Spin Correlations in the $\Lambda\Lambda$ and $\Lambda\bar{\Lambda}$ Systems Generated in Relativistic Heavy-Ion Collisions

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Spin correlations for the $\Lambda\Lambda$ and $\Lambda\bar{\Lambda}$ pairs, generated in relativistic heavy ion collisions, and related angular correlations at the joint registration of hadronic decays of two hyperons with non-conservation of space parity are analyzed. Within the conventional model of one-particle sources, correlations vanish at enough large relative momenta. However, under these conditions, in the case of two non-identical particles ($\Lambda\bar{\Lambda}$) a noticeable role is played by two-particle annihilation (two-quark, two-gluon) sources, which lead to the difference of the correlation tensor from zero. In particular, such a situation may arise when the system passes through the "mixed phase".

1 General Structure of the Spin Density Matrix of the Pairs $\Lambda\Lambda$ and $\Lambda\bar{\Lambda}$

Spin correlations for $\Lambda\Lambda$ and $\Lambda\bar{\Lambda}$ pairs, generated in relativistic heavy ion collisions, and respective angular correlations at joint registration of hadronic decays of two hyperons, in which space parity is not conserved, give important information on the character of multiple processes.

The spin density matrix of the $\Lambda\Lambda$ and $\Lambda\bar{\Lambda}$ pairs, just as the spin density matrix of two spin-1/2 particles in general, can be presented in the following form [1–3]:

$$\hat{\rho}^{(1,2)} = \frac{1}{4}\left[\hat{I}^{(1)} \otimes \hat{I}^{(2)} + (\hat{\boldsymbol{\sigma}}^{(1)}\mathbf{P}_1) \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes (\hat{\boldsymbol{\sigma}}^{(2)}\mathbf{P}_2) + \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ik}\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}\right]; \qquad (1)$$

in doing so, $tr_{(1,2)}\hat{\rho}^{(1,2)} = 1$.

Here $\hat{I}$ is the two-row unit matrix, $\boldsymbol{\sigma} = (\hat{\sigma}_x,\, \hat{\sigma}_y,\, \hat{\sigma}_z)$ is the vector Pauli operator $(x, y, z \to 1, 2, 3)$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are the polarisation vectors of first and second particle ($\mathbf{P}_1 = \langle\hat{\boldsymbol{\sigma}}^{(1)}\rangle$, $\mathbf{P}_2 = \langle\hat{\boldsymbol{\sigma}}^{(2)}\rangle$), $T_{ik} = \langle\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_k^{(2)}\rangle$ are the correlation tensor components. In the general case $T_{ik} \neq P_{1i}\, P_{2k}$. The tensor with components $C_{ik} = T_{ik} - P_{1i}P_{2k}$ describes the spin correlations of two particles.

2 Spin Correlations and Angular Correlations at Joint Registration of Decays of Two Λ Particles into the Channel $\Lambda \to p + \pi^-$

It is essential that any decay of an unstable particle may serve as an analyser of its spin state.

The normalised angular distribution at the decay $\Lambda \to p + \pi^-$ takes the form:

$$\frac{d\,w(\mathbf{n})}{d\,\Omega_{\mathbf{n}}} = \frac{1}{4\,\pi}(1 + \alpha_\Lambda \mathbf{P}_\Lambda\,\mathbf{n}) \tag{2}$$

Here $\mathbf{P}_\Lambda$ is the polarisation vector of the Λ particle, $\mathbf{n}$ is the unit vector along the direction of proton momentum in the rest frame of the Λ particle, α_Λ is the coefficient of P-odd angular asymmetry ($\alpha_\Lambda = 0.642$). The decay $\Lambda \to p+\pi^-$ selects the projections of spin of the Λ particle onto the direction of proton momentum; the analysing power equals $\boldsymbol{\xi} = \alpha_\Lambda \mathbf{n}$.

Now let us consider the double angular distribution of flight directions for protons formed in the decays of two Λ particles into the channel $\Lambda \to p + \pi^-$, normalised by unity (the analysing powers are $\boldsymbol{\xi}_1 = \alpha_\Lambda \mathbf{n}_1$, $\boldsymbol{\xi}_2 = \alpha_\Lambda \mathbf{n}_2$). It is described by the following formula [2,3]:

$$\frac{d^2\,w(\mathbf{n}_1, \mathbf{n}_2)}{d\,\Omega_{\mathbf{n}_1}\,d\,\Omega_{\mathbf{n}_2}} = \frac{1}{16\,\pi^2}\left[1 + \alpha_\Lambda \mathbf{P}_1\,\mathbf{n}_1 + \alpha_\Lambda \mathbf{P}_2\,\mathbf{n}_2 + \alpha_\Lambda^2 \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ik} n_{1i} n_{2k}\right], \tag{3}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ are polarisation vectors of the first and second Λ particle, T_{ik} are the correlation tensor components, $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors in the respective rest frames of the first and second Λ particle, defined in the common (unified) coordinate axes of the c.m. frame of the pair ($i, k = \{1, 2, 3\} = \{x, y, z\}$).

The polarisation parameters can be determined from the angular distribution of decay products by the method of moments [2,3] – as a result of averaging combinations of trigonometric functions of angles of proton flight over the double angular distribution.

The angular correlation, integrated over all angles except the angle between the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ and described by the formula [2 – 5]

$$d\,w(\cos\theta) = \frac{1}{2}\left(1 + \frac{1}{3}\alpha_\Lambda^2 T\,\cos\theta\right)\sin\theta\,d\theta = \frac{1}{2}\left[1 - \alpha_\Lambda^2\left(W_s - \frac{W_t}{3}\right)\cos\theta\right]\sin\theta\,d\theta, \tag{4}$$

is determined only by the "trace" of the correlation tensor $T = W_t - 3W_s$ (W_s and W_t are relative fractions of the singlet state and triplet states, respectively), and it does not depend on the polarisation vectors (single-particle states may be unpolarised).

3 Correlations at the Joint Registration of the Decays $\Lambda \to p + \pi^-$ and $\bar{\Lambda} \to \bar{p} + \pi^+$

Due to CP invariance, the coefficients of P-odd angular asymmetry for the decays $\Lambda \to p + \pi^-$ and $\bar{\Lambda} \to \bar{p} + \pi^+$ have equal absolute values and opposite signs: $\alpha_{\bar{\Lambda}} = -\alpha_\Lambda = -0.642$. The double angular distribution for this case is as follows [2,3]:

$$\frac{d^2\,w(\mathbf{n}_1, \mathbf{n}_2)}{d\,\Omega_{\mathbf{n}_1}\,d\,\Omega_{\mathbf{n}_2}} = \frac{1}{16\,\pi^2}\left[1 + \alpha_\Lambda \mathbf{P}_\Lambda\,\mathbf{n}_1 - \alpha_\Lambda \mathbf{P}_{\bar{\Lambda}}\,\mathbf{n}_2 - \alpha_\Lambda^2 \sum_{i=1}^{3}\sum_{k=1}^{3} T_{ik} n_{1i} n_{2k}\right], \tag{5}$$

(here $-\alpha_\Lambda = +\alpha_{\bar{\Lambda}}$ and $-\alpha_\Lambda^2 = +\alpha_\Lambda \alpha_{\bar{\Lambda}}$).

Thus, the angular correlation between the proton and antiproton momenta in the rest frames of the Λ and $\bar{\Lambda}$ particles is described by the expression:

$$d\,w(\cos\theta) = \frac{1}{2}\left(1 - \frac{1}{3}\alpha_\Lambda^2 T\,\cos\theta\right)\sin\theta\,d\theta = \frac{1}{2}\left[1 + \alpha_\Lambda^2\left(W_s - \frac{W_t}{3}\right)\cos\theta\right]\sin\theta\,d\theta, \tag{6}$$

where θ is the angle between the proton and antiproton momenta.

4 Spin Correlations at the Generation of $\Lambda\,\Lambda$ Pairs in Multiple Processes

Further we will use the model of one-particle sources [6], which is the most adequate one in the case of collisions of relativistic ions.

Two Λ particles are identical particles. Spin and angular correlations in this case arise due to the Fermi statistics and final-state interaction.

Indeed, it is easy to see that the Fermi-statistics effect leads not only to the momentum–energy $\Lambda\,\Lambda$-correlations at small relative momenta (correlation femtoscopy), but to the spin correlations as well.

The following relation holds, in consequence of the symmetrisation or industrialisation of the total wave function of any identical particles with nonzero spin (bosons as well as fermions) [7] :

$$(-1)^{S+L} = 1. \tag{7}$$

Here, S is the total spin and L is the orbital momentum in the c.m. frame of the pair. At the momentum difference $q = p_1 - p_2 \to 0$ the states with nonzero orbital momenta "die out", and only states with $L = 0$ and even total spin S survive.

Since the Λ-particle spin is equal to 1/2, at $q \to 0$ the $\Lambda\,\Lambda$ pair is generated only in the singlet state with $S = 0$.

Meantime, at the 4-momentum difference $q \neq 0$ there are also triplet states generated together with the singlet state.

Within the conventional model of one-particle sources emitting unpolarised particles, the triplet states with spin projections +1, 0, and −1 are produced with equal probabilities. If correlations are neglected, the singlet state is generated with the same probability – the relative "weights" are $\widetilde{W}_t = 3/4$, $\widetilde{W}_s = 1/4$.

When taking into account the Fermi statistics and s-wave final-state interaction, which is essential at close momenta (at orbital momenta $L \neq 0$ the contribution of final-state interaction is suppressed), the fractions of triplet states and the singlet state are renormalised.

We will perform here the analysis of spin $\Lambda\Lambda$ correlations in the c.m. frame of the $\Lambda\Lambda$ pair. In the c.m. frame, we have: $q = \{0, 2\mathbf{k}\}$, where q is the difference of 4-momenta of the Λ particles, $\mathbf{k}$ is the momentum of one of the particles. In doing so, the momentum $\mathbf{k}$ is connected with the relative momentum $\mathbf{q}$ in the laboratory frame by the Lorentz transformation [8] (we use the unit system with $\hbar = c = 1$):

$$\mathbf{k} = \frac{1}{2}\left[\mathbf{q} + (\gamma - 1)\frac{(\mathbf{q}\mathbf{v})\mathbf{v}}{|\mathbf{v}|^2} - \gamma\mathbf{v}\,q_0\right]; \tag{8}$$

here, $\mathbf{v} = (\mathbf{p}_1 + \mathbf{p}_2)/(\varepsilon_1 + \varepsilon_2)$ is the velocity of the $\Lambda\Lambda$ pair in the laboratory frame, $\gamma = (1 - v^2)^{-1/2}$ is the Lorentz factor, $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$ and $q_0 = \varepsilon_1 - \varepsilon_2$ are the laboratory relative momentum and energy, respectively.

The Lorentz transformations of 4-coordinates are given by the expressions:

$$\mathbf{r}^* = \mathbf{r} + (\gamma - 1)\,\frac{(\mathbf{r}\mathbf{v})\mathbf{v}}{|\mathbf{v}|^2} - \gamma\mathbf{v}\,t, \qquad t^* = \gamma(t - \mathbf{v}\mathbf{r}), \tag{9}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $t = t_1 - t_2$ are the differences of coordinates and times for one-particle sources in the laboratory frame.

The interference term connected with identity (quantum statistics) is determined by the expression:

$$\langle\cos 2\mathbf{k}\mathbf{r}^*\rangle = \int W_{\mathbf{v}}(\mathbf{r}^*)\cos(2\mathbf{k}\mathbf{r}^*)d^3\mathbf{r}^*, \tag{10}$$

where

$$W_{\mathbf{v}}(\mathbf{r}^*) = \int W(x)\,dt^* = \int W(\mathbf{r}^*, t^*)\,dt^*$$

is the distribution of coordinate difference between two sources in the c.m. frame of the $\Lambda\Lambda$ pair.

Meantime, the contribution of s-wave final-state interaction is expressed as follows (at the sizes of the generation region in the c.m. frame, exceeding the effective radius of interaction of two Λ particles):

$$B_{\rm int}(q) = B^{(\Lambda\Lambda)}(\mathbf{k}, \mathbf{v}) = \int W_{\mathbf{v}}(\mathbf{r}^*)\,b(\mathbf{k}, \mathbf{r}^*)d^3\mathbf{r}^*, \tag{11}$$

where the function $b(\mathbf{k}, \mathbf{r}^*)$ has the structure [2, 8, 9]:

$$b(\mathbf{k}, \mathbf{r}^*) = |f^{(\Lambda\Lambda)}(k)|^2\,\frac{1}{(r^*)^2} + 2\,\mathrm{Re}\left(f^{(\Lambda\Lambda)}(k)\,\frac{e^{ikr^*}\cos\mathbf{k}\mathbf{r}^*}{r^*}\right) - 2\,\pi\,|f^{(\Lambda\Lambda)}(k)|^2\,d_0^{(\Lambda\Lambda)}\delta^3(\mathbf{r}^*). \tag{12}$$

Here, $k = |\mathbf{k}|$, $r^* = |\mathbf{r}^*|$, $f^{(\Lambda\Lambda)}(k)$ is the amplitude of low-energy $\Lambda\Lambda$ scattering. In the framework of the effective radius theory [7, 10]:

$$f^{(\Lambda\Lambda)}(k) = a_0^{(\Lambda\Lambda)}\left(1 + \frac{1}{2}\,d_0^{(\Lambda\Lambda)}\,a_0^{(\Lambda\Lambda)}\,k^2 - i\,k\,a_0^{(\Lambda\Lambda)}\right)^{-1}, \tag{13}$$

where, by definition, $(-a_0^{(\Lambda\Lambda)})$ is the length of s-wave scattering and

$$d_0^{(\Lambda\Lambda)} = \frac{1}{k}\,\frac{d}{d\,k}\left(\mathrm{Re}\,\frac{1}{f^{(\Lambda\Lambda)}(k)}\right)$$

is the effective radius.

The integral (11), with expression (12) inside, approximately takes into account the difference of the true wave function of two interacting Λ particles with the momenta $\mathbf{k}$ and $(-\mathbf{k})$ at small distances from the asymptotic wave function of continuous spectrum [9, 11].

Information about the parameters of $\Lambda\Lambda$ scattering is contained in the works studying double hypernuclei and pair correlations in the reactions with formation of two Λ particles (see, for example, [12–14]). Analysis of the experimental data leads to the conclusion that the length of $\Lambda\Lambda$ scattering is comparable by magnitude ($\approx (-20)$ fm) with the length of neutron–neutron scattering [14].

5 Spin Correlations at the Generation of $\Lambda\bar{\Lambda}$ Pairs in Multiple Processes

In the framework of the model of independent one-particle sources, spin correlations in the $\Lambda\bar{\Lambda}$ system arise only on account of the difference between the interaction in the final triplet state ($S = 1$) and the interaction in the final singlet state. At small relative momenta, the s-wave interaction plays the dominant role as before, but, contrary to the case of identical particles ($\Lambda\Lambda$), in the case of non-identical particles ($\Lambda\bar{\Lambda}$) the total spin may take both the values $S = 1$ and $S = 0$ at the orbital momentum $L = 0$. In doing so, the interference effect, connected with quantum statistics, is absent.

If the sources emit unpolarised particles, then, in the case under consideration, the correlation function describing momentum-energy correlations has the following structure (in the c.m. frame of the $\Lambda\bar{\Lambda}$ pair):

$$R(\mathbf{k},\mathbf{v}) = 1 + \frac{3}{4}\, B_t^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v}) + \frac{1}{4}\, B_s^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v}). \tag{14}$$

The components of the correlation tensor for the $\Lambda\bar{\Lambda}$ pair are as follows:

$$T_{ik} = \frac{B_t^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v}) - B_s^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v})}{4 + 3\, B_t^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v}) + B_s^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v})}\, \delta_{ik}; \tag{15}$$

here the contributions of final-state triplet and singlet $\Lambda\bar{\Lambda}$ interaction are determined by the expression (analogously to Eqs. (11), (12) for the $\Lambda\Lambda$ interaction [2, 9], with the replacement $\cos \mathbf{kr}^* \to e^{i\,\mathbf{k}\,\mathbf{r}^*}$ in Eq. (12) owing to the non-identity of the particles Λ and $\bar{\Lambda}$ [8]):

$$B_{s(t)}^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v}) = \left| f_{s(t)}^{(\Lambda\bar{\Lambda})}(k) \right|^2 \left\langle \frac{1}{(r^*)^2} \right\rangle + 2\,\mathrm{Re}\left(f_{s(t)}^{(\Lambda\bar{\Lambda})}(k) \left\langle \frac{e^{ikr^*} e^{i\,\mathbf{k}\mathbf{r}^*}}{r^*} \right\rangle \right) -$$

$$- \frac{2\pi}{k} \left| f_{s(t)}^{(\Lambda\bar{\Lambda})}(k) \right|^2 \frac{d}{dk} \left(\mathrm{Re}\, \frac{1}{f_{s(t)}^{(\Lambda\bar{\Lambda})}(k)} \right) W_{\mathbf{v}}(0), \tag{16}$$

where $f_{s(t)}^{(\Lambda\bar{\Lambda})}(k)$ is the amplitude of the s-wave low-energy singlet (triplet) $\Lambda\bar{\Lambda}$ scattering.

At sufficiently large values of k, one should expect that [9]:

$$B_s^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v}) = 0, \qquad B_t^{(\Lambda\bar{\Lambda})}(\mathbf{k},\mathbf{v}) = 0.$$

In this case the angular correlations in the decays $\Lambda \to p + \pi^-$, $\bar{\Lambda} \to \bar{p} + \pi^+$, connected with the final-state interaction, are absent :

$$T_{ik} = 0, \qquad T = 0.$$

6 Angular Correlations in the Decays $\Lambda \to p + \pi^-$ and $\bar{\Lambda} \to \bar{p} + \pi^+$ and the "Mixed Phase"

Thus, at sufficiently large relative momenta (for $k \gg m_\pi$) one should expect that the angular correlations in the decays $\Lambda \to p + \pi^-$ and $\bar{\Lambda} \to \bar{p} + \pi^+$, connected with the interaction of the

Λ and $\bar{\Lambda}$ hyperons in the final state (i.e. with one-particle sources) are absent. But, if at the considered energy the dynamical trajectory of the system passes through the so-called "mixed phase", then the two-particle sources, consisting of the free quark and antiquark, start playing a noticeable role. For example, the process $s\,\bar{s} \to \Lambda\bar{\Lambda}$ may be discussed.

In this process, the charge parity of the pairs $s\bar{s}$ and $\Lambda\bar{\Lambda}$ is equal to $C = (-1)^{L+S}$, where L is the orbital momentum and S is the total spin of the fermion and antifermion. Meantime, the CP parity of the fermion–antifermion pair is $CP = (-1)^{S+1}$.

In the case of one-gluon exchange, $CP = 1$, and then $S = 1$, i.e. the $\Lambda\bar{\Lambda}$ pair is generated in the triplet state; in doing so, the "trace" of the correlation tensor $T = 1$.

Even if the frames of one-gluon exchange are overstepped, the quarks s and $\bar{s}$, being ultra-relativistic, interact in the triplet state ($S = 1$). In so doing, the primary CP parity $CP = 1$, and, due to the CP parity conservation, the $\Lambda\bar{\Lambda}$ pair is also produced in the triplet state. Let us denote the contribution of two-quark sources by x. Then at large relative momenta $T = x > 0$.

Apart from the two-quark sources, there are also two-gluon sources being able to play a comparable role. Analogously with the annihilation process $\gamma\gamma \to e^+e^-$, in this case the "trace" of the correlation tensor is described by the formula (the process $g\,g \to \Lambda\bar{\Lambda}$ is implied):

$$T = 1 - \frac{4\,(1-\beta^2)}{1 + 2\,\beta^2\,\sin^2\theta - \beta^4 - \beta^4\,\sin^4\theta}, \tag{17}$$

where β is the velocity of Λ (and $\bar{\Lambda}$) in the c.m. frame of the $\Lambda\bar{\Lambda}$ pair, θ is the angle between the momenta of one of the gluons and Λ in the c.m. frame (see [15]). At small β ($\beta \ll 1$) the $\Lambda\bar{\Lambda}$ pair is produced in the singlet state (total spin $S = 0$, $T = -3$), whereas at $\beta \approx 1$ – in the triplet state ($S = 1$, $T = 1$). Let us remark that at ultra-relativistic velocities β (i.e. at extremely large relative momenta of Λ and $\bar{\Lambda}$) both the two-quark and two-gluon mechanisms lead to the triplet state of the $\Lambda\bar{\Lambda}$ pair ($T = 1$).

In the general case, the appearance of angular correlations in the decays $\Lambda \to p + \pi^-$ and $\bar{\Lambda} \to \bar{p} + \pi^+$ with the *nonzero* values of the "trace" of the correlation tensor T at large relative momenta of the Λ and $\bar{\Lambda}$ particles may testify to the passage of the system through the "mixed phase".

7 Summary

So, it is advisable to investigate the spin correlations of $\Lambda\Lambda$ and $\Lambda\bar{\Lambda}$ pairs produced in relativistic heavy ion collisions.

The spin correlations are studied by the method of angular correlations – method of moments.

The spin correlations, as well as the momentum-energy ones, make it possible to determine the space-time characteristics of the generation region and, besides, the parameters of low-energy scattering of Λ on Λ and Λ on $\bar{\Lambda}$. They should be investigated jointly with the momentum-energy correlations.

References

[1] V.L. Lyuboshitz and M.I. Podgoretsky, Yad. Fiz. **60** 45 (1997)

[2] V.L. Lyuboshitz, Proceedings of XXXIV PNPI Winter School . Physics of Atomic Nuclei and Elementary Particles, St-Petersburg (2000), p.402.

[3] R. Lednicky and V.L. Lyuboshitz, Phys. Lett. **B508** 146 (2001).

[4] G. Alexander and H.J. Lipkin, Phys. Lett. **B352** 162 (1995) .

[5] R. Lednicky, V.V. Lyuboshitz and V.L. Lyuboshitz, Yad. Fiz. **66** 1007 (2003)

[6] M.I. Podgoretsky, Fiz. Elem. Chast. At. Yadra **20** 628 (1989)

[7] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics. Nonrelativistic Theory* (in Russian) (Nauka, Moscow, 1989); §§ 62, 133 .

[8] V.L. Lyuboshitz and V.V. Lyuboshitz, in: *Proceedings of XXXVII and XXXVIII Winter Schools of the Petersburg Institute of Nuclear Physics. Physics of Atomic Nuclei and Elementary Particles*, St.-Petersburg (2004), p.390 .

[9] R. Lednicky and V.L. Lyuboshitz, Yad. Fiz. **35** 1316 (1982)

[10] H.A. Bethe and P. Morrison, *Elementary Nuclear Theory* (New York, 1956); § 10 .

[11] V.L. Lyuboshitz, Yad. Fiz. **41** 820 (1985) [Sov. J. Nucl. Phys. **41** 529 (1985)].

[12] S. Iwao, M. Chako, I. Kazanava, Progr. Theor. Phys. **48** 1412 (1972).

[13] A.O. Onichi *et al.*, Nucl. Phys. **A670** 2970 (2000).

[14] R. Afnan, Nucl. Phys. **A639** 550 (1998).

[15] H. McMaster, Rev. Mod. Phys. **33** 8 (1961)