

SM and BSM physics at the LHC

Padé approximations and nonsinglet QCD analysis up to $N^3\text{LO}$

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Motivation

Deeply inelastic scattering (DIS) at large 4-momentum transfer Q^2 offers the possibility to probe the partonic sub-structure of the nucleon.

All calculations of high energy processes with initial hadrons, require parton distribution functions (PDF's) as an essential input.

Presently the next-to-leading-order is the standard approximation for most important processes in QCD.

For quantitatively reliable predictions of DIS and hard hadronic scattering processes, perturbative QCD corrections beyond the next-to-next-to-leading order (N^2LO) and the next-to-next-to-next-to-leading order (N^3LO) need to be taken into account.

Padé approximations and 4-loop anomalous dimensions

In spite of the unknown 4-loop anomalous dimensions, one can obtain the non-singlet parton distributions and Λ_{QCD} by estimating uncalculated fourth-order corrections to the non-singlet anomalous dimension.

Padé approximations have proved to be useful in many physics applications.

Padé approximations may be used either to predict the next term in some perturbative series, called a Padé approximation prediction, or to estimate the sum of the entire series, called Padé summation.

Padé approximations are rational functions chosen to equal the perturbative series to the order calculated:

$$[\mathcal{N}/\mathcal{M}] = \frac{a_0 + a_1x + \dots + a_{\mathcal{N}}x^{\mathcal{N}}}{1 + b_1x + \dots + b_{\mathcal{M}}x^{\mathcal{M}}},$$

to the series

$$S = S_0 + S_1x + \dots + S_{\mathcal{N}+\mathcal{M}}x^{\mathcal{N}+\mathcal{M}}$$

where we set

$$[\mathcal{N}/\mathcal{M}] = S + \mathcal{O}(x^{\mathcal{N}+\mathcal{M}+1})$$

and write an equation for the coefficients of each power of x .

A generic QCD anomalous dimension expansion in term of a_s then may be written in the form

$$\gamma(N) = a_s(\gamma^{(0)} + a_s^1\gamma^{(1)} + a_s^2\gamma^{(2)} + a_s^3\gamma^{(3)})$$

In N -Mellin space $\mathcal{M} \geq 1$ and $\mathcal{N} + \mathcal{M} = n$, where n stands for the maximal order in a_s at which the expansion coefficients $\gamma^{(n)}$ have been determined from an exact calculation.

This expansion then also provides the $[\mathcal{N}/\mathcal{M}]$ Padé approximations for the $(n+1)$ -th order quantities $\gamma^{(n+1)}$.

In N -Mellin space it is easy to obtain the following results for $\mathcal{M} = \mathcal{N} = 1$ and for $\mathcal{M} = 0, \mathcal{N} = 2$

$$\begin{aligned}\tilde{\gamma}^{[1/1]}(N) &= [1/1](N) = \frac{\gamma^{(2)^2}(N)}{\gamma^{(1)}(N)} \\ \tilde{\gamma}^{[0/2]}(N) &= [0/2](N) = \frac{2\gamma^{(1)}(N)\gamma^{(2)}(N)}{\gamma^{(0)}(N)} - \frac{\gamma^{(1)^3}(N)}{\gamma^{(0)^2}(N)}\end{aligned}$$

At $N^m LO$ the scale dependence of a_s is given by

$$\frac{d a_s}{d \ln Q^2} = \beta_{N^m LO}(a_s) = - \sum_{k=0}^m a_s^{k+2} \beta_k .$$

$$\beta_0 = 11 - 2/3 n_f ,$$

$$\beta_1 = 102 - 38/3 n_f ,$$

$$\beta_2 = 2857/2 - 5033/18 n_f + 325/54 n_f^2 ,$$

$$\beta_3 = 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1093/729 n_f^3$$

Up to $N^3 LO$ this expansion yields

$$\begin{aligned} a_s(Q^2) = & \frac{1}{\beta_0 L_\Lambda} - \frac{1}{(\beta_0 L_\Lambda)^2} b_1 \ln L_\Lambda + \frac{1}{(\beta_0 L_\Lambda)^3} [b_1^2 (\ln^2 L_\Lambda - \ln L_\Lambda - 1) + b_2] \\ & + \frac{1}{(\beta_0 L_\Lambda)^4} \left[b_1^3 \left(-\ln^3 L_\Lambda + \frac{5}{2} \ln^2 L_\Lambda + 2 \ln L_\Lambda - \frac{1}{2} \right) - 3b_1 b_2 \ln L_\Lambda + \frac{b_3}{2} \right] \end{aligned}$$

where $L_\Lambda \equiv \ln(Q^2/\Lambda^2)$, $b_k \equiv \beta_k/\beta_0$, and Λ is the QCD scale parameter. To be capable to compare with other measurement of Λ we adopt the matching of flavor thresholds at $Q^2 = m_c^2$ and $Q^2 = m_b^2$ with $m_c = 1.5$ GeV and $m_b = 4.5$ GeV.

The theoretical background of the QCD analysis

We choose the following parameterization for the valence quark densities in the input scale of $Q_0^2 = 4 \text{ GeV}^2$

$$xq_v(x, Q_0^2) = \mathcal{N}_q x^{a_q} (1-x)^{b_q} (1 + c_q \sqrt{x} + d_q x) ,$$

where $q = u, d$ and the normalizations \mathcal{N}_u and \mathcal{N}_d being fixed by $\int_0^1 u_v dx = 2$ and $\int_0^1 d_v dx = 1$, respectively.

In the above the x^a term controls the low- x behavior parton densities, and the $(1-x)^b$ terms the large x values. The remaining polynomial factor accounts for additional medium- x values.

A. N. Khorramian and S. A. Tehrani, Phys. Rev. D **78**, 074019 (2008)

The non-singlet (NS) parts of the structure functions $f_{p,d}^v(x, Q^2)$ in the LO approximation for $x > 0.3$, where valence quark dominance is adopted, given by

$$\begin{aligned} f_p^v(x, Q^2) &\equiv \frac{5}{18} [x(u - \bar{u}) + x(d - \bar{d})] \\ &\quad + \frac{1}{6} [x(u - \bar{u}) - x(d - \bar{d})] \\ &= \frac{4}{9} x u_v + \frac{1}{9} x d_v \end{aligned}$$

The combinations of parton densities in the valence region $x > 0.3$ for $f_d^v(x, Q^2)$ is

$$\begin{aligned} f_d^v(x, Q^2) &= \frac{5}{18} [x(u - \bar{u})(x, Q^2) + x(d - \bar{d})(x, Q^2)] \\ &= \frac{5}{18} x(u_v + d_v) \end{aligned}$$

where $d = (p + n)/2$.

In the region $x \leq 0.3$ for the difference of the proton and deuteron data we use

$$\begin{aligned} f^{\text{NS}}(x, Q^2) &= 2(F_2^{ep} - F_2^{ed}) \\ &= \frac{1}{3} [x(u + \bar{u}) - x(d + \bar{d})] + \frac{2}{3} x(\bar{u} - \bar{d}) \\ &= \frac{1}{3} x(u_v - d_v) + \frac{2}{3} x(\bar{u} - \bar{d}) \end{aligned}$$

where now sea quarks cannot be neglected for x smaller than about 0.3.

M. Gluck, E. Reya and C. Schuck, Nucl. Phys. B **754**, 178 (2006)

A. Vogt, Comput. Phys. Commun. **170**, 65 (2005)

The evolution equations are solved in Mellin- N space and the Mellin transforms of the above distributions are denoted by $f^{\text{NS}}(N, Q^2)$, $f_{p,d}^v(N, Q^2)$, respectively.

The non-singlet structure functions are given by

$$F_k(N, Q^2) = [1 + a_s(Q^2)C_1(N) + a_s^2(Q^2)C_2(N) + a_s^3(Q^2)C_3(N)] f_k(N, Q^2)$$

for the three cases above. Here $a_s(Q^2) = \alpha_s(Q^2)/(4\pi)$ denotes the strong coupling constant and $C_i(N)$ are the non-singlet Wilson coefficients in $O(a_s^i)$.

S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B **688** (2004) 101 [arXiv:hep-ph/0403192].

J. A. M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B **724** (2005) 3 [arXiv:hep-ph/0504242].

The solution of the non-singlet evolution equation for the parton densities to 4-loop order reads

$$\begin{aligned}
F_k(N, Q^2) = & F_k(N, Q_0^2) \left(\frac{a}{a_0} \right)^{-\hat{P}_0(N)/\beta_0} \\
& \left\{ 1 - \frac{1}{\beta_0} (a - a_0) \left[\hat{P}_1^+(N) - \frac{\beta_1}{\beta_0} \hat{P}_0 \right] \right. \\
& - \frac{1}{2\beta_0} (a^2 - a_0^2) \left[\hat{P}_2^+(N) - \frac{\beta_1}{\beta_0} \hat{P}_1^+ + \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \hat{P}_0(N) \right] \\
& + \frac{1}{2\beta_0^2} (a - a_0)^2 \left(\hat{P}_1^+(N) - \frac{\beta_1}{\beta_0} \hat{P}_0 \right)^2 \\
& - \frac{1}{3\beta_0} (a^3 - a_0^3) \left[\hat{P}_3^+(N) - \frac{\beta_1}{\beta_0} \hat{P}_2^+(N) + \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \hat{P}_1^+(N) \right. \\
& \left. + \left(\frac{\beta_1^3}{\beta_0^3} - 2 \frac{\beta_1 \beta_2}{\beta_0^2} + \frac{\beta_3}{\beta_0} \right) \hat{P}_0(N) \right] \\
& + \frac{1}{2\beta_0^2} (a - a_0) (a_0^2 - a^2) \left(\hat{P}_1^+(N) - \frac{\beta_1}{\beta_0} \hat{P}_0(N) \right) \\
& \times \left[\hat{P}_2(N) - \frac{\beta_1}{\beta_0} \hat{P}_1(N) - \left(\frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \hat{P}_0(N) \right] \\
& \left. - \frac{1}{6\beta_0^3} (a - a_0)^3 \left(\hat{P}_1^+(N) - \frac{\beta_1}{\beta_0} \hat{P}_0(N) \right)^3 \right\}.
\end{aligned}$$

Here, \hat{P}_k denote the Mellin transforms of the $(k + 1)$ -loop splitting functions.

One of the simplest and fastest possibilities in the SF reconstruction from the QCD predictions for its Mellin moments is Jacobi polynomials expansion.

S. I. Alekhin et al., Phys. Lett. B **452**, (1999) 402.
 A. L. Kataev, et al. Nucl. Phys. B **573**, (2000) 405.
 A. L. Kataev, et al. Phys. Part. Nucl. **34**, (2003) 20;

$$F_2^{N_{max}}(x, Q^2) = x^\alpha (1-x)^\beta \sum_{n=0}^{N_{max}} \Theta_n^{\alpha, \beta}(x) \sum_{j=0}^n c_j^{(n)}(\alpha, \beta) \times M_{F_2}(j+2, Q^2),$$

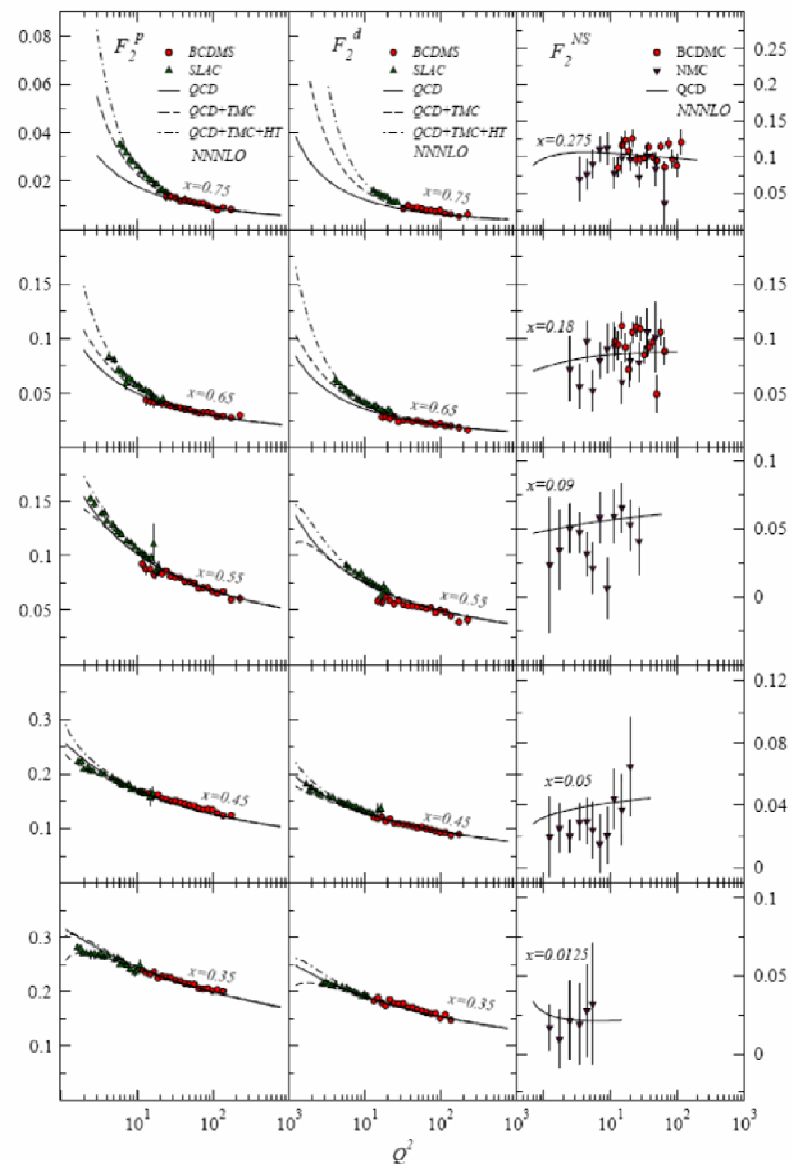
where N_{max} is the number of polynomials and $\Theta_n^{\alpha, \beta}(x)$ is Jacobi polynomials with the weight $x^\alpha (1-x)^\beta$.

In above $M_{F_2}(j+2, Q^2)$ are the moments of non singlet structure function. It is obvious that the Q^2 -dependence of the structure function is defined by the Q^2 -dependence of the moments.

S. Atashbar Tehrani and A. N. Khorramian JHEP **07** (2007) 048

A. N. Khorramian, A. Mirjalili and S. A. Tehrani, JHEP **0410** (2004) 062.

A. N. Khorramian and S. Atashbar Tehrani, JHEP **0703** (2007) 051



		NLO	NNLO	N ³ LO Padé[1/1]	N ³ LO Padé[0/2]
u_v	a_u	0.7434 ± 0.009	0.7772 ± 0.009	0.7917 ± 0.0106	0.7917 ± 0.00992
	b_u	3.8907 ± 0.040	4.0034 ± 0.033	4.0264 ± 0.0402	4.0268 ± 0.0327
	c_u	0.1620	0.1000	0.0940	0.0940
	d_u	1.2100	1.1400	1.1100	1.1100
d_v	a_d	0.7369 ± 0.040	0.7858 ± 0.043	0.8093 ± 0.0611	0.8093 ± 0.0405
	b_d	3.5051 ± 0.225	3.6336 ± 0.244	3.7685 ± 0.3499	3.7685 ± 0.2276
	c_d	0.3899	0.1838	0.13990	0.13990
	d_d	-1.3700	-1.2152	-1.1200	-1.1200
$\Lambda_{\text{QCD}}^{\text{N}_f=4}$, MeV		263.8 ± 30	239.9 ± 27	241.4 ± 29	241.5 ± 27
χ^2/ndf		$523/546 = 0.9578$	$506/546 = 0.9267$	$491.07/546 = 0.8994$	$491.12/546 = 0.8995$

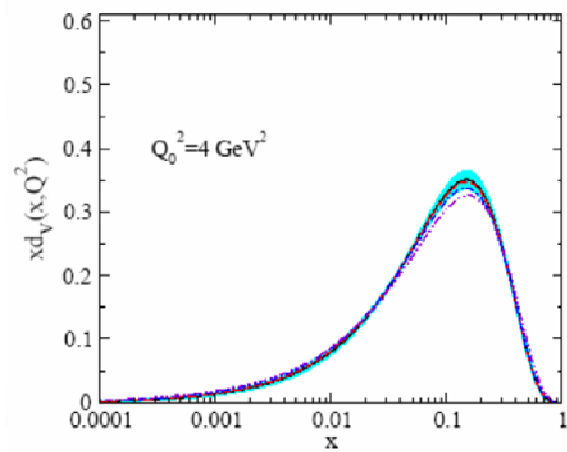
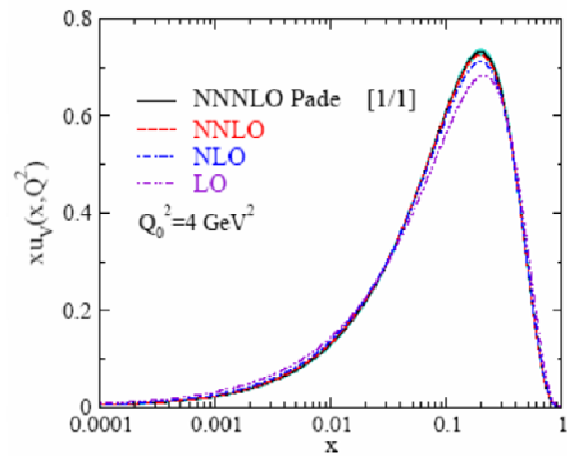
TABLE I: Parameter values of the NLO, N²LO and N³LO (Padé [1/1] and Padé [0/2]) nonsinglet QCD fit at $Q_0^2 = 4 \text{ GeV}^2$.

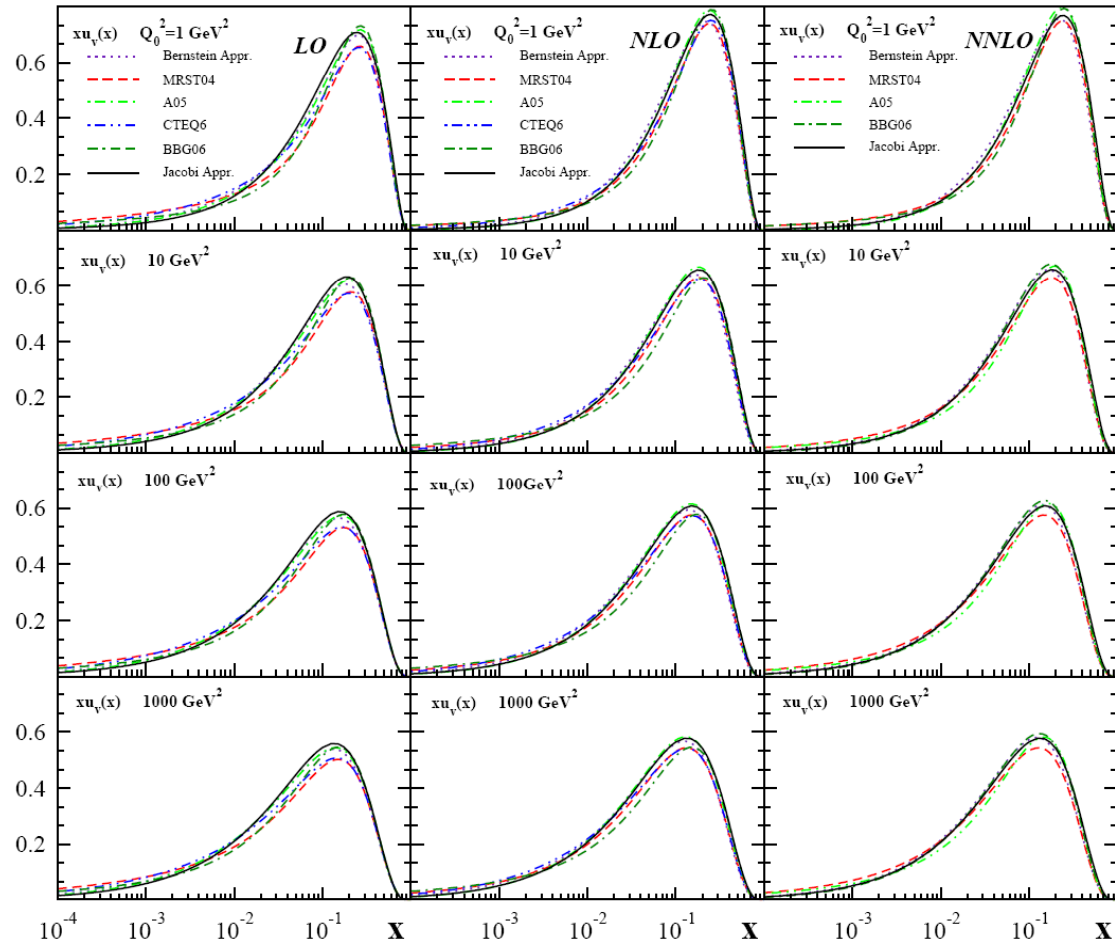
N^3LO Padé[1/1]	a_u	b_u	a_d	b_d	$\Lambda_{QCD}^{N_f=4}$
a_u	1.13×10^{-4}				
b_u	2.35×10^{-4}	1.62×10^{-3}			
a_d	1.09×10^{-4}	-1.59×10^{-3}	3.86×10^{-3}		
b_d	1.67×10^{-4}	-8.84×10^{-3}	2.11×10^{-2}	1.23×10^{-1}	
$\Lambda_{QCD}^{(4)}$	1.71×10^{-4}	-3.49×10^{-4}	5.04×10^{-4}	2.61×10^{-3}	8.65×10^{-4}
N^3LO Padé[0/2]	a_u	b_u	a_d	b_d	$\Lambda_{QCD}^{N_f=4}$
a_u	0.98×10^{-4}				
b_u	1.83×10^{-4}	1.07×10^{-3}			
a_d	-5.07×10^{-5}	-6.01×10^{-4}	1.66×10^{-3}		
b_d	-1.11×10^{-4}	-3.30×10^{-3}	8.58×10^{-3}	5.19×10^{-2}	
$\Lambda_{QCD}^{(4)}$	1.59×10^{-4}	-1.99×10^{-4}	1.94×10^{-4}	8.07×10^{-4}	7.53×10^{-4}

TABLE II: Our results for the covariance matrix of the N^3LO nonsinglet QCD fit for padé [1/1] and [0/2] at $Q_0^2 = 4 \text{ GeV}^2$ by using MINUIT[54].

f	N	BBG [44]	N ³ LO Padé[1/1]	N ³ LO Padé[0/2]
u_v	2	0.3006 ± 0.0031	0.3081 ± 0.0028	0.3081 ± 0.0026
	3	0.0877 ± 0.0012	0.0878 ± 0.0011	0.0878 ± 0.0010
	4	0.0335 ± 0.0006	0.0332 ± 0.0006	0.0332 ± 0.0005
d_v	2	0.1252 ± 0.0027	0.1249 ± 0.0028	0.1249 ± 0.0026
	3	0.0318 ± 0.0009	0.0301 ± 0.0008	0.0301 ± 0.0007
	4	0.0106 ± 0.0004	0.0099 ± 0.0004	0.0099 ± 0.0005
$u_v - d_v$	2	0.1754 ± 0.0041	0.1831 ± 0.0038	0.1831 ± 0.0034
	3	0.0559 ± 0.0015	0.0577 ± 0.0014	0.0577 ± 0.0013
	4	0.0229 ± 0.0007	0.0233 ± 0.0007	0.0233 ± 0.0006

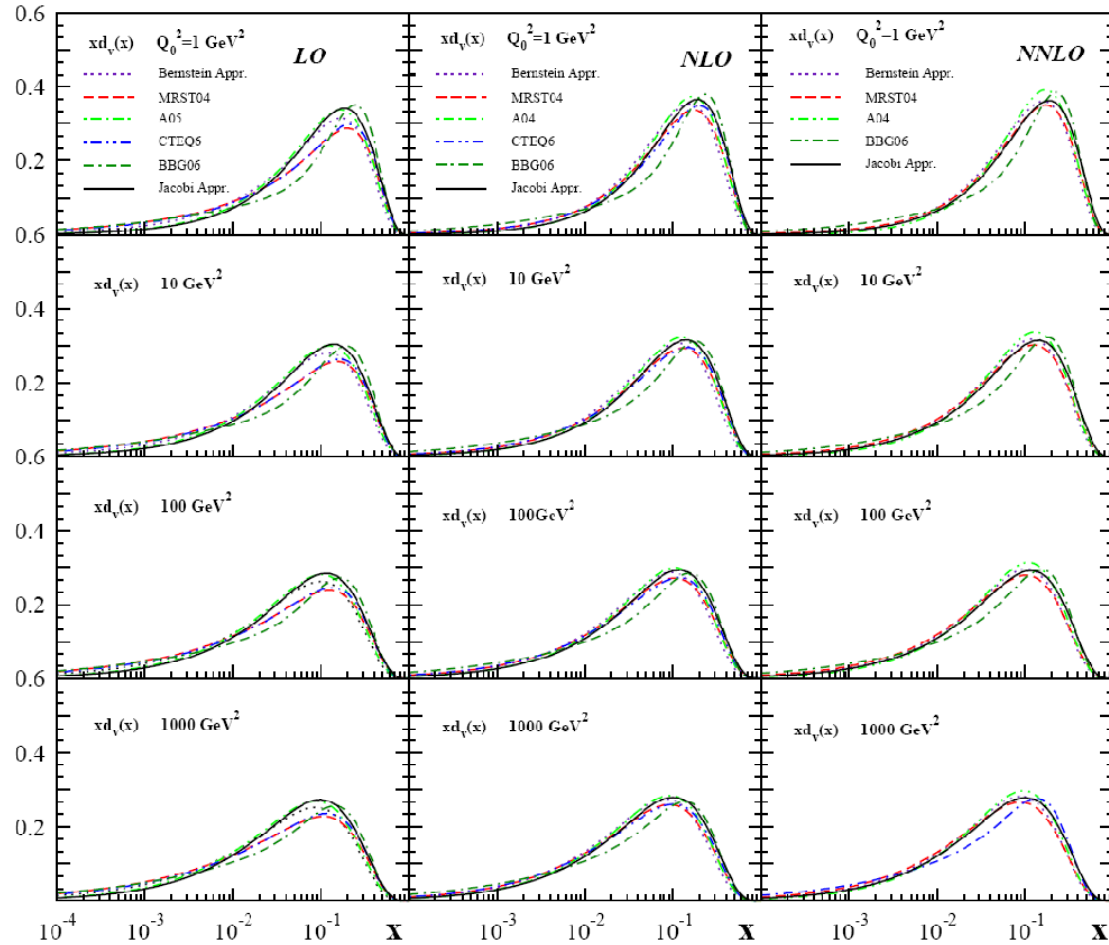
TABLE III: Comparison of low order moments from our nonsinglet N³LO QCD analysis at $Q_0^2 = 4 \text{ GeV}^2$ with the N³LO analysis BBG [44].





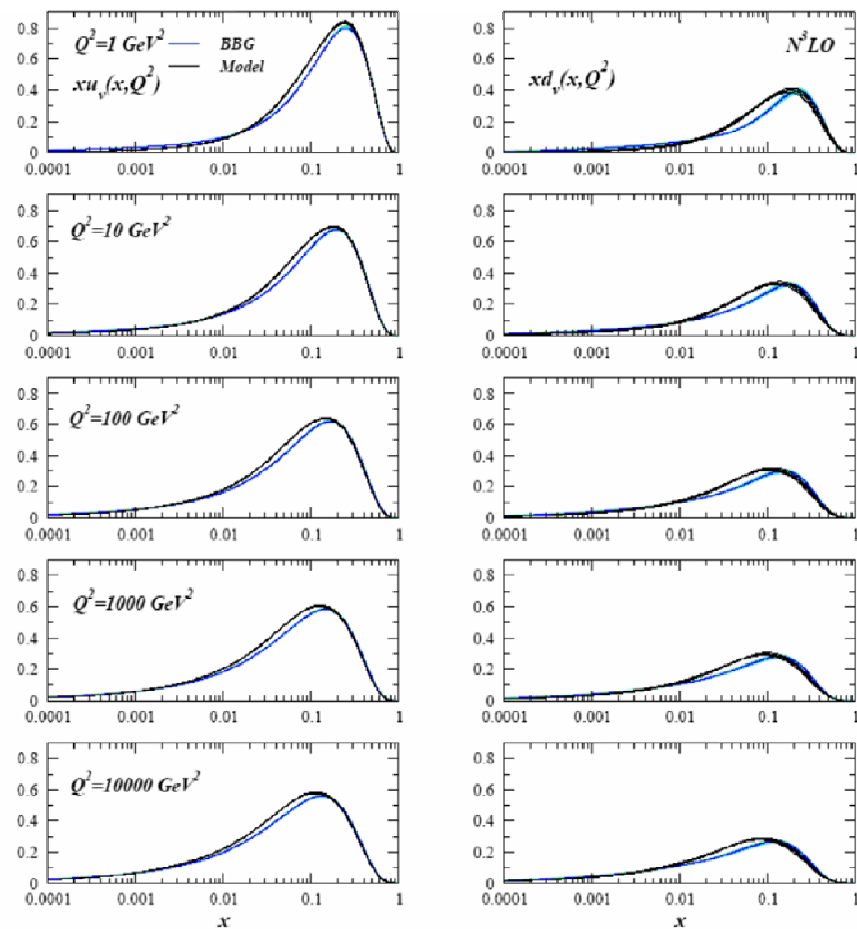
The parton distribution xu_v at some different values of Q^2 .

A. N. Khorramian and S. A. Tehrani, Phys. Rev. D **78**, 074019 (2008)



The parton distribution $x d_v$ at some different values of Q^2 .

A. N. Khorramian and S. A. Tehrani, Phys. Rev. D 78, 074019 (2008)



J. Blumlein, H. Bottcher and A. Guffanti, Nucl. Phys. B **774**, 182 (2007)

Conclusions

We performed QCD analysis of the non-singlet world data up to NNNLO by using Jacobi polynomials expansion and determined the valence quark distributions $xu_v(x, Q^2)$ and $xd_v(x, Q^2)$.

In spite of the unknown 4-loop anomalous dimensions, one can obtain the non-singlet parton distributions and Λ_{QCD} by estimating uncalculated fourth-order corrections to the non-singlet anomalous dimension.

$$\alpha_s(M_Z^2) = 0.1139 \pm 0.0020, \quad \text{N}^3\text{LO}.$$

$$\alpha_s(M_Z^2) = 0.1134 \quad {}^{+0.0019}_{-0.0021} \quad \text{BBG}$$

Parameterizations of these parton distribution functions were derived in a wide range of x and Q^2 as fit results at LO, NLO NNLO and NNNLO.

<http://particles.ipm.ir/links/QCD.htm>

<http://particles.ipm.ir/links/QCD.htm>

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KT08's Non-singlet PDFs

Author: **A. Khorramian** and **S. Atashbar Tehrani**

Package Version: 1 May 8, 2008

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A. N. Khorramian and S. A. Tehrani, Phys. Rev. D **78**, 074019 (2008)

EXTRAS

Experiment	x	Q^2 , GeV ²	F_2^p	$F_2^p \text{ cuts}$	\mathcal{N}
BCDMS (100)	0.35 – 0.75	11.75 – 75.00	51	29	1.005
BCDMS (120)	0.35 – 0.75	13.25 – 75.00	59	32	0.998
BCDMS (200)	0.35 – 0.75	32.50 – 137.50	50	28	0.998
BCDMS (280)	0.35 – 0.75	43.00 – 230.00	49	26	0.998
NMC (comb)	0.35 – 0.50	7.00 – 65.00	15	14	1.000
SLAC (comb)	0.30 – 0.62	7.30 – 21.39	57	57	1.013
H1 (hQ2)	0.40 – 0.65	200 – 30000	26	26	1.020
ZEUS (hQ2)	0.40 – 0.65	650 – 30000	15	15	1.007
proton			322	227	

(a) Number of F_2^p data points.

Experiment	x	Q^2 , GeV ²	F_2^d	$F_2^d \text{ cuts}$	\mathcal{N}
BCDMS (120)	0.35 – 0.75	13.25 – 99.00	59	32	1.001
BCDMS (200)	0.35 – 0.75	32.50 – 137.50	50	28	0.998
BCDMS (280)	0.35 – 0.75	43.00 – 230.00	49	26	1.003
NMC (comb)	0.35 – 0.50	7.00 – 65.00	15	14	1.000
SLAC (comb)	0.30 – 0.62	10.00 – 21.40	59	59	0.990
deuteron			232	159	

(b) Number of F_2^d data points.

Experiment	x	Q^2 , GeV ²	F_2^{NS}	$F_2^{NS} \text{ cuts}$	\mathcal{N}
BCDMS (120)	0.070 – 0.275	8.75 – 43.00	36	30	0.983
BCDMS (200)	0.070 – 0.275	17.00 – 75.00	29	28	0.999
BCDMS (280)	0.100 – 0.275	32.50 – 115.50	27	26	0.997
NMC (comb)	0.013 – 0.275	4.50 – 65.00	88	53	1.000
SLAC (comb)	0.153 – 0.293	4.18 – 5.50	28	28	0.994
nonsinglet			208	165	

(c) Number of F_2^{NS} data points.

TABLE I: Number of experimental data points (a) F_2^p , (b) F_2^d , and (c) F_2^{NS} for the nonsinglet QCD analysis with their x and Q^2 ranges. The name of different data set and range of x and Q^2 are given in the three first columns. The fourth column (F_2) contains the number of data points according to the cuts: $Q^2 > 4$ GeV², $W^2 > 12.5$ GeV², $x > 0.3$ for F_2^p and F_2^d and $x < 0.3$ for F_2^{NS} . The reduction of the number of data points by the additional cuts (see text) are given in the 5th column ($F_2 \text{ cuts}$). The normalization shifts are listed in the last column.

In the QCD analysis of the present paper we used three data sets: the structure functions $F_2^p(x, Q^2)$ and $F_2^d(x, Q^2)$ in the region of $x \geq 0.3$ and the combination of these structure functions $F_2^{\text{NS}}(x, Q^2)$ in the region of $x < 0.3$. Notice that we take into account the cuts $Q^2 > 4 \text{ GeV}^2$, $W^2 > 12.5 \text{ GeV}^2$ for our QCD fits to determine some unknown parameters.

$$W^2 \equiv \left(\frac{1}{x} - 1\right) Q^2 + m_{\text{N}}^2 > 12.5 \text{ GeV}^2$$

$$\chi_{\text{global}}^2 = \sum_n w_n \chi_n^2, \quad (n \text{ labels the different experiments})$$

$$\chi_n^2 = \left(\frac{1 - \mathcal{N}_n}{\Delta \mathcal{N}_n} \right)^2 + \sum_i \left(\frac{\mathcal{N}_n F_{2,i}^{\text{data}} - F_{2,i}^{\text{theor}}}{\mathcal{N}_n \Delta F_{2,i}^{\text{data}}} \right)^2 .$$

$$\sigma(f_q(x))^2 = \sum_{i=1}^{n_p} \sum_{j=1}^{n_p} \left(\frac{\partial f_q}{\partial p_i} \right) \left(\frac{\partial f_q}{\partial p_j} \right) \text{cov}(p_i, p_j) ,$$

$$\begin{aligned}
F_{2,\text{TMC}}^k(n, Q^2) &\equiv \int_0^1 x^{n-2} F_{2,\text{TMC}}^k(x, Q^2) dx \\
&= F_2^k(n, Q^2) + \frac{n(n-1)}{n+2} \left(\frac{m_N^2}{Q^2} \right) F_2^k(n+2, Q^2) \\
&\quad + \frac{(n+2)(n+1)n(n-1)}{2(n+4)(n+3)} \left(\frac{m_N^2}{Q^2} \right)^2 F_2^k(n+4, Q^2) + \mathcal{O} \left(\frac{m_N^2}{Q^2} \right)^3
\end{aligned}$$

$$F_2^{N_{max},k}(x, Q^2) = x^\beta (1-x)^\alpha \sum_{n=0}^{N_{max}} \Theta_n^{\alpha,\beta}(x) \times \sum_{j=0}^n c_j^{(n)}(\alpha, \beta) F_{2,\text{TMC}}^k(j+2, Q^2) ,$$

$$F_2^{\text{exp}}(x, Q^2) = O_{\text{TMC}}[F_2^{\text{HT}}(x, Q^2)] \cdot \left(1 + \frac{h(x, Q^2)}{Q^2 [\text{GeV}^2]} \right) ,$$

$$h(x) = a \left(\frac{x^b}{1-x} - c \right) .$$