1 - B_1}{B_1}(T^i)_{ad}(T^i)_{bc} = 0$$

$$(T^i)_{ab}(T^i)_{cd} - \frac{A_1 + iC_1}{B_1}(T^i)_{ac}(T^i)_{db} + \frac{A_1 - B_1}{B_1}(T^i)_{ad}(T^i)_{bc} = 0$$

An important observation is a solution to the above equation does NOT always exist.

However, when a solution does exist, we can use the traceless condition of the generators to arrive at

$$A_1 - 2B_1 = 0 \ , \qquad A_1 + B_1 + iC_1 = 0 \ ,$$

and the solution is

$$B_1 = \frac{A_1}{2} \ , \qquad C_1 = -\frac{3}{2} i A_1$$

Now the first equation becomes

$$(T^i)_{ab}(T^i)_{cd} + (T^i)_{ac}(T^i)_{db} + (T^i)_{ad}(T^i)_{bc} = 0$$

This is an important equation we call "Closure Condition:"

$$(T^i)_{ab}(T^i)_{cd} + (T^i)_{ac}(T^i)_{db} + (T^i)_{ad}(T^i)_{bc} = 0$$

It's important because

- It guarantees a solution for A_1, B_1, C_1.
- The solution can be written down without explicitly specifying H.

Thus the solution

$$B_1 = \frac{A_1}{2} \ , \qquad C_1 = -\frac{3}{2} i A_1$$

is universal for any H that satisfies the Closure Condition.

So when does the Closure Condition hold?

$$(T^i)_{ab}(T^i)_{cd} + (T^i)_{ac}(T^i)_{db} + (T^i)_{ad}(T^i)_{bc} = 0$$

If one stares at it for a while, it smells like a Jacobi identity….

So when does the Closure Condition hold?

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If one stares at it for a while, it smells like a Jacobi identity….

Indeed, for a symmetric coset G/H such that

$$[T^i, X^a] = if^{iab}X^b$$

The Jacobi identity among X^a, X^b, and X^c gives

$$f^{iab}f^{icd} + f^{ibc}f^{iad} + f^{ica}f^{ibd} = 0$$

So if we can identify

$$(T^i)_{ab} \sim f^{iab}$$

we would be golden…

To this end, we only have to recall some very basic facts about adjoint representation of a Lie algebra G, where $|X^a\rangle$ is the state vector upon which the generator acts on:

$$T^i|X^a\rangle = |[T^i, X^a]\rangle = -if^{iba}|X^b\rangle$$

which implies $-i f^{iba}$ is nothing but the matrix entry of T^i in the representation R of H, under which X^a furnishes:

$$(T^i)_{ab} = -if^{iab}$$

Given the convention that the structure constants are real, $(T^i)_{ab}$ is purely imaginary and anti-symmetric! This is the reason we chose this particular basis for the group generator.

This only requires R can be embedded in a symmetric coset. But in general the coset containing R does not have to be symmetric.

Some examples that can be embedded in a symmetric coset G/H:

- Fundamental representation of SO(N) can be embedded in SO(N+1)/SO(N).
- Any adjoint representation of G can be embedded in GxG/G
- Fundamental representation of SU(N) cannot be embedded in a symmetric coset, because SU(N+1)/SU(N) is not symmetric.

However, one can use fundamental rep. of SU(N)xU(1), which is the phenomenologically important case.

Now moving to higher orders in 1/f, our ansatz becomes

$$
\begin{aligned}
|\pi'\rangle &= |\pi\rangle + \sum_{n=0}^{\infty} \frac{A_n}{f^{2n}} \left(|T^i\pi\rangle\langle\pi T^i| \right)^n |\epsilon\rangle \ , \\
|\mathcal{D}\pi'\rangle &= \sum_{n=0}^{\infty} \frac{B_n}{f^{2n}} \left(|T^i\pi\rangle\langle\pi T^i| \right)^n |\partial\pi\rangle \ , \\
u^i(\pi,\epsilon) &= \langle\pi T^i| \sum_{n=1}^{\infty} \frac{C_n}{f^{2n-1}} \left(|T^i\pi\rangle\langle\pi T^i| \right)^n |\epsilon\rangle \ .
\end{aligned}
$$

It turns out the Closure Condition ensures a unique solution for all A_n, B_n and C_n , without having to specify explicitly G or H!

For example, at $1/f^4$ we have

$$
A_2 = -\frac{1}{5}A_1^2 \ , \qquad B_2 = \frac{3}{40}A_1^2 \ , \qquad C_2 = \frac{3i}{8}A_1^2
$$

In the end, one can obtain the complete CCWZ lagrangian:

$$|\mathcal{D}\pi\rangle = \left(\frac{\sin\sqrt{\mathcal{T}}}{\sqrt{\mathcal{T}}}\right)|\partial\pi\rangle \ , \qquad \mathcal{T} = \frac{1}{f^2}|T^i\pi\rangle\langle\pi T^i|$$

$$\mathcal{E}^i = \frac{2i}{f^2}\langle\partial\pi|\,\frac{1}{\mathcal{T}}\sin^2\frac{\sqrt{\mathcal{T}}}{2}\,|T^i\pi\rangle$$

The only knowledge required is the linear representation of H under which the Goldstones furnish. We do not need to know the broken group G in the UV!

- we derived a lagrangian that is universal for Goldstones transforming under a given linear representation of H, without making any reference to the broken group G in the UV.

- Different coset simply corresponds to different normalization for the decay constant "f".

- From the CCWZ perspective, this universality is rather surprising. From the IR perspective, the only essential ingredient is the Adler's zero condition.

- All the higher-order nonlinear interactions only serve to fulfill Adler's zero condition amid the constraints from the linearly realized H!

Last but not least, what's the implication of this universality for pNGB Higgs boson?

It suggests that we can write down a "minimal lagrangian" for all composite Higgs models by postulating:

- Four shift symmetries acting on the four real component of the Higgs doublet, subject to the constraint of electroweak SU(2)xU(1) symmetry.
- Adler's zero condition.

We can derive the lagrangian at leading order in 1/f:

$$H \mapsto H' = H + \epsilon + \frac{c_1}{f^2}\left[(\epsilon^\dagger H + H^\dagger \epsilon)H - 2(H^\dagger H)\epsilon\right]$$

$$\begin{aligned}\mathcal{L}^{(2)} &= \mathcal{D}_\mu H^\dagger \mathcal{D}^\mu H \\ &= \partial_\mu H^\dagger \partial^\mu H + \frac{c_H}{2f^2}\mathcal{O}_H + \frac{c_r}{2f^2}\mathcal{O}_r + \frac{c_T}{2f^2}\mathcal{O}_T\end{aligned}$$

again we have rescaled f: $f \to f/\sqrt{c_1}$

at leading order: $c_H = 1/6\ , \quad c_T = -1/2\ , \quad c_r = -4c_H\ .$

Again the signs depend on whether the coset is compact or non-compact.

An all-order expression is not difficult to obtain:

$$
\begin{aligned}
\mathcal{L}^{(2)} &= \mathcal{D}_\mu H^\dagger \mathcal{D}^\mu H \\
&= \partial_\mu H^\dagger \partial^\mu H + \frac{c_H}{2f^2}\mathcal{O}_H + \frac{c_r}{2f^2}\mathcal{O}_r + \frac{c_T}{2f^2}\mathcal{O}_T
\end{aligned}
$$

$$
c_H = \frac{f^2}{H^\dagger H}\left(1 - \frac{\sin\frac{\sqrt{H^\dagger H}}{f}}{\frac{\sqrt{H^\dagger H}}{f}}\right)
$$

$$
c_T = -2\frac{f^3}{(H^\dagger H)^{3/2}}\sin\frac{\sqrt{H^\dagger H}}{f}\ \sin^2\frac{\sqrt{H^\dagger H}}{2f}
$$

$$
c_r = -4c_H
$$

- if one would like to impose the custodial symmetry, the higgs is a vector under SO(4).

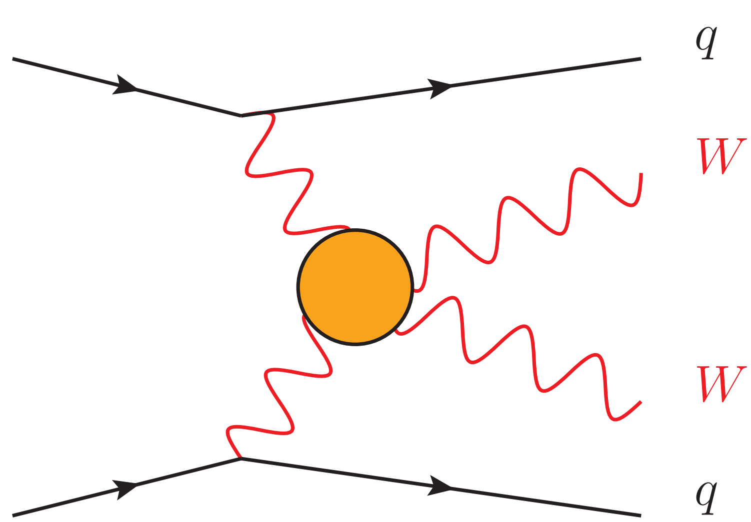
- two-derivative lagrangian is the same as SU(2)xU(1), by setting $c_T = 0$.

- the four-derivative lagrangian will differ, because of the SO(4) symmetry acting on the SU(2) and U(1) gauge fields.

What are the predictions of a minimally symmetric Higgs?

Worth emphasizing that only the G/H symmetry-preserving part of the lagrangian is universal. The symmetry-breaking part is model-dependent.

Therefore we need to look at very high energy scatterings of vector bosons:



Perhaps well above the scale of any composite resonances.

To this end, we can match to the effective lagrangian:

$$\frac{1}{2}\partial_\mu h \partial^\mu h - V(h) + \frac{v^2}{4}\mathrm{Tr}\left(D_\mu \Sigma^\dagger D^\mu \Sigma\right)\left[1 + 2a\frac{h}{v} + b\frac{h^2}{v^2} + b_3\frac{h^3}{v^2} + \cdots\right]$$

$$\Sigma = e^{i\sigma^a \pi^a(x)/v}$$

In the minimally symmetric Higgs, we have

$$a = \sqrt{1 - \frac{v^2}{2f^2}}\,, \qquad b = 1 - \frac{v^2}{f^2}\,, \qquad b_3 = -\frac{2}{3}\frac{v^2}{f^2}\sqrt{1 - \frac{v^2}{2f^2}}$$

These parameters enter into the vector boson scatterings:

Contino et. al.: 1309.7038

$$\mathcal{A}(W_L W_L \to Z_L Z_L) \approx \frac{s}{v^2}(1-a^2)$$
$$\mathcal{A}(W_L W_L \to hh) \approx \frac{s}{v^2}(b-a^2)$$
$$\mathcal{A}(V_L V_L \to hhh) = \frac{is}{v^3}(4ab-4a^3-3b_3)$$

The predictions from a minimally symmetric Higgs are:

$$\mathcal{A}(W_L W_L \to Z_L Z_L) = -\mathcal{A}(W_L W_L \to hh) = \frac{s}{2f^2} + \mathcal{O}(s^0)$$
$$\mathcal{A}(V_L V_L \to hhh) = 0$$

These predictions are universal in a PNGB Higgs!

summary:

- the self-interaction of goldstone bosons is an infrared property, dictated by the IR quantum number, instead of the symmetry breaking pattern in the UV.

- the nonlinear shift symmetry allows one to compute self-interactions of the goldstone boson, without ever referring to a particular UV symmetry breaking pattern.