Nonequilibrium dynamics of Φ^4 theory in the two-particle point-irreducible formalism

Andreas Heinen

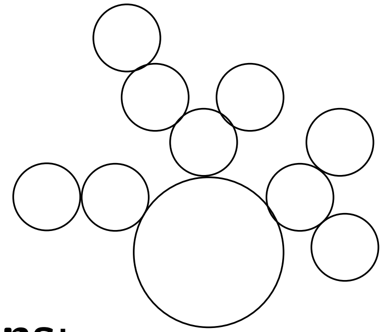
June 28, 2003

J. Baacke, A.H., Phys. Rev. **D67** 105020 (2003) [hep-ph/0212312]
J. Baacke, A.H., [hep-ph/0305220]

Universität Dortmund

• 1. Introduction •

- Applications of nonequilibrium quantum field theory: from **(p)reheating** during **cosmic inflation** to relativistic **heavy ion collisions**



- **Leading order** approximations (with self-consistent resummation of diagrams) → **large-**N and **Hartree** approximation

- Simulations beyond the leading orders in $1+1$ **dimensions**:
  - testing ground (renormalization easy, numerical simulations less involved)
  - **Hartree** approximation gives a **first order phase transition** → in contrast to the expectation of the absence of spontaneous symmetry breaking
  - Hartree approximation displays (almost) no dissipative behavior for the classical field → inflaton needs **dissipative dynamics**

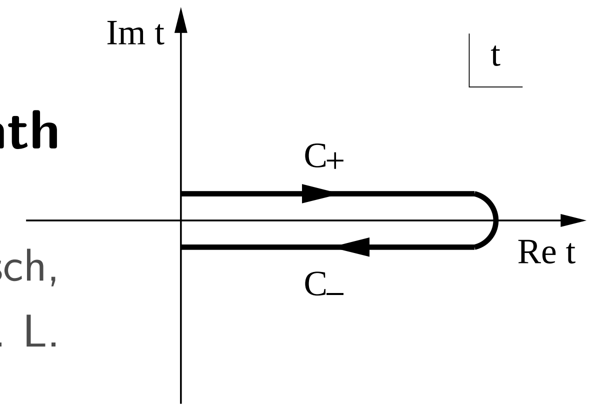
• 2. Nonequilibrium quantum field theory •

Action for the Φ^4 model:

$$S[\Phi] = \int d^D x \left[\frac{1}{2} \partial_\mu \Phi(x) \partial^\mu \Phi(x) - \frac{1}{2} m^2 \Phi^2(x) - \frac{\lambda}{4!} \Phi^4(x) \right] \qquad (1)$$

- Split Φ into classical expectation values $\phi = \langle \Phi \rangle$ and quantum parts via $\Phi(t, \mathbf{x}) = \phi(t) + \eta(t, \mathbf{x}) \rightarrow$ assume **spatially homogeneous** fields

- One needs **causal dynamics** $\rightarrow$ **closed time path** (CTP) formalism;
  J. Schwinger, J. Math. Phys. **2** 407-432(1961); L. V. Keldysch, Zh. Eksp. Teor. Fiz. **47** 1515-1527 (1964); E. Calzetta, B. L. Hu, Phys. Rev. **D37** 2878 (1988)



Approximation schemes

An approximation scheme for QFT out of thermal equilibrium has to be at least:

- non-perturbative ($\rightarrow$ **resumming** perturbative diagrams)

- **energy conserving** ($\rightarrow$ variational principle)

- **renormalizable** ($\rightarrow$ compare to the equilibrium case)

A reasonable choice: the **two-particle irreducible** (2PI) effective action or Cornwall-Jackiw-Tomboulis (CJT) action! $\rightarrow$ resummation of 2PI graphs.

J. M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. **D10**, 2428 (1974)

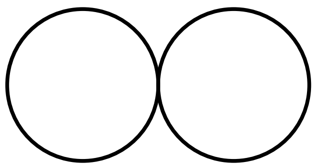
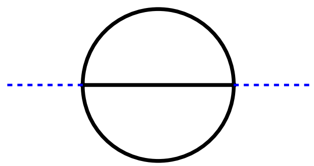
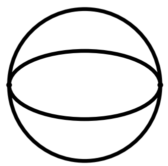
The 2PI effective action

Effective action from a Legendre transformation of $-i \ln Z[J, K]$ with sources $J(x)$ and $K(x, y)$

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}(D^{-1}G) + \Gamma^{2\mathrm{PI}}[\phi, G] \quad (2)$$

$$iD^{-1}(x, y) = \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)} \, . \quad (3)$$

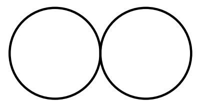
Sum over all 2PI diagrams:

$\Gamma^{2\mathrm{PI}}[\phi, G] =$  $+$  $+$  $+ \quad \ldots$

Equations of motion for the variational parameters ϕ and G

$$\frac{\delta \Gamma[\phi, G]}{\delta \phi} \;=\; 0 \quad ; \quad \frac{\delta \Gamma[\phi, G]}{\delta G} = 0 \Rightarrow G^{-1} = D^{-1}[\phi] - \underbrace{2i \frac{\delta \Gamma^{2\mathrm{PI}}[\phi, G]}{\delta G}}_{\Sigma[\phi, G]} \qquad (4)$$

Remarks:

- the **Schwinger-Dyson** equation $G^{-1} = D^{-1}[\phi] - \Sigma[\phi, G]$ resums 2PI graphs
- for the nonequilibrium case the Schwinger-Dyson equation is a **partial integro-differential equation** $\rightarrow$ solution in general numerically involved
- reproduces the **Hartree** (**large**-N) approximation if only (parts of)  is included $\rightarrow$ leading order
- renormalization beyond the leading orders?

• 3. The two-particle point-irreducible (2PPI) formalism •

H. Verschelde, M. Coppens, Phys. Lett. **B287**, 133 (1992)
M. Coppens, H. Verschelde, Z. Phys. **C58**, 319 (1993)

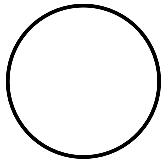
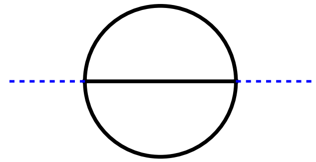
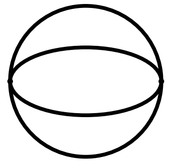
Idea: *local sources* $(K(x,y) \to K(x)\delta(x-y))$ for the 2PI effective action $\to$ 2PPI effective action.

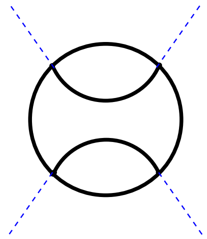
$$\begin{aligned}\Gamma[\phi, \mathcal{M}^2] &= \int d^D x \left[\frac{1}{2}\partial_\mu \phi(x)\, \partial^\mu \phi(x) - \frac{1}{2}\mathcal{M}^2(x)\phi^2(x) + \frac{\lambda}{12}\phi^4(x) \right. \\ &\qquad \left. + \frac{1}{2\lambda}\left(\mathcal{M}^2(x) - \mu^2\right)^2 \right] + \Gamma^{\mathrm{2PPI}}[\phi, \mathcal{M}^2] \qquad (5)\end{aligned}$$

Equations of motion:

$$\frac{\delta \Gamma[\phi, \mathcal{M}^2]}{\delta \mathcal{M}^2} = 0 \qquad ; \qquad \frac{\delta \Gamma[\phi, \mathcal{M}^2]}{\delta \phi} = 0 \qquad (6)$$

Sum over all 2PPI diagrams:

$\Gamma^{2\mathrm{PPI}}[\phi, \mathcal{M}^2] \quad =$  $+$  $+$ 

$+$  $+ \quad \ldots$

Gap equation (Schwinger-Dyson equation) from $\frac{\delta \Gamma}{\delta \mathcal{M}^2} = 0$:

$$\mathcal{M}^2(t) = m^2 + \frac{\lambda}{2}\phi^2(t) + \frac{\lambda}{2}\Delta(t) \tag{7}$$

$$\Delta(t) := -2\frac{\delta \Gamma^{2\mathrm{PPI}}[\phi, \mathcal{M}^2]}{\delta \mathcal{M}^2(t)} \tag{8}$$

Green's function (local equation):

$$G^{-1}(x,x') = i\left[\Box + \mathcal{M}^2(x)\right]\delta^D(x-x') \tag{9}$$

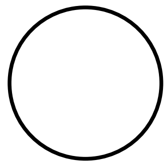
In momentum space **factorization** in **mode functions**

$$\begin{aligned} G_>(t,t';\mathbf{p}) &= \frac{1}{2\omega_p}(2n_p+1)f(t;\mathbf{p})f^*(t';\mathbf{p}) && (10)\\ 0 &= \ddot{f}(t;\mathbf{p}) + \left[\mathbf{p}^2 + \mathcal{M}^2(t)\right]f(t;\mathbf{p})\ , && (11)\end{aligned}$$

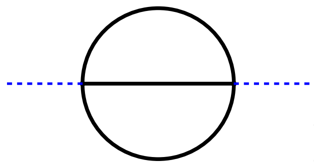
with $\omega_p = \sqrt{\mathbf{p}^2 + \mathcal{M}^2(0)}$.
$\rightarrow$ The Green's function G *is no variational parameter* of the theory. **The variational parameters are ϕ and $\mathcal{M}^2$!**

Equations of motion in the two-loop approximation

Two-loop approximation: $\Gamma^{2\text{PPI}} = \Gamma^{(1)} + \Gamma^{(2)}$ with

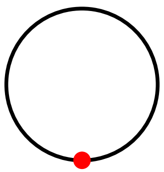


$$\Gamma^{(1)}[\mathcal{M}^2] = \frac{i}{2}\mathrm{Tr}\ln[G^{-1}(\mathcal{M}^2)] \qquad (12)$$



$$\Gamma^{(2)}[\phi,\mathcal{M}^2] = i\frac{\lambda^2}{12}\int d^{D-1}x\, d^{D-1}x' \int_{\mathrm{CTP}} dt\, dt' \times\phi(t)\phi(t')G^3(t,t';\mathbf{x},\mathbf{x}') \qquad (13)$$

One-loop part of Δ:



$$\Delta^{(1)}(t) = -2\frac{\delta\Gamma^{(1)}[\phi,\mathcal{M}^2]}{\delta\mathcal{M}^2(t)} = \int\frac{d^{D-1}p}{(2\pi)^{D-1}}G_>(t,t;\mathbf{p}) \qquad (14)$$

Specialize to $1+1$ dimensions. . . $\rightarrow$ renormalization:

$$m^2 \rightarrow m^2 + \delta m^2 \tag{15}$$

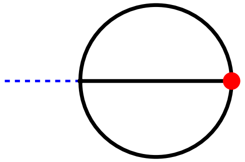
$$\Rightarrow \delta m^2 + \frac{\lambda}{2}\Delta^{(1)}(t) = \frac{\lambda}{8\pi}\ln\frac{|m^2|}{m_0^2} + \Delta^{(1)}_{\rm fin}(t) \tag{16}$$

Finite one-loop part:

$$\Delta^{(1)}_{\rm fin}(t) = \int \frac{dp}{2\pi}\left[G_>(t,t;p) - \frac{1}{2\omega_p}\right] = \int \frac{dp}{2\pi 2\omega_p}\left[(2n_p+1)\,|f(t,p)|^2 - 1\right] \tag{17}$$

Two-loop contributions:

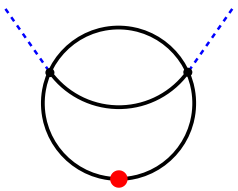
$$\mathcal{S}(t) = -\frac{\delta\Gamma^{(2)}[\phi,\mathcal{M}^2]}{\delta\phi(t)} \quad ; \quad \Delta^{(2)}(t) = -2\frac{\delta\Gamma^{(2)}[\phi,\mathcal{M}^2]}{\delta\mathcal{M}^2(t)} \tag{18}$$



$$\mathcal{S}(t) = -i\frac{\lambda^2}{6}\int_0^t dt'\phi(t')\int\prod_{\ell=1}^{3}\left(\frac{dp_\ell}{2\pi}\right)2\pi\delta\left(\sum_{\ell=1}^{3}p_\ell\right) \times\left[\prod_{\ell=1}^{3}G_>(t,t';p_\ell)-\prod_{\ell=1}^{3}G_>(t',t;p_\ell)\right] \tag{19}$$

and

$$\Delta^{(2)}(t) = -\lambda^2\int_0^t dt'\phi(t')\int_0^{t'}dt''\phi(t'')\int\prod_{\ell=1}^{3}\left(\frac{dp_\ell}{2\pi}\right)2\pi\delta\left(\sum_{\ell=1}^{3}p_\ell\right) \times\left[G_>(t,t';p_3)-G_>(t',t;p_3)\right] \times\left[G_>(t',t'';p_1)G_>(t',t'';p_2)G_>(t,t'';p_3) - G_>(t'',t';p_1)G_>(t'',t';p_2)G_>(t'',t;p_3)\right] . \tag{20}$$



Equations of motion

$$
\begin{aligned}
0 &= \ddot{\phi}(t) + \mathcal{M}^2(t)\phi(t) - \frac{\lambda}{3}\phi^3(t) + \mathcal{S}(t) && (21)\\
\mathcal{M}^2(t) &= m^2 + \frac{\lambda}{2}\left(\phi^2(t) + \Delta^{(1)}(t) + \Delta^{(2)}(t)\right)\ . && (22)
\end{aligned}
$$

Remarks:

- the **selfenergy** is local $\rightarrow$ **local mass** term $\mathcal{M}^2$

- G can be factorized in mode functions $\rightarrow$ the equations of motion are **ordinary differential equations**

- there are memory effects ($\rightarrow$ time integrations over the past of ϕ and G)

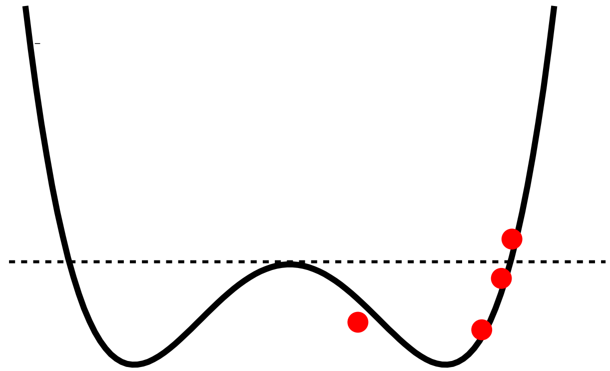
- there is re-scattering of the quanta ($\rightarrow$ two momentum integrations)

- less powerful in resumming diagrams (compared to the 2PI effective action)

- the problem of renormalization in $3+1$ dimensions has been solved in equilibrium QFT [explicit two-loop calculation at finite temperature: G. Smet et al., Phys. Rev. **D65** 045015 (2002)]

Initial conditions

- Gaussian initial density matrix $\rightarrow$ Fock space
  - nonzero value $\phi(0)$ for the mean field
  - $f(0,p) = 1,\ \dot{f}(0,p) = -i\omega_p$
  - initial ensemble n_p for the quanta $\rightarrow$ Bogoliubov rotated Fock space



- at $t = 0$: quanta like free particles with a mass $m_0^2 = \mathcal{M}^2(0)$; m_0^2 has to be determined self consistent

• 4. Numerical simulations in the two-loop approximation •

Numerical implementation:

- Runge-Kutta algorithm for the integration of the differential equations with stepsize $\Delta t = 0.001$—0.005

- Momentum discretization with $p_{\max} = 15$—20 and $dp \leq 0.05$

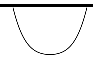
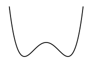
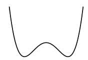
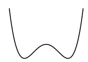
- Energy conserved to at least five significant digits

- Wronskians constant with a relative precision of 10^{-8}

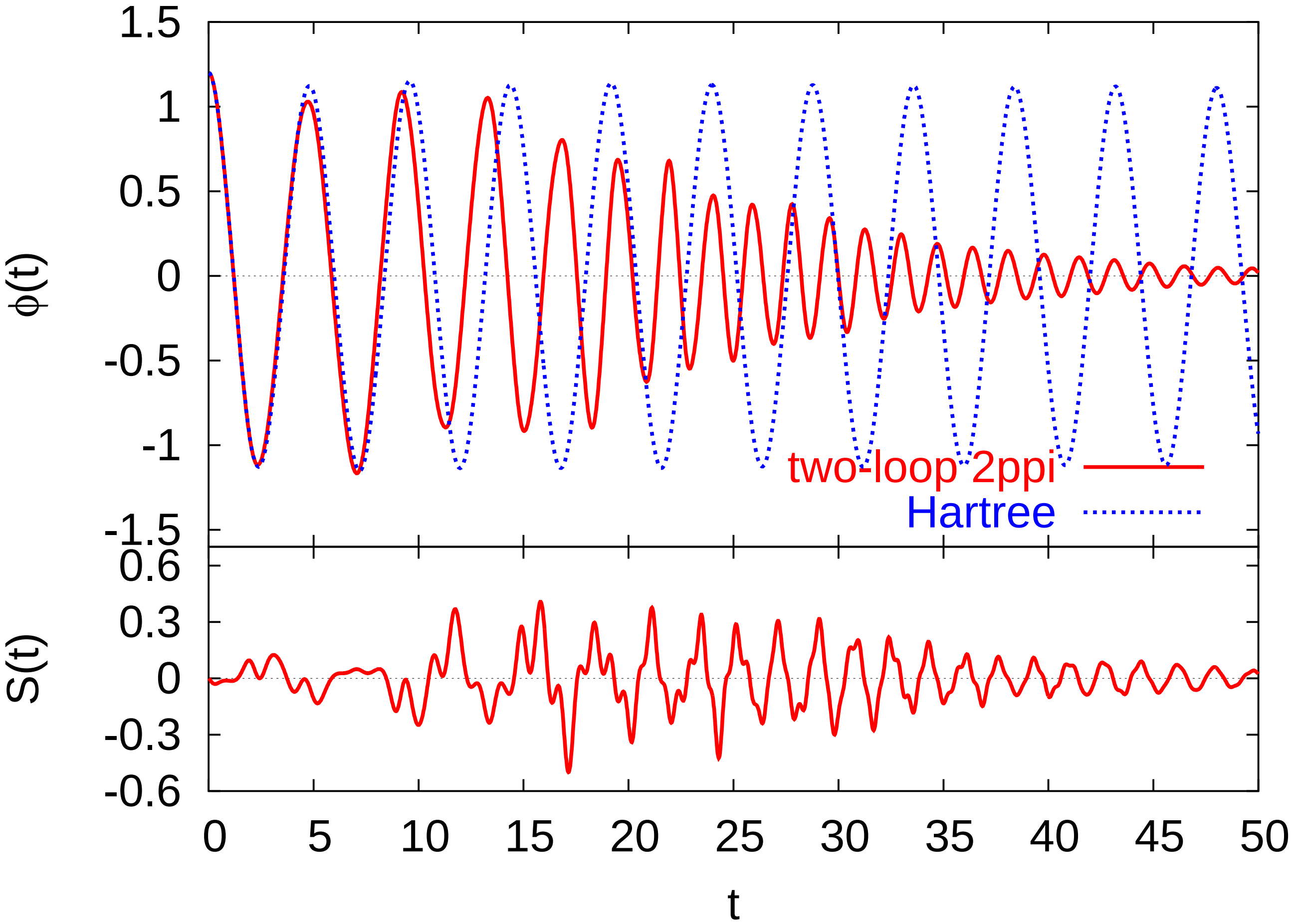
$\rightarrow$ Results compared to the **Hartree** approximation (**one-loop 2PPI**)

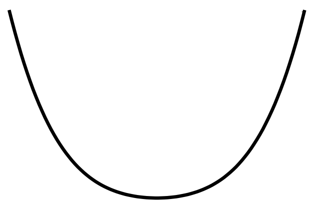
Parameters: Distribution function (Bose-Einstein) for the quanta at $t = 0$

$$n_p = \frac{1}{e^{\omega_p/T_0} - 1} \tag{23}$$

Set	Potential		Initial conditions		λ	m^2
			$\phi(0)$	T_0		
0	symmetric		1.2	0	6	$+1$
1	double well		1.4	0	1	-1/6
2	double well		1.2	0	1	-1/6
3	double well		0.4	0.1	21.9	-1

Symmetric potential: *parameter set 0*: $\phi(t)$ and $\mathcal{S}(t)$



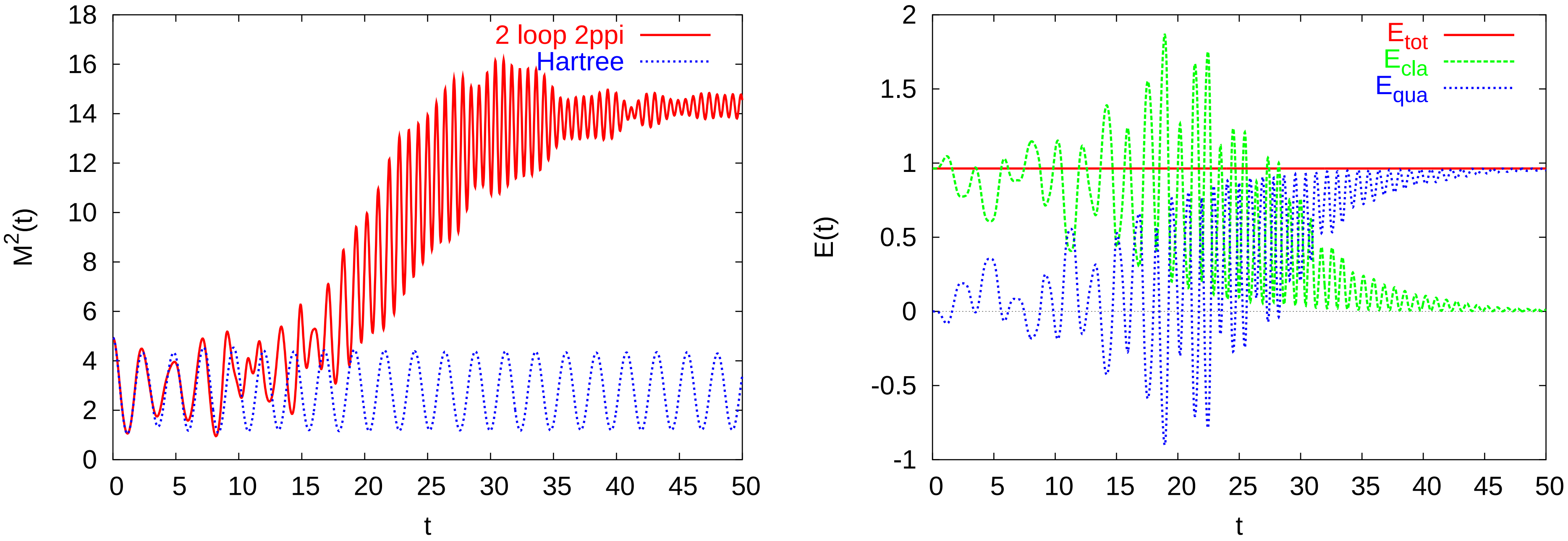


Parameters:

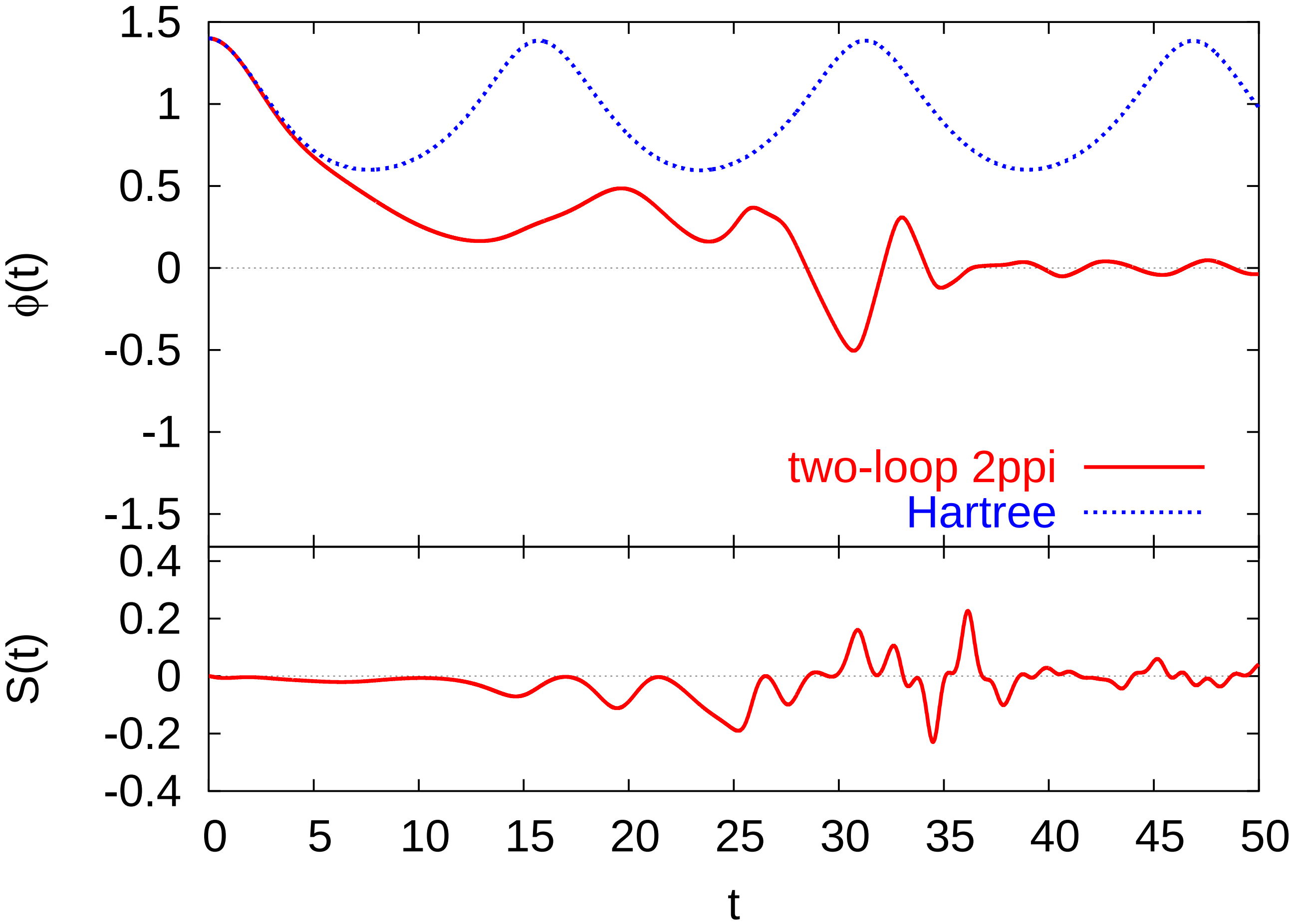
$\phi(0) = 1.2,\ T_0 = 0$

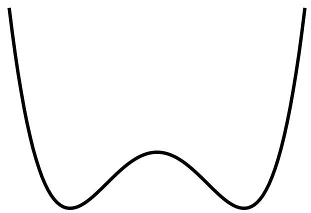
$\lambda = 6,\ m^2 = 1$

Effective mass $\mathcal{M}^2(t)$ (left);
Energy contributions with $E_{\text{cla}} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(m^2 + \delta m^2_{\text{fin}})\phi^2 + \frac{\lambda}{4!}\phi^4$ (right)



Double well potential: *parameter set 1*: near the critical value $\phi(0) < \sqrt{2}$

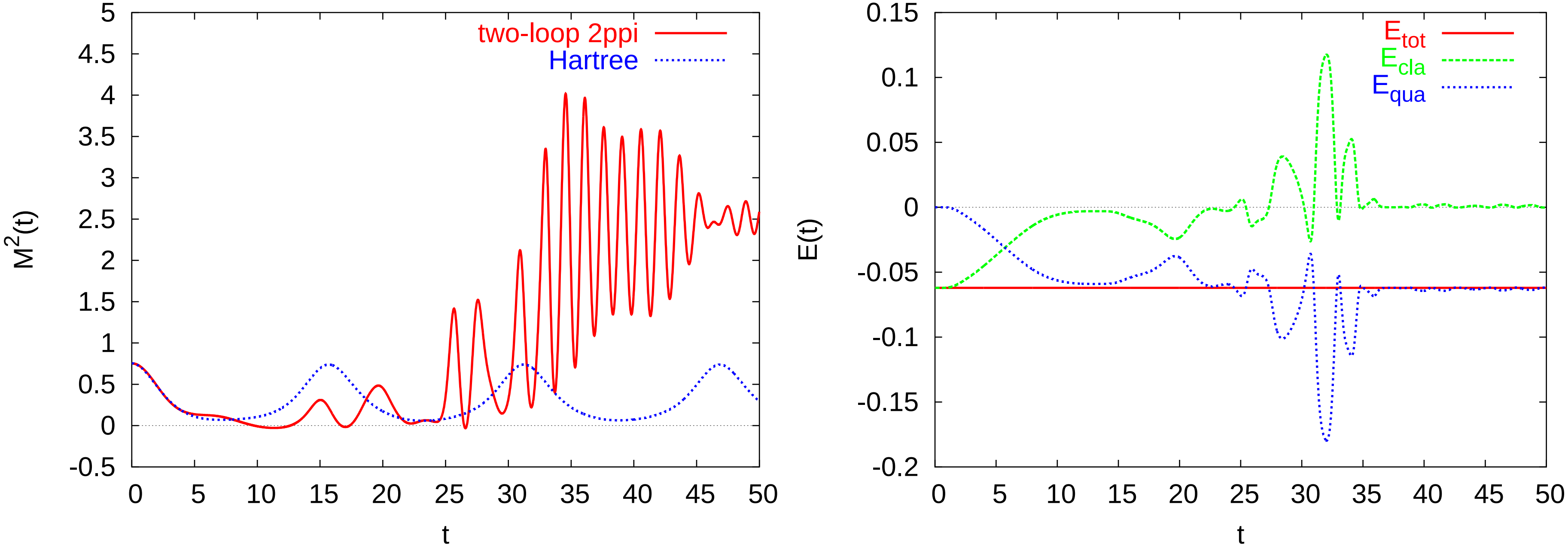




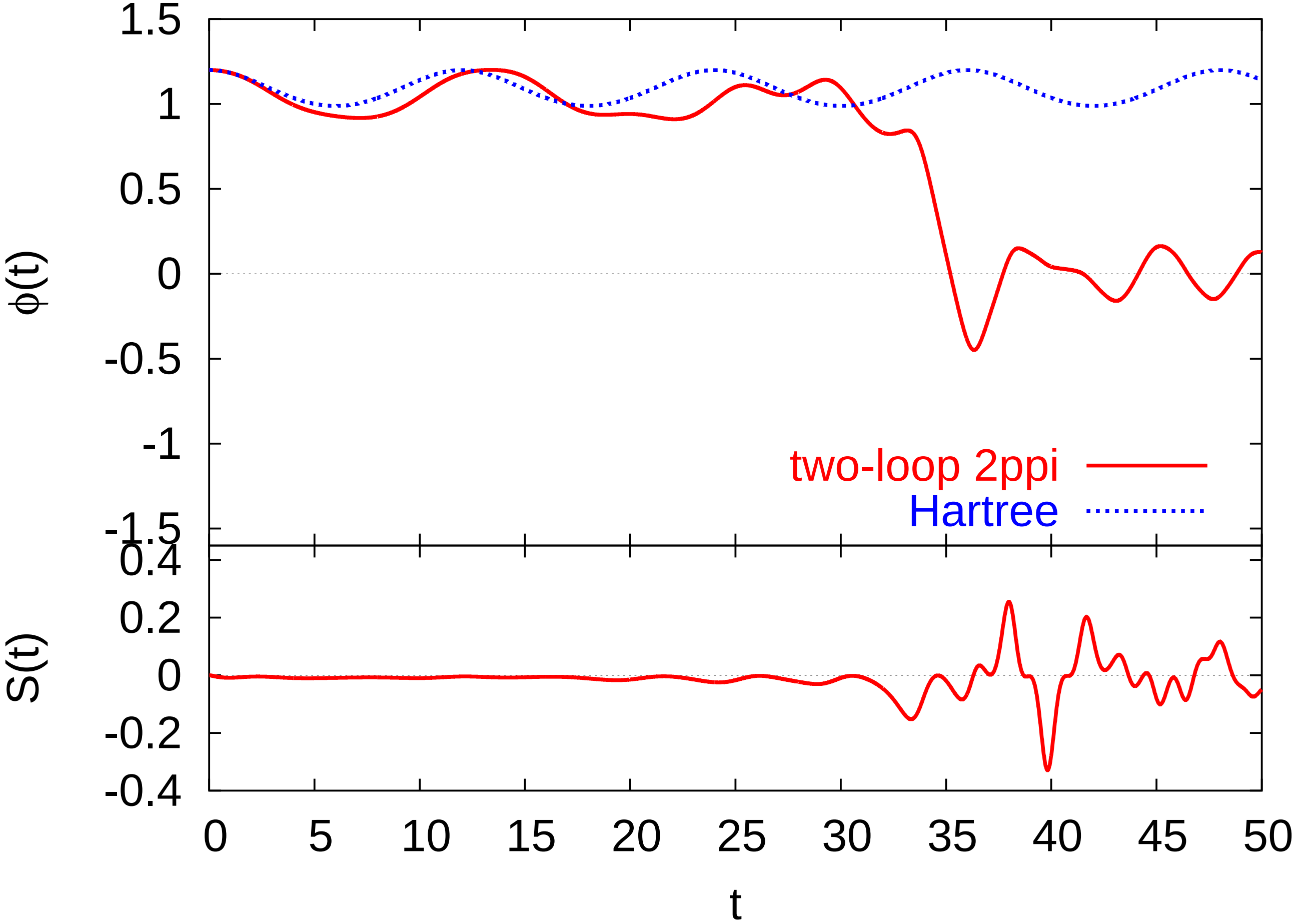
Parameters:

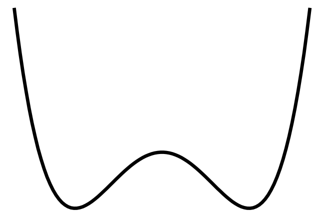
$\phi(0) = 1.4$, $T_0 = 0$

$\lambda = 1$, $m^2 = -\frac{1}{6}$


two-loop 2ppi
Hartree
M²(t)
t
E(t)
E_tot
E_cla
E_qua
0
5
10
15
20
25
30
35
40
45
50
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0.15
0.1
0.05
-0.05
-0.1
-0.15
-0.2

parameter set 2: $\phi(0) = 1.2$

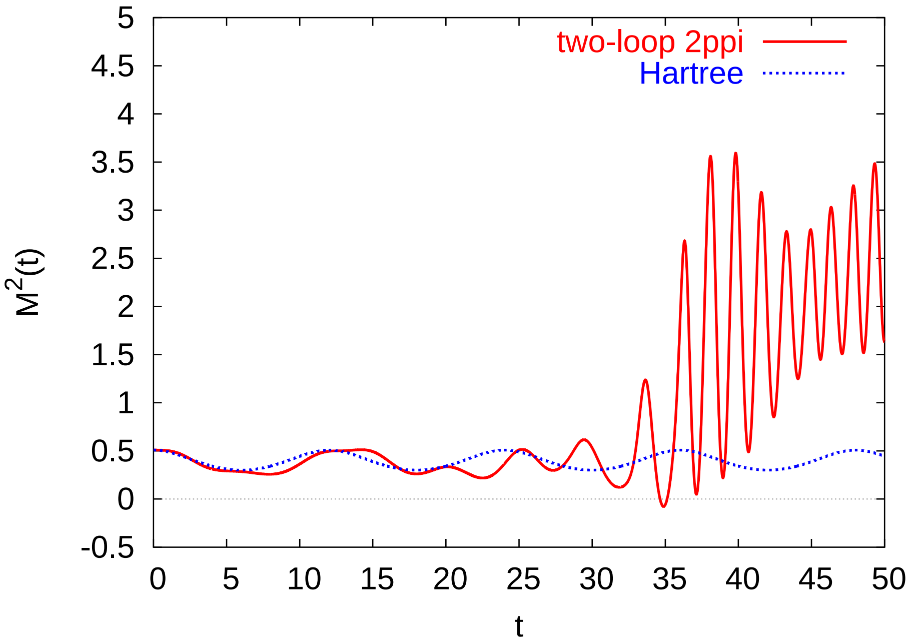




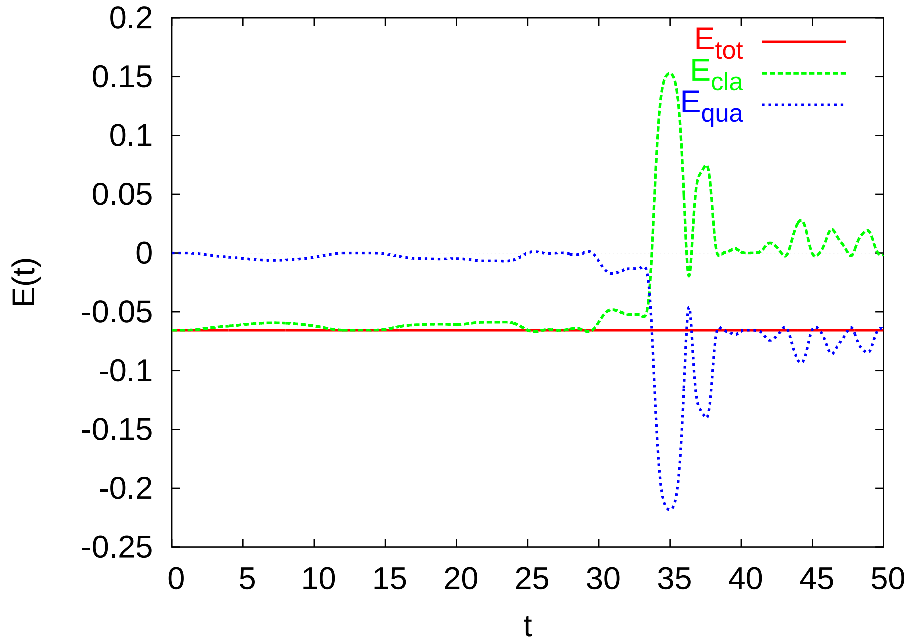
Parameters:

$\phi(0) = 1.2,\ T_0 = 0$

$\lambda = 1,\ m^2 = -\frac{1}{6}$

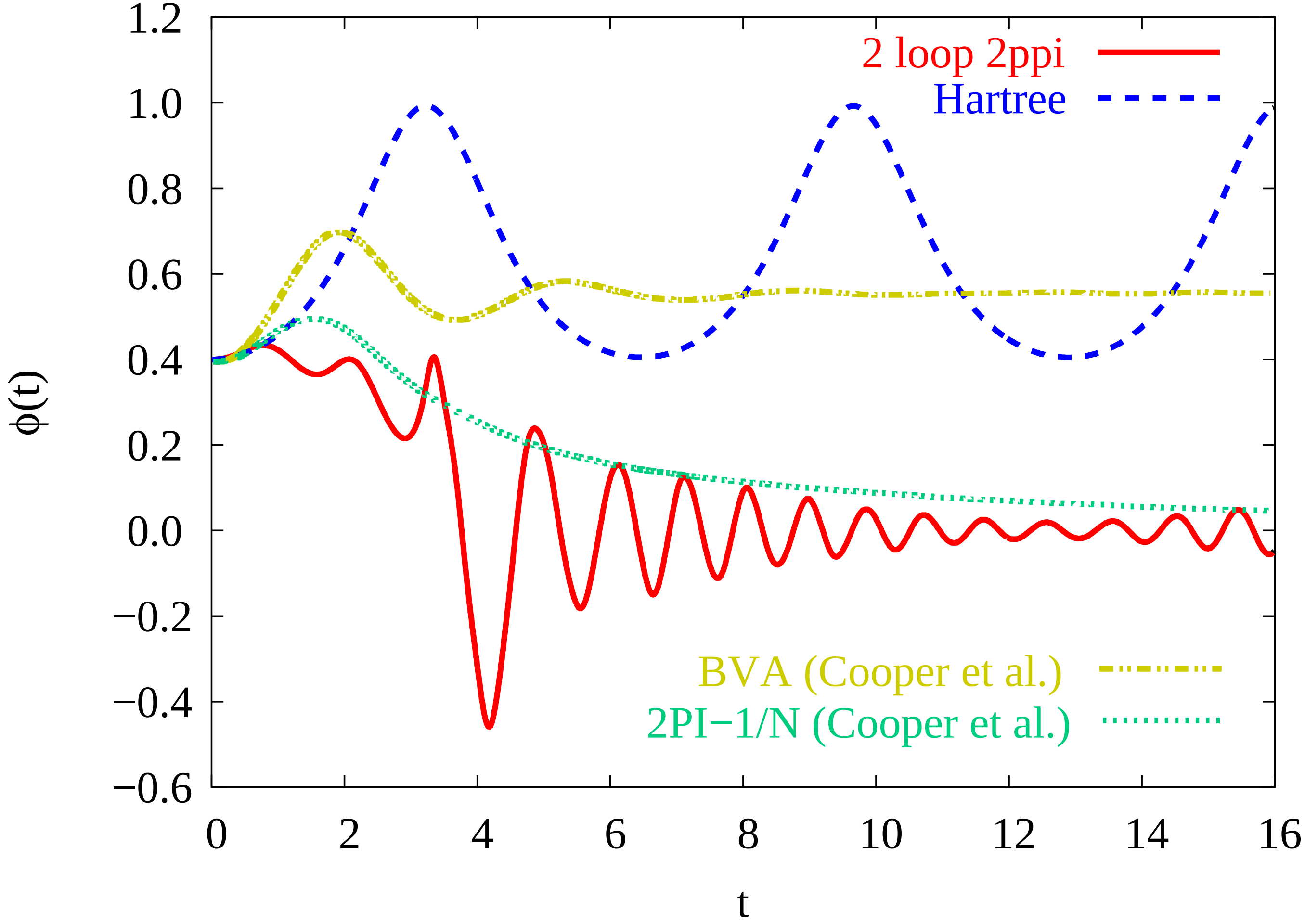


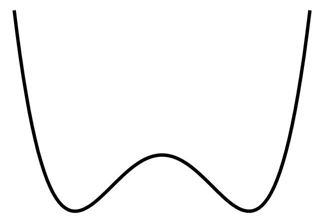
two-loop 2ppi
Hartree
M^2(t)
t



E_tot
E_cla
E_qua
E(t)
t

parameter set 3: a large coupling constant





Parameters:
$\phi(0) = 0.4$, $T_0 = 0.1$
$\lambda = 21.9$, $m^2 = -1$

• 5. Summary and Outlook •

Summary:

- We have presented the **generalization** of the **2PPI** scheme to **nonequilibrium** quantum field theory

- Compared to other methods computationally less involved framework for simulations beyond the large-N or Hartree approximation

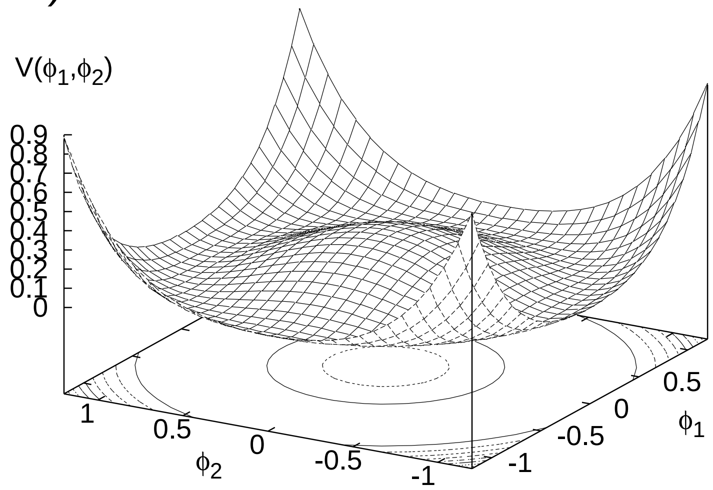
- Numerical simulations in $1 + 1$ dimensions in the two-loop approximation displayed:

  – the **absence of symmetry breaking** as expected from the exact theory
  – **normal dissipative dynamics** for the mean field ϕ
  – this both cures (some) obvious deficits of the Hartree (and large-N) approximation

Outlook:

- **Extension** of the presented formalism **to** $3{+}1$ **dimensions** as well as to O(N) models $\longrightarrow$ the techniques for the **separation of the divergent parts** are available
- Application of the formalism to **hybrid potentials** with additional scalar fields $\longrightarrow$ scenario for the **(p)reheating stage of cosmic inflation**

O(N)



hybrid