Intro to SUSY II: SUSY QFT

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PRE-SUSY 2016 SCHOOL
27 JUNE -1 JULY 2016, MELBOURNE

Recap from the first lecture:

- N=1 SUSY algebra $P_\mu,\ M_{\mu\nu},\ Q_\alpha,\ \bar{Q}^{\dot\alpha},\ R$

$$[P_\mu, P_\nu] = 0 \ ;$$
$$[P_\mu, M_{\rho\sigma}] = -i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho) \ ;$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})$$

$$[P_\mu, Q_\alpha] = [P_\mu, \bar{Q}^{\dot\alpha}] = 0,$$
$$[M_{\mu\nu}, Q_\alpha] = -(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta,\ [M_{\mu\nu}, \bar{Q}^{\dot\alpha}] = -(\bar\sigma^{\mu\nu})_{\dot\beta}{}^{\dot\alpha}\bar{Q}^{\dot\beta},$$
$$\{Q_\alpha, \bar{Q}^{\dot\beta}\} = 2(\sigma^\mu)_\alpha{}^{\dot\alpha}P_\mu,$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} = 0,$$
$$[Q_\alpha, R] = Q_\alpha,\ [\bar{Q}^{\dot\alpha}, R] = -\bar{Q}^{\dot\alpha}\ .$$

$$Q_\alpha = -i\partial_\alpha - (\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu$$
$$\bar{Q}^{\dot\alpha} = i\bar\partial^{\dot\alpha} + \theta^\beta(\sigma^\mu)_\beta{}^{\dot\alpha}\partial_\mu$$

Recap from the first lecture:

- 8-dimensional N=1 superspace $X^M = (x^\mu,\ \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$

$$dS^2 = G_{MN} dX^M dX^N$$

$$G_{MN} = (\eta_{\mu\nu}, \epsilon^{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}})$$

- Covariant derivatives $D_M = (\partial_\mu,\ D_\alpha,\ \bar{D}_{\dot{\alpha}})$

$$D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \ ,$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu \ ,$$

- Non-zero torsion $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i(\sigma)_{\alpha\dot{\alpha}}\partial_\mu$

Recap from the first lecture:

- Superspace integration $\int d\theta\theta = 1$ and $\int d\theta = 0.$

$\int d\theta \frac{df}{d\theta} = 0$; $\delta(\theta) = \theta$ - Grassmann delta-function;

$\int d\theta f(\theta) = f_1 = \frac{d}{d\theta} f(\theta)$

Grassmann integration is equivalent to differentiation

$$d^2\theta = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta},\ d^2\bar{\theta} = -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}},\ d^4\theta = d^2\theta d^2\bar{\theta}$$

$$\int d^2\theta\ \theta^2 = \int d^2\bar{\theta}\ \bar{\theta}^2 = \int d^4\theta\ \bar{\theta}^2\theta^2 = 1$$

Outline of Part II: SUSY QFT

- Basic consequences of superalgebra
- Superfields
  - Chiral superfield
  - Vector superfield. Super-gauge invariance
- Nonrenormalisation theorems

Basic consequences of the superalgebra

i. Inspect,

$$[P_\mu, Q_\alpha] = \left[P_\mu, \bar{Q}^{\dot{\alpha}}\right] = 0$$

$$\Longrightarrow \quad [P^\mu P_\mu, Q_\alpha] = \left[P^\mu P_\mu, \bar{Q}^{\dot{\alpha}}\right] = 0$$

P^2 is a quadratic Casimir operator of the super-Poincaré algebra with eigenvalues m^2.

That is, each irreducible representation of superalgebra contains fields that are degenerate in mass.

Basic consequences of the superalgebra

ii. Inspect, $\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\} = 2(\sigma^\mu)_\alpha{}^{\dot{\alpha}} P_\mu.$

Since P_μ is an invertible operator, so is $\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\}$.

Then, the action

$$\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\}|B\rangle = Q_\alpha\bar{Q}^{\dot{\alpha}}|B\rangle + \bar{Q}^{\dot{\alpha}}Q_\alpha|B\rangle = Q_\alpha|\bar{F}\rangle + Q^{\dot{\alpha}}|F\rangle = |B'\rangle$$

implies one-to-one correspondence between fermionic and bosonic states. Thus, each irreducible representation of superalgebra contains equal number of fermionic and bosonic states.

Basic consequences of the superalgebra

iii. Take sum over spinor indices in

$$\{Q_\alpha, Q^*_{\dot\alpha}\} = 2(\sigma^\mu)_\alpha{}^{\dot\alpha} P_\mu .$$

$$P_0 \equiv H = \frac{1}{4}\left[Q_1 Q_1^* + Q_2 Q_2^* + Q_1^* Q_1 + Q_2^* Q_2\right],$$

$$H|E\rangle = E|E\rangle \ .$$

Basic consequences of the superalgebra

iiia. Total energy of an arbitrary supersymmetric system is positive definite:

$$E \geqslant 0$$

iiib The vacuum energy of a supersymmetric system is 0,

$$E_{\mathrm{vac}} = 0 \quad [Q_\alpha|\mathrm{vac}\rangle = 0]$$

iiic Supersymmetry is broken if

$$E_{\mathrm{vac}} \neq 0 \quad [Q_\alpha|\mathrm{vac}\rangle \neq 0]$$

Basic consequences of the superalgebra

- Compute 1-loop vacuum energy of a system of particles with various spins S and corresponding masses m_S:

$$E_{\text{vac}} = \frac{1}{2}\sum_{(S)}(-1)^{2S}(2S+1)\int d^3\vec{q}\sqrt{\vec{q}^2+m_S^2} =$$
$$\frac{1}{2}\sum_{(S)}(-1)^{2S}(2S+1)\int d^3\vec{q}\sqrt{\vec{q}^2}\left[1+\frac{m_S^2}{2\vec{q}^2}-\frac{m_S^4}{\vec{q}^4}+\ldots\right]$$

$$\mathrm{Str}\, m_S^0 \equiv \sum_{(S)}(-1)^{2S}(2S+1) = 0 \;\; \text{(no quart. div.)}$$

$$\mathrm{Str}\, m_S^2 \equiv \sum_{(S)}(-1)^{2S}(2S+1)m_S^2 = 0 \;\; \text{(no quadr. div.)}$$

$$\mathrm{Str}\, m_S^4 \equiv \sum_{(S)}(-1)^{2S}(2S+1)m_S^4 = 0 \;\; \text{(no log. div.)}$$

Basic consequences of the superalgebra

- Compute 1-loop correction to the mass of a scalar field (SUSY adjustment of couplings is assumed):

$$\delta m_H^2 \;\propto\; \sum_{(S)}(-1)^2(2S\,+\,1)\int d^4p_E\frac{1}{p_E^2+m_S^2} \;=$$

$$\sum_{(S)}(-1)^2(2S+1)\int d^4p_E\frac{1}{p_E^2}\left[1-\frac{m_S^2}{p_E^2}+...\right]$$

$$\mathrm{Str}m_S^0\equiv\sum_{(S)}(-1)^{2S}(2S+1)=0\ \text{ (no quadr. div.)}$$

$$\mathrm{Str}m_S^2\equiv\sum_{(S)}(-1)^{2S}(2S+1)m_S^2=0\ \text{ (no log. div.)}$$

Basic consequences of the superalgebra

- Cancellation of divergences follow automatically from superalgebra! The above examples follow from the general all-loop perturbative non-renormalization theorem in quantum field theories with supersymmetry.

- Absence of quadratic divergences in scalar masses is the main phenomenological motivation for supersymmetry: supersymmetry may ensure stability of the electroweak scale against quantum corrections (the hierarchy problem)

Superfields

- Superfield is a function of superspace coordinates. A generic scalar superfield can be expanded in a form of a Taylor series expansion with respect to Grassmannian coordinates:

$$S(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi_1(x) + \bar{\theta}\bar{\psi}_2(x) + \theta^2 M(x) + \bar{\theta}^2 N(x)$$
$$\theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta^2\bar{\theta}\bar{\lambda}_1(x) + \bar{\theta}^2\theta\lambda_2(x) + \theta^2\bar{\theta}^2 D(x)$$

- This generic scalar superfield is in a reducible SUSY representation. We may impose covariant constraints to obtain irreducible superfields.

Chiral superfield

- Consider, e.g., the following covariant condition:

$$\bar{D}_{\dot{\alpha}} S(x, \theta, \bar{\theta}) = 0.$$

- The solution to the above constraint is known as the (left-handed) chiral superfield.

Chiral superfield

- To solve the constraint let us introduce new bosonic coordinates:

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

- One verifies that

$$\bar{D}_{\dot{\alpha}} y^\mu = \left(\bar{\partial}_{\dot{\alpha}} + i\theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu\right)(x^\mu + i\theta\sigma^\mu\bar{\theta}) = 0$$

Exercise: Check this.

Chiral superfield

- Therefore, the solution is:

$$S(x,\theta,\bar{\theta}) = \Phi(y,\theta)$$

- Taylor expansion of the chiral superfield reads:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$$
$$= \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x)$$
$$+ i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu\phi(x)$$

Exercise: Obtain this expansion

Chiral superfield

- Supersymmetry transformations:

$$\delta\Phi(y,\theta) = \left(i\epsilon Q + i\bar{\epsilon}\bar{Q}\right)\Phi(y,\theta) =$$
$$\sqrt{2}\epsilon\psi(x) + \sqrt{2}\theta\left(\epsilon F(x) + i\sigma^{\mu}\bar{\epsilon}\partial_{\mu}\phi(x)\right) - i\sqrt{2}\theta^{2}(\partial_{\mu}\psi(x))\sigma^{\mu}\bar{\epsilon}$$

Exercise: Verify this

- Thus, we have:

$$\delta\phi(x) = \sqrt{2}\epsilon\psi(x)$$
$$\delta\psi(x) = \sqrt{2}\left(\epsilon F(x) + i\sigma^{\mu}\bar{\epsilon}\partial_{\mu}\phi(x)\right)$$
$$\delta F(x) = -i\sqrt{2}(\partial_{\mu}\psi(x))\sigma^{\mu}\bar{\epsilon}$$

Anti-chiral superfield

- In similar manner, we can define anti-chiral superfield through the constraint:

$$D_\alpha S(x, \theta, \bar{\theta}) = 0$$

which posses the solution

$$S(x, \theta, \bar{\theta}) = \Phi^{+}(y, \theta) = \Phi(y^{+}, \bar{\theta})$$

$$\text{where } y^{+} = (y)^{+} = x^\mu - i\theta\sigma^\mu\bar{\theta}.$$

Chiral field Lagrangian - Superpotential

- Superpotential $W(\Phi)$ $[W(\Phi^{+})]$ is a holomorphic function of a chiral (anti-chiral) superfields
- Superpotential itself is a (composite) chiral (anti-chiral) superfield:

$$\bar{D}_{\dot{\alpha}} W(\Phi) = \frac{\partial W}{\partial \Phi} \bar{D}_{\dot{\alpha}} \Phi = 0$$

- Consider, $\int d\theta^2 \ W = W|_{\theta^2} + \text{total derivatives}$

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y)$$
$$= \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x)$$
$$+ i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi(x) + \frac{i}{\sqrt{2}}\theta^2\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\partial_{\mu}\partial^{\mu}\phi(x)$$

Chiral field Lagrangian - Superpotential

- Since F-terms $W(\Phi)\big|_{\theta^2}$, $W(\Phi^+)\big|_{\bar{\theta}^2}$ transform as total derivatives under SUSY transformations,

$$\int d\theta^2 \; W(\Phi) + \int d\bar{\theta}^2 \; W(\Phi^+)$$

is SUSY invariant Lagrangian density!

Chiral field Lagrangian – Kähler potential

- Consider product of chiral and anti-chiral superfields,

$$\Phi^{+}\Phi$$

which is a generic scalar superfield with the reality condition imposed.

$$S(x,\theta,\bar{\theta}) = \phi(x) + \theta\psi_1(x) + \bar{\theta}\bar{\psi}_2(x) + \theta^2 M(x) + \bar{\theta}^2 N(x)$$
$$\theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta^2\bar{\theta}\bar{\lambda}_1(x) + \bar{\theta}^2\theta\lambda_2(x) + \theta^2\bar{\theta}^2 D(x)$$

- SUSY transformation of D-term

$$\delta D = \left(i\epsilon Q + i\bar{\epsilon}\bar{Q}\right) S\Big|_{\theta^2\bar{\theta}^2} = \text{total derivative}$$

Chiral field Lagrangian - Kähler potential

- Hence,

$$\int d\theta^2 d\bar{\theta}^2 \; K(\Phi^+\Phi)$$

is SUSY invariant Lagrangian density!

- $K(\Phi^+\Phi)$ Kähler potential

Wess-Zumino model

- A chiral superfield, which contains a complex scalar (2 dofs both on-shell and off-shell), Majorana fermion (2 dofs on-shell, 4-dofs off-shell), a complex auxiliary field (0 dof on-shell, 2 dofs off-shell)

$$\mathcal{L}_{WZ} = \int d\theta^2 d\bar{\theta}^2 \ \Phi^+\Phi + \int d\theta^2 \ W(\Phi) + h.c.$$

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$$

Exercise: Express WZ Lagrangian in component form

J. Wess and B. Zumino, "Supergauge transformations in four Dimensions", Nuclear Physics B 70 (1974) 39

Vector (real) superfield

$$S(x,\theta,\bar\theta) = \phi(x) + \theta\psi_1(x) + \bar\theta\bar\psi_2(x) + \theta^2 M(x) + \bar\theta^2 N(x)$$
$$\theta\sigma^\mu\bar\theta A_\mu(x) + \theta^2\bar\theta\bar\lambda_1(x) + \bar\theta^2\theta\lambda_2(x) + \theta^2\bar\theta^2 D(x)$$

- Reality condition: $S^+(x,\theta,\bar\theta) = S(x,\theta,\bar\theta)$

- Solution is:

$$V(x,\theta,\bar\theta) = \phi(x) + \theta\psi + \bar\theta\bar\psi + \theta^2 M(x) + \bar\theta^2 M^*(x)$$
$$+ \theta\sigma^\mu\bar\theta A_\mu(x) + \theta^2\bar\theta\bar\lambda + \bar\theta^2\theta\lambda + \theta^2\bar\theta^2 D(x)$$

Vector (real) superfield

- Let's use the vector superfield to describe a SUSY gauge theory, e.g. super-QED
- Supergauge transformations

$$V' = V + i(\Lambda^+ - \Lambda)$$

- Arbitrary chiral superfield –

$$\Lambda = \alpha(x) + \sqrt{2}\theta\xi(x) + \theta^2 f(x)$$
$$+ i\theta\sigma^\mu\bar{\theta}\partial_\mu\alpha(x) + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\xi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu\alpha(x)$$

- Standard gauge transformations

$$A'_\mu = A_\mu + \partial_\mu(\alpha + \alpha^*)$$

Vector (real) superfield

- We can fix $\alpha(x),\ \xi(x), f(x)$ to remove $\phi(x),\ \psi(x),\ M(x)$ (Wess-Zumino gauge)

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + \theta^2\bar{\theta}\bar{\lambda} + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}^2 D(x)$$

- Note, in the Wess-Zumino gauge SUSY is not manifest.

- We can generalize this construction to super-Yang-Mills:

$$V \to V^a T^a$$

Strength tensor superfield

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V, \ \bar{W}_{\dot{\alpha}} = -\frac{1}{4}D^2 \bar{D}_{\dot{\alpha}} V$$

Exercise: Prove that these are chiral and anti-chiral superfields, respectively.

- In the WZ gauge:

$$W_\alpha = \lambda_\alpha + \theta_\alpha D + \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha F_{\mu\nu} + i\theta^2 (\sigma^\mu \partial_\mu \bar{\lambda})_\alpha$$

Super-QED

$$\mathcal{L}_{SQED} = \int d\theta^2 \ \frac{1}{4} W^{\alpha} W_{\alpha} + h.c.$$

Exercise: Rewrite this Lagrangian in component form.

- Matter couplings:

$$\Phi' = \mathrm{e}^{ig\Lambda}\Phi, \ \ \Phi^{+\prime} = \Phi^{+}\mathrm{e}^{-ig\Lambda^{+}}$$

- Gauge and SUSY invariant 'kinetic' term

$$\int d\theta^2 d\bar{\theta}^2 \ \Phi^{+}\mathrm{e}^{gV}\Phi$$

Nonrenormalisation theorems

M.T. Grisaru, W. Siegel and M. Rocek, ``Improved Methods for Supergraphs,” Nucl. Phys. B159 (1979) 429

$$\mathcal{L} = K\left[\Phi^{+} \mathrm{e}^{gV} \Phi\right]\Big|_{\theta^2\bar{\theta}^2}$$
$$+\, W(\Phi)|_{\theta^2} + h.c.$$
$$+\, f(\Phi) W^{\alpha} W_{\alpha}|_{\theta^2} + h.c.$$

- Kahler potential $K\left[\Phi^{+} \mathrm{e}^{gV} \Phi\right]$ receives corrections order by order in perturbation theory

- Only 1-loop corrections for $f(\Phi)$

- $W(\Phi)$ is not renormalised in the perturbation theory!

Nonrenormalisation theorems

N. Seiberg, ``Naturalness versus supersymmetric nonrenormalization theorems,” Phys. Lett. B318 (1993) 469

- Consider just Wess-Zumino model:

$$W(\Phi) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$$

- R-symmetry and U(1) charges:

‘field’	Φ	m	λ
U(1)	1	–2	–3
$U(1)_R$	1	0	–1

Nonrenormalisation theorems

N. Seiberg, ``Naturalness versus supersymmetric nonrenormalization theorems," Phys. Lett. B318 (1993) 469

- Quantum corrected superpotential:

$$W_{eff}(\Phi) = m\Phi^2 f\left(\frac{\lambda\Phi}{m}\right) = \sum_{n\geq 0} c_n \lambda^n m^{1-n}\Phi^{n+2}$$

- Consider $\lambda \to 0 \to n \geq 0$
- Consider $m \to 0 \to n \leq 1$

- Hence, $W_{eff}(\Phi) = W(\Phi)$

Summary of Part II

- Basic consequences of SUSY algebra
- Chiral superfield. Superpotential and Kahler potential. Wess-Zumino model
- Vector superfield. Wess-Zumino gauge. Super-QED and matter coupling
- Nonrenormalisation theorems